A new proof of the sharpness of the phase transition for Bernoulli percolation on $\mathbb{Z}^d$

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Abstract

We provide a new proof of the sharpness of the phase transition for nearest-neighbour Bernoulli percolation on $\mathbb{Z}^d$. More precisely, we show that

- for $p < p_c$, the probability that the origin is connected by an open path to distance n decays exponentially fast in n.
- for $p > p_c$, the probability that the origin belongs to an infinite cluster satisfies the mean-field lower bound $\theta(p) \geq \frac{p-p_c}{p(1-p_c)}$.

In [DT15], we give a more general proof which covers long-range Bernoulli percolation (and the Ising model) on arbitrary transitive graphs. This article presents the argument of [DT15] in the simpler framework of nearest-neighbour Bernoulli percolation on $\mathbb{Z}^d$.

1 Statement of the result

Motivation. Bernoulli percolation was introduced by Broadbent and Hammersley [BH57] as a model for liquid poured in a porous medium. Since then, Bernoulli percolation has found many applications in statistical physics and beyond, and it has been one of the most studied models of random graph.

In this model, each edge of the lattice $\mathbb{Z}^d$ is open with probability p, and closed with probability $1-p$, thus giving us a random graph ω_p given by the vertices of $\mathbb{Z}^d$ and the open edges. Of special interest for mathematicians and physicists are the connectivity properties of ω_p. For $d \geq 2$, one may show that there exists a critical parameter $p_c = p_c(d) \in (0,1)$ separating a supercritical phase $p > p_c$ where ω_p almost surely contains an infinite connected component from a subcritical phase $p < p_c$ where the connected components of ω_p are almost surely finite.

The definition of the subcritical phase implies that for $p < p_c$, the probability of 0 being connected to distance n by edges in ω_p decays to 0. This result was refined in the following way: [Men86] and [AB87] proved that the probability is in fact smaller than $\exp(-cn)$ where $c = c(p) > 0$. This result, sometimes referred to as the sharpness of the phase transition, is a milestone in the area (many of the delicate properties of the subcritical phase are based on this property). In this paper, we provide an alternative (short) proof of this result.

Notation. Fix an integer $d \geq 2$. We consider the d-dimensional hypercubic lattice $(\mathbb{Z}^d, \mathbb{E}^d)$. Let $\Lambda_n = \{-n, \ldots, n\}^d$, and let $\partial\Lambda_n := \Lambda_n \setminus \Lambda_{n-1}$ be its vertex-boundary. Throughout this paper, S *always stands for a finite set of vertices containing the origin.* Given such a set, we denote its edge-boundary by ΔS, defined by all the edges $\{x, y\}$ with $x \in S$ and $y \notin S$.

Consider the Bernoulli bond percolation measure $\mathbb{P}_p$ on $\{0,1\}^{\mathbb{E}^d}$ for which each edge of $\mathbb{E}^d$ is declared *open* with probability p and closed otherwise, independently for different edges.

Two vertices x and y are *connected in* $S \subset V$ if there exists a path of vertices $(v_k)_{0\leq k\leq K}$ in S such that $v_0 = x$, $v_K = y$, and $\{v_k, v_{k+1}\}$ is open for every $0 \leq k < K$. We denote this event by $x \overset{S}{\longleftrightarrow} y$. If $S = \mathbb{Z}^d$, we drop it from the notation. We set $0 \longleftrightarrow \partial\Lambda_n$ if 0 is connected to a vertex in $\partial\Lambda_n$, and $0 \longleftrightarrow \infty$ if $0 \longleftrightarrow \partial\Lambda_n$ holds for all n.

