MORPHISMS OF GENERALIZED AFFINE BUILDINGS

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Abstract. We define a notion of morphism between generalized affine buildings, generalizing existing definitions appearing in the literature. For buildings equipped with a transitive group action (such as Bruhat–Tits buildings, homogeneous buildings, lattice buildings and norm buildings), we provide sufficient conditions under which a morphism of apartments extends to a morphism of buildings. As an application, we show relationships between these different types of buildings via our notion of morphism and prove functoriality results for homogeneous buildings under base field extensions and group homomorphisms.

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1. Introduction

The goal of this article is to unify the study of generalized affine buildings and their morphisms. Generalized affine buildings appear in a wide range of mathematical contexts, including the asymptotic geometry of symmetric spaces [KL97], the structure theory of Kac–Moody groups over valued fields [Tit87, Ŕ99], and compactifications of character varieties [BIPP23]. Buildings and their generalizations have been extensively studied in the literature, with their beginning in the foundational work of Bruhat–Tits [BT84], then generalized in the non-discrete setting to Euclidean buildings [KL97, Rou04], to $\mathbb{R}$-buildings [BT72, Par00] and then finally, to Λ-buildings, which were introduced by Bennett in [Ben94] and have since then been extensively studied by many authors [KT02, Ben09, BS14, HIL23, App24]. However a unified theory of morphisms of the most general notion of buildings that applies to a large class of examples and with good functoriality properties has not yet been established. The central challenge is that different contexts naturally give rise to different models of buildings of different types, making it difficult to construct meaningful morphisms or discuss subbuildings systematically. In this article we propose a new notion of morphism for generalized affine buildings that addresses this challenge. It is especially well-suited for buildings endowed with group actions. In this setting, morphisms can be constructed under some mild conditions and we find additional conditions on when such morphisms are injective, surjective, or isomorphisms see Theorem 1.3. The main advantage of our approach is that the conditions in Theorem 1.3 are easy to verify in explicit examples. This result can be applied to show certain functoriality properties under subgroups (Theorem 1.4), group morphisms (Theorem 1.5) and field extensions (Theorem 1.6) for a certain family of buildings. We then use this new notion to relate examples of generalized affine buildings in the literature (Figure 1), notably the norm buildings, the lattice buildings, the Bruhat–Tits buildings and the homogeneous buildings; see Section 3.2. Let us now explain the results in more detail.

1.1. **A new notion of morphisms of apartments and buildings.** We begin by defining the new notions of morphisms of apartments and buildings. Recall that a generalized affine building B is a set together with an atlas of maps $\mathcal{A}$ from the model apartment $\mathbb{A}$ to B satisfying certain compatibility axioms; see Definition 3.1 for the precise conditions. The *model apartment* is given by $\mathbb{A} = \operatorname{Span}_{\mathbb{Q}}(\Phi) \otimes_{\mathbb{Q}} \Lambda$, where Φ is a crystallographic root system and Λ an ordered abelian group, together with an action of the *affine Weyl group* defined by $W_{T,\Phi} := W_{\mathrm{s}}(\Phi) \ltimes T$. Here $W_{\mathrm{s}}(\Phi)$ is the spherical Weyl group associated to Φ and T is a subgroup of $\mathbb{A} \cong \Lambda^n$ which acts by translation on $\mathbb{A}$. When T and Φ can be deduced from the context, we write W_{a} for the affine Weyl group. In this case the apartment $\mathbb{A}$ (and the building $(B, \mathcal{A})$ as well) is said to be *of type* $\mathbb{A}(\Phi, \Lambda, T)$. The main motivation for our definition is to be able to account for buildings of different types. For example, for buildings associated to algebraic groups, a subgroup generally has a different root system than the ambient group, which makes it difficult to talk about subbuildings. We would like to propose a solution to this.

Let now Φ, Φ' be two crystallographic root systems and Λ, Λ' two ordered abelian groups, that are also $\mathbb{Q}$-vector spaces.

Definition 1.1 (Definition 4.4)**.** Let $\mathbb{A} = \operatorname{Span}_{\mathbb{Q}}(\Phi) \otimes_{\mathbb{Q}} \Lambda$ and $\mathbb{A}' = \operatorname{Span}_{\mathbb{Q}}(\Phi') \otimes_{\mathbb{Q}} \Lambda'$ be two model apartments. Let $L\colon \operatorname{Span}_{\mathbb{Q}}(\Phi) \to \operatorname{Span}_{\mathbb{Q}}(\Phi')$ be a $\mathbb{Q}$-linear map and $\gamma\colon \Lambda \to \Lambda'$ a morphism of ordered abelian groups. The map

$$L \otimes_{\mathbb{Q}} \gamma \colon \mathbb{A} \to \mathbb{A}'$$

is called a *morphism of apartments* if it is equivariant for the action of the affine Weyl groups, i.e. if there exists a map $\sigma\colon W_{T,\Phi} \to W_{T',\Phi'}$ such that for all $a \in \mathbb{A}$ and $w \in W_{T,\Phi}$ we have

$$(L \otimes_{\mathbb{Q}} \gamma)(w.a) = \sigma(w).((L \otimes_{\mathbb{Q}} \gamma)(a)).$$

Even though two buildings might have the same apartment $\mathbb{A}$, we do not always use the identity map $\mathbb{A} \to \mathbb{A}$ as the apartment morphism, but instead the inversion $i\colon \mathbb{A} \to \mathbb{A}$, $x \mapsto -x$, see e.g. Section 7.1. Moreover, we cannot in general ask for the map σ to be a group homomorphism, as shown in Figure 4 in the case $\Phi \neq \Phi'$. The definition might seem restrictive as we do not allow L to be an affine map (but only linear), but we do not loose any generality.

Since a building is made out of copies of the model apartment, we use the notion of morphism of apartments to define morphisms of buildings.

Definition 1.2 (Definition 4.6)**.** Let $B = (B, \mathcal{A})$ be an affine building of type $\mathbb{A} = \mathbb{A}(\Phi, \Lambda, T)$ and $B' = (B', \mathcal{A}')$ an affine building of type $\mathbb{A}' = \mathbb{A}(\Phi', \Lambda', T')$. A *morphism of generalized affine buildings* is a collection of maps

$$\psi\colon B \to B', \quad \varphi\colon \mathcal{A} \to \mathcal{A}', \quad \tau\colon \mathbb{A} \to \mathbb{A}',$$

where τ is a morphism of apartments, such that for all $f \in \mathcal{A}$ it holds

$$\psi \circ f = \varphi(f) \circ \tau.$$

On the one hand, this definition provides considerable flexibility compared to existing notions of building morphisms in the literature, that will be discussed in Section 4.3. The key feature is that it allows to change the root system, which is the main new thing our notion of morphism brings to the table. On the other hand, the notion is still rigid enough to conclude that the only morphism from the $\mathbb{R}$-building $\mathbb{R}$ to the $\mathbb{Q}$-building $\mathbb{Q}$ is the trivial morphism, since there is no non-trivial order-preserving group homomorphism from $\mathbb{R}$ to $\mathbb{Q}$.

Notions of isomorphisms of buildings have been studied since they were first introduced by Tits and Bruhat–Tits, see e.g. [Tit74, Tit86, BT72], and [Sch09] in the affine case. In the discrete case, i.e. when Λ is a discrete subgroup of $\mathbb{R}$, and the building has hence a simplicial structure, notions of morphisms of buildings and subbuildings are suggested and studied in [Lan00, KP23]. A more metric approach is undertaking in [KL97, Sections 3.10 and 4.7]. Rousseau defines in [Rou04, Definitions 1.1.4.1 and 2.1.13.1] notions of (weak) morphisms of apartments and $\mathbb{R}$-buildings. There has been a notion of morphism of apartments defined in [LN04], which however does not allow for flexibility in changing the root system. More generally, in [SS11] Schwer–Struyve show that an order-preserving homomorphism from $\Lambda \to \Lambda'$, where Λ and Λ' are ordered abelian groups, induces a natural map from a Λ-building to a Λ'-building. However still in this case, the two model apartments are modeled on the same root system.

1.2. **Extension of morphisms of apartments to morphisms of buildings.** The goal now is to show that we can construct morphisms of generalized affine buildings easily for those buildings that are endowed with an action of a group that is transitive “enough”. The idea is to define only a morphism between the standard apartments and then move this map around using the group action. Let us now make this idea precise.

Let G be a group and $(B, \mathcal{A})$ an affine Λ-building of type $\mathbb{A} = \mathbb{A}(\Phi, \Lambda, T)$. We say that B is a *G-building* if G acts on both B and $\mathcal{A}$ in a compatible way, i.e. for all $g \in G$, $f \in \mathcal{A}$ and $a \in \mathbb{A}$ we have

$$(g.f)(a) = g.(f(a)).$$

Note that we do not ask G to act by morphisms of buildings. On the contrary, we use this action and certain conditions on subsets of G to verify when we are able to extend a morphism of apartments to a morphism of buildings using this action. Namely, for $f \in \mathcal{A}$ and $w \in W_{\mathrm{a}}$, we define

$$A_{f,w} = \{g \in G \colon g.f = f \circ w\} \subseteq G.$$

In other words, it is the subset of those elements of G that act on a chart f the same as precomposition by the element w of the affine Weyl group.

If G' is another group, a morphism from a G-building to a G'-building is a *morphism of* (G, G')*-buildings* if there exists a group homomomorphism $\rho\colon G \to G'$ for which the morphism is equivariant. The main result about our notions of morphisms is the following theorem, which gives sufficient conditions on when to extend morphisms of apartments to (injective or surjective) morphisms of (G, G')-buildings.

Theorem 1.3 (Theorem 5.12)**.** *Let* $(B, \mathcal{A})$ *be a* G*-building of type* $\mathbb{A} = \mathbb{A}(\Phi, \Lambda, T)$ *with a transitive* G*-action on* $\mathcal{A}$*, and* $(B', \mathcal{A}')$ *a* G'*-building of type* $\mathbb{A}' = \mathbb{A}(\Phi', \Lambda', T')$*. Let* $\tau = (L, \gamma, \sigma)\colon \mathbb{A} \to \mathbb{A}'$ *be a morphism of apartments and* $\rho\colon G \to G'$ *a group homomorphism.*

If there exist charts $f \in \mathcal{A}$ *and* $f' \in \mathcal{A}'$ *such that*

(1) $\rho(\mathrm{Stab}_G(f(a))) \subseteq \mathrm{Stab}_{G'}(f'((L \otimes \gamma)(a)))$ *for all* $a \in \mathbb{A}$*, and*
(2) $\rho(A_{f,w}) \subseteq A_{f',\sigma(w)}$ *for all* $w \in W_{\mathrm{a}}$*,*

then there exists a morphism (ψ, φ, τ) *of* (G, G')*-buildings from* B *to* B' *extending* τ*, that is* ρ*-equivariant. Moreover,*

(a) if τ *and* ρ *are injective and* $\rho(\mathrm{Stab}_G(f)) = \mathrm{Stab}_{G'}(f')$*, then* (ψ, φ, τ) *is injective;*
(b) if G' *acts transitively on* $\mathcal{A}'$*, and* τ *and* ρ *are surjective, then* (ψ, φ, τ) *is surjective;*
(c) if G' *acts transitively on* $\mathcal{A}'$*,* ρ *is an isomorphism of groups,* τ *is an isomorphism, and the two inclusions (1) and (2) are equalities, then there exists an inverse morphism. That is,* $(B, \mathcal{A})$ *and* $(B', \mathcal{A}')$ *are isomorphic.*

We say that the morphism of (G, G')-buildings constructed in the above theorem is *induced* by the group homomorphism ρ. The construction of the map $\varphi\colon \mathcal{A} \to \mathcal{A}'$ on the level of atlases is straight-forward, as by assumption G acts transitively on $\mathcal{A}$, and (1) applied to $w = \mathrm{Id} \in W_{\mathrm{a}}$ implies that $\mathrm{Stab}_G(f) \subseteq \mathrm{Stab}_G(f')$, so φ is well-defined. To define $\psi\colon B \to B'$ on the underlying building we use again that G acts transitively on $\mathcal{A}$ and that every point in B is in the image of some chart (see axiom (A3) in Definition 3.1). To check that ψ is well-defined, one combines (1) and (2), and the axioms on buildings (A1) and (A2). It is then easy to verify that $\psi \circ f = \varphi(f) \circ (L \otimes f)$ using the transitivity of the action of G on $\mathcal{A}$. The ρ-equivariance follows from the construction of the maps. The proofs of (a) and (b) use only axiom (A3). To prove (c) we construct an inverse morphism. The strength of this theorem is that in concrete examples, an apartment morphism can be easily constructed, and the conditions (1) and (2), as well as (a)-(c) are often directly verifiable. We observe this in the proofs of the following results, which concern applications of the notion of morphism, where we see the above theorem in action.

1.3. Applications and examples.

The main applications of the above result are two-fold. First it allows to relate existing models of apartments in the literature. Secondly, in the example of one of such family of buildings, we can prove certain functoriality properties.

1.3.1. *Relationship between different models of buildings.* In this article we focus on four models of (families of) buildings, namely the norm building B_N, the lattice building B_L, the Bruhat–Tits buildings B_{BT} (for split algebraic groups) and the homogeneous buildings B_H as defined by Kramer–Tent in [KT02]. Their precise definitions are given in Section 3.2. It would be interesting to compare these to the even more general model given in [HIL23], that generalizes Bruhat–Tits buildings for higher rank Λ.

In Section 5.2 we first show that all these buildings constitute examples of G-buildings for some appropriate group G. In the lattice and norm buildings this group will be $\mathrm{SL}_n(\mathbb{F})$ for $\mathbb{F}$ a field endowed with a valuation, see Section 2.1. One can think of the homogeneous buildings and the Bruhat–Tits buildings as the generalizations of the lattice respectively norm building to Lie groups different from $\mathrm{SL}_n(\mathbb{F})$, where Bruhat–Tits buildings are defined for general valued fields and split algebraic groups, and the homogeneous buildings for real closed valued fields and (semi-)algebraic groups (that are not necessarily split). The relations between these buildings are summarized in the following diagram. For the precise statements of the results we refer to Section 7.

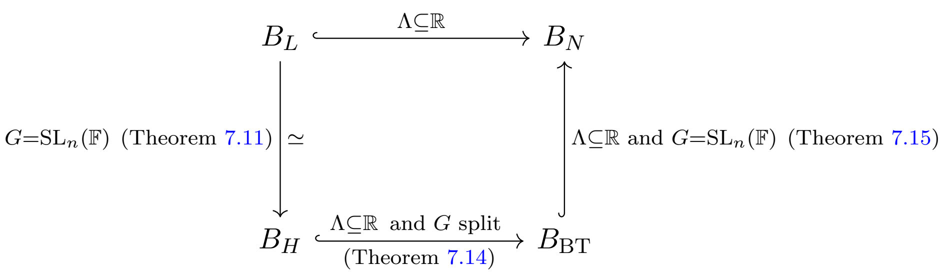

FIGURE 1. Morphisms between different buildings in the literature. When $\Lambda = \mathbb{R}$, all morphisms are isomorphisms.

Through personal communication, we know that the isomorphism between B_{BT} and B_N (see [Par00]) when $\Lambda = \mathbb{R}$ and $G = \mathrm{SL}_n(\mathbb{F})$ has already been known to Anne Parreau.

1.3.2. *Functoriality properties.* We now use the notion of morphism to address certain functiorality properties. For this we restrict our attention to homogeneous buildings, but equivalent question have been discussed for Bruhat–Tits buildings for example in [Lan00]. Homogeneous buildings are special generalized affine buildings associated to the $\mathbb{F}$-points of semisimple linear algebraic groups $\mathbf{G}$ defined over $\mathbb{Q}$, where $\mathbb{F}$ is a *real closed valued* field. This means that $\mathbb{F}$ is endowed with a total order such that $\mathbb{F}[\sqrt{-1}]$ is algebraically closed, as well as a rank one valuation $v\colon \mathbb{F}^\times \to \mathbb{R}_{\geq 0}$, that is compatible with this order, meaning that $0 < x \leq y$ implies that $v(x) \geq v(y)$ for all $x, y \in \mathbb{F}$. For a precise definition of homogeneous buildings we refer to Example 3.5. In this special case we apply Theorem 1.3 to prove certain functoriality properties. A natural question is whether a (surjective, injective) group morphism of algebraic groups $\rho\colon \mathbf{G} \to \mathbf{G}'$, where $\mathbf{G}'$ is a semisimple linear algebraic group defined over $\mathbb{Q}$, induces a natural (surjective, injective) morphism of the associated homogeneous buildings B, respectively B'. This question was answered positively for (discrete) Bruhat–Tits buildings over quasi-local fields by Landvogt in [Lan00]. We now study this question in the context of homogeneous buildings for real closed fields. The first example is when $\mathbf{G}$ is a subgroup of $\mathbf{G}'$ and ρ is the inclusion. In this case we have the following result.

Theorem 1.4 (Theorem 6.7)**.** *Let $\mathbf{G} < \mathbf{G}' < \mathrm{SL}_n$ be two semisimple self-adjoint linear algebraic $\mathbb{Q}$-groups. Let $\mathbb{F}$ be a non-Archimedean real closed* *field and $v\colon \mathbb{F}^\times \to \Lambda$ an order-compatible valuation. Let B (resp. B') be the associated homogeneous $\mathbf{G}(\mathbb{F})$- (resp. $\mathbf{G}'(\mathbb{F})$-) building. Then the inclusion $\mathbf{G}(\mathbb{F}) \hookrightarrow \mathbf{G}'(\mathbb{F})$ induces an injective morphism $B \to B'$ of generalized affine buildings.*

In this theorem the full flexibility of our notion of morphism of apartments becomes apparent, since we need to be able to change the root system in the type of the model apartment. Namely, the idea is to show that there is an inclusion of the model apartments, see Lemma 6.3, which is in fact an injective morphism of apartments. The main difficulty in this step is the existence of a map $\sigma\colon W_{\mathrm{a}} \to W'_{\mathrm{a}}$, which is the content of **??**. We then conclude using Theorem 1.3 (a). For more general group morphisms $G \to G'$ we have the following result generalizing the above.

Theorem 1.5 (Theorem 6.11)**.** *Let $\mathbf{G}$, $\mathbf{G}' < \mathrm{SL}_n$ be two semisimple selfadjoint linear algebraic $\mathbb{Q}$-groups. Let $\mathbb{F}$ be a non-Archimedean real closed field and $v\colon \mathbb{F}^\times \to \Lambda$ an order-compatible valuation. Let B (resp. B') be the $\mathbf{G}(\mathbb{F})$- (resp. $\mathbf{G}'(\mathbb{F})$-) homogeneous building defined as in Example 3.5. If there exists an injective morphism of groups $\mathbf{G}(\mathbb{F}) \to \mathbf{G}'(\mathbb{F})$, then it induces an injective morphism of buildings $B \to B'$.*

The idea is to use Theorem 1.4. Now the main step towards the proof of the above result is to show that if $\mathbf{G}$ and $\mathbf{G}'$ are isomorphic, then there is an isomorphism between their associated homogeneous buildings, see Theorem 6.9. This first needs to be established for the associated model apartments (Lemma 6.8), and then we apply again Theorem 1.3 (c). We also obtain certain functoriality properties under real closed valued field extensions.

Theorem 1.6 (Theorem 6.12)**.** *Let $\mathbf{G} < \mathrm{SL}_n$ be a semisimple self-adjoint linear algebraic $\mathbb{Q}$-group, $\mathbb{K}, \mathbb{F}$ non-Archimedean real closed fields with order-compatible valuations $v_{\mathbb{K}}, v_{\mathbb{F}}$ and B, B' the homogeneous affine buildings associated to $\mathbf{G}(\mathbb{K})$ and $\mathbf{G}(\mathbb{F})$ respectively (see Example 3.5). Suppose there exists a morphism of valued fields $\eta\colon \mathbb{K} \to \mathbb{F}$, that is $v_{\mathbb{F}}(\eta(x)) = v_{\mathbb{K}}(x)$ for every $k \in \mathbb{K}^\times$, then there exists a building morphism $B \to B'$.*

The proof proceeds by constructing the building morphism in three steps. First, we show that the valuation-preserving property of η allows us to define a well-behaved morphism of ordered groups γ between the value groups. Second, we use this to construct a morphism σ between the affine Weyl groups, which in turn yields a morphism of apartments $\tau = (\mathrm{Id}, \gamma, \sigma)$. Finally, we define maps between the buildings and between the atlases by applying η entrywise to matrix entries. The key technical challenge is then to verify that the maps commute.

Remark 1.7. We believe that most of these functoriality properties can be extended to other models of buildings, such as for example Bruhat–Tits buildings.

Another loose end are spherical buildings at infinity. One may ask whether a morphism of generalized affine buildings induces a simplicial morphism of their respective buildings at infinity or, equivalently, their local buildings. The answer in general is no, as soon as the generalized affine buildings are not modeled on the same root system, see the example above of $\mathrm{Sp}_4(\mathbb{F}) < \mathrm{SL}_4(\mathbb{F})$. In this case, a chamber of the smaller apartment is not sent to any Weyl simplex (of any dimension) in the larger apartment. Correspondingly, the sectorpanels in the building associated to $\mathrm{Sp}_4(\mathbb{F})$ are not being sent to sectorpanels in the building associated to $\mathrm{SL}_4(\mathbb{F})$, hence no map from the building at infinity or the local building can be defined. However, as soon as $\Phi = \Phi'$ we believe that our notion of morphism between generalized affine buildings induces a simplicial morphism between the buildings at infinity.

1.4. **Structure of the paper.** The article is organized as follows. We give preliminaries on valued and ordered fields in Section 2. In Section 3 we recall the definition of generalized affine buildings. We then continue to present all the examples of buildings that will be studied throughout this article in Section 3.2. We define morphisms of apartments and buildings in Section 4. In order to prove Theorem 1.3 in Section 5.4, we introduce G-buildings in Section 5, and we investigate the notion of morphisms and the above examples of buildings within this new context. We then apply Theorem 1.3 in Section 6 and Section 7 to prove Theorem 1.4, Theorem 1.5, Theorem 1.6 and the results announced in Figure 1 on the relations between the different examples. Necessary background on real algebraic geometry, which is only needed for the example of the homogeneous buildings, is summarized in Appendix A and can be consulted at any moment.

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2. Preliminaries

2.1. **Valued fields.** For a thorough introduction to the theory of valuations and valued fields we refer to [EP05]. Let $\mathbb{K}$ be a field and Λ an *ordered abelian group*, i.e. Λ is an abelian group together with a total order that is compatible with the group operations. A map $v\colon \mathbb{K} \to \Lambda \cup \{\infty\}$ is called a Λ-*valuation* (or short just *valuation*) if v is surjective and satisfies the following three conditions for all $x, y \in \mathbb{K}$:

(1) $v(x) = \infty \implies x = 0$,
(2) $v(xy) = v(x) + v(y)$,
(3) $v(x + y) \geq \min\{v(x), v(y)\}$.

If $\Lambda = \{0\}$, we call v the *trivial valuation*; if Λ has rank 1 (i.e. it is isomorphic as an ordered abelian group to a subgroup of $\mathbb{R}$), we call v a *rank*-1 *valuation*. More generally, we define the *rank* of v as the rank (as an abelian group) of the value group $\Lambda = v(\mathbb{K}^\times)$. The subset

$$\mathcal{O} := \{x \in \mathbb{K}^\times \mid v(x) \geq 0\}$$

forms a subring, which is a *valuation ring* of $\mathbb{K}$, i.e. a subring of $\mathbb{K}$ such that for all $x \in \mathbb{K}^\times$ we have $x \in \mathcal{O}$ or $x^{-1} \in \mathcal{O}$. A field together with a valuation is called a *valued field*.

Example 2.1. Examples of valued fields are the p-adic numbers. Whenever $\mathbb{F}$ is any field, the field of rational functions with coefficients in $\mathbb{F}$ is naturally a valued field, where the valuation is given by $v\colon \mathbb{F}(X) \to \mathbb{Z}$, $\frac{P}{Q} \mapsto \deg Q - \deg P$.

Ordered fields are naturally valued, and they play an important role in some of the examples we consider in the following.

