ETH zürich

On the real spectrum compactification of Hitchin components

Doctoral defense

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ETH Zürich

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Introduction

What is geometry?

γεωμετρία (geōmetría) = land measurement

Branch of mathematics that studies properties of space such as

distance, angle, shape, size, relative position, ...

Figure 1: Euclid (300 BC), Gauss (1777–1855), Riemann (1826–1866) from left to right

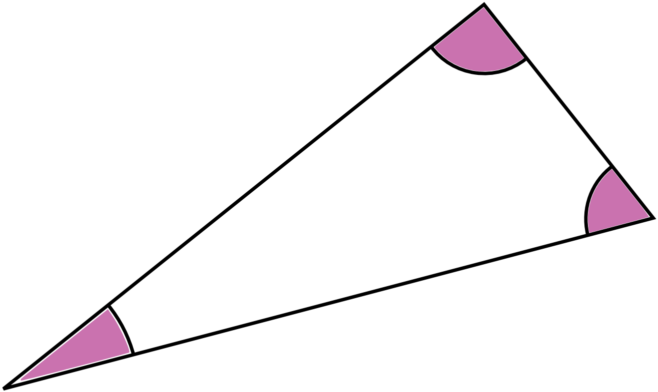


Figure 2: The angle sum is 180°.

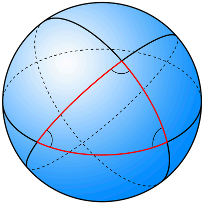


Figure 3: [1]The angle sum is $> 180°$.

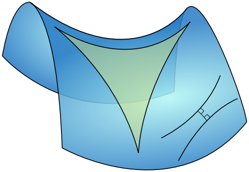


Figure 4: [2]The angle sum is $< 180°$.

[1]Source: https://www.mezzacotta.net/100proofs/archives/450
[2]Source: Wikipedia

Surfaces and geometry

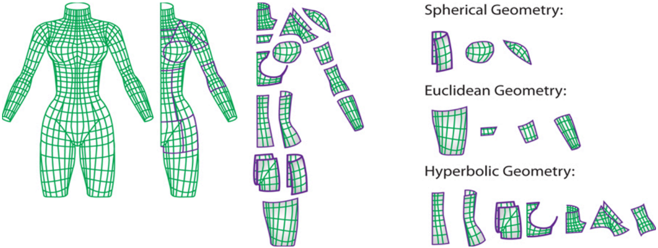


Figure 5: [3] Different geometries of the surface of a body.

[3] Source: http://www.drmarkliu.com/noneuclidean

Surfaces

Figure 6: [4]Surfaces of genus 0, 1, 2 and 3.

[4]Source: https://dmargalit7.math.gatech.edu/about.shtml

Teichmüller space

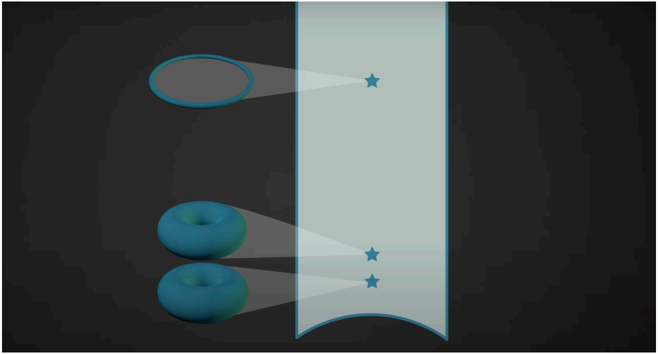


Figure 7: [5] Teichmüller space of a donut.

[5] Source: Secrets of the Surface: The Mathematical Vision of Maryam Mirzakhani

As Bers put it:

> *There are two ways to send a Riemann surface to infinity in Teichmüller space: by pinching it, or by wringing its neck.*

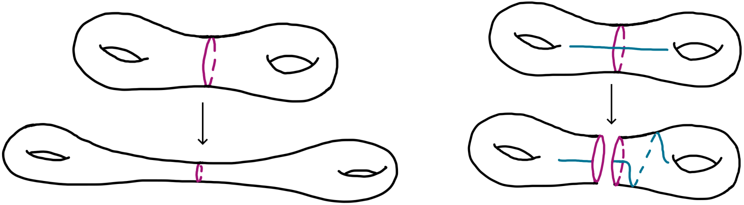


Figure 8: Pinching and twisting

Question: What should "points at infinity" (of Teichmüller space) be?