Phase transition. The critical parameter for Bernoulli percolation is usually defined by

$$p_c := \sup\{p \in [0,1] \text{ s.t. } \mathbb{P}_p[0 \longleftrightarrow \infty] = 0\}.$$

A new idea of this paper is to use a different definition of the critical parameter. This new definition relies on the following quantity. For $p \in [0,1]$ and $0 \in S \subset \mathbb{Z}^d$, define

$$\varphi_p(S) := p \sum_{\{x,y\}\in\Delta S} \mathbb{P}_p[0 \overset{S}{\longleftrightarrow} x].$$

This can be interpreted as the expected number of open edges at the boundary of S, that are connected to 0 in S. Based on this new quantity, we introduce:

$$\tilde{p}_c := \sup\Big\{p \in [0,1] \text{ s.t. there exists a finite set } 0 \subset S \subset \mathbb{Z}^d \text{ with } \varphi_p(S) < 1\Big\}.$$

We are now in a position to state our main result.

Theorem 1.1. *For any $d \geq 1$, $\tilde{p}_c = p_c$. Furthermore,*

1. *For $p < p_c$, there exists $c = c(p) > 0$ such that for every $n \geq 1$,*

$$\mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_n] \leq e^{-cn}.$$

2. For $p > p_c$,

$$\mathbb{P}_p[0 \longleftrightarrow \infty] \geq \frac{p - p_c}{p(1 - p_c)}.$$

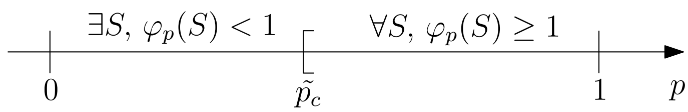
Remarks.

1. On $\mathbb{Z}$, we easily find that $p_c = \tilde{p}_c = 1$, and Item 2 is then irrelevant.

2. We refer to [DT15] for a detailed bibliography, and for a version of the proof valid in greater generality. The aim of this paper is to provide a proof in the simplest possible framework.

3. Theorem 1.1 was proved by Aizenman and Barsky [AB87] in the more general framework of long-range percolation. In their proof, they consider an additional parameter h corresponding to an external field, and they derive the results from differential inequalities satisfied by the thermodynamical quantities of the model. A different proof, based on the geometric study of the pivotal edges, was obtained at the same time by Menshikov [Men86]. These two proofs are also presented in [Gri99].

4. In the definition of $\tilde{p}_c$, the set of parameters p such that there exists a finite set $0 \subset S \subset \mathbb{Z}^d$ with $\varphi_p(S) < 1$ is an open subset of $[0,1]$. Thus, $\tilde{p}_c$ do not belong to this set, as illustrated below.

   Therefore, we obtain that the expected size of the cluster of the origin satisfies for every $p \geq p_c$,

   $$\sum_{x \in \mathbb{Z}^d} \mathbb{P}_p[0 \longleftrightarrow x] \geq \frac{1}{dp} \sum_{n \geq 0} \varphi_p(\Lambda_n) = +\infty.$$

   This result was originally proved in [AN84].

5. Since $\varphi_p(\{0\}) = 2dp$, we obtain $p_c \geq 1/2d$.

6. Item 2 is called the mean-field lower bound for the infinite cluster density. This is due to the fact that $\theta(p) \asymp p - p_c$ holds as $p \searrow p_c$ for Bernoulli percolation on a regular tree. This mean-field behavior is expected to hold for Bernoulli percolation on $\mathbb{Z}^d$ when $d \geq 6$ (it is proved for $d \geq 19$ [HS90]).

7. On the square lattice, the inequality $p_c \geq 1/2$ was first obtained by Harris in [Har60] (see also the short proof of Zhang presented in [Gri99]). The other inequality $p_c \leq 1/2$ was first proved by Kesten in [Kes80] using a delicate geometric construction involving crossing events. Since then, many other proofs invoking exponential decay in the subcritical phase (see [Gri99]) or sharp threshold arguments (see e.g. [BR06]) have been found. Here, Theorem 1.1 provides a short proof of exponential decay and therefore a short alternative to these proofs. For completeness, let us sketch how exponential decay implies that $p_c \leq 1/2$: Item 1 implies that for $p < p_c$ the crossing probability for a n by n square tends to 0 as n goes to infinity. But self-duality implies that this does not happen when $p = 1/2$, thus implying that $p_c \leq 1/2$.

2 Proof of the theorem

It is sufficient to show Items 1 and 2 with p_c replaced by $\tilde{p}_c$ (since it immediately implies the equality $p_c = \tilde{p}_c$).