2.2. **Ordered fields.** Let $\mathbb{F}$ be an ordered field, i.e. a field together with a total order that is compatible with the field operations. One says that $\mathbb{F}$ is *non-Archimedean* if there exists $x \in \mathbb{F}$ with $x > n$ for all $n \in \mathbb{N}$.

An absolute value v on $\mathbb{F}$ is *order-compatible* if for all $x, y \in \mathbb{F}$ with $0 < x \leq y$ we have $v(x) \geq v(y)$. There is a general construction to define order-compatible absolute values on ordered fields.

Example 2.2 (Order valuation)**.** Let $\mathbb{F}$ be an ordered field. We say that two elements $x, y \in \mathbb{F}$ are in the same *Archimedean class* if there exists $n, m \in \mathbb{N}$ such that $|x| < n|y|$ and $|y| < m|x|$. The set of Archimedean classes forms an ordered abelian group Λ, where addition and the order are induced from the ones in $\mathbb{F}$. The map that assigns to an element $0 \neq x \in \mathbb{F}$ its Archimedean class is a valuation, called the *order valuation*. It is furthermore order-compatible. Note that $\Lambda \neq \{0\}$, i.e. the order valuation is non-trivial, if and only if $\mathbb{F}$ is non-Archimedean.

An ordered field can admit many order-compatible valuations, where the order valuation is in some sense the "coarsest" one. For example, if $\mathbb{F}$ has a big element b, i.e. for all $x \in \mathbb{F}$ there exists $n \in \mathbb{N}$ with $x < b^n$, then one can define an order-compatible rank-1 valuation $v_b \colon \mathbb{F} \to \mathbb{R}_{\geq 0} \cup \infty$ by setting

$$v_b(x) \coloneqq -\inf \left\{ \frac{p}{q} \in \mathbb{Q} \mid x^q < b^p \right\},$$

mimicking the definition of the standard logarithm. In fact, any order-compatible rank-1 valuation on $\mathbb{F}$ is a positive scalar multiple of v_b for some big element $b \in \mathbb{F}$. Note that there are ordered fields that do not admit big elements, e.g. the hyperreals.

An ordered field is *real closed* if every positive element is a square and every odd degree polynomial has a root. Note that every ordered field has a *real closure*, that means an algebraic field extension that is real closed and whose order extends the original one [BCR98, §1.3].

Example 2.3. The real numbers $\mathbb{R}$ and the real algebraic numbers $\overline{\mathbb{Q}}^r$ are both real closed. The field of *real Puiseux series* is the set of expressions

$$\mathbb{R}(X)^\wedge \coloneqq \Big\{ \sum_{k=k_0}^{\infty} c_k X^{k/m} \;\Big|\; k_0 \in \mathbb{Z},\, m \in \mathbb{N} \setminus \{0\},\, c_k \in \mathbb{R} \Big\},$$

together with formal addition and multiplication. An element $\sum_{k=k_0}^{\infty} c_k X^{k/m}$ is positive if $c_{k_0} > 0$. With this order $\mathbb{R}(X)^\wedge$ is real closed, see e.g. [BPR06, Theorem 2.91]. The real closure of $\mathbb{Q}$ is $\overline{\mathbb{Q}}^r$. The real closure of $\mathbb{R}(X)$ (together with the order $X > 0$ but $X < \lambda$ for all $\lambda \in \mathbb{R}_{>0}$) is the field of real Puiseux series that are algebraic over $\mathbb{R}(X)$.

Real closed fields play a crucial role in real algebraic geometry. Since real algebraic geometry only appears in one of the models for buildings and are not the subject of this article, we summarize the basics needed throughout this article in Appendix A.

3. Generalized affine buildings

In this section we recall the definition of generalized affine buildings, and then give several examples.

3.1. **Definition.** For more details and a thorough introduction to generalized affine buildings we recommend for example [Ben94, Sch09, App24].

Let Φ be a crystallographic[1] root system in a Euclidean vector space $(V, \langle\cdot,\cdot\rangle)$ and Λ a non-trivial ordered abelian group. As is usually done, we assume that Λ is a $\mathbb{Q}$-vector space, as otherwise we may replace Λ by $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$. In particular, both Λ and $\mathrm{Span}_{\mathbb{Q}}(\Phi)$ have the structure of $\mathbb{Q}$-vector spaces and we define the *model apartment* as

$$\mathbb{A} := \mathrm{Span}_{\mathbb{Q}}(\Phi) \otimes_{\mathbb{Q}} \Lambda.$$

If $\Delta \subseteq \Phi$ is a basis of Φ, then a model for the apartment is given by

$$\mathbb{A} = \left\{ \sum_{\alpha\in\Delta} \lambda_\alpha \alpha \colon \lambda_\alpha \in \Lambda \right\},$$

so that $\mathbb{A}$ is isomorphic as a group to Λ^n for $n = |\Delta| \in \mathbb{N}$, which is called the *dimension of the apartment*. Moreover, the root system Φ defines a spherical Weyl group W_{s}. Let T be a subgroup of $\mathbb{A} \cong \Lambda^n$ which acts by translation on $\mathbb{A}$ and define the *affine Weyl group with respect to* T as $W_{\mathrm{a}} := T \rtimes W_{\mathrm{s}}$. Formally, we call the combined data (Φ, Λ, T) an *apartment*. An apartment determines the model apartment $\mathbb{A}$ together with the action of the affine Weyl group W_{a}. It is customary to write $\mathbb{A} = \mathbb{A}(\Phi, \Lambda, T)$. The scalar product $\langle\cdot,\cdot\rangle$ on V extends to a bilinear pairing

$$\langle\cdot,\cdot\rangle \colon \mathbb{A} \times \mathrm{Span}_{\mathbb{Q}}(\Phi) \to \Lambda, \quad \left\langle \sum_{\alpha\in\Delta} \lambda_\alpha \alpha, \sum_{\delta\in\Delta} \lambda_\delta \delta \right\rangle := \sum_{\alpha,\delta\in\Delta} \lambda_\alpha \lambda_\delta \langle\alpha,\delta\rangle,$$

which in general cannot be extended to all of $\mathbb{A} \times \mathbb{A}$, since Λ may not have a multiplication. Every root $\alpha \in \Phi$ determines a reflection $r_\alpha \colon \mathbb{A} \to \mathbb{A}$ defined by

$$r_\alpha \left(\sum_{\delta\in\Delta} \lambda_\delta \delta \right) := \sum_{\delta\in\Delta} \lambda_\delta r_\alpha(\delta) = \sum_{\delta\in\Delta} \lambda_\delta \left(\delta - 2\frac{\langle\delta,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha \right).$$

Elements of the affine Weyl group W_{a} that are conjugate to r_α are called *reflections*. Every reflection r determines a hyperplane

$$H_r := \{x \in \mathbb{A} \colon r(x) = x\},$$

which is also called a *wall*. Associated to each wall there are two *half-apartments* of $\mathbb{A}$ which are of the form

$$H^+_{\alpha,k} := \{x \in \mathbb{A} \colon \langle x,\alpha\rangle \geq k\} \quad \text{and} \quad H^-_{\alpha,k} := \{x \in \mathbb{A} \colon \langle x,\alpha\rangle \leq k\}$$

for $\alpha \in \Phi$ and $k \in \Lambda$. The *fundamental Weyl chamber* associated to a basis $\Delta \subseteq \Phi$ is given by

$$C_0 := \bigcap_{\alpha\in\Delta} H^+_{\alpha,0}.$$

A *Weyl-chamber*, or *sector*, of $\mathbb{A}$ is any of the sets $w(C_0)$ for $w \in W_{\mathrm{a}}$. If a sector s is a subset of another sector s', then s is called a *subsector* of s'. Following [Ben94, §2.5], we say that a subset $\Omega \subset \mathbb{A}$ is *convex*, if it is an intersection of half-apartments. A convex set $\Omega \subset \mathbb{A}$ is *closed*, if it is the intersection of finitely many half-apartments. With these definitions we define affine Λ-buildings as in [Ben94, §3.1], based on ideas of Tits [Tit86] and generalizing the notion of Λ-trees in Morgan–Shalen [MS84].

Definition 3.1. Let B be a set and $\mathcal{A}$ a set of maps from $\mathbb{A}$ to B. We say that $(B, \mathcal{A})$ is an *affine* Λ*-building of type* $\mathbb{A} = \mathbb{A}(\Phi, \Lambda, T)$, if it satisfies the following six axioms:

(A1) For all $f \in \mathcal{A}$ and $w \in W_{\mathrm{a}}$ we have $f \circ w \in \mathcal{A}$.

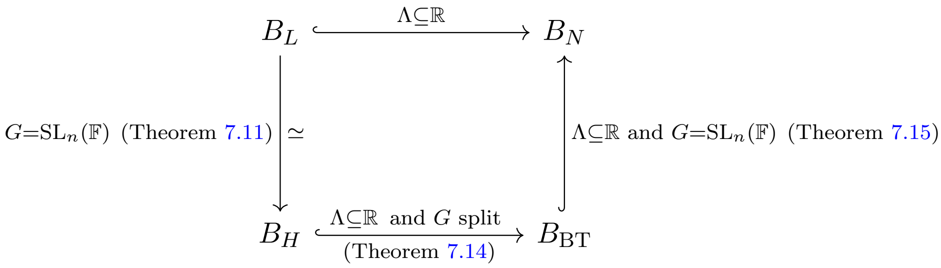
[1]A root system is crystallographic if $\alpha(H_\beta) \in \mathbb{Z}$ for all roots α, β. It turns out all root systems coming from Lie groups are crystallographic. An example of a non-crystallographic root system is the one of type $I(m)$ for m large enough.

(A2) For all $f, f' \in \mathcal{A}$, the set $\Omega := f^{-1}(f(\mathbb{A}) \cap f'(\mathbb{A})) \subseteq \mathbb{A}$ is a closed convex set and there exists $w \in W_{\mathrm{a}}$ such that $f|_{\Omega} = f' \circ w|_{\Omega}$.
(A3) For any two points in B there is an apartment containing both. That is, there exists a $f \in \mathcal{A}$ such that $x, y \in f(\mathbb{A})$.
(A4) Given sectors $S_1, S_2 \subseteq \mathbb{A}$, then there exist sectors $S_1', S_2' \subseteq \mathbb{A}$ contained in S_1 respectively S_2, such that $S_1' \cup S_2' \subseteq f(\mathbb{A})$.
(A5) If the images of three maps $f_1, f_2, f_3 \in \mathcal{A}$ pairwise intersect in a halfapartment, then they intersect.
(A6) For any chart $f \in \mathcal{A}$ and any point $p \in f(\mathbb{A})$, there is a distance-diminishing retraction $r_{f,p} \colon B \to f(\mathbb{A})$ with $(r_{f,p})^{-1}(p) = p$.

Note that axiom (A6) makes sense since axioms (A1)-(A3) imply that we can define a Λ-distance on B, that is a function $d \colon B \times B \to \Lambda$ satisfying all conditions of the definition of a Λ-metric but the triangle inequality, see [Ben94, Remarks 3.1 and 3.2]. Axiom (A6) implies that d in fact satisfies the triangle inequality, hence defines a metric on B.

If Λ can be inferred from the context, we say that B is a *generalized affine building*. The set $\mathcal{A}$ is called the *atlas* of the generalized affine building B, and its elements are called *charts*.

3.2. **Examples.** We now give examples of generalized affine buildings from the literature, namely the norm building B_N, the lattice building B_L, homogeneous buildings B_H and Bruhat–Tits buildings B_{BT}. All these constructions use a reductive algebraic group G over a field $\mathbb{F}$ with valuation $v \colon \mathbb{F} \to \Lambda \cup \{\infty\}$. In Section 7 we will relate these different models in specific contexts, as illustrated in the following diagram from the introduction, see Figure 1.



In the special case $\Lambda = \mathbb{R}$, all models are isomorphic. Let us now define the buildings in question.

Example 3.2 (Norm building B_N)**.** This model for an affine building has been studied in various settings in [GI63, Ger81, BT84]; we follow [Par23, §3].

Let $\mathbb{F}$ be any field with a non-Archimedean rank one valuation $v \colon \mathbb{F}^\times \to \Lambda < \mathbb{R}$. For $a \in \mathbb{F}$, set $|a| := \exp(-v(a)) \in \mathbb{R}$ and let $V := \mathbb{F}^n$. An *ultrametric norm* is a function $\eta \colon V \to \mathbb{R}_{\geq 0}$ that satisfies for all $a \in \mathbb{F}$ and $v, w \in V$

(a) $\eta(v) = 0$ if and only if $v = 0$,
(b) $\eta(av) = |a|\eta(v)$, and
(c) $\eta(v + w) \leq \max\{\eta(v), \eta(w)\}$.

An ultrametric norm η is *adapted* to a basis $\mathcal{E} = \{e_1, \ldots, e_n\}$ of V if

$$\eta\left(\sum_{i=1}^{n} a_i e_i\right) = \max\{|a_1|\eta(e_1), \ldots, |a_n|\eta(e_n)\},$$

and η is *adaptable* if there exists a basis to which it is adapted. The *norm building* B_N is the set of all homothety classes of adaptable ultrametric norms.

The model apartment can be identified with

$$\mathbb{A} \cong \mathbb{R}^n / \mathbb{R}(1, \ldots, 1) \cong \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \colon \sum_{i=1}^{n} x_i = 0 \right\}$$

and the spherical Weyl group is the symmetric group on n letters acting on $\mathbb{A}$ by permuting the entries. To a basis $\mathcal{E}$ and an ultrametric norm η adapted to $\mathcal{E}$, we associate a *chart* $f_{[\eta],\mathcal{E}} \colon \mathbb{A} \to B_N$ by

$$f_{[\eta],\mathcal{E}}((x_1, \ldots, x_n)) \left(\sum_{i=1}^{n} a_i e_i \right) := \max_i \left\{ e^{-x_i} |a_i| \eta(e_i) \right\}.$$

Let $\mathcal{A}$ denote the set of charts. The pair $(B_N, \mathcal{A})$ then is an affine $\mathbb{R}$-building of type $(A_{n-1}, \mathbb{R}, \mathbb{R}^n/\mathbb{R}(1, \ldots, 1))$, see [Par23, Sections 3B - 3F]. We note that B_N also admits a different atlas $\mathcal{A}'$, so that $(B_N, \mathcal{A}')$ is an affine $\mathbb{R}$-building of type $(A_{n-1}, \mathbb{R}, \Lambda^n/\Lambda(1, \ldots, 1))$, see [Par23, Remark in Section 3B4].

Example 3.3 (Lattice building B_L)**.** We now recall what we call the *lattice building*, i.e. the space of homethety classes of lattices for $\mathbb{F}^n$. This was defined in [Ron09, Section 9.2] in the discrete case, and in [Ben94, Example 3.2] in general. We follow the latter exposition.

Let Λ be an ordered abelian group (not necessarily of rank one) and $\mathbb{F}$ a field with a Λ-valuation $v \colon \mathbb{F}^\times \to \Lambda$ (in particular non-Archimedean). Denote by $\mathcal{O} := \{x \in \mathbb{F} \mid v(x) \geq 0\}$ the valuation ring. A *lattice* (sometimes called $\mathcal{O}$-lattice) of $\mathbb{F}^n$ is the set $\mathcal{O}e_1 + \ldots + \mathcal{O}e_n$, where $\{e_1, \ldots, e_n\}$ is a basis of $\mathbb{F}^n$. Two lattices L_1 and L_2 are *homothetic* if there exists $x \in \mathbb{F}$ such that $xL_1 = L_2$. We write $[L]$ for the homothety class of a lattice L. The *lattice building* B_L is the set of all homothethy classes of lattices in $\mathbb{F}^n$, i.e.

$$B_L := \{[L] \mid L \text{ is a lattice}\}.$$

The model apartment can be identified with $\mathbb{A} \cong \Lambda^{n-1}$. A basis $\mathcal{E} = \{e_1, \ldots, e_n\}$ determines the lattice $\mathcal{O}^n$ of $\mathbb{F}^n$. To a basis we define a chart $f_\mathcal{E} \colon \Lambda^{n-1} \to B_L$ as follows. For $(\lambda_1, \ldots, \lambda_{n-1}) \in \Lambda^{n-1}$, we set

$$f_\mathcal{E}((\lambda_1, \ldots, \lambda_{n-1})) := [\mathcal{O}(x_{\lambda_1} e_1) + \mathcal{O}(x_{\lambda_2 - \lambda_1} e_2) + \ldots + \mathcal{O}(x_{\lambda_{n-1} - \lambda_{n-2}} e_{n-1}) + \mathcal{O}(x_{-\lambda_{n-1}} e_n)],$$

where $x_{\lambda_i} \in \mathbb{F}$ with $v(x_{\lambda_i}) = \lambda_i$. Note that the so obtained lattice is independent of the choices of x_{λ_i}.

Let $\mathcal{A}$ denote the set of charts. The pair $(B_L, \mathcal{A})$ is a generalized affine building of type $(A_{n-1}, \Lambda, \Lambda^{n-1})$, see [Ben94, Example 3.2].

Example 3.4 (Bruhat–Tits buildings B_{BT})**.** In their seminal works [BT72, BT84] Bruhat–Tits construct affine buildings from reductive algebraic groups that are quasi-split with respect to a valued field $\mathbb{F}$ with value group $\Lambda \subseteq \mathbb{R}$. More general cases in which the Bruhat–Tits building exists have been investigated over the years for example in [MSVM14, Str14]. More recently, this construction was generalized to value groups Λ that are not necessarily a subgroup of $\mathbb{R}$ [HIL23]. We restrict our setting to the case when the group is semisimple and split, but we allow for value groups that are not necessarily a subgroup of $\mathbb{R}$. Our main reference when $\Lambda \subseteq \mathbb{R}$ is [BT72, Chapters 6 and 7] and for general Λ [HIL23].

Let $\mathbb{F}$ be a field with a valuation $\omega \colon \mathbb{F}^\times \to \Lambda$. The *rank* $\mathrm{rk}(\Lambda)$ of Λ is the (totally ordered) set of Archimedean equivalence classes of Λ. Hahn's embedding theorem states that Λ can be viewed as a subgroup of the additive group

$$\mathfrak{R} := \{(x_r)_{r \in \mathrm{rk}(\Lambda)} \in \mathbb{R}^{\mathrm{rk}(\Lambda)} \colon \operatorname{supp}((x_r)_{r \in \mathrm{rk}(\Lambda)}) \text{ is a well-ordered subset of } \mathrm{rk}(\Lambda)\}$$

endowed with a lexicographical order (see e.g. [Gra56]). In the classical case [BT72], $\Lambda \subseteq \mathbb{R} = \mathfrak{R}$ and $\mathrm{rk}(\Lambda)$ consists of a single element. Let $\mathbf{G}$ a semisimple, connected, simply-connected algebraic $\mathbb{F}$-group that is split over $\mathbb{F}$ (such as SL_n). Let $\mathbf{S}$ be a maximal $\mathbb{F}$-split torus. For a root $\alpha \in \Phi$ in the relative root system $\Phi := {}_{\mathbb{F}}\Phi$, we consider the root group $\mathbf{U}_\alpha$. Let $G = \mathbf{G}(\mathbb{F})$, $U_\alpha = \mathbf{U}_\alpha(\mathbb{F})$, $T = \mathrm{Cent}_{\mathbf{G}}(\mathbf{S})(\mathbb{F})$ and $N = \mathrm{Nor}_{\mathbf{G}}(\mathbf{S})(\mathbb{F})$. From [BT72, (6.1.3)b)] and [HIL23, Section 7.19], we get that there are subgroups M_α such that $(T, (U_\alpha, M_\alpha)_{\alpha\in\Phi})$ is a *generating root group datum* that admits a *valuation* $\varphi_\alpha \colon U_\alpha \to \mathbb{R} \cup \{0\}$ for all $\alpha \in \Phi$, see [BT72, (6.2.3)b)], [BT84, (4.1.19)(ii)], and [HIL23, Proposition 7.24]; for the classical groups see [BT72, Chapter 10, page 208ff].

We consider the Euclidean vector space $(V, \langle\cdot,\cdot\rangle)$ such that the dual space $(V^\star, \langle\cdot,\cdot\rangle)$ is spanned by Φ. The group N acts on the root groups by conjugation $nU_\alpha n^{-1} = U_{n.\alpha}$ and this action descends to the action of the spherical Weyl group ${}_{\mathbb{F}}W := N/T$ on $\Phi \subseteq V^\star$. The dual root system $\Phi^\vee \subseteq V$ of $\Phi \subset V^\star$ consists of the dual roots $\alpha^\vee \in \Phi^\vee$ defined by

$$\langle \alpha^\vee, v\rangle = \frac{2\,\alpha(v)}{\langle\alpha,\alpha\rangle} \quad \text{for all } v \in V$$

where $\alpha \in \Phi$. We note that the linear maps $\alpha \in V^\star$ extend to linear maps on the $\mathfrak{R}$-vector space $V \otimes_{\mathbb{Q}} \mathfrak{R}$, and the action of the spherical Weyl group ${}_{\mathbb{F}}W$ extends to $V \otimes_{\mathbb{Q}} \mathfrak{R}$. The space $V \otimes_{\mathbb{Q}} \mathfrak{R} \cong \mathrm{Span}_{\mathbb{Q}}(\Phi^\vee) \otimes_{\mathbb{Q}} \mathfrak{R}$ can be identified with the *affine space of root group valuations*

$$A = \{(\varphi^v_\alpha)_\alpha \colon \exists\, v \in V \otimes_{\mathbb{Q}} \mathfrak{R},\ \forall \alpha \in \Phi,\ \forall u \in U_\alpha \colon \varphi^v_\alpha(u) = \varphi_\alpha(u) + \alpha(v), \}$$

and we abbreviate the root group valuation $(\varphi^v_\alpha)_\alpha$ by $\varphi + v$. For $n \in N$,

$$(n.\varphi)_\alpha(u) = \varphi_{n^{-1}.\alpha}(n^{-1}un)$$

defines a root group valuation and an action $\nu \colon N \times A \to A$ by

$$\nu(n)(\varphi + v) \coloneqq n.\varphi + n.v,$$

where the $n.v$ comes from the action of the spherical Weyl group on $V \otimes_{\mathbb{Q}} \mathfrak{R}$. The kernel of this action is denoted by H and N/H is called the affine Weyl group. For $x \in A \cong V$, let

$$P_x = \langle u \in U_\alpha \colon \exists\, \lambda \in \Lambda \text{ such that } \varphi_\alpha(u) \geq \lambda \geq -\alpha(x)\rangle \cdot H.$$

The *Bruhat–Tits building* is now defined as the quotient $B_{\mathrm{BT}} \coloneqq (G \times A)/{\sim}$ for the equivalence relation $(g, x) \sim (h, y)$ when

$$\exists\, n \in N \colon g^{-1}hn \in P_x \quad \text{and} \quad \nu(n)(x) = y.$$

The Bruhat–Tits building is a generalized affine building of type $\mathbb{A}(\Phi^\vee, \mathfrak{R}, T/H)$ [HIL23, Theorem 3.30]. The apartment $\mathbb{A}$ is modeled on A and we have the map $f_0 \colon A \to B_{\mathrm{BT}}$, $x \mapsto [\mathrm{Id}, x]$. The natural action of G on $G \times A$ descends to an action on B_{BT} given by $g.[h, x] = [gh, x]$. The atlas of B_{BT} is given by $\mathcal{A} \coloneqq \{g.f_0 \mid g \in G\}$, where $g.f(x) \coloneqq [g, x]$ for all $x \in A$. Indeed, the affine Weyl group W_{a} is defined as $\nu(N) = W_{\mathrm{a}}$, $N < G$, and thus only translations by Λ-valued vectors are possible.

We remark that the root group valuations $(\varphi_\alpha)_\alpha$ are *compatible with* the field valuation ω, see [BT84, 4.2.7(2)] and [HIL23, Section 7.44, Fact 7.12], for all $t \in T$

$$(\star) \qquad \varphi_\alpha(tut^{-1}) = \varphi_\alpha(u) + \omega(\alpha(t)),$$

which means that if $t \in T$ and $x \in A \cong V$ satisfy $(-\omega)(\alpha(t)) = \alpha(x)$, then $[t, 0] = [\mathrm{id}, x] \in I$.