Compactification

Definition (intuitively)

A ***compactification*** *of a space is obtained by adding points at infinity.*

Examples:

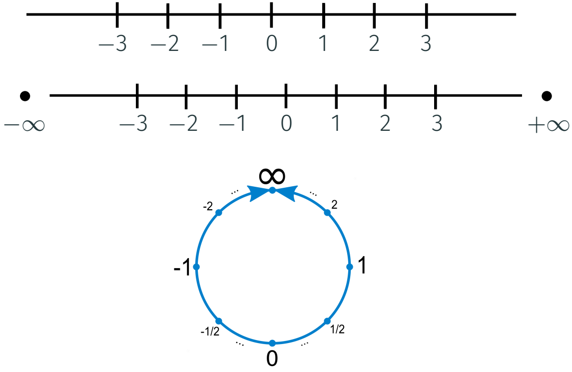


Figure 9: $\mathbb{R}$ and its two-point and one-point[6] compactifications

[6] Source: Wikipedia

Summary

Goal 1: Compactify Teichmüller space & its generalizations

Tool: Representation theory and real algebraic geometry

Goal 2: Interpret points at infinity geometrically

Tool: Spaces of flags and positivity

Main objects

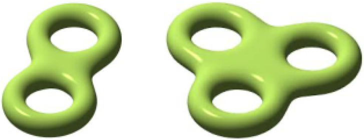


Figure 10: [7]Surfaces of genus ≥ 2

S	–	closed, connected, orientable surface of genus $g \geq 2$
$\pi_1(S)$	–	fundamental group of S
$\mathrm{PSL}(n, \mathbb{R})$	–	projective special linear group

$\rightsquigarrow$ $\chi(S, n) := \mathrm{Hom}(\pi_1(S), \mathrm{PSL}(n, \mathbb{R}))/\mathrm{PSL}(n, \mathbb{R})$, the character variety.

[7]Source: Wikipedia

Definition

$$T(S) = \{\text{hyperbolic structures on } S\}$$

Remark: $T(S) \hookrightarrow \chi(S, 2)$

Theorem (Goldman [Gol88])

$\chi(S, 2)$ has $4g-3$ connected components, two of which are $\cong \mathbb{R}^{6g-6}$ and are a copy of the Teichmüller space $T(S)$ of S.

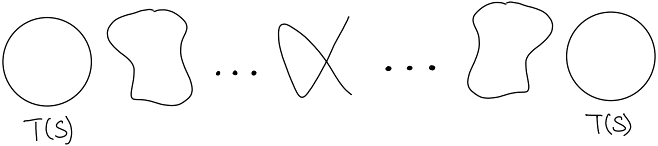


Figure 11: $\chi(S, 2)$ and its 4g-3 connected components

Goal: Compactify the character variety

The Hitchin component

Theorem ([Hit92]) For $n \geq 3$, $\chi(S, n)$ has

$$\begin{cases} \text{3 connected components, one of which is } \cong \mathbb{R}^{(n^2-1)(2g-2)}, & \text{if } n \text{ is odd,} \\ \text{6 connected components, two of which are } \cong \mathbb{R}^{(n^2-1)(2g-2)}, & \text{if } n \text{ is even.} \end{cases}$$

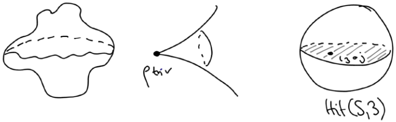


Figure 12: $\chi(S, 3)$ and its 3 connected components

Definition

The **Hitchin component** $\mathrm{Hit}(S, n)$ *is the connected component(s) of* $\chi(S, n)$ *homeomorphic to* $\mathbb{R}^{(n^2-1)(2g-2)}$.