2.1 Proof of Item 1

The proof of Item 1 (with $\tilde{p}_c$ in place of p_c) can be derived from the BK-inequality [vdBK85]. We present here an exploration argument, similar to the one in [Ham57], which does not rely on the BK-inequality. Let $p < \tilde{p}_c$. By definition, one can fix a finite set S containing the origin, such that $\varphi_p(S) < 1$. Let $L > 0$ such that $S \subset \Lambda_{L-1}$.

Let $k \geq 1$ and assume that the event $0 \longleftrightarrow \partial\Lambda_{kL}$ holds. Let

$$\mathscr{C} = \{z \in S \text{ s.t. } 0 \overset{S}{\longleftrightarrow} z\}.$$

Since $S \cap \partial\Lambda_{kL} = \emptyset$, there exists an edge $\{x,y\} \in \Delta S$ such that the following events occur:

- 0 is connected to x in S,
- $\{x,y\}$ is open,
- y is connected to $\partial\Lambda_{kL}$ in $\mathscr{C}^c$.

Using first the union bound, and then a decomposition with respect to possible values of $\mathscr{C}$, we find

$$\begin{aligned}
&\mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_{kL}] \\
&\leq \sum_{\{x,y\}\in\Delta S} \sum_{C\subset S} \mathbb{P}_p\big[\{0 \overset{S}{\longleftrightarrow} x, \mathscr{C} = C\} \cap \{\{x,y\} \text{ is open}\} \cap \{y \overset{\mathbb{Z}^d\setminus C}{\longleftrightarrow} \partial\Lambda_{kL}\}\big] \\
&= p \sum_{\{x,y\}\in\Delta S} \sum_{C\subset S} \mathbb{P}_p\big[0 \overset{S}{\longleftrightarrow} x, \mathscr{C} = C\big]\mathbb{P}_p\big[y \overset{\mathbb{Z}^d\setminus C}{\longleftrightarrow} \partial\Lambda_{kL}\big].
\end{aligned}$$

In the second line, we used the fact that the three events depend on different sets of edges: the first event $\{0 \overset{S}{\longleftrightarrow} x, \mathscr{C} = C\}$ depends on edges between a vertex of C and one of S (which may be in C too), the second on the state of $\{x,y\}$ only and the third on the state of edges not sharing any endpoint with C (this excludes the edge $\{x,y\}$ or the edges involved in the first event). As a consequence, these three events are independent. Since $y \in \Lambda_L$, one can bound $\mathbb{P}_p[y \overset{\mathbb{Z}^d \setminus C}{\longleftrightarrow} \partial\Lambda_{kL}]$ by $\mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_{(k-1)L}]$ in the last expression. Hence, we find

$$\mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_{kL}] \le \varphi_p(S)\mathbb{P}_p\big[0 \longleftrightarrow \partial\Lambda_{(k-1)L}\big]$$

which by induction gives

$$\mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_{kL}] \le \varphi_p(S)^{k-1}.$$

This proves the desired exponential decay.

2.2 Proof of Item 2

Let us start by the following lemma providing a differential inequality valid for every p.

Lemma 2.1. *Let $p \in [0,1]$ and $n \ge 1$,*

$$\frac{d}{dp}\mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_n] \ge \frac{1}{p(1-p)} \cdot \inf_{\substack{S \subset \Lambda_n \\ 0 \in S}} \varphi_p(S) \cdot \big(1 - \mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_n]\big). \tag{2.1}$$

Let us first see how it implies Item 2 of Theorem 1.1. Let $n \ge 1$ and set $f(p) = \mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_n]$. For $p \in (\tilde{p}_c, 1)$, $\varphi_p(S) \ge 1$ for every $S \ni 0$ so that the differential inequality (2.1) becomes

$$\frac{f'(p)}{1 - f(p)} \ge \frac{1}{p(1-p)}.$$

Integrating this inequality between $\tilde{p}_c$ and $p > \tilde{p}_c$ gives

$$\frac{1 - f(\tilde{p}_c)}{1 - f(p)} \ge \frac{p}{1-p} \cdot \frac{1 - \tilde{p}_c}{\tilde{p}_c}.$$

Using the trivial bound $f(\tilde{p}_c) \ge 0$, we find

$$\mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_n] = f(p) \ge 1 - \frac{\tilde{p}_c(1-p)}{p(1-\tilde{p}_c)} = \frac{p - \tilde{p}_c}{p(1-\tilde{p}_c)}.$$

By letting n tend to infinity, we obtain the desired bound on $\mathbb{P}_p[0 \longleftrightarrow \infty]$.