Example 3.5 (Homogeneous buildings B_H)**.** In [KT02], for more details see [App24], Kramer–Tent define a building associated to the following data. Let $\mathbf{G}$ be a semisimple self-adjoint linear algebraic $\mathbb{Q}$-group $\mathbf{G} < \mathrm{SL}_n$ for some $n \in \mathbb{N}$ and let $\mathbb{F}$ be a real closed field. To define a building, we need $\mathbb{F}$ to be non-Archimedean with an order-compatible valuation $v \colon \mathbb{F}^\times \to \Lambda$ (not necessarily of rank one), but for now let $\mathbb{F}$ just be a real closed field, possibly $\mathbb{F} = \mathbb{R}$. Denote by $G := \mathbf{G}(\mathbb{F})$ the $\mathbb{F}$-points of the algebraic group $\mathbf{G}$.

The group $\mathrm{SL}_n(\mathbb{F})$ acts transitively on the set

$$P_1(n, \mathbb{F}) = \{M \in \mathbb{F}^{n\times n} : M = M^T, \det(M) = 1, M \gg 0\}$$

of positive definite symmetric matrices of determinant one by congruence, i.e. $g.M := gMg^T$. Let $X_\mathbb{F}$ be the orbit $G.\mathrm{Id}$ of $\mathrm{Id} \in P_1(n, \mathbb{F})$. If $\mathbb{F} = \mathbb{R}$, $X_\mathbb{R}$ is a model of a symmetric space of non-compact type that is a totally geodesic submanifold of the symmetric space $P_1(n, \mathbb{R})$. When $\mathbb{F}$ is non-Archimedean, we call $X_\mathbb{F}$ a *non-standard symmetric space*. Given an order-preserving valuation $v \colon \mathbb{F}^\times \to \Lambda$, it is then possible to define a G-invariant, Λ-valued pseudo-distance $d\colon X_\mathbb{F} \times X_\mathbb{F} \to \Lambda$ whose quotient $B_H = X_\mathbb{F}/\!\sim$ after identifying all points of distance 0 is an affine Λ-building in the following sense.

Let $\mathbf{K} := \mathbf{G} \cap \mathrm{SO}_n$. Let $\mathbf{S} < \mathbf{G}$ be a maximal $\mathbb{R}$-split torus that is self-adjoint. Let $A_\mathbb{F}$ be the semi-algebraically connected component of the identity in $\mathbf{S}_\mathbb{F}$. When $\mathbb{F} = \mathbb{R}$, $\mathfrak{g} := \mathrm{Lie}(\mathbf{G}_\mathbb{R})$ admits a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ given by the Cartan involution $\theta\colon X \mapsto -X^T$. The group $A_\mathbb{R}$ is a connected real Lie group whose Lie algebra $\mathfrak{a} \subseteq \mathfrak{p}$ can be used to obtain the restricted root space decomposition

$$\mathrm{Lie}(\mathbf{G}_\mathbb{R}) = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha,$$

where $(\Phi, \mathfrak{a})$ is a root system with spherical Weyl group

$$W_\mathrm{s} = \mathrm{Nor}_{\mathbf{K}_\mathbb{R}}(A_\mathbb{R}) / \mathrm{Cent}_{\mathbf{K}_\mathbb{R}}(A_\mathbb{R}).$$

Associated to every root $\alpha \in \Phi$, there is an algebraic character $\chi_\alpha\colon A_\mathbb{F} \to \mathbb{F}_{>0}$. The Cartan decomposition $\mathbf{G}_\mathbb{F} = \mathbf{K}_\mathbb{F}\mathbf{A}_\mathbb{F}\mathbf{K}_\mathbb{F}$ can then be used to define a *Cartan projection* $\delta_\mathbb{F}\colon X_\mathbb{F} \to C_\mathbb{F}$, where $C_\mathbb{F} := \{a \in A_\mathbb{F} : \chi_\alpha(a) \geq 1,\ \forall \alpha \in \Delta\}$, where Δ is a basis of Φ, see e.g. [App24, Lemma 7.2]. Then, $N_\mathbb{F}\colon A_\mathbb{F} \to \mathbb{F}_{>0}$ defined by

$$N_\mathbb{F}(a) := \prod_{\alpha\in\Phi} \max\left\{\chi_\alpha(a), \chi_\alpha(a)^{-1}\right\}$$

is a semi-algebraic multiplicative $\mathbb{F}_{\geq 1}$-valued G-invariant norm on $A_\mathbb{F}$. When $\mathbb{F}$ is non-Archimedean, $N_\mathbb{F}$ and $\delta_\mathbb{F}$ together with an order-compatible valuation v, gives a pseudo-metric $d := -v \circ N_\mathbb{F} \circ \delta_\mathbb{F}$ on $X_\mathbb{F}$, and a metric on $B_H = X_\mathbb{F}/\!\sim$. Let $o = [\mathrm{Id}] \in B_H$ be a base point. It is then possible to show that the apartment $\mathbb{A} := A_\mathbb{F}.o$ is a $\mathbb{Q}$-vector space isomorphic to $\mathrm{Span}_\mathbb{Q}(\Phi^\vee) \otimes_\mathbb{Q} \Lambda$, where $\Phi^\vee$ is the dual root system, of dimension $r = \dim(\mathfrak{a}) = \mathrm{rank}(\mathbf{G}_\mathbb{R})$. Taking $T = \mathbb{A} \cong \Lambda^r$, B_H is an affine Λ-building of type $(\Phi^\vee, \Lambda, \Lambda^r)$ [App24, Theorem 8.1] in the case that $\Phi^\vee$ is reduced. If $f_0\colon \mathbb{A} \to B_H$ denotes the inclusion, the atlas is given by $\mathcal{A} = \{g.f_0\colon \mathbb{A} \to B_H \mid g \in G_\mathbb{F}\}$. The following compatibility condition is useful [App24]: for every point $x \in \mathbb{A}$ and $a \in A_\mathbb{F}$, $f_0(x) = a.o$ if and only if

$$(-v)(\chi_\alpha(a)) = \alpha(x). \tag{$\star\star$}$$

The descriptions of the following stabilizers will be useful.

Proposition 3.6 (Theorems 7.11 and 8.19 in [App24])**.** *Let $\mathcal{O} = \{x \in \mathbb{F} : v(x) \leq 0\}$ be the valuation ring of $\mathbb{F}$. Let $T = \mathrm{Cent}_G(\mathbf{S}(\mathbb{F}))$*[2]*. Then $\mathrm{Stab}_G(o) = G(\mathcal{O}) := G \cap \mathcal{O}^{n\times n}$ and*

$$\mathrm{Stab}_G(f_0) = \{g \in G : g.a.o = a.o \textit{ for all } a \in A_\mathbb{F}\} = T(\mathcal{O}) := T \cap \mathcal{O}^{n\times n}.$$

[2]Here we defined $T = \mathrm{Cent}_G(\mathbf{S}(\mathbb{F}))$, but in [App24], $T = \mathrm{Cent}_G(A)$, where A is the semi-algebraically connected component of $\mathbf{S}(\mathbb{F})$.

Remark 3.7. It was announced in [KT02, Theorem 5.7], and proven in [App24], that in the setting of the above example, there is another description of the homogeneous building. Using the transitive action of G on B_H, the underlying set of the homogeneous building B_H can be identified with $\mathbf{G}(\mathbb{F})/\mathbf{G}(\mathcal{O})$. The model apartment $\mathbb{A}$ is given by $A(\mathbb{F})/A(\mathcal{O})$, and an atlas of charts by

$$\mathcal{A} = \{f_g \colon A(\mathbb{F})/A(\mathcal{O}) \to \mathbf{G}(\mathbb{F})/\mathbf{G}(\mathcal{O}), [a] \mapsto g.[a] \mid g \in G(\mathbb{F})\}.$$

4. Morphisms of generalized affine buildings

The goal of this section is to introduce the definition of morphisms of generalized affine buildings. Before we can do so we define morphism of apartments.

4.1. **Morphisms of apartments.** Let Φ and Φ' be two crystallographic root systems with respective basis Σ and Σ'. Let Λ and Λ' be ordered abelian groups and $\mathbb{A} = \mathrm{Span}_{\mathbb{Q}}(\Phi) \otimes_{\mathbb{Q}} \Lambda$ and $\mathbb{A}' = \mathrm{Span}_{\mathbb{Q}}(\Phi') \otimes_{\mathbb{Q}} \Lambda'$ the corresponding model apartments. For $T < \mathbb{A}$ and $T' < \mathbb{A}'$, we have the respective affine Weyl groups with respect to T and T', denoted by $W_{\mathrm{a}} = T \rtimes W_{\mathrm{s}}(\Phi)$ respectively $W'_{\mathrm{a}} = T' \rtimes W_{\mathrm{s}}(\Phi')$. There is a natural action of the affine Weyl groups on their respective model apartments. We will write $\mathbb{A} = \mathbb{A}(\Phi, \Lambda, T)$ to highlight that the apartment comes equipped with the action of the affine Weyl group W_a.

Definition 4.1. Let $\mathbb{A} = \mathbb{A}(\Phi, \Lambda, T)$ and $\mathbb{A}' = \mathbb{A}(\Phi', \Lambda', T')$ be two model apartments. Let $L \colon \mathrm{Span}_{\mathbb{Q}}(\Phi) \to \mathrm{Span}_{\mathbb{Q}}(\Phi')$ be a $\mathbb{Q}$-linear map, $\gamma \colon \Lambda \to \Lambda'$ a morphism of ordered abelian groups (order preserving group homomorphism) and $\sigma \colon W_{\mathrm{a}} \to W'_{\mathrm{a}}$ a function (not necessarily a group homomorphism). The triple (L, γ, σ) is called a *morphism of apartments* if for all $a \in \mathbb{A}$ and $w \in W_{\mathrm{a}}$ we have

$$(L \otimes_{\mathbb{Q}} \gamma)(w.a) = \sigma(w).((L \otimes_{\mathbb{Q}} \gamma)(a)).$$

This means that the following diagram commutes for every $w \in W_{\mathrm{a}}$.

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{L\otimes\gamma} & \mathbb{A}' \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle \sigma(w)} \\ \mathbb{A} & \xrightarrow[L\otimes\gamma]{} & \mathbb{A}' \end{array}$$

We will denote morphisms of apartments by $\tau = (L, \gamma, \sigma)$, and when it is convenient we will write by slight abuse of notation $\tau \colon \mathbb{A} \to \mathbb{A}'$ and $\tau = L \otimes \gamma$.

We call τ *injective (resp. surjective)* if L and γ are injective (resp. surjective). It is a linear algebra exercise to see that τ is injective (resp. surjective) if and only if $L \otimes_{\mathbb{Q}} \gamma$ is injective (resp. surjective).

Example 4.2. Suppose $\Phi = \Phi'$ and $\gamma \colon \Lambda \to \Lambda'$ is a morphism of ordered abelian groups such that $\gamma(T) \subset T'$. Then $\mathrm{Id} \otimes_{\mathbb{Q}} \gamma \colon \mathbb{A} \to \mathbb{A}'$ defines a morphism of apartments. Furthermore, this recovers the definition of morphisms in [SS11].

Remark 4.3. The identity morphism is given by $L = \mathrm{Id}_{\mathrm{Span}_{\mathbb{Q}}(\Phi)}$, $\gamma = \mathrm{Id}_\Lambda$ with $\sigma = \mathrm{Id}_{W_{\mathrm{a}}}$ and composition of morphisms is given by composition of L, γ and σ. It is not hard to check that with these notions, model apartments form a category. A morphism $\tau \colon \mathbb{A} \to \mathbb{A}'$ then is an *isomorphism of apartments* if there is a morphism $\tau^{-1} \colon \mathbb{A}' \to \mathbb{A}$ with $\tau^{-1} \circ \tau = \mathrm{Id}_{\mathbb{A}}$ and $\tau \circ \tau^{-1} = \mathrm{Id}_{\mathbb{A}'}$.

Definition 4.4. Let $\mathbb{A} = \mathbb{A}(\Phi, \Lambda, T)$ and $\mathbb{A}' = \mathbb{A}(\Phi', \Lambda', T')$ be two model apartments. Let $L \colon \mathrm{Span}_{\mathbb{Q}}(\Phi) \to \mathrm{Span}_{\mathbb{Q}}(\Phi')$ be a $\mathbb{Q}$-linear map, $\gamma \colon \Lambda \to \Lambda'$ a morphism of ordered abelian groups (order preserving group homomorphism) and $\sigma\colon W_{\mathrm{a}} \to W'_{\mathrm{a}}$ a group homomorphism. The triple (L, γ, σ) is called a *morphism of apartments* if $L \otimes \gamma(T) \subseteq T'$ and for all $a \in \mathbb{A}$ and $w \in W_{\mathrm{s}}$ we have

$$(L \otimes_{\mathbb{Q}} \gamma)(w.a) = \sigma(w).((L \otimes_{\mathbb{Q}} \gamma)(a)).$$

This means that the following diagram commutes for every $w \in W_{\mathrm{s}}$.

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{L\otimes\gamma} & \mathbb{A}' \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle \sigma(w)} \\ \mathbb{A} & \xrightarrow[L\otimes\gamma]{} & \mathbb{A}' \end{array}$$

We will denote morphisms of apartments by $\tau = (L, \gamma, \sigma)$, and when it is convenient we will write by slight abuse of notation $\tau\colon \mathbb{A} \to \mathbb{A}'$ and $\tau = L \otimes \gamma$.

We call τ *injective (resp. surjective)* if L and γ are injective (resp. surjective). It is a linear algebra exercise to see that τ is injective (resp. surjective) if and only if $L \otimes_{\mathbb{Q}} \gamma$ is injective (resp. surjective). If τ is injective, so is σ, but if τ is surjective, σ may not be surjective (for instance in the inclusion of type A_2 in type G_2).

Lemma 4.5. *If (L, γ, σ) is a morphism of apartments $\mathbb{A} \to \mathbb{A}'$, then $\sigma\colon W_{\mathrm{s}} \to W'_{\mathrm{s}}$ and $L \otimes \gamma|_T \colon T \to T'$ extend to a group homomorphism $\overline{\sigma}\colon W_{\mathrm{a}} \to W'_{\mathrm{a}}$ and*

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{L\otimes\gamma} & \mathbb{A}' \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle \overline{\sigma}(w)} \\ \mathbb{A} & \xrightarrow[L\otimes\gamma]{} & \mathbb{A}' \end{array}$$

commutes for all $w \in W_{\mathrm{a}}$.

Proof. We denote by $t^x\colon \mathbb{A} \to \mathbb{A}$ the translation by $x \in \mathbb{A}$. Let $t^x, t^{x_1}, t^{x_2} \in T$, $w, w_1, w_2 \in W_{\mathrm{s}}$. We define a map $\overline{\sigma}\colon W_{\mathrm{a}} \to W'_{\mathrm{a}}$ by $\overline{\sigma}(t^x w) := t^{L\otimes\gamma(x)}\sigma(w)$. By the semidirect product structure of $W_{\mathrm{a}} = T \rtimes W_{\mathrm{s}}$, $w_1 t^{x_2} = t^{w_1(x_2)} w_1$, so

$$\begin{aligned} \overline{\sigma}(t^{x_1}w_1 \cdot t^{x_2}w_2) &= \sigma\left(t^{x_1}t^{w_1(x_2)}w_1w_2\right) = t^{L\otimes\gamma(x_1+w_1(x_2))}\sigma(w_1w_2) \\ &= t^{L\otimes\gamma(x_1)}t^{L\otimes\gamma(w_1(x_2))}\sigma(w_1)\sigma(w_2) \\ &= t^{L\otimes\gamma(x_1)}t^{\sigma(w_1)(L\otimes\gamma(x_2))}\sigma(w_1)\sigma(w_2) \\ &= t^{L\otimes\gamma(x_1)}\sigma(w_1)t^{L\otimes\gamma(x_2)}\sigma(w_2) = \overline{\sigma}(t^{x_1}w_1) \cdot \overline{\sigma}(t^{x_2}w_2) \end{aligned}$$

shows that $\overline{\sigma}$ is a group homomorphism. If $y \in \mathbb{A}$, we have

$$\begin{aligned} L \otimes \gamma(t^x w(y)) &= L \otimes \gamma(x + w(y)) = L \otimes \gamma(x) + L \otimes \gamma(w(y)) \\ &= t^{L\otimes\gamma(x)}(\sigma(w)(y)) = \overline{\sigma}(t^x w)(y), \end{aligned}$$

so the diagram commutes. □

4.2. **Morphisms of buildings.** We use the notion of morphisms of apartments to define a notion of morphisms of generalized affine buildings.

Definition 4.6. Let $B = (B, \mathcal{A})$ be an affine Λ-building of type $\mathbb{A} = \mathbb{A}(\Phi, \Lambda, T)$ and $B' = (B', \mathcal{A}')$ an affine Λ'-building of type $\mathbb{A}' = \mathbb{A}(\Phi', \Lambda', T')$. A *morphism of generalized affine buildings* is a collection of maps

$$\psi\colon B \to B', \quad \varphi\colon \mathcal{A} \to \mathcal{A}', \quad \tau\colon \mathbb{A} \to \mathbb{A}',$$

where τ is a morphism of apartments, such that the following diagram commutes for all $f \in \mathcal{A}$.

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{\;f\;} & B \\ {\scriptstyle\tau}\big\downarrow & & \big\downarrow{\scriptstyle\psi} \\ \mathbb{A}' & \xrightarrow[\varphi(f)]{} & B' \end{array}$$

We denote morphisms of generalized affine buildings by $m = (\psi, \varphi, \tau)$, and when it is convenient we will write by slight abuse of notation $m\colon (B, \mathcal{A}) \to (B', \mathcal{A}')$ or $m\colon B \to B'$. We say that a morphism (ψ, φ, τ) is *injective* (resp. *surjective*) if ψ, φ and τ are injective (resp. surjective).

Example 4.7. Since there is no order-preserving group homomorphism from $\mathbb{R}$ to $\mathbb{Q}$, we obtain with this definition that the only morphism from the $\mathbb{R}$-building $\mathbb{R}$ to the $\mathbb{Q}$-building $\mathbb{Q}$ is the trivial morphism.

Remark 4.8. The identity morphism is given by $\psi = \mathrm{Id}_{\mathcal{A}}$, $\varphi = \mathrm{Id}_B$ and $\tau = \mathrm{Id}_{\mathbb{A}}$ and composition of morphisms is given by the composition of all three ψ, φ and τ. It is not hard to check that with these notions, generalized affine buildings form a category. A morphism $m = (\psi, \varphi, \tau)\colon B \to B'$ is then called an *isomorphism* if there exists $m^{-1}\colon B' \to B$ with $m^{-1} \circ m = \mathrm{Id}_B$ and $m \circ m^{-1} = \mathrm{Id}_{B'}$.

4.3. **Relation to existing notions in the literature.** We finish this section by explaining how our definition relates to existing notions of morphisms and isomorphisms of apartments and (generalized affine) buildings already present in the literature. Already in [Tit74, Tit86], Tits defines a notion of isomorphism of discrete affine buildings, and classifies them up to isomorphism. Similarly, still in the discrete case, Gérardin states that the norm building "is" the Bruhat–Tits building [Ger81, Theorem in §2.3.6]. He defines in §1.3.7 an *isomorphism of apartments* as a linear map that preserves distances and that exchanges the walls. This is stronger than our definition, as we do not ask a morphism of apartments to preserve distances.

More generally, when the valuation takes values in a discrete subset of $\mathbb{R}$, there are a several notions of morphisms, see e.g. [Lan00, KP23]. We briefly explain Landvogt's notion [Lan00] and his result on Bruhat–Tits buildings, as it can be compared to Theorem 1.5. Landvogt showed a functoriality property, namely that a homomorphism of algebraic k-groups, where k is a quasi-local field, induces an equivariant continuous map between the associated Bruhat–Tits buildings, that is *toral*—a notion that ensures that apartments are mapped to apartments. Furthermore, after suitable normalization of the metric, this map is an isometry. In the case of inclusion of connected, reductive k-subgroups the author also examines the set of all maps with the above properties. A toral map should be thought of as the analog of our notion of morphism of apartments, and it would be interesting to compare these two notions in the discrete setting.

A comparison to the notion in [KP23] would also be desirable. However in general it is not clear how properties of a Λ-buildings for Λ a discrete subgroup of $\mathbb{R}$ relate to properties of affine buildings.

In the non-discrete case, but in the setting of Euclidean buildings, Rousseau defines notions of (weak) morphisms of apartments, buildings and sub-buildings [Rou04]. A *weak morphism of apartments* (endowed with a Euclidean distance) is an affine map $\varphi\colon \mathbb{A} \to \mathbb{A}'$ that satisfies three axioms related to the affine Weyl group actions, the walls of the apartments and their Euclidean distances

[Rou04, Definition 1.1.4.1]. However the composition of two weak morphisms is in general not a weak morphism. Rousseau then defines a *morphism of apartments* to be a metric weak morphism, meaning that φ preserves distances up to the kernel of the linear part of φ. Note that our definition does à priori not take the metrics on $\mathbb{A}$ and $\mathbb{A}'$ into account, and allows thus for more flexibility. The notion of *(weak) morphisms of buildings* [Rou04, Definition 2.1.13.1] is then directly defined as a map between two Euclidean buildings that maps an apartment $\mathbb{A}$ to an apartment $\mathbb{A}'$, and that when restricted to $\mathbb{A}$ is a (weak) morphism of apartments from $\mathbb{A}$ to $\mathbb{A}'$.

Similarly, still in the setting of Euclidean buildings, Kleiner–Leeb define a *subbuilding* as a metric subspace of a Euclidean building which admits a Euclidean building structure [KL97, Subsections 4.7]. It is not clear, whether any subbuilding is the image of a morphism of buildings in our sense. However we believe that the image of an injective morphism of buildings gives rise to a subbuilding.

Several notions of morphisms of root systems have been developed. In [LN04] and similarly in [Dye09] morphisms are linear maps $L\colon \mathrm{Span}_{\mathbb{R}}(\Phi) \to \mathrm{Span}_{\mathbb{R}}(\Phi')$ such that $L(\Phi) \subseteq \Phi'$, possibly with some extra conditions. In our definition of morphisms of apartments, we also have a linear map L, but roots may not be sent to roots. This happens for instance in the context of functoriality under subgroups $G < G'$ (see Section 6.2, in particular the example in Figure 4), where the linear map L does not send roots to roots. A morphism of root systems in the category **RCE** defined in [LN04] satisfies the compatibility condition on the Weyl groups [LN04, Theorem 5.7], and thus defines a morphism of apartments (when $\Lambda = \mathbb{R}$) in our setting. This means that our notion is a generalisation of the one in [LN04]. The notion in [Dye09] is not stronger or weaker than ours.

The most general and at the same time closest to our notions are probably [Sch09] and [SS11], where the latter generalizes the former. Let us discuss these two now in more detail.

In [Sch09, Definition 5.5], Schwer defines a notion of *isomorphism* of generalized affine buildings. This definition agrees with Definition 4.6 where the author considers inverse maps to ψ and φ, and requires that τ is an automorphism of apartment, which according to personal communication with Petra Schwer coincides with our definition. Thus the definition of isomorphism given in [Sch09] agrees with Definition 4.6.