Theorem (Fock–Goncharov [FG06], Labourie [Lab06])

The Hitchin component consists only of injective representations with discrete image.

Definition

*$\Omega \subset \mathbb{RP}^2 = \{$lines in $\mathbb{R}^3$ through $0\}$ is **strictly convex** if it is bounded and strictly convex in an affine chart.*

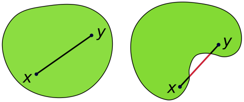


Figure 13: [8]A convex set and a non-convex set.

Theorem (Choi–Goldman [CG93])

$$\mathrm{Hit}(S, 3) = \{\text{strictly convex real projective structures on } S\}$$

[8]Source: Wikipedia

$n = 2 \rightsquigarrow$ Thurston's compactification $\overline{T(S)}$ of Teichmüller space

Properties:

- $T(S)$ is open and dense in $\overline{T(S)}$
- $\mathrm{MCG}(S) \curvearrowright \overline{T(S)} \cong \mathbb{B}^{6g-6}$
- points in $\partial\overline{T(S)} \leftrightarrow$ (small) actions on real trees

Question: Compactification for $\mathrm{Hit}(S, n)$ with "good" geometric properties?

Answer: Yes! The real spectrum compactification.

(Brumfiel [Bru88] for Teichmüller space,
Burger–Iozzi–Parreau–Pozzetti [BIPP21] for higher rank Lie groups)

Real spectrum compactification

Definition

*An ordered field is **real closed** if every positive element is a square and every odd degree polynomial has a root.*

Examples: $\mathbb{R}$, $\overline{\mathbb{Q}} \cap \mathbb{R}$,

$$\text{real Puiseux series} = \left\{ \sum_{k=-\infty}^{k_0} c_k X^{\frac{k}{m}} \;\middle|\; k_0, m \in \mathbb{Z},\ m > 0,\ c_k \in \mathbb{R},\ c_{k_0} \neq 0 \right\}$$

with the order $X > \lambda$ for all $\lambda \in \mathbb{R}$

Non-examples: $\mathbb{C}$, finite fields, $\mathbb{Q}$, $\mathbb{R}(X)$

Definition/Theorem ([BIPP21])

The **real spectrum compactification $\mathrm{RSp}(\chi(S, n))$** is

$$\mathrm{RSp}(\chi(S,n)) = \{(\rho, \mathbb{F}) \mid \rho \colon \pi_1(S) \to \mathrm{PSL}(n, \mathbb{F}),\ \mathbb{R} \subseteq \mathbb{F} \text{ real closed field}\}/_{\sim}.$$

Idea

Replace coefficients tending to $+\infty$ by variables X with $X > \lambda$ for all $\lambda \in \mathbb{R}$

Example

$\pi_1(S) \underset{\text{f. i.}}{<} \Delta := (3,3,4)\text{-triangle group } \langle a, b \mid a^3 = b^3 = (ab)^4 = 1\rangle$

$$\rho_t \colon \Delta \to \mathrm{PSL}(3,\mathbb{R}),$$
$$a \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 1 & 2-t+t^2 & 3+t^2 \\ 0 & -2+2t-t^2 & -1+t-t^2 \\ 0 & 3-3t+t^2 & (t-1)^2 \end{pmatrix}$$

$$\rho \colon \Delta \to \mathrm{PSL}(3,\overline{\mathbb{R}(X)}^r),$$
$$a \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 1 & 2-X+X^2 & 3+X^2 \\ 0 & -2+2X-X^2 & -1+X-X^2 \\ 0 & 3-3X+X^2 & (X-1)^2 \end{pmatrix}$$

Theorem (Long-Reid-Thistlethwaite [LRT11])

For all $t \in \mathbb{R}$, the representations ρ_t are Hitchin.

$\rightsquigarrow \rho := \lim_{t\to\infty} \rho_t \in \overline{\mathrm{Hit}(S,3)} \subset \mathrm{RSp}(\chi(S,3))$.