Proof of Lemma 2.1. Recall that $\{x,y\}$ is pivotal for the configuration ω and the event $\{0 \longleftrightarrow \partial\Lambda_n\}$ if $\omega_{\{x,y\}} \notin \{0 \longleftrightarrow \partial\Lambda_n\}$ and $\omega^{\{x,y\}} \in \{0 \longleftrightarrow \partial\Lambda_n\}$. (The configuration $\omega_{\{x,y\}}$, resp. $\omega^{\{x,y\}}$, coincides with ω except that the edge $\{x,y\}$ is closed, resp. open.) By Russo's formula (see [Gri99, Section 2.4]), we have

$$\begin{aligned}\frac{d}{dp}\mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_n] &= \sum_{e\subset\Lambda_n} \mathbb{P}_p\big[e \text{ is pivotal}\big] \\ &= \frac{1}{1-p}\sum_{e\subset\Lambda_n} \mathbb{P}_p\big[e \text{ is pivotal}, 0 \not\longleftrightarrow \partial\Lambda_n\big].\end{aligned}$$

Define the following random subset of Λ_n:

$$\mathscr{S} := \{x \in \Lambda_n \text{ s.t. } x \not\longleftrightarrow \partial\Lambda_n\}.$$

The boundary of $\mathscr{S}$ corresponds to the outmost blocking surface (which can be obtained by exploring from the outside the set of vertices connected to the boundary). When 0 is not connected to $\partial\Lambda_n$, the set $\mathscr{S}$ is always a subset of Λ_n containing the origin. By summing over the possible values for $\mathscr{S}$, we obtain

$$\frac{d}{dp}\mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_n] = \frac{1}{1-p}\sum_{\substack{S\subset\Lambda_n\\ 0\in S}}\sum_{e\subset\Lambda_n} \mathbb{P}_p\big[e \text{ is pivotal}, \mathscr{S} = S\big].$$

Observe that on the event $\mathscr{S} = S$, the pivotal edges are the edges $\{x,y\} \in \Delta S$ such that 0 is connected to x in S. This implies that

$$\frac{d}{dp}\mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_n] = \frac{1}{1-p}\sum_{\substack{S\subset\Lambda_n\\ 0\in S}}\sum_{\{x,y\}\in\Delta S} \mathbb{P}_p\big[0 \overset{S}{\longleftrightarrow} x, \mathscr{S} = S\big].$$

The event $\{\mathscr{S} = S\}$ is measurable with respect to the state of edges having one endpoint in $\mathbb{Z}^d \setminus S$, while $\{0 \overset{S}{\longleftrightarrow} x\}$ depends by definition on edges with both endpoints in S. As a consequence, they are independent. We obtain

$$\begin{aligned}\frac{d}{dp}\mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_n] &= \frac{1}{1-p}\sum_{\substack{S\subset\Lambda_n\\ 0\in S}}\sum_{\{x,y\}\in\Delta S} \mathbb{P}_p\big[0 \overset{S}{\longleftrightarrow} x\big]\mathbb{P}_p\big[\mathscr{S} = S\big] \\ &= \frac{1}{p(1-p)}\sum_{\substack{S\subset\Lambda_n\\ 0\in S}} \varphi_p(S)\mathbb{P}_p\big[\mathscr{S} = S\big] \\ &\geq \frac{1}{p(1-p)}\inf_{\substack{S\subset\Lambda_n\\ 0\in S}} \varphi_p(S)\cdot\mathbb{P}_p\big[0 \not\longleftrightarrow \partial\Lambda_n\big],\end{aligned}$$

as desired. □

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