Schwer–Struyve consider in [SS11, Sections 3 and 5] a generalized affine building $(B, \mathcal{A})$ of type $\mathbb{A} := \mathbb{A}(\Phi, \Lambda, T)$ and an order-preserving surjective (resp. injective) group morphism $\gamma\colon \Lambda \to \Lambda'$. From this data, they construct a model apartment $\mathbb{A}' := \mathbb{A}(\Phi, \Lambda', T')$, where T' is the component-wise image of T under γ. In the first paragraph of Section 3, the authors construct a surjective (resp. injective) map $\mathbb{A} \to \mathbb{A}'$, which in our notation corresponds to two maps $\mathrm{Id} \otimes \gamma\colon \mathbb{A} \to \mathbb{A}'$ and $\mathrm{Id}\colon W_{\mathrm{a}} \to W_{\mathrm{a}}$. In fact, they show that these maps define a morphism of apartments in the sense of Definition 4.4. Note however that the root system Φ is the same. Let us denote by τ this morphism of apartments. Secondly, Schwer–Struyve construct a topological space B', a surjective (injective) map $\phi\colon B \to B'$ and for each $f \in \mathcal{A}$ a map $f'\colon \mathbb{A}' \to B'$. If we denote by $\mathcal{A}'$ the set of maps f' constructed, then we get a map $\varphi\colon \mathcal{A} \to \mathcal{A}'$. Then they show in [SS11, Sections 3 and 5] that $(B', \mathcal{A}')$ is a building and that $m = (\phi, \varphi, \tau)$ is a morphism (in the sense of Definition 4.6) between the buildings $(B, \mathcal{A})$ and $(B', \mathcal{A}')$ such that ϕ, φ, τ are surjective (resp. injective). Furthermore, in [SS11, Theorem 1.1] the authors prove that the respective spherical buildings of B and B' at infinity are isomorphic, and this isomorphism is induced by the morphism of buildings m.

In [HIL23, §7], the authors associate to a quasi-split reductive algebraic group $\mathbf{G}$ and a Henselian field $\mathbb{F}$, equipped with a valuation $v\colon \mathbb{F}^\times \to \Lambda$, a generalized affine building, denoted by $I(\mathbb{F}, v, \mathbf{G})$. They then construct, given a surjective morphism of totally ordered abelian groups $f\colon \Lambda \to \Lambda'$, a projection map $I(\mathbb{F}, v, \mathbf{G}) \to I(\mathbb{F}, f \circ v, \mathbf{G})$, which is surjective and compatible with the action of $\mathbf{G}(\mathbb{F})$. It would be interesting to investigate whether the projection map they construct is a surjective morphism of G-buildings in the sense of Definition 5.8.

5. G-buildings and their morphisms

The goal of this section is to construct morphisms of generalized affine buildings eqipped with group actions. Under sufficient transitivity assumptions this allows to define a morphism of generalized affine buildings by specifying a morphism of one apartment, and then moving this map around under the action of the group. This motivates the following definition.

Definition 5.1. Let G be a group and $(B, \mathcal{A})$ an affine Λ-building. We call B a *G-building* if G acts on both B and the atlas $\mathcal{A}$ such that the actions are compatible, i.e.

$$(g.f)(a) = g.(f(a)) \quad \forall\, g \in G,\ f \in \mathcal{A},\ a \in \mathbb{A}.$$

Note that we always have $\operatorname{Stab}_G(f) \subseteq \operatorname{Stab}_G(f(0))$. Often we may want to require transitivity of the action of G on $\mathcal{A}$ or on B.

5.1. **Transitivity properties of G-buildings.** Here are some direct consequences of the definition of G-buildings.

Proposition 5.2. *Let $(B, \mathcal{A})$ be a G-building of type $\mathbb{A} = \mathbb{A}(\Phi, \Lambda, T)$. If G acts transitively on B, then $T = \Lambda^n$.*

Proof. Let $f \in \mathcal{A}$ and $a \in \mathbb{A}$. By transitivity, there is $g \in G$ such that $f(0) = g.f(a)$. Recall axiom (A2), that says that for all f, $f' \in \mathcal{A}$, the set $\Omega := f^{-1}(f(\mathbb{A}) \cap f'(\mathbb{A}))$ is W_{a}-convex and there exists $w \in W_{\mathrm{a}}$ such that $f|_\Omega = f \circ w|_\Omega$. We apply it to f and $g.f$. We note that $0 \in \Omega$. From the second part we get $w \in W_{\mathrm{a}}$ such that $f(0) = g.f \circ w(0)$. Applying g^{-1} we get $f(w(0)) = g^{-1}.f(0) = f(a)$. Since f is injective we have $w(0) = a$. The element $w \in W_{\mathrm{a}} = T \rtimes W_{\mathrm{s}}$ can be written as $w = (t, w_s)$ with $t \in T$ and $w_s \in W_{\mathrm{s}}$. Since $w_s(0) = 0$, we have $w(0) = t(0) = a$, hence $t = w_a$. This shows that for all $a \in \mathbb{A}$, $w_a \in T$, and hence $T = \Lambda^n$. □

Conversely we have the following.

Proposition 5.3. *Let $(B, \mathcal{A})$ be a G-building of type $\mathbb{A} = \mathbb{A}(\Phi, \Lambda, T)$. If $T = \Lambda^n$ is the full translation group and G acts transitively on $\mathcal{A}$, then G also acts transitively on B.*

Proof. We use axioms

(A1) $\forall f \in \mathcal{A}\ \forall w \in W_{\mathrm{a}}\colon f \circ w \in \mathcal{A}$, and
(A3) $\forall x, y \in B\ \exists f \in \mathcal{A}\colon x, y \in f(\mathbb{A})$.

Let $x, y \in B$. By axiom (A3), there is a $f \in \mathcal{A}$ and $a, b \in \mathbb{A}$ such that $x = f(a), y = f(b)$. Since we assumed that $T = \Lambda^n$ is the full translation group, we have $w = b - a \in W_{\mathrm{a}} = T \rtimes W_{\mathrm{s}}$. By axiom (A1), we have that $f \circ w \in \mathcal{A}$. Since G acts transitively on $\mathcal{A}$, there is a $g \in G$, such that $g.f = f \circ w$. In particular $g.x = g.f(a) = f \circ w(a) = f(b) = y$, and hence G acts transitively on B. □

5.2. **Examples of G-buildings.** All examples discussed in Section 3.2 fall in fact in the framework of G-buildings for some appropriate group G. Let us now explain this and discuss certain transitivity properties in more detail.

Example 5.4 (Norm building, Example 3.2 revisited)**.** Recall that the norm building B_N associated to $V = \mathbb{F}^n$, where $\mathbb{F}$ is a field with a rank one valuation $v\colon \mathbb{F}^\times \to \Lambda < \mathbb{R}$, is an $\mathbb{R}$-building of type $(A_{n-1}, \mathbb{R}, \mathbb{R}^n/\mathbb{R}(1,\dots,1))$. We claim that it is also a G-building for $G = \mathrm{GL}_n(\mathbb{F})$. Indeed, for $g \in \mathrm{GL}_n(\mathbb{F})$ the action on B_N given by $g.\eta \coloneqq \eta \circ g^{-1}$ for η (a homothety class of) an adaptable ultrametric norm on V, and the action on $\mathcal{A}$ given by $g.f_{[\eta],\mathcal{E}} \coloneqq f_{[g.\eta],g\mathcal{E}}$ are compatible; see also [Par23, Sections 3A and 3B2]. Note that $\mathrm{GL}_n(\mathbb{F})$ acts transitively the set of apartments (the images of charts in $\mathcal{A}$), but not on B_N and $\mathcal{A}$ unless the valuation v is surjective onto $\mathbb{R}$.

Example 5.5 (Lattice building, Example 3.3 revisited)**.** The lattice building B_L associated to $\mathbb{F}^n$, where $\mathbb{F}$ is a field with a valuation $v\colon \mathbb{F}^\times \to \Lambda$ (not necessarily of rank one), is an affine Λ-building of type $(A_{n-1}, \Lambda, \Lambda^{n-1})$. It is also a $G = \mathrm{SL}_n(\mathbb{F})$-building. Indeed, B_L consists of homothety classes of lattices L, which are of the form $\mathcal{O}e_1 + \mathcal{O}e_2 + \ldots + \mathcal{O}e_n$, where $\{e_1, \ldots, e_n\}$ is a basis of $\mathbb{F}^n$. Thus $\mathrm{SL}_n(\mathbb{F})$ acts on a lattice by acting on the basis, i.e. $g.L = \mathcal{O}ge_1 + \ldots + \mathcal{O}ge_n$. We also define an action of $\mathrm{SL}_n(\mathbb{F})$ on a chart $f_{\mathcal{E}}$ for $\mathcal{E}$ a basis of $\mathbb{F}^n$ by setting $g.f_{\mathcal{E}} = f_{g\mathcal{E}}$. Then these two actions commute.

The atlas $\mathcal{A}$ consists of all charts $f_{\mathcal{E}}$ for $\mathcal{E}$ a basis of $\mathbb{F}^n$. Since $\mathbb{F}$ is real closed, the group $\mathrm{SL}_n(\mathbb{F})$ acts transitively on the set of homothety classes of unordered bases of $\mathbb{F}^n$. Thus $\mathrm{SL}_n(\mathbb{F})$ acts transitively on $\mathcal{A}$. Furthermore, $\mathrm{SL}_n(\mathbb{F})$ acts thus also transitively on the set of lattices up to homothety, and thus $\mathrm{SL}_n(\mathbb{F})$ acts transitively on the building B_L.

Note that B_L is also a $\mathrm{GL}_n(\mathbb{F})$-buildings with the same transitivity properties.

Example 5.6 (Bruhat–Tits building, Example 3.4 revisited)**.** Let B_{BT} be the Bruhat–Tits building associated to $\mathbf{G}(\mathbb{F}) =: G$, where $\mathbf{G}$, $\mathbb{F}$, Λ, and Φ are as in Example 3.4. Then B_{BT} is a G-building of type $(\Phi^\vee, \mathfrak{R}, \Lambda^n)$. Recall that $B_{\mathrm{BT}} = (G \times A)/{\sim}$, where A is the affine space of root group valuations. We already saw in Example 3.4, that G acts on B_{BT} via $g.[h,x] = [gh,x]$. There is a chart $f_0\colon \mathbb{A} \to B_{\mathrm{BT}}$, $a \mapsto [\mathrm{Id}, a]$ and the atlas $\mathcal{A}$ is the orbit of this chart under the action of G. Thus $\mathcal{A}$ is naturally endowed with a G-action and these actions are compatible, hence B_{BT} is a G-building of type $(\Phi^\vee, \mathfrak{R}, \Lambda^n)$.

Note that by definition G acts transitively on the atlas $\mathcal{A}$. However when $\Lambda \neq \mathbb{R}$, then the action of G on B_{BT} is not transitive, see e.g Proposition 5.2.

Example 5.7 (Homogeneous building, Example 3.5 revisited)**.** We claim that the homogeneous building B_H associated to a semisimple self-adjoint linear algebraic $\mathbb{Q}$-group $\mathbf{G} < \mathrm{SL}_n$ and a real closed field $\mathbb{F}$ endowed with a Λ-valuation not necessarily of rank one, as defined in Example 3.5, is an example of a G-building, where $G \coloneqq \mathbf{G}(\mathbb{F})$.

Recall that G acts on $X_{\mathbb{F}} = G.\mathrm{Id} \subseteq P_1(n, \mathbb{F})$ by congruence, and thus on $B_H = X_{\mathbb{F}}/\sim$. It is left to define the action on the atlas $\mathcal{A}$ and to show that these actions are compatible. Fix the base point $o = [\mathrm{Id}] \in B_H$. The standard apartment is identified with $\mathbb{A} = A_{\mathbb{F}}.o$, where $A_{\mathbb{F}}$ is the $\mathbb{F}$-extension of $\exp(\mathfrak{a})$, for $\mathfrak{a}$ a Cartan subalgebra of the Lie algebra of G. Denote by f_0 the inclusion from $\mathbb{A}$ to B_H. The atlas $\mathcal{A}$ is the set $\{g.f_0\colon \mathbb{A} \to B_H \mid g \in G\}$, which is naturally endowed with a left G-action. Clearly, the two actions are compatible, and hence B_H is a G-building of type $(\Phi^\vee, \Lambda, \Lambda^n)$.

By definition, the action of G on $\mathcal{A}$ is transitive. Since $T = \Lambda^n$ is the full translation group, it follows from Proposition 5.2 that the action on B_H is transitive as well.

5.3. **Morphisms of G-buildings.** We have a natural notion of morphisms for G-buildings.

Definition 5.8. Let $(B, \mathcal{A})$ be a G-building, $(B', \mathcal{A}')$ a G'-building and $\rho\colon G \to G'$ a group homomorphism. A morphism of generalized affine buildings

$$(\psi\colon B \to B', \varphi\colon \mathcal{A} \to \mathcal{A}', \tau\colon \mathbb{A} \to \mathbb{A}')$$

is *ρ-equivariant* if for all $g \in G$, $x \in B$ and $f \in \mathcal{A}$ we have

$$\psi(g.x) = \rho(g).\psi(x), \text{ and } \varphi(g.f) = \rho(g).\varphi(f).$$

The goal of this section is to construct morphisms between buildings on which groups act sufficiently transitively. To do so we find conditions for when one can extend a morphism of apartments to a morphism of the buildings, and check when it is injective, surjective or bijective.

Definition 5.9. Let G be a group and $(B, \mathcal{A})$ a G-building of type (Φ, Λ, T). For $f \in \mathcal{A}$ and $w \in W_{\mathrm{a}} = W_{\mathrm{s}}(\Phi) \ltimes T$, we define the subset $A_{f,w}$ of G consisting of *w-translations of f* by

$$A_{f,w} \coloneqq \{g \in G\colon g.f = f \circ w\} \subseteq G.$$

In other words, it is the subset of those elements of G that act on a chart f the same as precomposition by the element w of the affine Weyl group.

Note that $A_{f,w}$ can be empty. However if G acts transitively on the atlas $\mathcal{A}$, then axiom (A1) is equivalent to asking that $A_{f,w} \neq \emptyset$ for all $f \in \mathcal{A}$ and $w \in W_{\mathrm{a}}$.

The following properties about the sets $A_{f,w}$ are a consequence of the compatibility of the actions of G on the atlas and on the building.

Lemma 5.10. *For every $f \in \mathcal{A}$, the set $A_f \coloneqq \bigcup_{w \in W_{\mathrm{a}}} A_{f,w}$ forms a subgroup of G.*

Proof. Let $g \in A_{f,w}$ and $g' \in A_{f,w'}$ with $w, w' \in W_{\mathrm{a}}$, i.e. $g.f = f \circ w$ and $g'.f = f \circ w'$. Then for all $a \in \mathbb{A}$ we have

$$\begin{aligned}((gg').f)(a) &= g.(g'.f)(a) = g.(f \circ w')(a)\\ &= (g.f)(w'(a)) = f \circ w(w'(a)) = (f \circ (ww'))(a),\end{aligned}$$

thus $gg' \in A_{f,ww'}$. Similarly we have $g^{-1} \in A_{f,w^{-1}}$, since for all $a \in \mathbb{A}$ we have

$$(g^{-1}.f)(a) = g^{-1}.(f \circ w)(w^{-1}a) = g^{-1}g.f(w^{-1}a) = f \circ w^{-1}(a). \qquad \square$$

Note that when S is a subgroup of W_{a}, for example $S = T$, then the same arguments as in the proof of the above lemma show that $A_{f,S} \coloneqq \bigcup_{w \in S} A_{f,w}$ is a subgroup of G.

Proposition 5.11. *With the above notations, the sets $A_{f,w}$ satisfy the following properties.*

(1) For all $g \in G$, $f \in \mathcal{A}$ and $w \in W_{\mathrm{a}}$, we have $A_{gf,w} = gA_{f,w}g^{-1}$.
(2) For all $f \in \mathcal{A}$ and $w, w' \in W_{\mathrm{a}}$, if $A_{f,w}$, $A_{f,w'}$ and $\mathrm{Stab}_G(f)$ are non-empty, then $A_{f,ww'} = A_{f,w}A_{f,w'}\,\mathrm{Stab}_G(f) = A_{f,w'w}$.
(3) If $g \in A_{f,w}$ for some $f \in \mathcal{A}$ and $w \in W_{\mathrm{a}}$, then $A_{gf,e} = A_{f,w}g^{-1}$.

Proof.

(1) Let $h \in A_{gf,w}$ then $ghg^{-1}f = g^{-1}gfw = fw$ so that $A_{gf,w} \subseteq gA_{f,w}g^{-1}$. For the other direction, let $h \in A_{f,w}$, then $ghg^{-1}(gf) = ghf = g(hf) = gfw = (gf)w$ so that $gA_{f,w}g^{-1} = A_{gf,w}$.

(2) Let $a \in A_{f,w}$, $b \in A_{f,w'}$ and $c \in A_{f,e}$, then $abcf = abf = afw' = fww'$ so that $A_{f,w}A_{f,w'}\operatorname{Stab}_G(f) \subseteq A_{f,ww'}$. Furthermore, if $A_{f,w}$,$A_{f,w'}$ and $\operatorname{Stab}_G(f)$ are non-empty and $g \in A_{f,ww'}$, then $gf = fww' = afw'$ for some $a \in A_{f,w}$. Thus for some $b \in A_{f,w'}$, $afw' = abf$ so that $g^{-1}ab \in \operatorname{Stab}_G f$. Hence $A_{f,ww'} = A_{f,w}A_{f,w'}\operatorname{Stab}_G(f)$ as desired.

(3) Let $h \in A_{f,w}$ then $hg^{-1}(gf) = hf = fw = gf$ so that $A_{f,w}g^{-1} \subset A_{gf,e}$. Moreover, for $h \in A_{gf,e}$ it holds $h(gf) = gf = fw$ so that $hg \in A_{f,w}$. Thus $A_{gf,e} = A_{f,w}g^{-1}$. □

5.4. **Extending morphisms of apartments.** The goal of this section is to prove Theorem 1.3.

Let Λ, Λ' be ordered abelian groups, Φ, Φ' crystallographic root systems of rank n, m respectively with spherical Weyl groups W_{s}, W'_{s}, $\mathbb{A}$, $\mathbb{A}'$ the associated model apartments and consider $T < \mathbb{A}$, $T' < \mathbb{A}'$ the translation subgroups of the affine Weyl groups $W_{\mathrm{a}} = T \rtimes W_{\mathrm{s}}$, $W'_{\mathrm{a}} = T' \rtimes W'_{\mathrm{s}}$. If $(B, \mathcal{A})$ is a G-building, let as in Definition 5.9 $A_{f,w} \coloneqq \{g \in G\colon g.f = f \circ w\}$ for $g \in G$, $w \in W_{\mathrm{a}}$.

Theorem 5.12 (Theorem 1.3)**.** *Let $(B, \mathcal{A})$ be a G-building of type $\mathbb{A} = \mathbb{A}(\Phi, \Lambda, T)$ with a transitive G-action on $\mathcal{A}$, and $(B', \mathcal{A}')$ a G'-building of type $\mathbb{A}' = \mathbb{A}(\Phi', \Lambda', T')$. Let $\tau = (L, \gamma, \sigma)\colon \mathbb{A} \to \mathbb{A}'$ be a morphism of apartments and $\rho\colon G \to G'$ a group homomorphism. If there exist charts $f \in \mathcal{A}$ and $f' \in \mathcal{A}'$ such that*

(1) $\rho(\operatorname{Stab}_G(f(a))) \subseteq \operatorname{Stab}_{G'}(f'(L \otimes \gamma(a)))$ *for all* $a \in \mathbb{A}$*, and*
(2) $\rho(A_{f,w}) \subseteq A_{f',\sigma(w)}$ *for all* $w \in W_{\mathrm{a}}$*,*

then there exists a morphism (ψ, φ, τ) of (G, G')-buildings from B to B' extending τ, that is ρ-equivariant.

If, in addition,

(a) L and γ are injective, then ψ is injective. Also, if ρ is injective and $\rho(\operatorname{Stab}_G(f)) = \operatorname{Stab}_{G'}(f')$, then φ is injective;
(b) G' acts transitively on $\mathcal{A}'$, and the maps ρ, L and γ are surjective, then φ and ψ are surjective;
(c) G' acts transitively on $\mathcal{A}'$, ρ is an isomorphism of groups, τ is an isomorphism of apartments, and the two inclusions (1) and (2) are equalities, then there exists an inverse morphism. That is, $(B, \mathcal{A})$ and $(B', \mathcal{A}')$ are isomorphic.

Proof. We start with the construction of $\varphi\colon \mathcal{A} \to \mathcal{A}'$. Indeed, since G acts transitively on $\mathcal{A}$, every element of $\mathcal{A}$ is of the form $g.f$ for some $g \in G$, so we define a ρ-equivariant function $\varphi\colon \mathcal{A} \to \mathcal{A}'$ by

$$\varphi(g.f) \coloneqq \rho(g).f.$$

The map φ is well defined: indeed, if $g.f = g'.f$ for some $g, g' \in G$, then $g^{-1}g'.f = f$ so that using (2) for $w = \mathrm{Id}$, $\rho(g^{-1}g') \in \rho(\operatorname{Stab}_G(f)) = \rho(A_{f,\mathrm{Id}}) \subseteq A_{f',\mathrm{Id}} = \operatorname{Stab}_G(f')$ and

$$\varphi(g.f) = \rho(g).f' = \rho(g').f' = \varphi(g'.f).$$

Next we define $\psi\colon B \to B'$. Any element in B is of the form $g.f(a)$ for some $g \in G$ and $a \in \mathbb{A}$ — this follows from axiom (A3) and the fact that G acts transitively on $\mathcal{A}$. For every $g \in G$ and $a \in \mathbb{A}$, we define ψ by

$$\psi\left(g.f\left(a\right)\right) \coloneqq \rho(g).f'\left(L \otimes \gamma\left(a\right)\right).$$

We check that ψ is well defined. Let $g, g' \in G$ and $a, a' \in \mathbb{A}$ such that $g.f(a) = g'.f(a')$. From axiom (A2) there exists $w \in W_{\mathrm{a}}$ such that

$$(g.f)|_{\Omega} = \left(g'.f \circ w\right)|_{\Omega} \text{ where } \Omega \coloneqq (g.f)^{-1}\left(g.f\left(\mathbb{A}\right) \cap g'.f\left(\mathbb{A}\right)\right).$$

Note that $a \in \Omega$ since $g.f(a) = g'.f(a')$. In particular

$$g'.f(a') = g.f(a) = g'.f(w(a)),$$

so that $w(a) = a'$ by injectivity of f. The set $A_{f,w}$ is non-empty by transitivity of the action of G on $\mathcal{A}$ and axiom (A1), so we can consider $g_w \in A_{f,w}$. By definition of $A_{f,w}$, it holds

$$g.f(a) = g'.f(w(a)) = g' g_w.f(a),$$

so that $\rho(g^{-1}g'g_w) \in \rho(\mathrm{Stab}_G(f(a))) \subseteq \mathrm{Stab}_G(f'(L \otimes \gamma(a)))$ using (1) and

$$\rho(g).f'(L \otimes \gamma(a)) = \rho(g'g_w).f'(L \otimes \gamma(a)).$$

By (2) $\rho(g_w) \in \rho(A_{f,w}) \subseteq A_{f',\sigma(w)}$ so that

$$\begin{aligned}\rho(g).f'(L \otimes \gamma(a)) &= \rho(g'g_w).f'(L \otimes \gamma(a)) = \rho(g').f'\left(\sigma\left(w\right)\left(L \otimes \gamma\left(a\right)\right)\right) \\ &= \rho(g').f'(L \otimes \gamma(w(a))) = \rho(g').f'(L \otimes \gamma(a')).\end{aligned}$$

This shows that ψ is well defined. By construction, φ and ψ are ρ-equivariant.

To show that (ψ, φ, τ) is a morphism of (G, G')-buildings, it remains to check that the following diagram commutes for every chart $h \in \mathcal{A}$

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{\;h\;} & B \\ {\scriptstyle L\otimes\gamma}\downarrow & & \downarrow{\scriptstyle \psi} \\ \mathbb{A}' & \xrightarrow[\varphi(h)]{} & B'. \end{array}$$

Let $h \in \mathcal{A}$ and $a \in \mathbb{A}$. By transitivity of the G-action on $\mathcal{A}$, there exists $g \in G$ with $g.f = h$. Then

$$\begin{aligned}(\psi \circ h)(a) = \psi(g.f(a)) &= \rho(g).f'(L \otimes \gamma(a)) \\ &= \varphi(g.f)(L \otimes \gamma(a)) = \varphi(h)(L \otimes \gamma(a)),\end{aligned}$$

so the diagram commutes.

We now prove the three additional statements (a)-(c).