$\overline{\mathbb{R}(X)}^r$ = the real closure of $\mathbb{R}(X)$ together with the order $X > \lambda$ for all $\lambda \in \mathbb{R}$.

Characterisation of Hitchin boundary points & main result

Question: Let $(\rho, \mathbb{F}) \in \mathrm{RSp}(\chi(S, n))$. When is $(\rho, \mathbb{F}) \in \overline{\mathrm{Hit}(S, n)}$?

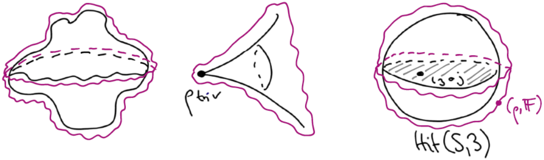


Figure 14: $\mathrm{RSp}(\chi(S, 3))$

Theorem (Fock–Goncharov [FG06])

Let $\rho\colon \pi_1(S) \to \mathrm{PSL}(n, \mathbb{R})$. *Then* ρ *is Hitchin* $\iff$ ρ *is positive.*

Theorem (F. [Fla22])

$(\rho, \mathbb{F}) \in \overline{\mathrm{Hit}(S, n)} \iff \rho$ *is* $\mathbb{F}$*-positive and weakly dynamics preserving.*

Definition

*A **flag** is a nested sequence of $n+1$ subspaces of $\mathbb{F}^n$ of strictly increasing dimension, i.e.*

$$F = (V_0 \subset V_1 \subset \ldots \subset V_{n-1} \subset V_n).$$

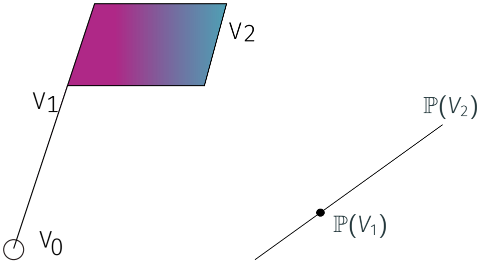


Figure 15: [9] A flag in $\mathbb{F}^3$ and its projectivization in $\mathbb{FP}^2$

[9] Source: Wikipedia

Let $\mathrm{Fix}(S) \subset \partial\mathbb{H}^2 \cong S^1$.

Remark: $\pi_1(S) \curvearrowright \mathrm{Fix}(S)$ and $\mathrm{PSL}(n,\mathbb{F}) \curvearrowright \mathrm{Flag}(\mathbb{F}^n)$

Definition

*A representation $\rho\colon \pi_1(S) \to \mathrm{PSL}(n,\mathbb{F})$ is **$\mathbb{F}$-positive** if there exists a map $\xi_\rho\colon \mathrm{Fix}(S) \to \mathrm{Flag}(\mathbb{F}^n)$ (called the **limit map**), that is*

- *ρ-equivariant, i.e. $\xi_\rho(\gamma x) = \rho(\gamma)\xi_\rho(x)$ for all $x \in \mathrm{Fix}(S)$ and $\gamma \in \pi_1(S)$*
- *tuples of cyclically ordered points $\mapsto$ positive tuples of flags.*

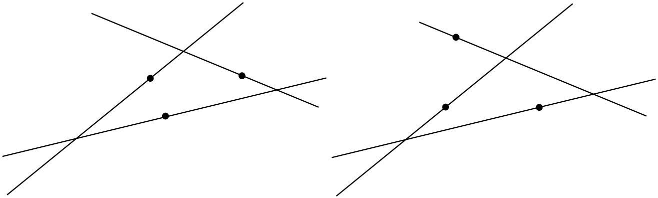


Figure 16: A positive and a negative triple.

Tools and techniques

Recall that we would like to prove

Theorem (F. [Fla22])

$(\rho, \mathbb{F}) \in \overline{\mathrm{Hit}(S, n)} \iff \rho$ *is* $\mathbb{F}$*-positive and weakly dynamics preserving.*

Idea: Transferring properties from the Hitchin component to the boundary.