Proof of (a): For every $b \in B$, there exists $g \in G$ and $a \in \mathbb{A}$ such that $g.f(a) = b$. So to show that ψ is injective, consider $g, g' \in G$ and $a, a' \in \mathbb{A}$ such that $\psi(g.f(a)) = \psi(g'.f(a'))$ and we check that $g.f(a) = g'.f(a')$. From axiom (A3) and the fact that G acts transitively on $\mathcal{A}$, there exists $g'' \in G$ and a_1, $a_1' \in \mathbb{A}$, so that

$$g''.f(a_1) = g.f(a) \text{ and } g''.f(a_1') = g'.f(a').$$

We compose with ψ and obtain

$$\begin{aligned}\rho(g'').f'(L \otimes \gamma(a_1)) = \psi(g''.f(a_1)) &= \psi(g.f(a)) = \psi(g'.f(a')) \\ &= \psi(g''.f(a_1')) = \rho(g'').f'(L \otimes \gamma(a_1')).\end{aligned}$$

Thus by the injectivity of $L \otimes \gamma$ and the injectivity of the charts in the atlas, it holds $a_1 = a_1'$ so that

$$g.f(a) = g''.f(a_1) = g'.f(a')$$

as desired. So ψ is injective.

Now suppose ρ is injective and $\rho(\mathrm{Stab}_G(f)) = \mathrm{Stab}_{G'}(f')$. For $h, h' \in \mathcal{A}$ there exist g, $g' \in G$ such that $h = g.f$ and $h' = g.f$. If $\varphi(h) = \varphi(h')$, then $\rho(g).f' = \rho(g').f'$, so

$$\rho(g^{-1}g') \in \mathrm{Stab}_{G'}(f') = \rho(\mathrm{Stab}_G(f)).$$

So there exists $g'' \in \mathrm{Stab}_G(f)$ such that $\rho(g^{-1}g') = \rho(g'')$ and by injectivity of ρ, $g^{-1}g' = g''$ so that $g^{-1}g' \in \mathrm{Stab}_G(f)$ and $h = g.f = g'.f = h'$ so that φ is injective.

Proof of (b): Let $h' \in \mathcal{A}'$. By transitivity of the G'-action on $\mathcal{A}'$, there exists $g' \in G'$ such that $h' = g'.f'$. Moreover ρ is surjective such that $g' = \rho(g)$ for some $g \in G$. Hence $\varphi(g.f) = g'.f' = h'$, so φ is surjective.

We now prove surjectivity of ψ. Let $b' \in B'$. By axiom (A3) of buildings, there exists $a' \in \mathbb{A}'$ and $h' \in \mathcal{A}'$ such that $b = h'(a')$. Since the action of G' on $\mathcal{A}'$ is transitive and ρ is surjective, there exists $g \in G$ with $b' = h'(a') = \rho(g).f'(a')$. By surjectivity of the maps L and γ, there exists $a \in \mathbb{A}$ such that $a' = L \otimes \gamma(a)$. Hence

$$\psi(g.f(a)) = \rho(g).f'(L \otimes \gamma(a)) = h'(a') = b'$$

and both φ and ψ are surjective.

Proof of (c): Suppose that $\tau\colon \mathbb{A} \to \mathbb{A}'$ is an isomorphism of apartments, G' acts transitively on $\mathcal{A}'$, ρ is an isomoprhism and the two inclusions in the conditions of the theorem are equalities. We construct, as above, a ρ^{-1}-equivariant morphism $(\psi', \varphi', \tau^{-1})$ from $(B', \mathcal{A}')$ to $(B, \mathcal{A})$, such that $\varphi'(f') = f$. By ρ- and ρ^{-1}-equivariance we have for every $g \in G$ and $a \in \mathbb{A}$

$$\psi' \circ \psi(g.f(a)) = \psi'(\rho(g).f'(a)) = g.f(a)$$
$$\varphi' \circ \varphi(g.f) = \varphi'(\rho(g).f') = g.f,$$

and similarly $\psi \circ \psi' = \mathrm{id}_{B'}$ and $\varphi \circ \varphi' = \mathrm{id}_{\mathcal{A}'}$. Hence B and B' are isomorphic as generalized affine buildings. □

6. Functoriality for homogeneous buildings

We would like to apply our notion of morphism of generalized affine buildings and their construction using Theorem 5.12 to prove certain natural functoriality properties. For this we put ourself in the setting of homogeneous buildings (Example 3.5), but we expect similar results also in the context of the other models of buildings introduced in Section 3.2.

6.1. **Containment of root systems.** Before working with Lie groups, we consider just the case of root systems. Let V, V' be Euclidean vector spaces sucht that $V < V'$ and let $V^\star$, $(V')^\star$ be their duals. Let $(\Phi, V^\star)$, $(\Phi', (V')^\star)$ be root systems and let W, W' denote their (spherical) Weyl groups. We think of W, W' as acting on V, V'. The goal of this subsection is Proposition 6.2, where under certain conditions we find an injective homomorphism $\sigma\colon W \to W'$ such that $\sigma(w)|_V = w$ for all $w \in W$.

Example 6.1. Let V be a two-dimensional vector space and $\Phi' \subseteq (V')^\star$ a root system of type A_2. For every one-dimensional subspace $V \subseteq V'$ there are root systems in $V^\star$ of type A_1, but only when V is perpendicular to a hyperplane of Φ' are the actions of the Weyl groups compatible, see Figure 2. The arrangement on the left of Figure 2 appears when considering subgroups $\mathrm{SL}_2 < \mathrm{SL}_3$. More examples of root systems where Proposition 6.2 applies can be found in Figure 3 and Figure 4.

Proposition 6.2. *Let $V < V'$ be Euclidean vector spaces and $(\Phi, V^\star)$, $(\Phi', (V')^\star)$ root systems with Weyl groups W, W'. Let C_0, C_0' be fundamental Weyl cones in V, V' such that C_0' contains a regular point of C_0. Assume that for all $w \in W$, $w' \in W'$ we have*

(1) for all $\alpha \in \Phi$ there exists $\alpha' \in \Phi'$ such that $\alpha'|_V = \alpha$,
(2) for all $x \in C_0 \cap C_0'$, if $w(x) \in w'(C_0')$, then $w(x) = w'(x)$.

Then there is an injective group homomorphism $\sigma\colon W \to W'$ such that $\sigma(w)|_V = w$ for all $w \in W$.

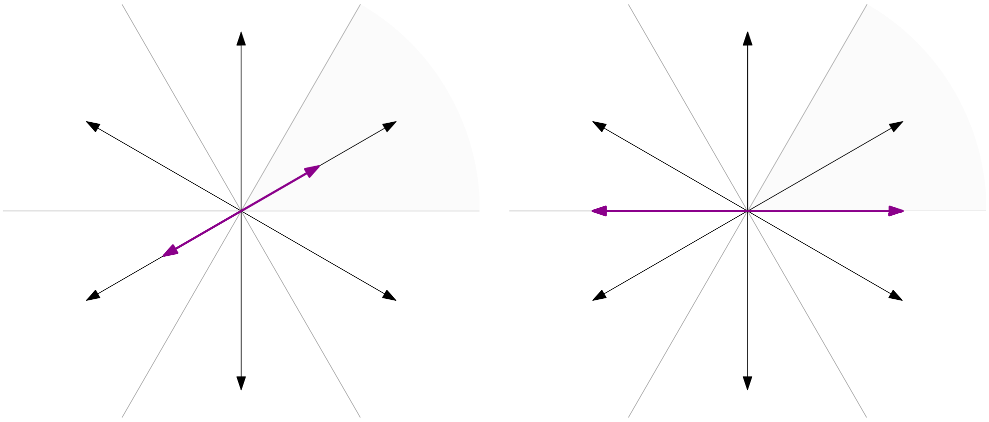


FIGURE 2. The black and purple arrows depict the elements dual to the elements of root systems Φ and Φ' of type A_1 and A_2. The Weyl groups W and W' are generated by reflections along the hyperplanes (depicted in gray for W'). On the left, W and W' are compatible as in Proposition 6.2. For the choice of Φ on the right, there is no Weyl group element of W' that restricts to the reflection in W.

Proof. Let $p \in V$ be a regular point in C_0 that lies in C_0' and is maximally regular in C_0'. Let $\Phi_{>0}$ (resp. $\Phi'_{>0}$) be the set of positive roots and let Δ (resp. Δ') the basis with respect to the fundamental Weyl cells. If $\alpha \in \Phi_{>0}$ and $\alpha' \in \Phi'$ with $\alpha'|_V = \alpha$ as in (1), then $\alpha' \in \Phi_{>0}$, since $\alpha'(p) = \alpha(p) \geq 0$. This implies

$$C_0' \cap V \subseteq C_0,$$

since for all $\alpha \in \Phi_{>0}$ and $x \in C_0' \cap V$, $\alpha(x) = \alpha'(x) \geq 0$, this is illustrated in Figure 3. Let $\Delta_0' \coloneqq \{\alpha' \in \Delta' \colon \alpha'(p) = 0\}$. Since p is maximally regular in V'

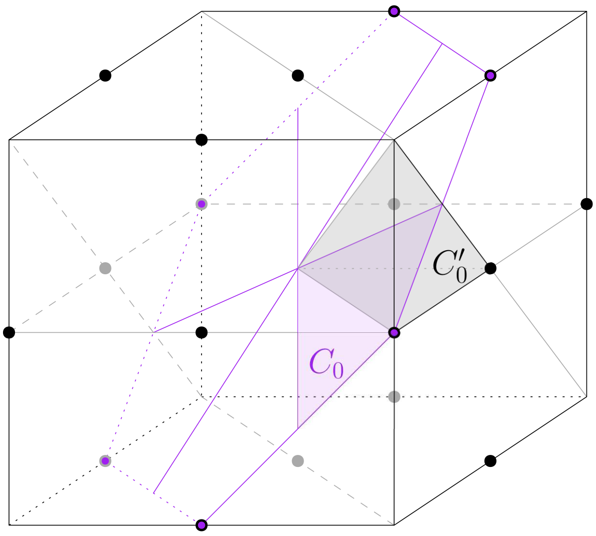


FIGURE 3. A coroot system of type A_2 lying in type B_3 is illustrated by purple and black dots. The fundamental Weyl cones C_0 and C_0' are chosen so that C_0' contains a regular point of C_0. The walls encasing the fundamental chambers are shown in purple and gray.

and all $\alpha' \in \Delta_0'$ are linear (and thus determined by a neighborhood around

p), all $\alpha' \in \Delta'_0$ satisfy $\alpha'(v) = 0$ for all $v \in V$,

$$\Delta'_0 = \left\{ \alpha'_0 \in \Delta' \colon \alpha'_0(V) = 0 \right\}.$$

There may be multiple cells in V' containing p, but they are fully determined by the signs of $\alpha' \in \Delta'_0$. The same holds for cells containing points $w(p) \in V$ for $w \in W$. The fundamental chamber can be described as

$$C'_0 = \bigcap_{\alpha' \in \Delta} H^+_{\alpha'} = \bigcap_{\alpha'_0 \in \Delta'_0} H^+_{\alpha'_0} \cap \bigcap_{\substack{\alpha' \in \Phi' \\ \alpha'(p) > 0}} H^+_{\alpha'},$$

where $H^+_{\alpha'} = \{v' \in V' \colon \alpha'(v') \geq 0\}$. For $w \in W$, there may similarly exist multiple cells in V' containing $w(p)$, but we define the unique cell

$$C'_w \coloneqq \bigcap_{\alpha'_0 \in \Delta'_0} H^+_{\alpha'_0} \cap \bigcap_{\substack{\alpha' \in \Phi' \\ \alpha'(w(p)) > 0}} H^+_{\alpha'}$$

containing $w(p)$ and having the same positive signs as C'_0 for elements in Δ'_0. For every $w \in W$ there is a unique $w' \in W'$ with $w'(C'_0) = C'_w$, which we denote by $\sigma(w) \coloneqq w'$. By condition (2), $w(p) = \sigma(w)(p)$. Actually for $x \in C_0$ near p we still have $w(x) \in C'_w = w'(C'_0)$ because p is regular in C_0, so $w(x) = \sigma(w)(x)$. Since w and $\sigma(w)|_V$ act linearly on V, we then have that $w = \sigma(w)|_V$. It remains to show that $\sigma \colon W \to W'$ is an injective group homomorphism.

Recall that W' acts on the Φ' by $w'(\alpha') \coloneqq \alpha' \circ w'^{-1}$, so that for all $v' \in V'$ and $w' \in W'$ we have $w'(\alpha')(w'(v')) = \alpha(v')$. Note that

$$w'(H^+_{\alpha'}) = H^+_{w'(\alpha')}$$

for all $w' \in W'$. Moreover, for all $w \in W$ and $\alpha'_0 \in \Delta'_0$ we have $\sigma(w)(\alpha'_0) \in \Delta'_0$, since for all $v \in V$, $(\sigma(w)(\alpha'_0))(v) = \alpha'_0(\sigma(w)^{-1}(v)) = 0$, because $\sigma(w)^{-1}(v) = w^{-1}(v) \in V$. This means

$$\sigma(w)(\Delta'_0) = \Delta'_0.$$

Let $w, \overline{w} \in W$, we have

$$\begin{aligned}
\sigma(w)\sigma(\overline{w})(C'_0) = \sigma(w)(C'_{\overline{w}}) &= \bigcap_{\alpha'_0 \in \Delta'_0} H^+_{\alpha'_0} \cap \bigcap_{\substack{\alpha' \in \Phi' \\ \alpha'(\overline{w}(p)) > 0}} \sigma(w) \left(H^+_{\alpha'}\right) \\
&= \bigcap_{\alpha'_0 \in \Delta'_0} H^+_{\alpha'_0} \cap \bigcap_{\substack{\beta' \in \Phi' \\ \beta'(\sigma(w)\overline{w}(p)) > 0}} H^+_{\beta'} = C'_{w\overline{w}}
\end{aligned}$$

where we used the substitution $\beta' = \sigma(w)(\alpha')$, $\alpha' = \sigma(w)^{-1}(\beta')$ and the fact that $\sigma(w)(\overline{w}(p)) = w\overline{w}(p)$ since $\overline{w}(p) \in V$. By the definition of σ, this means $\sigma(w\overline{w}) = \sigma(w)\sigma(w')$. We note that $\ker(\sigma) = \{\mathrm{Id}_V\}$, since all other elements $w \in W \setminus \{\mathrm{Id}_V\}$ send p to $w(p) = \sigma(w)(p) \neq p$, since p is regular in V. This concludes the proof that $\sigma \colon W \to W'$ is an injective group homomorphism with $\sigma(w)|_V = w$. □

6.2. **Functoriality under subgroups.** In this section, we show that the homogeneous buildings constructed from subgroups are related by an injective morphism of buildings. Examples include $\mathrm{SL}_m < \mathrm{SL}_n$ for $m \leq n$ or $\mathrm{Sp}_{2n} < \mathrm{SL}_{2n}$ (see Figure 4), as well as inclusions $\mathrm{SL}_2 < G$ or $\mathrm{PGL}_2 < G$ arising from the Jacobson–Morozov theorem [BT72, (6.1.3.b.2), (6.2.3.b)] and [App24, Section 6.8].

Let $\mathbf{G} < \mathbf{G}' < \mathrm{SL}_n$ be two Zariski-connected semisimple selfadjoint linear algebraic $\mathbb{Q}$-groups. Let $\mathbf{S} < \mathbf{S}'$ be selfadjoint maximal $\mathbb{R}$-split tori of $\mathbf{G}, \mathbf{G}'$ and let $A_{\mathbb{F}}, A'_{\mathbb{F}}$ be the semi-algebraically connected components of the $\mathbb{F}$-extensions $\mathbf{S}_{\mathbb{F}}, \mathbf{S}'_{\mathbb{F}}$ of $\mathbf{S}$ respectively $\mathbf{S}'$ that contain the identity. Let $\mathbb{F}$ be a

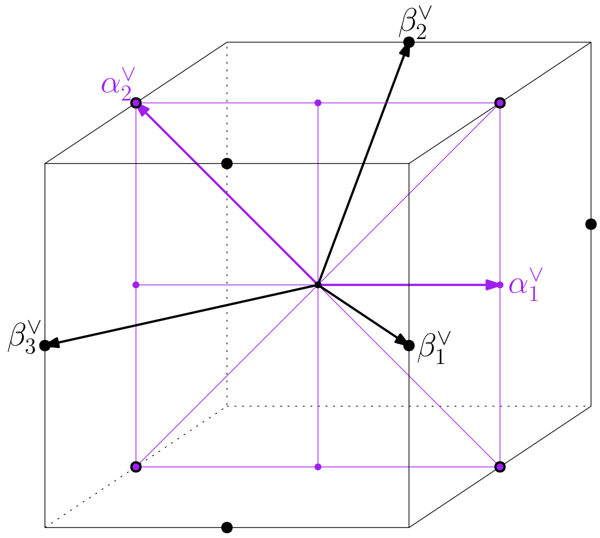


FIGURE 4. In the inclusion $\mathbf{G} = \mathrm{Sp}_4 < \mathrm{SL}_4 = \mathbf{G}'$, the root system $\Sigma^\vee \subseteq \mathfrak{a}$ of type B_2 is not a subset of $(\Sigma')^\vee \subseteq \mathfrak{a}'$ of type A_3. For $H = \mathrm{Diag}(a,b,c,d) \in \mathfrak{a}'$, we have $a+b+c+d = 0$ and a basis $\Delta' = \{\beta_1, \beta_2, \beta_3\}$ of Σ' is given by the roots $\beta_1(H) = a-b$, $\beta_2(H) = b-c$, $\beta_3(H) = c-d$. For $H \in \mathfrak{a}$, we have additionally $a = -c$, $b = -d$ and a basis $\Delta = \{\alpha_1, \alpha_2\}$ of Σ is given by $\alpha_1(H) = a-b$ and $\alpha_2(H) = 2b$. The restrictions $\alpha_1 = \beta_1|_{\mathfrak{a}}$ and $\alpha_2 = (\beta_2+\beta_3)|_{\mathfrak{a}}$ illustrate Lemma 6.3.

non-Archimedean real closed field and $v \colon \mathbb{F}^\times \to \Lambda$ an order-compatible valuation. In this section we will show that the inclusion $\mathbf{G}(\mathbb{F}) < \mathbf{G}'(\mathbb{F})$ induces an injective morphism from the homogeneous building B associated to $\mathbf{G}(\mathbb{F})$, to B' associated to $\mathbf{G}'(\mathbb{F})$. Let W_{s}, W'_{s} denote the (spherical) Weyl groups of the root systems $\Sigma \subseteq \mathfrak{a}^\star$, $\Sigma' \subseteq (\mathfrak{a}')^\star$, where $\mathfrak{a}$, $\mathfrak{a}'$ are the Lie algebras of $A_{\mathbb{R}}$, $A'_{\mathbb{R}}$. For more detailed definitions see Example 3.5.

The following Lemmas are used to constuct a morphism of apartments $\mathbb{A} \to \mathbb{A}'$ in Proposition 6.6, in particular, we verify the assumptions of Proposition 6.2. We would like to express our gratitude for communication with Anne Parreau which lead to some of the ideas in this section.

Lemma 6.3. *For every $\alpha \in \Sigma$ there exists $\alpha' \in \Sigma'$ such that $\alpha'|_{\mathfrak{a}} = \alpha$. In fact, for all $\alpha \in \Sigma$,*

$$\mathfrak{g}_\alpha = \bigoplus_{\substack{\alpha' \in \Sigma' \\ \alpha'|_{\mathfrak{a}} = \alpha}} \mathfrak{g}'_{\alpha'} \cap \mathfrak{g}.$$

Proof. Let $\mathfrak{g} = \mathrm{Lie}(\mathbf{G}(\mathbb{R}))$ and $\mathfrak{g}' = \mathrm{Lie}(\mathbf{G}(\mathbb{R}'))$ and

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha, \quad \mathfrak{g}' = \mathfrak{g}'_0 \oplus \bigoplus_{\alpha' \in \Sigma'} \mathfrak{g}'_{\alpha'}$$

the root decompositions. Let $\alpha \in \Sigma$ and $X \in \mathfrak{g}_\alpha \setminus \{0\}$, in particular $X \in \mathfrak{g} \subseteq \mathfrak{g}'$. Let $X_0 \in \mathfrak{g}'_0$ and $X'_\alpha \in \mathfrak{g}'_{\alpha'}$ for $\alpha' \in \Sigma'$ such that

$$X = X_0 + \sum_{\alpha' \in \Sigma'} X_{\alpha'}.$$

For any $H \in \mathfrak{a} \subseteq \mathfrak{a}'$, we now have

$$\begin{aligned}[H, X] &= \alpha(H)X = \alpha(H)X_0 + \sum_{\alpha' \in \Sigma'} \alpha(H)X_{\alpha'} \\ &= [H, X_0] + \sum_{\alpha' \in \Sigma'} [H, X_{\alpha'}] = \sum_{\alpha' \in \Sigma'} \alpha'(H)X_{\alpha'}\end{aligned}$$

and note that the non-zero elements in $\{X_0\} \cup \{X_{\alpha'} : \alpha' \in \Sigma'\}$ are linearly independent. In particular, $X_0 = 0$ and for at least one $\alpha' \in \Sigma'$, $X_{\alpha'} \neq 0$. We have shown that for all $\alpha \in \Sigma$, there exists $\alpha' \in \Sigma'$ such that $\alpha'|_{\mathfrak{a}} = \alpha$. □

Lemma 6.4. *Let C_0 and C_0' be fundamental Weyl cones in $\mathfrak{a}$, $\mathfrak{a}'$ such that C_0' contains a regular point of C_0 and let $w \in W_{\mathrm{s}}$, $w' \in W_{\mathrm{s}}'$. If $x \in C_0 \cap C_0'$ satisfies $w(x) \in w'(C_0')$, then $w(x) = w'(x)$.*

Proof. Let $p \in \mathfrak{a}$ be a regular point in C_0 that lies in C_0' and is maximally regular in C_0'. Let $w \in W_{\mathrm{s}}$, $w' \in W_{\mathrm{s}}'$ and $x \in C_0 \cap C_0'$ such that $w(x) \in w'(C_0')$. The Weyl groups are identified with $W_{\mathrm{s}} \cong \operatorname{Nor}_{K_{\mathbb{R}}}(A_{\mathbb{R}}) / \operatorname{Cen}_{K_{\mathbb{R}}}(A_{\mathbb{R}})$ and $W_{\mathrm{s}}' \cong \operatorname{Nor}_{K_{\mathbb{R}}'}(A_{\mathbb{R}}') / \operatorname{Cen}_{K_{\mathbb{R}}'}(A_{\mathbb{R}}')$, in particular there exists $k \in K_{\mathbb{R}} \subseteq K_{\mathbb{R}}'$ and $k' \in K_{\mathbb{R}}'$ such that $k.\exp(H) = \exp(w(H))$ for all $H \in \mathfrak{a}$ and $k'.\exp(H') = \exp(w'(H'))$ for all $H' \in \mathfrak{a}'$ [Kna02, Proposition 7.32]. Recall that the Cartan projection $\delta \colon X_{\mathbb{R}}' \to C_0'$ is invariant under the action of $K_{\mathbb{R}}'$ in the sense that for all $q \in X_{\mathbb{R}}'$, $k \in K_{\mathbb{R}}'$ we have $\delta(k.q) = \delta(q)$, and $\delta(\exp(y')) = y'$ for all $y' \in C_0'$. We have $(w')^{-1}w(x) \in C_0'$ and so

$$(w')^{-1}w(x) = \delta((w')^{-1}w(x)) = \delta((k')^{-1}k.\exp(x)) = \delta(\exp(x)) = x,$$

so $w(x) = w'(x)$, which concludes the proof. □

Recall that the building B associated to $\mathbf{G}(\mathbb{F})$ has type $\mathbb{A} = \mathbb{A}(\Sigma^\vee, \Lambda, T)$ with full translation group $T = \mathbb{A}$ and the building B' associated to $\mathbf{G}'(\mathbb{F})$ has type $\mathbb{A}' = \mathbb{A}((\Sigma')^\vee, \Lambda, T')$ with full translation group $T' = \mathbb{A}'$.