Definition

*A **semi-algebraic set** is a finite union of subsets of $\mathbb{R}^m$ defined by finitely many polynomial equalities and inequalities.*

Example

The circle $x^2 + y^2 - 1 = 0$ is semi-algebraic, as well as its inside and outside.

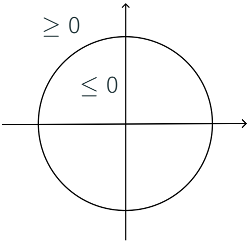


Figure 17: Semi-algebraic sets defined by the polynomial $x^2 + y^2 - 1$.

("$\Rightarrow$"): Tarski–Seidenberg transfer principle

Let $X \subseteq \mathbb{R}^{m+1}$ be semi-algebraic and $p\colon \mathbb{R}^{m+1} \to \mathbb{R}^m$ the projection onto the first m coordinates.

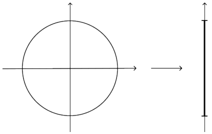


Figure 19: Projection onto a coordinate.

Theorem (Tarski–Seidenberg)

- *$p(X) \subseteq \mathbb{R}^m$ is semi-algebraic.*
- *If $\mathbb{R} \subseteq \mathbb{F}$ real closed, its* ***$\mathbb{F}$-extension $X_\mathbb{F}$*** *—the subset of $\mathbb{F}^m$ satisfying the polynomial equalities and inequalities defining X—is well-defined.*
- $$\begin{array}{ccc} X & \xhookrightarrow{\mathbb{F}\text{-extension}} & X_\mathbb{F} \\ \downarrow p & & \downarrow p_\mathbb{F} \\ p(X) & \xhookrightarrow{\mathbb{F}\text{-extension}} & p(X)_\mathbb{F} = p_\mathbb{F}(X_\mathbb{F}) \end{array}$$

(“$\Rightarrow$”): Tarski–Seidenberg transfer principle

Fact: $(\rho, \mathbb{F}) \in \overline{\mathrm{Hit}(S,n)} \iff \rho \in \mathrm{Hit}(S,n)_{\mathbb{F}}$

Proposition (F. [Fla22])

Let $(\rho, \mathbb{F}) \in \overline{\mathrm{Hit}(S,n)}$. Then ρ is injective.

Proof: For $\mathrm{Id} \neq \gamma \in \pi_1(S)$ consider $X_\gamma = \{\rho \in \mathrm{Hit}(S,n) \mid \rho(\gamma) \neq \mathrm{Id}\} \subset \mathbb{R}^M$.
Then X_γ is semi-algebraic and $X_\gamma = \mathrm{Hit}(S,n)$.
Tarski–Seidenberg $\Longrightarrow$ $(X_\gamma)_{\mathbb{F}} = \mathrm{Hit}(S,n)_{\mathbb{F}}$, so ρ is injective. $\square$

Proposition (F. [Fla22])

Let $(\rho, \mathbb{F}) \in \overline{\mathrm{Hit}(S,n)}$. Then $\rho(\gamma)$ has distinct, positive eigenvalues for all $\mathrm{Id} \neq \gamma \in \pi_1(S)$.

Definition

Let $M \in \mathrm{GL}(n, \mathbb{F})$ with distinct, positive eigenvalues $\lambda_1 > \ldots > \lambda_n > 0$ and corresponding eigenspaces $\ell_1, \ldots, \ell_n$. Its ***stable flag*** F_M^+ *is*

$$F_M^+ = \left(\{0\} \subset \ell_1 \subset \ell_1 \oplus \ell_2 \subset \ldots \subset \ell_1 \oplus \ldots \oplus \ell_{n-1} \subset \mathbb{F}^n\right).$$

Recall that we would like to prove

Theorem (F. [Fla22])

$(\rho, \mathbb{F}) \in \overline{\mathrm{Hit}(S,n)} \iff \rho$ *is* $\mathbb{F}$*-positive and weakly dynamics preserving.*

Proof of (“⇒”).