Lemma 6.5. *The inclusion $\mathfrak{a} \to \mathfrak{a}'$ restricts to an inclusion* $\operatorname{Span}_{\mathbb{Q}}(\Sigma^\vee) \subseteq \operatorname{Span}_{\mathbb{Q}}((\Sigma')^\vee)$.

Proof. Since we are in the context of algebraic groups, the coroot lattice $\operatorname{Span}_{\mathbb{Q}}(\Sigma^\vee)$ can be identified with the cocharacter lattice $X^\star(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{Q}$ (for the root lattice and the characters, this identification is spelled out in [App24, Section 6.2]). Cocharacters $t \in X^\star(\mathbf{S})$ are algebraic homomorphisms $\mathbb{G}_m \to \mathbf{S}$. Every cocharacter of $\mathbf{S}$ gives a cocharacter of $\mathbf{S}$ by postcomposing with the inclusion $\mathbf{S} < \mathbf{S}'$, so $\operatorname{Span}_{\mathbb{Q}}(\Sigma^\vee) \cong X^\star(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{Q} \subseteq X^\star(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \operatorname{Span}_{\mathbb{Q}}((\Sigma')^\vee)$. □

Proposition 6.6. *There is an injective morphism of apartments $\mathbb{A} \to \mathbb{A}'$.*

Proof. Let $L \colon \operatorname{Span}_{\mathbb{Q}}(\Sigma^\vee) \to \operatorname{Span}_{\mathbb{Q}}((\Sigma')^\vee)$ be the inclusion map from Lemma 6.5 and $\gamma \colon \Lambda \to \Lambda$ the identity. By Lemmas 6.3 and 6.4, the conditions for proposition 6.2 are satisfied, so we obtain a group homomorphism $\sigma \colon W_{\mathrm{s}} \to W_{\mathrm{s}}'$ such that $\sigma(w)|_{\operatorname{Span}_{\mathbb{Q}}(\Sigma^\vee)} = w$ for all $w \in W_{\mathrm{s}}$. The diagram

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{L \otimes \gamma} & \mathbb{A}' \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle \sigma(w)} \\ \mathbb{A} & \xrightarrow[L \otimes \gamma]{} & \mathbb{A}' \end{array}$$

commutes for all $w \in W_{\mathrm{s}}$, so (L, γ, σ) is a morphism of apartments. Both L and γ (and even σ) are injective, so the morphism is injective. □

Theorem 6.7. *Let $\mathbf{G}$, $\mathbf{G}' < \mathrm{SL}_n$ be two semisimple selfadjoint linear algebraic $\mathbb{Q}$-groups. Let $\mathbb{F}$ be a non-Archimedean real closed field and $v\colon \mathbb{F}^\times \to \Lambda$ an order-compatible valuation. Let B (resp. B') be the associated homogeneous $\mathbf{G}(\mathbb{F})$- (resp. $\mathbf{G}'(\mathbb{F})$-)building as in Example 3.5. If $\mathbf{G}(\mathbb{F}) < \mathbf{G}'(\mathbb{F})$, then the inclusion induces a morphism $B \to B'$.*

Proof. Let $o = [\mathrm{Id}] \in B$ and $o' = [\mathrm{Id}] \in B'$ be the base points. By Proposition 6.6, the inclusion $\tau\colon \mathbb{A} \to \mathbb{A}'$ is an injective morphism of apartments. We consider the standard charts $f_0\colon \mathbb{A} \to B$ and $f_0'\colon \mathbb{A}' \to B'$. Recall that for $x \in \mathbb{A}$, $a \in A_{\mathbb{F}}$, $f_0(x) = a.o$ whenever $(-v)(\chi_\alpha(a)) = \alpha(x)$ for all $\alpha \in \Sigma$. Analogously, for $x' \in \mathbb{A}'$, $a' \in A'_{\mathbb{F}}$, $f_0'(x') = a'.o'$ whenever $(-v)(\chi_{\alpha'}(a')) = \alpha'(x')$ for all $\alpha' \in \Sigma'$. By definition, $\mathcal{A} = \mathbf{G}(\mathbb{F}).f_0$ and $\mathcal{A}' = \mathbf{G}'(\mathbb{F}).f_0'$. For $g.f_0 \in \mathcal{A}$, let $\varphi(g.f_0) := g.f_0' \in \mathcal{A}'$. Let $\psi\colon B \to B'$ be the map $g.o \mapsto g.o'$, which is well defined since $\mathrm{Stab}_{\mathbf{G}(\mathbb{F})}(o) = \mathbf{G}(\mathcal{O}) \subseteq \mathbf{G}'(\mathcal{O}) = \mathrm{Stab}_{\mathbf{G}'(\mathbb{F})}(o')$ [App24]. We now verify that the diagram

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{\ \tau\ } & \mathbb{A}' \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f_0'} \\ B & \xrightarrow[\ \psi\ ]{} & B' \end{array}$$

commutes. Let $x \in \mathbb{A}$. Then there exists $a \in A_{\mathbb{F}}$ with $a.o = f_0(x)$. Consider $\alpha' \in \Sigma'$. Then $\alpha'|_{\mathrm{Span}_{\mathbb{Q}}(\Sigma^\vee)} = \sum_{i=1}^r q_i\alpha_i$ for some $\alpha_i \in \Sigma$, since in the theory of algebraic groups, the restriction of a character is still a character. Then

$$(-v)(\chi_{\alpha'}(a)) = (-v)\left(\prod_{i=1}^r \chi_{\alpha_i}(a)^{q_i}\right) = \sum_{i=1}^r q_i(-v)(\chi_{\alpha_i}(a)) = \sum_{i=1}^r q_i\alpha_i(x) = \alpha'(x)$$

shows that $a.o' = f_0'(\tau(x))$, so the diagram commutes. More generally, for all $g \in \mathbf{G}(\mathbb{F})$, the diagram

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{\ \tau\ } & \mathbb{A}' \\ {\scriptstyle g.f_0}\downarrow & & \downarrow{\scriptstyle g.f_0'} \\ B & \xrightarrow[\ \psi\ ]{} & B' \end{array}$$

from Definition 4.6 commutes, since for $x \in \mathbb{A}$

$$\psi(g.f_0(x)) = \psi(g.a.o) = g.a.o' = g.f_0'(x) = g.f_0'(\tau(x)).$$

Thus (ψ, φ, τ) is a morphism of buildings. The map $\psi\colon B \to B'$ is injective, since if $g, h \in \mathbf{G}(\mathcal{O})$ satisfy $g.o' = h.o'$, then $h^{-1}g \in \mathrm{Stab}_{\mathbf{G}'(\mathbb{F})}(o') = \mathbf{G}'(\mathcal{O})$, see Proposition 3.6, but also $h^{-1}g \in \mathbf{G}(\mathbb{F})$, so $h^{-1}g \in \mathbf{G}(\mathcal{O}) = \mathrm{Stab}_{\mathbf{G}(\mathbb{F})}(o)$, so $g.o = h.o$. Similarly, the map $\varphi\colon \mathcal{A} \to \mathcal{A}'$ is injective because $\mathrm{Stab}_{\mathbf{G}(\mathbb{F})}(f_0) = T(\mathcal{O}) \subseteq T'(\mathcal{O}) = \mathrm{Stab}_{\mathbf{G}'(\mathbb{F})}(f_0')$, see Proposition 3.6. Finally τ is an injective morphism of apartments, so (ψ, φ, τ) is an injective morphism of buildings. □

6.3. **Functoriality under group morphisms.** We use Theorem 5.12 to strengthen Theorem 6.7 to any injective morphism instead of restricting it to subgroups. To do this, we first show that changing the maximal flat in the homogeneous building, see Example 3.5, gives rise to isomorphic buildings.

Lemma 6.8. *Let $\mathbb{F}$ be a non-Archimedean real closed field and $v\colon \mathbb{F}^\times \to \Lambda$ an order-compatible valuation. Let $\rho\colon \mathbf{G} \to \mathbf{G}'$ be an isomorphism of semisimple linear algebraic $\mathbb{Q}$-groups. Let $\mathbf{S}$ be a maximal $\mathbb{R}$-split tori of $\mathbf{G}$ and $\mathbb{A} = \mathbb{A}(\Phi^\vee, \Lambda, \Lambda^n)$ the model apartment of the homogeneous building of $\mathbf{G}(\mathbb{F})$, where $\Phi^\vee$ is the dual root system of $\mathbf{G}$ associated to $\mathbf{S}$.*

Then ρ induces an isomorphism of model apartments from $\mathbb{A} \to \mathbb{A}'$, where $\mathbb{A}' = \mathbb{A}'((\Phi')^\vee, \Lambda, \Lambda^n)$ is the model apartment of the homogeneous building of

$\mathbf{G}'(\mathbb{F})$ and Φ' is the dual root system of $\mathbf{G}'$ associated to the maximal $\mathbb{R}$-split torus $\rho(\mathbf{S})$ of $\mathbf{G}'$.

Proof. We abbreviate $\mathbf{S}' := \rho(\mathbf{S})$. Consider $\tau = (L, \gamma, \sigma)$, where

$$L\colon \operatorname{Span}_{\mathbb{Q}}(\Phi) \to \operatorname{Span}_{\mathbb{Q}}(\Phi'), \quad \sum \lambda_\alpha \alpha \mapsto \sum \lambda_\alpha (\alpha \circ D\rho),$$

$\sigma\colon W_{\mathrm{a}} \to W'_{\mathrm{a}}$ sends $[w] \in W_{\mathrm{a}} \cong \operatorname{Nor}_K(A)/\operatorname{Cent}_K(A)$ to $[\rho(w)] \in W'_{\mathrm{a}} \cong \operatorname{Nor}_{K'}(A')/\operatorname{Cent}_{K'}(A')$ where $K' := \rho(K)$ and $\gamma\colon \Lambda \to \Lambda$ is the identity. We claim that τ is a morphism of apartments. For this we only need to show that the diagram in Definition 4.4 commutes. If $\sum \lambda_\alpha \chi_\alpha \in \operatorname{Span}_{\mathbb{Q}}(\Phi) \otimes \Lambda$ and $(w, t) \in W_{\mathrm{a}}$, then

$$\begin{aligned} L \otimes \gamma \left((w,t) \sum \lambda_\alpha \chi_\alpha \right) &= L \otimes \gamma \left(\sum t(\lambda_\alpha) \chi_{w(\alpha)} \right) = \sum t(\lambda_\alpha) \chi_{\rho(w)(\alpha \circ \rho)}, \\ \sigma(w) L \otimes \gamma \left(\sum \lambda_\alpha \chi_\alpha \right) &= \sigma(w) \sum \lambda_\alpha \chi_{\alpha \circ \rho} = \sum t(\lambda_\alpha) \chi_{\rho(w)(\alpha\circ\rho)}. \end{aligned}$$

Hence τ is a morphism of apartments. Inverting the roles of G and G' using ρ^{-1}, we show that τ is an isomorphism of apartments. □

The main difficulty in constructing a building morphism from an apartment morphism arises, in our study, from understanding the charts in the atlases of our buildings. For this reason, the following results on buildings are given for the homogeneous buildings only, even though we think it is possible to generalize them to other models of buildings, where the action describes the building structure sufficiently well.

Theorem 6.9. *Let $\mathbf{G}, \mathbf{G}' < \mathrm{SL}_n$ be semisimple algebraic linear $\mathbb{Q}$-groups, $\mathbb{F}$ be a non-Archimedean closed real field and $v\colon \mathbb{F}^\times \to \Lambda$ an order compatible valuation. If $\rho\colon \mathbf{G} \to \mathbf{G}'$ is an isomorphism of algebraic groups, then the homogeneous buildings B and B' of $\mathbf{G}(\mathbb{F})$ and $\mathbf{G}'(\mathbb{F})$ with respect to maximal $\mathbb{R}$-split tori $\mathbf{S}$ and $\rho(\mathbf{S})$ are isomorphic.*

Proof. As in Lemma 6.8, consider $\mathbf{S}$ a maximal $\mathbb{R}$-split tori of $\mathbf{G}$ and $\mathbb{A} = \mathbb{A}(\Phi, \Lambda, \Lambda^n)$ its model apartment where Φ is the root system of $\mathbf{G}$ associated to $\mathbf{S}$. Define $\mathbf{S}' := \rho(\mathbf{S})$ a maximal $\mathbb{R}$-split tori of $\mathbf{G}'$ and $\mathbb{A} = \mathbb{A}(\Phi', \Lambda, \Lambda^n)$ its model apartment where Φ' is the root system of $\mathbf{G}'$ associated to $\mathbf{S}'$. Consider the isomorphism of apartments $\tau = (L, \gamma, \sigma)$ as in Lemma 6.8 and

$$\begin{array}{cccc} \xi_\rho\colon & \mathbb{A} := A(\mathbb{F})/A(\mathcal{O}) & \longrightarrow & A'(\mathbb{F})/A'(\mathcal{O}) =: \mathbb{A}'; \\ & [b]_A & \longmapsto & [\rho(b)]_{A'} . \end{array}$$

Claim: If gf_Λ^{-1} is an affine Weyl group equivariant isomorphism of groups as defined in [App24, Proposition 7.9], then the following diagram is commutative

$$\begin{array}{ccc} A_{\mathbb{F}}/A_{\mathcal{O}} & \xrightarrow{\ \xi_{\rho^{-1}}\ } & A'_{\mathbb{F}}/A'_{\mathcal{O}} \\ {\scriptstyle gf_\Lambda^{-1}}\big\downarrow & & \big\downarrow{\scriptstyle gf_\Lambda^{-1}} \\ \operatorname{Span}_{\mathbb{Q}}(\Phi) \otimes_{\mathbb{Q}} \Lambda & \xrightarrow[L\otimes\gamma]{} & \operatorname{Span}_{\mathbb{Q}}(\Phi') \otimes_{\mathbb{Q}} \Lambda. \end{array}$$

Proof: To do this, we first describe $\xi_\rho(\prod_{\delta\in\Delta} t_\delta^\Lambda(\lambda_\delta))$ for t_δ^Λ as described in [App24, Lemma 6.2, Proposition 7.6]. We start with the real case. For $\prod_{\delta\in\Delta} t_\delta^{\mathbb{R}}(\lambda_\delta) \in A_{\mathbb{R}}$ it holds

$$\xi_\rho \prod_{\delta\in\Delta} t_\delta^{\mathbb{R}}(\lambda_\delta) = \prod_{\delta\in\Delta} \rho\left(\exp\left(\log(\lambda_\delta) x_\delta\right)\right) = \prod_{\delta\in\Delta} \left(\exp\left(\log(\lambda_\delta) \mathrm{d}_{\mathrm{Id}}\rho(x_\delta)\right)\right)$$

where $x_\delta = \frac{2}{B_\theta(H_\delta, H_\delta)} H_\delta$ with $H_\delta \in \mathfrak{a}$, the Lie algebra of A, such that $B_\theta(H_\delta, H) = \mathrm{d}_{\mathrm{Id}}\delta(H)$ for every $H \in \mathfrak{a}$. Thus

$$\mathrm{d}_{\mathrm{Id}}\rho(x_\delta) = \frac{2}{B_\theta(\mathrm{d}_{\mathrm{Id}}\rho(H_\delta), \mathrm{d}_{\mathrm{Id}}\rho(H_\delta))} \mathrm{d}_{\mathrm{Id}}\rho(H_\delta)$$

because $\mathrm{d}_{\mathrm{Id}}\rho$ is an isomorphism of Lie algebra and B_θ is invariant under Lie algebras isomorphisms. Moreover, for the same reason

$$B_\theta(\mathrm{d}_{\mathrm{Id}}\rho(H_\delta), H) = B_\theta\left(H_\delta, \mathrm{d}_{\mathrm{Id}}\rho^{-1}(H)\right) = \mathrm{d}_{\mathrm{Id}}\delta\left(\mathrm{d}_{\mathrm{Id}}\rho^{-1}H\right) = \mathrm{d}_{\mathrm{Id}}\left(\delta\rho^{-1}\right)(H)$$

Hence, we obtain

$$\xi_\rho \prod_{\delta\in\Delta} t_\delta^{\mathbb{R}}(\lambda_\delta) = \prod_{\delta\in\Delta} t_{\delta\circ\rho^{-1}}^{\mathbb{R}}(\lambda_\delta).$$

By the transfer principle (Theorem A.7), the same algebraic equation holds for $A_{\mathbb{F}}$ and $t_\delta^{\mathbb{F}}$. Finally by commutativity of the diagram in [App24, Proposition 7.6], the same algebraic equation holds for A_Λ and t_δ^Λ. Now we can prove the claim. Let $\sum_{\delta\in\Delta} \delta\otimes\lambda_\delta \in \mathrm{span}_{\mathbb{Q}}(\Phi)$ such that

$$\begin{aligned} gf_\Lambda^{-1}\left(\xi_\rho f_\Lambda g^{-1}\sum_{\delta\in\Delta}\delta\otimes\lambda_\delta\right) &= gf_\Lambda^{-1}\left(\xi_\rho\prod_{\delta\in\Delta} t_\delta^\Lambda(\lambda_\delta)\right) \\ &= gf_\Lambda^{-1}\left(\prod_{\delta\in\Delta}\rho\left(t_\delta^\Lambda(\lambda_\delta)\right)\right) \\ &= gf_\Lambda^{-1}\left(\prod_{\delta\in\Delta} t_{\delta\circ\rho^{-1}}^\Lambda(\lambda_\delta)\right) \\ &= \sum_{\delta\in\Delta}\delta\circ\rho^{-1}\otimes\lambda_\delta \end{aligned}$$

where we use in the third equality the above computation.

Thus in the following, we write $L\otimes\gamma$ for ξ_ρ when we work at the level of homogeneous models. We use this identification and Theorem 5.12 to construct an isomorphism between the two buildings B and B'. Consider the two charts $f_{\mathrm{Id}} : \mathbb{A} \to B$, which sends $[b]_A$ to $[b]_B \in B$, and $f'_{\mathrm{Id}} : \mathbb{A}' \to B'$, which sends $[b]_{A'}$ to $[b]_{B'} \in B'$ in $\mathcal{A}$ and $\mathcal{A}'$ respectively. By definition of ρ and the atlas in the homogeneous models, ρ is an isomorphism of groups and $\mathbf{G}(\mathbb{F}), \mathbf{G}'(\mathbb{F})$ act transitively on both $\mathcal{A}$ and $\mathcal{A}'$. Moreover, for $[b]_A \in \mathbb{A}$ and $k \in \mathrm{Stab}_{\mathbf{G}(\mathbb{F})}(f_{\mathrm{Id}}([b]_A))$

$$\begin{aligned} \rho(k).f'_{\mathrm{Id}}(L\otimes\gamma[b]_A) &= \rho(k).\left[\rho(b)\right]_{B'} = \left[\rho(kbk^{-1})\right]_{B'} \\ &= [\rho(b)]_{B'} = f'_{\mathrm{Id}}(L\otimes\gamma[b]_A) \end{aligned}$$

Hence $\rho(\mathrm{Stab}_{\mathbf{G}(\mathbb{F})}(f_{\mathrm{Id}}([b]_A))) \subset \mathrm{Stab}_{\mathbf{G}'(\mathbb{F})}(f'_{\mathrm{Id}}(L\otimes\gamma[b]_A))$ and the reverse inclusion is similar using that ρ has an inverse morphism such that we obtain the equality $\rho(\mathrm{Stab}_{\mathbf{G}(\mathbb{F})}(f_g([b]_A))) = \mathrm{Stab}_{\mathbf{G}(\mathbb{F})}(f_{\mathrm{Id}}(L\otimes\gamma[b]_A))$. Finally, for every $w \in W_{\mathrm{a}}$, $k \in A_{f_{\mathrm{Id}},w}$ and $[b']_{A'} \in \mathbb{A}'$ it holds

$$\begin{aligned} \rho(k).f'_{\mathrm{Id}}\left([b']_{A'}\right) &= \rho(k).\left([\rho(b)]_{B'}\right) = \left[\rho(kbk^{-1})\right]_{B'} \\ &= [\rho(wb)]_{B'} = [\sigma(w)\rho(b)]_{B'} = f'_{\mathrm{Id}}\sigma(w)\left([b']_{A'}\right), \end{aligned}$$

where the third equality is due to the definition of $A_{f_{\mathrm{Id}},w}$ and the fourth one to the fact that ξ_ρ induces a morphism of apartments. Thus $\rho(A_{f_{\mathrm{Id}},w}) \subset A_{f'_{\mathrm{Id}},\sigma(w)}$. Using that σ is an isomorphism of affine Weyl groups and ρ an isomorphism of groups, they have inverses. Hence the reverse inclusion also holds and $\rho(A_{f_{\mathrm{Id}},w}) = A_{f_{\mathrm{Id}},\sigma(w)}$ for every $w \in W_{\mathrm{a}}$. Hence by condition (c) in Theorem 5.12, there exists an isomorphism of buildings between B and B'. □

With this, we can now show that an isomorphism of algebraic groups induces an isomorphism of their associated buildings for the homogeneous model.

Corollary 6.10. *Let $\mathbf{G} < \mathrm{SL}_n$ be a semisimple linear algebraic $\mathbb{Q}$-group, $\mathbb{F}$ a non-Archimedean real closed field with an order-compatible valuation $v\colon \mathbb{F}^\times \to \Lambda$. If $\mathfrak{a}, \mathfrak{a}' \subset \mathfrak{p}$ are two maximal abelian subalgebras and $\mathbb{A}, \mathbb{A}'$ are their respective model apartments. Then the homogeneous buildings B and B' defined using $\mathbb{A}$, $\mathbb{A}'$ are isomorphic.*

Proof. From symmetric space theory, see e.g. [Hel68, Theorem 5.2], there exists $k \in \mathbb{K}$ such that $\mathfrak{a}' = \mathrm{Ad}(k)\mathfrak{a}$. So, $A' = \exp(\mathfrak{a}') = \exp(\mathrm{Ad}(k)\mathfrak{a}) = kAk^{-1}$. Thus the result follows from Theorem 6.9 using the group isomorphim $\rho\colon G \to G$ given by conjugation by k. □

As a corollary, we strengthen our result in Theorem 6.7. Indeed, we no longer need to consider a functoriality for inclusion, but now have a functoriality for any injective group morphism.

Theorem 6.11. *Let $\mathbf{G}$, $\mathbf{G}' < \mathrm{SL}_n$ be two semisimple selfadjoint linear algebraic $\mathbb{Q}$-groups. Let $\mathbb{F}$ be a non-Archimedean real closed field and $v\colon \mathbb{F}^\times \to \Lambda$ an order-compatible valuation. Let B (resp. B') be the $\mathbf{G}(\mathbb{F})$- (resp. $\mathbf{G}'(\mathbb{F})$-) homogeneous building defined as in Example 3.5. If there exists an injective morphism of groups $\mathbf{G}(\mathbb{F}) \to \mathbf{G}'(\mathbb{F})$, then it induces an injective morphism of buildings $B \to B'$.*

6.4. **Functoriality under field extensions.** We now discuss functoriality for homogeneous buildings under valued field extensions $\mathbb{K} \subseteq \mathbb{F}$. This is a generalization of [BIPP21, Corollary 5.19] for when the valuation is given by a big element.