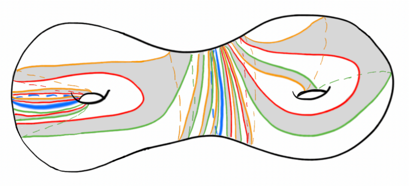
- Define an equivariant limit map for $\rho \in \mathrm{Hit}(S,n)_{\mathbb{F}}$ by

$$\xi_\rho \colon \mathrm{Fix}(S) \to \mathrm{Flag}(\mathbb{F}^n),\ \gamma^+ \mapsto F^+_{\rho(\gamma)}$$

- Use Tarski–Seidenberg $\implies$ ξ_ρ is positive. □

(“$\Leftarrow$”): Bonahon–Dreyer coordinates

Fix a maximal geodesic lamination L on S.



$\rightsquigarrow$ ideal triangulation of $\tilde{S}$

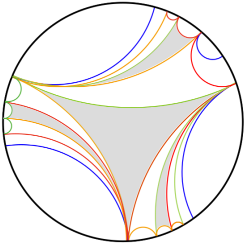


Figure 20: The lift $\tilde{L}$ of L to $\tilde{S}$.

(“$\Leftarrow$”): Bonahon–Dreyer coordinates

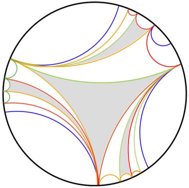


Figure 21: The lift $\tilde{L}$ of L to $\tilde{S}$.

Theorem (Bonahon–Dreyer [BD14])

The map $\mathrm{Hit}(S, n) \to \mathbb{R}^N$ *that assigns to a Hitchin representation* ρ *with limit map* $\xi_\rho \colon \mathrm{Fix}(S) \to \mathrm{Flag}(\mathbb{R}^n)$

- *the triangle invariants of* $(\xi_\rho(x), \xi_\rho(y), \xi_\rho(z))$ *for every ideal triangle with vertices* x, y, z*, and*
- *the shear invariants of* $(\xi_\rho(x), \xi_\rho(y), \xi_\rho(z), \xi_\rho(w))$ *for every geodesic with adjacent ideal triangles with vertices* x, y, z, w

is a homeomorphism onto an explicit semi-algebraic subset $X \subset \mathbb{R}^N$*.*

Definition

A matrix in $\mathrm{GL}(n, \mathbb{F})$ *is* ***totally positive****, if all its minors are positive.*

Example

$M_1 = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$, $M_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 4 & 8 \end{pmatrix}$ *are totally positive.*

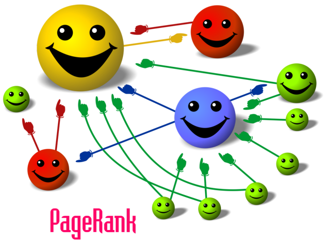


Figure 22: [10] PageRank algorithm.

Theorem (Perron–Frobenius)

A positive matrix has a largest positive eigenvalue.

Theorem (Gantmacher–Krein [GK02])

A totally positive matrix has distinct, positive eigenvalues.

[10] Source: Wikipedia

Theorem (Fock–Goncharov [FG06])

Let $(F_1, \dots, F_k)$ be a positive k-tuple of flags, (F'_1, F'_2, F'_3) a positive subtriple (distinct from (F_1, F_2, F_3)), and assume $g(F_1, F_2, F_3) = (F'_1, F'_2, F'_3)$ for some $g \in \mathrm{PGL}(n, \mathbb{F})$. Then g is conjugate to a totally positive matrix.

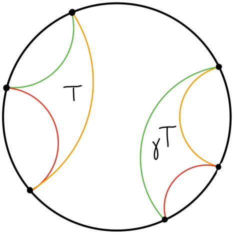


Figure 23: Six points cyclically ordered in $\mathrm{Fix}(S)$.

Proposition (F. [Fla22])

Let ρ be $\mathbb{F}$-positive. Then $\rho(\gamma)$ has distinct, positive eigenvalues for all non-trivial $\gamma \in \pi_1(S)$.

Thank you

Thank you for your attention!

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