Theorem 6.12. *Let $\mathbf{G} < \mathrm{SL}_n$ be a semisimple self-adjoint linear algebraic $\mathbb{Q}$-group, $\mathbb{K}, \mathbb{F}$ non-Archimedean real closed fields with order-compatible valuations $v_\mathbb{K}, v_\mathbb{F}$ and B, B' the homogeneous affine buildings associated to $\mathbf{G}(\mathbb{K})$ and $\mathbf{G}(\mathbb{F})$ respectively (see Example 3.5). Suppose there exists a morphism of valued fields $\eta\colon \mathbb{K} \to \mathbb{F}$, that is $v_\mathbb{F}(\eta(k)) = v_\mathbb{K}(k)$ for every $k \in \mathbb{K}^\times$, then there exists a building morphism $B \to B'$.*

Proof. Denote by Φ the root system of G, by $\mathbb{A} := \mathbb{A}(\Phi^\vee, \Lambda_\mathbb{K}, T)$ the apartments of B, and by $\mathbb{A}' := \mathbb{A}(\Phi^\vee, \Lambda_\mathbb{F}, T')$ the apartments of B'. Since valuations are surjective, we define a map

$$\gamma : \begin{array}{ccc} \Lambda_\mathbb{K} & \longrightarrow & \Lambda_\mathbb{F}; \\ v_\mathbb{K}(k) & \longmapsto & v_\mathbb{F}(\eta(k)), \end{array}$$

which we verify is a well defined ordered group morphism. Suppose $v_\mathbb{K}(k_1) = v_\mathbb{K}(k_2)$. Then $v(k_1/k_2) = 0$, so that $v_\mathbb{F}(\eta(k_1)/\eta(k_2)) = 0$ since η preserves the valuation. Hence $v_\mathbb{F}(\eta(k_1)) = v_\mathbb{F}(\eta(k_2))$ and γ is well defined. Next, for $v_\mathbb{K}(k_1), v_\mathbb{K}(k_2) \in \Lambda_\mathbb{K}$, it holds

$$\gamma\left(v_\mathbb{K}\left(k_1\right) + v_\mathbb{K}(k_2)\right) = \gamma(v_\mathbb{K}(k_1k_2)) = v_\mathbb{F}(\eta(k_1k_2)) = v_\mathbb{F}(\eta(k_1)) + v_\mathbb{F}(\eta(k_2)).$$

Thus $\gamma\left(v_\mathbb{K}\left(k_1\right) + v_\mathbb{K}(k_2)\right) = \gamma(v_\mathbb{K}(k_1)) + \gamma(v_\mathbb{K}(k_2))$ as wanted. Finally, we check that γ preserves the order. If $v_\mathbb{K}(k_1) \leq v_\mathbb{K}(k_2)$, then $v_\mathbb{K}(k_1/k_2) \leq 0$. Since η preserves the valuations, it holds $v_\mathbb{F}(\eta(k_1/k_2)) \leq 0$ so that $v_\mathbb{F}(\eta(k_1)) \leq v_\mathbb{F}(\eta(k_2))$ as wanted.

Using this group morphism, define

$$\sigma : \begin{array}{ccc} W_\mathrm{a} & \longrightarrow & W'_\mathrm{a}; \\ (w, (t_1, \ldots, t_n)) & \longmapsto & (w, (\gamma(t_1), \ldots, \gamma(t_n))). \end{array}$$

By construction, for every $w \in W_{\mathrm{a}}$, the following diagram is commutative

$$\begin{array}{ccc} \mathrm{Span}_{\mathbb{Q}}(A_{n-1}) \otimes_{\mathbb{Q}} \Lambda & \xrightarrow{w} & \mathrm{Span}_{\mathbb{Q}}(A_{n-1}) \otimes_{\mathbb{Q}} \Lambda \\ {\scriptstyle \mathrm{Id} \otimes \gamma}\downarrow & & \downarrow{\scriptstyle \mathrm{Id} \otimes \gamma} \\ \mathrm{Span}_{\mathbb{Q}}(A_{n-1}) \otimes_{\mathbb{Q}} \Lambda' & \xrightarrow[\sigma(w)]{} & \mathrm{Span}_{\mathbb{Q}}(A_{n-1}) \otimes_{\mathbb{Q}} \Lambda' \end{array}$$

so that $\tau := (\mathrm{Id}, \gamma, \sigma)$ is a morphism of apartments between $\mathbb{A}$ and $\mathbb{A}'$. If ρ denotes the inclusion $\mathrm{SL}_n(\mathbb{K}) \to \mathrm{SL}_n(\mathbb{F})$ induced by η entrywise on matrices, then

$$\begin{array}{rccc} \psi: & \mathbf{G}(\mathbb{K})/\mathbf{G}(\mathcal{O}_{\mathbb{K}}) & \longrightarrow & \mathbf{G}(\mathbb{F})/\mathbf{G}(\mathcal{O}_{\mathbb{F}}); \\ & [g]_{B_{\mathbb{K}}} & \longmapsto & [\rho(g)]_{B_{\mathbb{F}}}, \end{array}$$

is well defined. Indeed, let $h \in \mathbf{G}(\mathcal{O}_{\mathbb{K}})$ so that its matrix entries lie in $\mathcal{O}_{\mathbb{K}}$. Applying the field homomorphism η entrywise to h, we obtain $\rho(h) \in \mathbf{G}(\mathbb{F})$. Because η is valuation-preserving $\eta(\mathcal{O}_{\mathbb{K}}) \subseteq \mathcal{O}_{\mathbb{F}}$ so that $\rho(h) \in \mathbf{G}(\mathcal{O}_{\mathbb{F}})$ as wanted. Finally, with the notation from Example 3.5, define

$$\varphi\colon \mathcal{A} \to \mathcal{A}', \quad f_g \mapsto f_{\rho(g)}.$$

<u>Claim:</u> In our setting, the following diagram commutes:

$$\begin{array}{ccc} A(\mathbb{K})/A(\mathcal{O}_{\mathbb{K}}) & \xrightarrow{\xi_\rho} & A(\mathbb{F})/A(\mathcal{O}_{\mathbb{F}}) \\ {\scriptstyle f_\Lambda^{-1}}\downarrow & & \downarrow{\scriptstyle f_{\Lambda'}^{-1}} \\ \mathrm{Span}_{\mathbb{Q}}(\Phi) \otimes_{\mathbb{Q}} \Lambda & \xrightarrow[\mathrm{Id} \otimes \gamma]{} & \mathrm{Span}_{\mathbb{Q}}(\Phi) \otimes_{\mathbb{Q}} \Lambda' \end{array} \quad ,$$

where $\xi_\rho([b]_{A_\Lambda}) := [\rho(b)]_{A_{\Lambda'}}$, and the maps $f_\Lambda^{-1}, f_{\Lambda'}^{-1}$ are as in [App24, Proposition 7.9].
Proof: Let $\Delta \subset \Phi$ be a fixed simple root basis. Consider an element

$$\sum_{\alpha \in \Delta} r_\alpha \alpha \in \mathbb{K}_{>0} \otimes \mathrm{Span}_{\mathbb{Q}}(\Delta).$$

Using the isomorphism presented in [App24, Proposition 7.7], we have on the one hand

$$f_{\mathbb{F}} \circ \mathrm{Id} \otimes \eta \left(\sum_{\alpha \in \Delta} r_\alpha \alpha \right) = f_{\mathbb{F}} \left(\sum_{\alpha \in \Delta} \eta(r_\alpha) \alpha \right) = \prod_{\alpha \in \Delta} t_\alpha^{\mathbb{F}}(\eta(r_\alpha)),$$

and on the other hand

$$\rho \circ f_{\mathbb{K}} \left(\sum_{\alpha \in \Delta} r_\alpha \alpha \right) = \rho \left(\prod_{\alpha \in \Delta} t_\alpha^{\mathbb{K}}(r_\alpha) \right) = \prod_{\alpha \in \Delta} \rho \left(t_\alpha^{\mathbb{K}}(r_\alpha) \right).$$

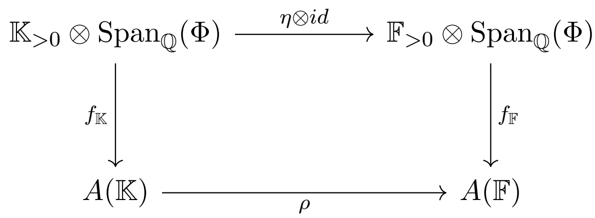
Since t_α is described entrywise by algebraic formulas [App24, Lemma 6.2], there exists for every $\alpha \in \Delta$ a polynomial with $t_\alpha^{\mathbb{L}}(s_\alpha) = T_\alpha(s_\alpha)$ for any field $\mathbb{L}$. So componentwise, we obtain

$$\rho \left(t_\alpha^{\mathbb{K}}(r_\alpha)_{i,j} \right) = \eta(T_\alpha(s_\alpha)) = T_\alpha(\eta(s_\alpha)) = t_\alpha^{\mathbb{F}}(\eta(r_\alpha))_{i,j}.$$

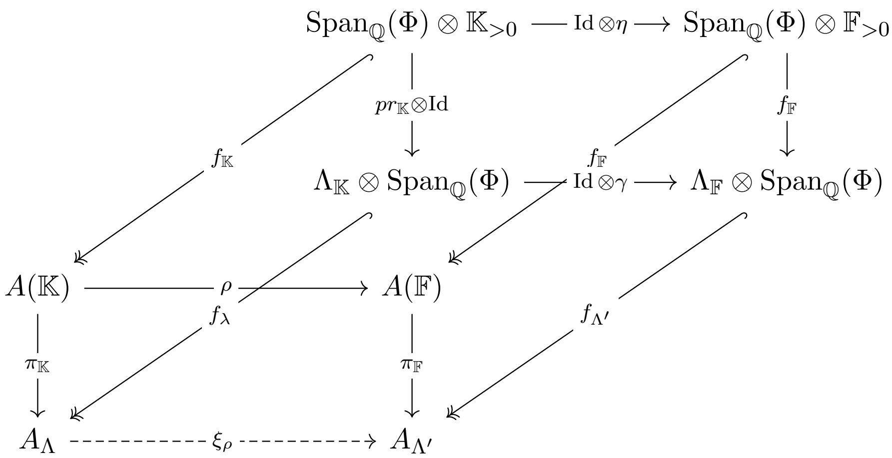
Thus,

$$\rho \circ f_{\mathbb{K}} \left(\sum_{\alpha \in \Delta} r_\alpha \alpha \right) = \prod_{\alpha \in \Delta} t_\alpha^{\mathbb{F}}(\eta(r_\alpha)),$$

so that the following diagram is commutative:



Now, using [App24, Theorem 7.8], we obtain the following commutative diagram:



Since $f_{\mathbb{K}}, f_{\mathbb{F}}, f_{\Lambda}$ and $f_{\Lambda'}$ are bijective, there exists a unique map $\xi_\rho \colon A_\Lambda \to A_{\Lambda'}$ that makes the whole diagram commutative and by commutativity

$$\xi_\rho([b]_{A_\Lambda}) := [\rho(b)]_{A_{\Lambda'}}$$

as wanted.

Thus in the following, we write ξ_ρ for $L \otimes \gamma$ when we work at the level of homogeneous models. Now, if $f_g \in \mathcal{A}$ and $[b]_{\mathbb{A}} \in \mathbb{A}$, then

$$\varphi(f_g)(\xi_\rho[b]_{\mathbb{A}}) = f_{\rho(g)}[\rho(b)]_{\mathbb{A}'} = \left[\rho\left(gbg^{-1}\right)\right]_{B'} = \psi(g[b]_{B'}) = \psi(f_g[b]_{\mathbb{A}})$$

so that (ψ, φ, σ) is a morphism of affine buildings. □

7. Relations between the different buildings

The goal of this section is to prove the existence of morphisms as discussed in Figure 1. All morphisms are injective morphisms, and when $\Lambda = \mathbb{R}$ (the valuation is surjective to $\mathbb{R}$), all morphisms are isomorphisms. When $\Lambda \subseteq \mathbb{R}$ concatenating the morphisms yields an injective morphism from the lattice building B_L to the norm building B_N.

7.1. **Lattice and homogeneous buildings.** In this section we construct an isomorphism from the lattice building B_L (Example 3.3) to the homogeneous building B_H (Example 3.5), in the case where $\mathbf{G} = \mathrm{SL}_n$ and $\mathbb{F}$ is a non-Archimedean real closed field with an order-compatible valuation $v \colon \mathbb{F}^\times \to \Lambda = \mathbb{F}^\times/\mathcal{O}^\times$. Both buildings are of type $\mathbb{A}(\Phi, \Lambda, \Lambda^{n-1})$, where the underlying root system of type A_{n-1} is

$$\Phi = \{x_{ij} \in V \colon x_{ij} = e_i - e_j \in V\}$$

where $V = \mathbb{R}^n$ is the standard Euclidean space with standard basis $\{e_1, \ldots, e_n\}$. The basis $\Delta = \{x_{i,i+1} \colon i \in \{1, 2, \ldots, n-1\}\}$ of Φ induces an isomorphism

$$\mathrm{Span}_{\mathbb{Q}}(\Phi) \otimes \Lambda \to \Lambda^{n-1}, \quad \sum_{i=1}^{n-1} x_{i,i+1} \otimes \lambda_i \mapsto (\lambda_1, \ldots, \lambda_{n-1})$$

of the apartment $\mathbb{A} = \mathrm{Span}_{\mathbb{Q}}(\Phi) \otimes \Lambda$. The spherical Weyl group W_{s} is the symmetric group S_n on n elements and acts by $\sigma(x_{ij}) = x_{\sigma(i)\sigma(j)}$ for $\sigma \in W_{\mathrm{s}}$ on Φ and by linear extension on $\mathbb{A}$. The translation group T is the full translation group $T \cong \Lambda^{n-1}$.

7.1.1. *Setup for the lattice building.* Recall from Example 3.3 that lattices are of the form $\mathcal{O}v_1 + \mathcal{O}v_2 + \ldots + \mathcal{O}v_n$ where $\{v_1, \ldots, v_n\}$ is a basis of $\mathbb{F}^n$, the lattice building B_L is the set of homothety classes of lattices, and that $\mathrm{SL}_n(\mathbb{F})$ acts transitively on B_L (Example 5.5). Viewing $\mathrm{SL}_n(\mathbb{F}) \subseteq \mathbb{F}^{n\times n}$ we define $\mathrm{SL}_n(\mathcal{O}) \coloneqq \mathrm{SL}_n(\mathbb{F}) \cap \mathcal{O}^{n\times n}$. The lattice $L_0 = \mathcal{O}^n$ corresponding to the standard basis is called the *standard lattice* and we call $[L_0]$ the *base point* of B_L. The following is well known in the discrete case.

Proposition 7.1. *We have* $\mathrm{Stab}_{\mathrm{SL}_n(\mathbb{F})}([L_0]) = \mathrm{SL}_n(\mathcal{O})$.

Proof. If $g \in \mathrm{SL}_n(\mathcal{O})$, $g(e_i) \in \mathcal{O}e_1 + \ldots + \mathcal{O}e_n$, so $g.[L_0] = [L_0]$.

For the converse we first establish the following fact: If $\mathcal{E}$ and $\mathcal{E}'$ are bases of $\mathbb{F}^n$ that represent the same $\mathcal{O}$-lattice L, then $v(\det(\mathcal{E})) = v(\det(\mathcal{E}')) \in \Lambda = \mathbb{F}^\times/\mathcal{O}^\times$. Indeed if $M, M' \in \mathrm{GL}_n(\mathbb{F})$ represent the bases $\mathcal{E}$ and $\mathcal{E}'$, they satisfy $M(\mathcal{O}^n) = M'(\mathcal{O}^n)$. Since $M^{-1}M'(\mathcal{O}^n) \subseteq \mathcal{O}^n$, we have $M^{-1}M' \in \mathrm{GL}_n(\mathcal{O})$ (because the elements of the standard basis of $\mathbb{F}^n$ lie in $\mathcal{O}^n$). Since the determinant is a polynomial we obtain $\det(M^{-1}M') \in \mathcal{O}$, so $v(\det(M)^{-1}\det(M')) \geq 0$ and $v(\det(\mathcal{E}')) \geq v(\det(\mathcal{E}))$. In fact these are equalities since also $(M')^{-1}M \in \mathrm{GL}_n(\mathcal{O})$.

Now if $g \in \mathrm{SL}_n(\mathbb{F})$ fixes $[L_0]$, there exists $\lambda \in \mathbb{F}^\times$ such that $g(\mathcal{O}^n) = \lambda\mathcal{O}^n$. The basis given by the columns of g and the basis λe_i are two bases that represent the same lattice, so by the fact, $v(\det(g)) = v(\lambda^n) = nv(\lambda)$. Since $g \in \mathrm{SL}_n(\mathbb{F})$, $v(\det(g)) = 0$, so $\lambda = 0$ and $g(\mathcal{O}^n) = \mathcal{O}^n$. Since the standard basis is part of $\mathcal{O}^n$, the columns of g all lie in $\mathcal{O}^n$, so $g \in \mathrm{SL}_n(\mathcal{O})$. □

Recall that the chart given by [Ben94] corresponding to the standard basis is given by

$$\begin{aligned} f_{\mathcal{E}} \colon \mathbb{A} \cong \Lambda^{n-1} &\to B_L \\ (\lambda_1, \ldots, \lambda_{n-1}) &\mapsto \left[\mathcal{O}x_{\lambda_1}e_1 + \mathcal{O}\frac{x_{\lambda_2}}{x_{\lambda_1}}e_2 + \ldots + \mathcal{O}\frac{x_{\lambda_{n-1}}}{x_{\lambda_{n-2}}}e_{n-1} + \mathcal{O}\frac{1}{x_{\lambda_{n-1}}}e_n\right], \end{aligned}$$

where $x_\lambda \in \mathbb{F}^\times$ such that $v(x_\lambda) = \lambda$.

Lemma 7.2. *For* $a = \mathrm{Diag}(a_1, \ldots, a_n) \in \mathrm{SL}_n(\mathbb{F})$ *and* $\lambda_k = v\left(\prod_{i=1}^k a_i\right)$, *we have*

$$f_{\mathcal{E}}((\lambda_1, \ldots, \lambda_{n-1})) = a.[L_0].$$

Every point in the image of $f_{\mathcal{E}}$ *is of the form* $a.[L_0]$ *for some* $a = \mathrm{Diag}(a_1, \ldots, a_n)$ *with* $a_i > 0$.

Proof. We take $x_{\lambda_k} = \prod_{i=1}^k a_i$, so that $x_{\lambda_k}/x_{\lambda_{k-1}} = a_k$ for $k \in \{2, 3, \ldots, n-1\}$ and $1/x_{\lambda_{n-1}} = a_n$ since $\det(a) = 1$. The first description then follows directly from the definition of $f_{\mathcal{E}}$. If we start with some $\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \in \Lambda^{n-1}$, we can choose $a_1 \in \mathbb{F}_{>0}$ with $v(a_1) = \lambda_1$, and then iteratively $a_k \in \mathbb{F}_{>0}$ with $v(a_k) = \lambda_k - \lambda_{k-1}$ for all $k \in \{2, 3, \ldots, n-1\}$. Finally define $a_n \coloneqq 1/(a_1 \cdots a_{n-1}) \in \mathbb{F}_{>0}$. Then

$$v\left(\prod_{i=1}^k a_i\right) = \sum_{i=1}^k v(a_i) = \lambda_1 + (\lambda_2 - \lambda_1) + \ldots + (\lambda_k - \lambda_{k-1}) = \lambda_k,$$

so $f_{\mathcal{E}}(\lambda_1, \ldots, \lambda_{n-1}) = a.[L_0]$. □

The dual roots $\alpha_{ij} \in V^\star$, $(v_1, \dots, v_n) \mapsto v_j - v_i$ extend to linear maps

$$\alpha_{ij} \colon \operatorname{Span}_{\mathbb{Q}}(\Phi) \otimes \Lambda \to \Lambda, \quad \sum_{k=1}^{n-1} x_{k,k+1} \otimes \lambda_k \mapsto \sum_{k=1}^{n-1} \lambda_k \alpha_{ij}(x_{k,k+1}),$$

that can be used to characterize the diagonal element a in Lemma 7.2.

Lemma 7.3. *A point $x \in \mathbb{A}$ satisfies $f_{\mathcal{E}}(x) = a.[L_0]$ if and only if $\alpha_{ij}(x) = v(a_i/a_j)$ for all $\alpha_{ij} \in {}_{\mathbb{F}}\Phi$.*

Proof. Let $a \in A_{\mathbb{F}}$. We will describe the point $x \in \mathbb{A}$ that corresponds to $a.[L_0]$ via the identifications

$$\begin{array}{ccccc} \mathbb{A} & \leftarrow & \Lambda^n & \rightarrow & A_{\mathbb{F}}.[L_0] \\ \sum_{k=1}^{n-1} x_{k,k+1} \otimes \lambda_i & \leftarrow\!\shortmid & (\lambda_k)_{k=1}^{n-1} & \mapsto & f_{\mathcal{E}}(\lambda_1, \dots, \lambda_{n-1}) \end{array}$$

By Lemma 7.2, $\lambda_k = \sum_{\ell=1}^{k} v(a_\ell)$, so

$$x = \sum_{k=1}^{n-1} x_{k,k+1} \otimes v\left(\prod_{\ell=1}^{k} a_\ell\right).$$

Now applying α_{ij} and using $\alpha_{ij}(x_{k,k+1}) = \delta_{ik} + \delta_{j,k+1} - \delta_{i,k+1} - \delta_{jk}$, where δ is the Kronecker-symbol, we obtain

$$\begin{aligned} \alpha_{ij}(x) &= \sum_{k=1}^{n-1} v\left(\prod_{\ell=1}^{k} a_\ell\right) \alpha_{ij}(x_{k,k+1}) \\ &= v\left(\prod_{\ell=1}^{i} a_\ell\right) + v\left(\prod_{\ell=1}^{j-1} a_\ell\right) - v\left(\prod_{\ell=1}^{i-1} a_\ell\right) - v\left(\prod_{\ell=1}^{j} a_\ell\right) \\ &= v(a_i/a_j). \end{aligned}$$

On the other hand, if we know that $\alpha_{ij}(x) = v(a_i/a_j)$ for some $x = \sum_{k=1}^{n-1} x_{k,k+1} \otimes \lambda_k$, then we know by the same calculation and by uniqueness of the λ_k that $\lambda_k = v\left(\prod_{\ell=1}^{k} a_\ell\right)$. Then by Lemma 7.2, x corresponds to $a.[L_0]$. □

Proposition 7.4. *The pointwise stabilizer of the standard appartment in B_L is given by*

$$\operatorname{Stab}_{\mathrm{SL}_n(\mathbb{F})}(f_{\mathcal{E}}(\mathbb{A})) = \{\operatorname{Diag}(a_1, \dots, a_n) \in \mathrm{SL}_n(\mathbb{F}) \colon v(a_i) = 0\}.$$

Proof. A point $p \in f_{\mathcal{E}}(\mathbb{A})$ is an homothety class of lattices of the form $p = [\sum_{i=1}^{n} \mathcal{O} x_i e_i]$ where $x_i \in \mathbb{F}^\times$. Acting by $g = \operatorname{Diag}(a_1, \dots, a_n)$ with $v(a_i) = 0$ gives $g.p = [\sum_{i=1}^{n} \mathcal{O} a_i x_i e_i] = p$ since $\mathcal{O} a_i = \mathcal{O}$ when $a_i \in \mathcal{O}^\times$. On the other hand if

$$g = \begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & \ddots & \vdots \\ g_{n1} & \cdots & g_{nn} \end{pmatrix} \in \mathrm{SL}_n(\mathbb{F})$$

fixes all points $p \in f_{\mathcal{E}}(\mathbb{A})$, then writing $p = a.[L_0]$ for $a = \operatorname{Diag}(a_1, \dots, a_n)$ (using Lemma 7.2) this means that $g.a.[L_0] = a.[L_0]$, so $a^{-1} g a \in \operatorname{Stab}_{\mathrm{SL}_n(\mathbb{F})}([L_0]) = \mathrm{SL}_n(\mathcal{O})$ by Proposition 7.1. In coordinates $g_{ij} a_j / a_i \in \mathcal{O}$ for all such a. For $i \neq j$, this implies $g_{ij} = 0$, so g has to be diagonal and the diagonal entries have to satisfy $g_{ii} = g_{ii} a_i / a_i \in \mathcal{O}$. Since the stabilizer is closed under inverses, also $g_{ii}^{-1} \in \mathcal{O}$, so $v(g_{ii}) = 0$. □

7.1.2. *Setup for the homogeneous building.* For $G = \mathrm{SL}_n(\mathbb{F})$, we have $K_\mathbb{F} = \mathrm{SO}_n(\mathbb{F})$ and for the maximal torus given by the diagonal subgroup its semi-algebraically connected component of the identity is

$$A_\mathbb{F} \coloneqq \{\mathrm{diag}(a_1, \dots, a_n) \in \mathrm{SL}_n(\mathbb{F}) \colon a_i > 0\}\,.$$

The root system relative to the maximal torus can be identified with

$${}_\mathbb{F}\Phi = \{\alpha_{ij} \in V^\star \colon \alpha_{ij}(x_1, \dots, x_n) = x_j - x_i \text{ for all } (x_1, \dots, x_n) \in V\}\,,$$

so that the dual root system ${}_\mathbb{F}\Phi^\vee$ is Φ as defined earlier. Similar to Lemma 7.3, we set up a characterization of those $a \in A_\mathbb{F}$, where $a.o$ corresponds to a point $x \in \mathbb{A}$.

Lemma 7.5. *A point $x \in \mathbb{A}$ satisfies $f_0(x) = a.o$ for $a \in A_\mathbb{F}$ if and only if $\alpha_{ij}(x) = (-v)(a_i/a_j)$ for all $\alpha_{ij} \in {}_\mathbb{F}\Phi$.*

Proof. From the compatiblity condition ($\star\star$) we get that for $\alpha \in {}_\mathbb{F}\Phi$,

$$(-v)(\chi_\alpha(a)) = \alpha(x).$$

In our specific case, if $a.o$ corresponds to $x \in \mathbb{A}$, setting $\alpha = \alpha_{ij}$ we obtain exactly $(-v)(a_i/a_j) = \alpha_{ij}(x)$. On the other hand, any $x = \sum_{k=1}^{n-1} x_{i,i+1} \otimes \lambda_k$ is uniquely determined by the $\lambda_i \in \Lambda$. So if $(-v)(a_i/a_j) = \alpha_{ij}(x)$, then

$$(-v)(a_i/a_j) = \sum_{k=1}^{n-1} \lambda_k \alpha_{ij}(x_{k,k+1}) = \lambda_i + \lambda_{j-1} - \lambda_{i-1} - \lambda_j$$

(with the convention $\lambda_0 = 0$). This results in the system of linear equations

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 2 & -1 \\ & \cdots & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \\ \lambda_k \end{pmatrix} = \begin{pmatrix} (-v)(a_1/a_2) \\ (-v)(a_2/a_3) \\ \vdots \\ \\ (-v)(a_{n-1}/a_n) \end{pmatrix},$$

which determines the solution uniquely since the matrix is invertible. □

The stabilizers for B_L in Propositions 7.1 and 7.4 coincide with the stabilizers for the action on B_H.

Proposition 7.6. *The stabilizer of the base point $o \in B_H$ is $\mathrm{Stab}_{\mathrm{SL}_n(\mathbb{F})}(o) = \mathrm{SL}_n(\mathcal{O})$ and the pointwise stabilizer of the standard apartment is*

$$\mathrm{Stab}_{\mathrm{SL}_n(\mathbb{F})}(f_0(\mathbb{A})) = \{\mathrm{Diag}(a_1, \dots, a_n) \in \mathrm{SL}_n(\mathbb{F}) \colon v(a_i) = 0\}\,.$$

Proof. For o this is [App24, Theorem 7.11], and for the standard apartment the statement follows from [App24, Theorem 8.19]. Indeed, the latter shows that $\mathrm{Stab}_{\mathrm{SL}_n(\mathbb{F})}(f_0(\mathbb{A})) = M_\mathbb{F} A_\mathbb{F}(\mathcal{O})$, where $M_\mathbb{F} \coloneqq \mathrm{Cent}_{\mathrm{SO}_n(\mathbb{F})}(A_\mathbb{F})$ consists of all diagonal matrices in $\mathrm{SL}_n(\mathbb{F})$ with entries ± 1. □

7.1.3. *Isomorphism between lattice and homogeneous building.* In this subsection we will show that the lattice building B_L is isomorphic to the homogeneous building B_H. Even though these two buildings have the same apartment $\mathbb{A}$, we will not use the identity map $\mathbb{A} \to \mathbb{A}$ as our apartment morphism, but instead the inversion $i \colon \mathbb{A} \to \mathbb{A}, x \mapsto -x$. In the discrete setting this corresponds to an isomorphism that does not preserve the type of the vertices.

Lemma 7.7. *Let $L \colon \mathrm{Span}_\mathbb{Q}(\Phi) \to \mathrm{Span}_\mathbb{Q}(\Phi)$, $x \mapsto -x$, $\gamma = \mathrm{id} \colon \Lambda \to \Lambda$ and $\sigma \colon W_\mathrm{a} \to W_\mathrm{a}$, $w \mapsto i \circ w \circ i$. Then $\tau = (L, \gamma, \sigma)$ is an isomorphism of apartments $\mathbb{A} \to \mathbb{A}$ of type $(\Phi, \Lambda, \Lambda^{n-1})$.*

Proof. By definition, L is linear, γ is a group homomorphism, and L, γ and σ clearly verify

$$(L \otimes_{\mathbb{Q}} \gamma)(w.a) = \sigma(w).((L \otimes_{\mathbb{Q}} \gamma)(a))$$

for all $w \in W_{\mathrm{a}}$. Thus τ is a morphism of apartments. Since τ is also its own inverse, it is an isomorphism. □

In the following three lemmas we investigate the action of the spherical and affine Weyl groups on $\mathbb{A}$. Then we are ready to prove that the lattice building B_L is isomorphic to B_H.

Lemma 7.8. *For every $w \in W_{\mathrm{s}} \cong S_n$, $x \in \mathbb{A}$ and $a \in (\mathbb{F}^\times)^n$, we have*

$$\forall i,j\colon \alpha_{ij}(x) = v\left(\frac{a_i}{a_j}\right) \qquad \Longleftrightarrow \qquad \forall i,j\colon \alpha_{ij}(w(x)) = v\left(\frac{a_{w^{-1}(i)}}{a_{w^{-1}(j)}}\right).$$

Proof. Recall that $\alpha_{ij}(x_{k,k+1}) = \delta_{ik} + \delta_{j,k+1} - \delta_{i,k+1} - \delta_{jk}$, where δ is the Kronecker-symbol. So if $x = \sum_{k=1}^{n-1} x_{k,k+1} \otimes \lambda_k \in \mathbb{A}$, then

$$\alpha_{ij}(x) = \sum_{k=1}^{n-1} \lambda_k \alpha_{ij}(x_{k,k+1}) = \lambda_i + \lambda_{j-1} - \lambda_{i-1} - \lambda_j,$$

with the convention that $\lambda_0 = 0$. Moreover

$$\begin{aligned}
\alpha_{ij}(w(x)) &= \sum_{k=1}^{n-1} \lambda_k \alpha_{ij}(x_{w(k),w(k+1)}) = \sum_{k=1}^{n-1} \lambda_{w^{-1}(k)} \alpha_{ij}(x_{k,k+1}) \\
&= \lambda_{w^{-1}(i)} + \lambda_{w^{-1}(j-1)} - \lambda_{w^{-1}(i-1)} - \lambda_{w^{-1}(j)} \\
&= \alpha_{w^{-1}(i),w^{-1}(j)}(x) = v\left(\frac{a_{w^{-1}(i)}}{a_{w^{-1}(j)}}\right)
\end{aligned}$$

where the last equality holds if and only if $\alpha_{ij}(x) = v(a_i/a_j)$ for all i,j. □

Lemma 7.9. *For every $w \in W_{\mathrm{s}} \cong S_n$ there exists $k \in \mathrm{SO}_n(\mathbb{F})$ such that for all $a \in A_{\mathbb{F}}$*

(1) *if $f_{\mathcal{E}}(x) = a.[L_0]$, then $f_{\mathcal{E}}(w(x)) = k.f_{\mathcal{E}}(x)$,*
(2) *if $f_0(x) = a.o$, then $f_0(w(x)) = k.f_0(x)$.*

Proof. For the permutation $w \in W_{\mathrm{s}} = S_n$ consider a permutation matrix $k \in \mathrm{SO}_n(\mathbb{F})$ defined by $k_{ij} = \pm\delta_{i,w(j)}$ (choose $+1$ or -1 so that $\det(k) = 1$). Then

$$\begin{aligned}
(kak^{-1})_{ij} &= \sum_{k,\ell=1}^{n} k_{ik} a_{k\ell} (k^{-1})_{\ell j} = \sum_{k,\ell=1}^{n} \left(\pm\delta_{i,w(k)}\right) a_{k\ell} \left(\pm\delta_{w(\ell),j}\right) \\
&= \sum_{k,\ell=1}^{n} \left(\pm\delta_{w^{-1}(i),k}\right) a_{k\ell} \left(\pm\delta_{\ell,w^{-1}(j)}\right) = a_{w^{-1}(i),w^{-1}(j)},
\end{aligned}$$

where we note that only when $i = j$ is $a_{w^{-1}(i),w^{-1}(j)}$ non-zero, and in that case the ± 1 cancel. Now if $f_{\mathcal{E}}(x) = a.[L_0]$, then $\alpha_{ij}(x) = v(a_i/a_j)$ by Lemma 7.3 and

$$f_{\mathcal{E}}(w(x)) = kak^{-1}.[L_0] = ka.[L_0] = k.f_{\mathcal{E}}(x)$$

by Lemmas 7.8 and 7.3 and the fact that $k^{-1} \in \mathrm{SO}_n(\mathbb{F}) \subseteq \mathrm{SL}_n(\mathcal{O})$, so $k.[L_0] = [L_0]$ by Proposition 7.1. If $f_0(x) = a.o$, then $\alpha_{ij}(x) = -v(a_i/a_j)$ by Lemma 7.5 and

$$f_0(w(x)) = kak^{-1}.o = ka.o = k.f_0(x)$$

by Lemmas 7.8 and 7.5 and the fact that $k^{-1} \in \mathrm{SO}_n(\mathbb{F}) \subseteq \mathrm{SL}_n(\mathcal{O})$, so $k.[L_0] = [L_0]$ by Proposition 7.6. □

Lemma 7.10. *Let $t \in T = \mathbb{A}$ be the translation of $\mathbb{A}$ by a vector $y \in \mathbb{A}$ that satisfies $a_{ij}(y) = v(a_i/a_j)$ for some $a = \operatorname{diag}(a_1, \ldots, a_n) \in A_{\mathbb{F}}$. Then*

$$f_{\mathcal{E}}(t(x)) = a.f_{\mathcal{E}}(x) \quad \text{and} \quad f_0(t(x)) = a^{-1}.f_0(x).$$

Proof. Let $b = \operatorname{diag}(b_1, \ldots, b_n) \in A_{\mathbb{F}}$ such that $x \in \mathbb{A}$ satisfies $\alpha_{ij}(x) = v(b_i/b_j)$. Then $\alpha_{ij}(x + y) = \alpha_{ij}(x) + \alpha_{ij}(y) = v(a_i b_i/a_j b_j)$, so by Lemma 7.3,

$$f_{\mathcal{E}}(t(x)) = f_{\mathcal{E}}(x + y) = ab.[L_0] = a.f_{\mathcal{E}}(x).$$

Similarly

$$f_0(t(x)) = f_0(x + y) = (ab)^{-1}.o = a^{-1}.b^{-1}.o = a^{-1}.f_0(x).$$

using Lemma 7.5. □

Theorem 7.11. *There is an isomorphism of affine Λ-buildings between the lattice building B_L and the homogeneous building B_H.*

Proof. We use Theorem 5.12 as $\mathrm{SL}_n(\mathbb{F})$ acts transitively on B_L, see Example 5.5. Let $G = G' = \mathrm{SL}_n(\mathbb{F})$ and ρ the identity map. We take the apartment isomorphism $\tau = (L, \gamma, \sigma)$ from Lemma 7.7. We consider the charts

$$f = f_{\mathcal{E}} \colon \mathbb{A} \to B_L \quad \text{and} \quad f' = f_0 \colon \mathbb{A} \to B_H.$$

For $x \in \mathbb{A}$, let $a \in A_{\mathbb{F}}$ such that $\alpha_{ij}(x) = v(a_i/a_j)$ as in Lemma 7.3. Then by Lemma 7.5 we have $a.o = f'(-x) = f'(\tau(x))$. The Propositions 7.1 and 7.6 imply

$$\begin{aligned}
\rho\left(\mathrm{Stab}_{\mathrm{SL}_n(\mathbb{F})}(f(x))\right) &= \mathrm{Stab}_{\mathrm{SL}_n(\mathbb{F})}(a.[L_0]) = a^{-1}\,\mathrm{Stab}_{\mathrm{SL}_n(\mathbb{F})}([L_0])a \\
&= a^{-1}\,\mathrm{SL}_n(\mathcal{O})a = a^{-1}\,\mathrm{Stab}_{\mathrm{SL}_n(\mathbb{F})}(o)a = \mathrm{Stab}_{\mathrm{SL}_n(\mathbb{F})}(a.o) \\
&= \mathrm{Stab}_{\mathrm{SL}_n(\mathbb{F})}(f'(-x)) = \mathrm{Stab}_{\mathrm{SL}_n(\mathbb{F})}(f'(\tau(x))
\end{aligned}$$

as required for condition (1).

For condition (2) we have to show that $A_{f,w} = \rho(A_{f',\sigma(w)})$ for all $w \in W_{\mathrm{a}}$, where $A_{f,w} = \{g \in \mathrm{SL}_n(\mathbb{F}) \colon g.f = f \circ w\}$. Recall that if $w = t \circ w_s \in T \rtimes W_{\mathrm{s}} = W_{\mathrm{a}}$, then $\sigma(w) = i \circ w \circ i = i \circ t \circ i \circ w_s = t^{-1} \circ w_s$ since elements of W_{s} commute with the inversion i. The statement $g \in A_{f,w}$ means that for all $x \in \mathbb{A}$ we have $g.f(x) = f(w(x))$. By Lemmas 7.9 and 7.10 there are suitable $k \in \mathrm{SO}_n(\mathbb{F})$ and $a \in A_{\mathbb{F}}$ such that $f(w(x)) = f(t(w_s(x))) = a.f(w_s(x)) = ak.f(x)$. so $g \in A_{f,w}$ is equivalent to $(ak)^{-1}g.f(x) = f(x)$ for all $x \in \mathbb{A}$, which is equivalent to $(ak)^{-1}g \in \mathrm{Stab}_{\mathrm{SL}_n(\mathbb{F})(f(\mathbb{A}))} = M_{\mathbb{F}}A_{\mathbb{F}}(\mathcal{O})$ by Proposition 7.4. But by Proposition 7.6 this is in turn equivalent to $(ak)^{-1}g.f'(x) = f'(x)$ for all $x \in \mathbb{A}$, or $g.f'(x) = ak.f'(x) = a.f'(w_s(x)) = f'(t^{-1}(w_s(x))) = f'(\sigma(w)(x))$ by Lemmas 7.9 and 7.10 with the same k and a as above. Thus $A_{f,w} = A_{f',\sigma(w)}$.

In fact, $\mathrm{SL}_n(\mathbb{F})$ also acts transitively on B_H (Example 5.7), $\rho = \mathrm{Id}$ is an isomorphism of groups, L, γ and σ are injective, and the inclusions in the conditions (1) and (2) are equalities. By (c) we can conclude that B_L and B_H are isomorphic. □

7.2. **Homogeneous and Bruhat–Tits buildings.** In this section we show that there is an injective morphism from the homogeneous building B_H to the Bruhat-Tits building B_{BT}. When $\Lambda = \mathbb{R}$, it is an isomorphism. Let $\mathbb{F}$ be a real closed field with an order-compatible valuation $v \colon \mathbb{F}^\times \to \Lambda \subseteq \mathfrak{R}$ and denote by $\mathcal{O}$ the valuation ring of v. Set $G = \mathbf{G}(\mathbb{F})$, where $\mathbf{G}$ is a semi-simple, connected, self-adjoint, $\mathbb{F}$-split algebraic group $\mathbf{G} < \mathrm{SL}_n$. Let $\mathbf{S}$ be a maximal ($\mathbb{F}$-split) torus and assume that the root system $\Phi = {}_{\mathbb{F}}\Phi$ is reduced. These conditions ensure that both the Bruhat–Tits building B_{BT} (Example 3.4) and the homogeneous building B_H (Example 3.5) are defined.

Let $K = G \cap \mathrm{SO}_n(\mathbb{F})$, $T = \mathrm{Cent}_G(\mathbf{S}(\mathbb{F}))$ and U_α the root groups for $\alpha \in \Phi$.

Lemma 7.12. *For the base points* $[\mathrm{Id},0] \in B_{\mathrm{BT}}$ *and* $o \in B_H$*, we have* $\mathrm{Stab}_G([\mathrm{Id},0]) = \mathrm{Stab}_G(o) = \mathbf{G}(\mathcal{O}) := G \cap \mathcal{O}^{n\times n}$*.*

Proof. We will first show the statement for $g \in N = \mathrm{Nor}_{\mathbf{G}}(\mathbf{S})$. The action $\nu\colon N \to \mathrm{Aff}(V \otimes \mathfrak{R})$ of N as the affine Weyl group decomposes as a semi-direct product $N = A \cdot \mathrm{Nor}_K(A)$, where $A \subseteq T$ is the semi-algebraically connected component of the identity of $\mathbf{S}(\mathbb{F})$ [App24, Proposition 6.4]. By the compatibility

$$\varphi_\alpha(aua^{-1}) = \varphi_\alpha(u) + \omega(\alpha(a))$$

for all $u \in U_\alpha$, $\alpha \in \Phi$ and $a \in A$, we have that

$$(1) \qquad \nu(a) = \mathrm{Id}_{V\otimes\mathfrak{R}} \quad \text{if and only if} \quad \forall \alpha \in \Phi,\ \omega(\alpha(a)) = 0.$$

If $g \in \mathrm{Stab}_G([\mathrm{Id},0])$, then $\exists n \in N$ with $g^{-1}n \in P_0$ and $\nu(n)(0) = 0$, and since $g^{-1}n \in N \cap P_0 =: N_0$, we have $\nu(g^{-1}n)(0) = (0)$ [BT72, (7.1.8)][HIL23, Corollary 5.15], and thus $\nu(g)(0) = 0$. Now if $g = ak \in A \cdot \mathrm{Nor}_K(A)$, $\nu(a) = \mathrm{id}_V$, since $\nu(k)(0) = 0$. By (1) and [App24, Proposition 7.10], $a \in \mathrm{Stab}_G(o)$. Moreover $k \in K \subseteq \mathrm{Stab}_G(o)$ [App24, Theorem 7.11 and Corollary 7.12], and so $g \in \mathrm{Stab}_G(o)$.

If however $g \in \mathrm{Stab}_G(\mathcal{O})$, then let $g = ak \in A\cdot\mathrm{Nor}_K(A)$ and where $a \in A \cap \mathcal{O}^{n\times n}$ [App24, Corollary 7.12]. Then by [App24, Proposition 7.10], $\omega(\alpha(a)) = 0$ for all $\alpha \in \Phi$, so $\nu(a) = \mathrm{Id}_V$. To prove $g.[\mathrm{Id},0] = [\mathrm{Id},0]$, we can take $n := a \in N$ to obtain $g^{-1}n = k^{-1} \in \hat{N}_0 := \{n \in N\colon \nu(n)(0) = 0\}$ and $\nu(n)(0) = 0$. By definition of B_{BT} in [BT72, (7.4.1)][HIL23, Definition 6.1], this means that $g \in \mathrm{Stab}_G([\mathrm{Id},0])$, when $g \in N$.

For general $g \in G$, we use the objects $U_{\alpha,0} = \{u \in U_\alpha\colon \varphi_\alpha(u) \geq 0\}$ where φ_α is the root group valuation defined via the Jacobson–Morozov maps $\mathrm{SL}_2 \to \mathbf{G}$ [BT72, (6.1.3.b.2), (6.2.3.b)][HIL23, Notation 7.22]. We would like to emphasize that while $U_{\alpha,0}$ is defined differently in [App24], the two concepts agree due to [App24, Lemmas 8.22 and 8.26]. Recall that

$$P_0 = \langle u \in U_{\alpha,0},\ h \in H := \nu^{-1}(\mathrm{Id}_{V\otimes\mathfrak{R}})\rangle.$$

If $g \in \mathrm{Stab}_G([\mathrm{Id},0])$, there exists $n \in N$ such that $g^{-1}n \in P_0$ and $\nu(n)(0) = 0$. Since $n \in \mathrm{Stab}_G([\mathrm{Id},0]) \cap N$, $n \in \mathrm{Stab}_G(o)$. Any $h \in H = \nu^{-1}(\mathrm{Id}_V)$ satisfies $h.[\mathrm{Id},0] = [\mathrm{Id},\nu(h)(0)] = [\mathrm{Id},0]$ and by [App24, Lemma 8.22] $U_{\alpha,0} \subseteq \mathrm{Stab}_G(o)$, so $g = n(g^{-1}n)^{-1} \in \mathrm{Stab}_G(o)$.

If we start with $g \in \mathrm{Stab}_G(o)$, then we use [App24, Theorem 8.45] for

$$g \in \langle N_0, U_{\alpha,0}\rangle \subseteq \hat{P}_0,$$

so we can take $n = \mathrm{Id} \in N$ with $g^{-1}n \in \hat{P}_0$ and $\nu(n)(0) = 0$ to obtain $g \in \mathrm{Stab}_G([\mathrm{Id},0])$. □

Remark 7.13. In [BT84, Corollaire (4.6.7)][HIL23, Proposition 7.48], it is shown that $\mathrm{Stab}([\mathrm{Id},0]) = \mathfrak{G}(\mathcal{O})$, where $\mathfrak{G}$ is a group $\mathcal{O}$-scheme defined in terms of some Chevalley basis. In our definition $\mathbf{G}$ is just a group of matrices and the definition $\mathbf{G}(\mathcal{O}) = G \cap \mathcal{O}^{n\times n}$ avoids any algebraic geometry.

We can now prove the existence of an injective morphism from B_H to B_{BT}.

Theorem 7.14. *There is an injective morphism of G-buildings from the homogeneous G-building $(B_H, \mathcal{A})$ of type $\mathbb{A}(\Phi^\vee, \Lambda, \Lambda^n)$, see Example 5.7, to the Bruhat–Tits G-building $(B_{\mathrm{BT}}, \mathcal{A}')$ of type $\mathbb{A}'(\Phi^\vee, \mathbb{R}, \Lambda^n)$, see Example 5.6. If $\Lambda = \mathbb{R}$, this morphism is an isomorphism $B_H \cong B_{\mathrm{BT}}$.*

Proof. We apply Theorem 5.12 (a), as G acts transitively on $\mathcal{A}$, see Example 5.7. Let $\gamma\colon \Lambda \to \mathfrak{R}$ be the inclusion, $L = \mathrm{Id}_{\mathrm{Span}_{\mathbb{Q}}(\Phi)}$, $\sigma = \mathrm{Id}_{W_{\mathrm{a}}}$ and $\rho = \mathrm{Id}_G$. For all $w \in W_{\mathrm{a}}$ and $a \in \mathbb{A}$ we have $(L\otimes\gamma)(w(a)) = \sigma(w)\left((L\otimes\gamma)\right)(a)$ and thus $L\otimes\gamma\colon \mathbb{A} \to \mathbb{A}'$ is a morphism of apartments, see Definition 4.4.

We choose $f_0\colon \mathbb{A} \to B_H \in \mathcal{A}$ as in Example 3.5, see also [App24, Section 8]. By Equation ($\star\star$), for all $x \in \mathbb{A}$ there exists $t \in T$ such that $f_0(x) = t.o$ with $(-\omega)(\chi_\alpha(t)) = \alpha(x)$ for all $\alpha \in \Phi$. For B_{BT} we choose $f_0'\colon \mathbb{A}' \to B_{\mathrm{BT}}$ by $f_0'(x') = [\mathrm{Id}, x']$. If $f_0(x) = t.o$ as above, then we have by Equation ($\star$) of Example 3.4 that $[t, 0] = [\mathrm{Id}, L \otimes \gamma(x)] \in B_{\mathrm{BT}}$. Using Lemma 7.12 we can then verify condition (1) of Theorem 5.12 by calculating

$$\begin{aligned}\rho\left(\mathrm{Stab}_G(f_0(x))\right) &= \mathrm{Stab}_G(t.o) = t\,\mathrm{Stab}_G(o)t^{-1} = t\,\mathrm{Stab}_G([\mathrm{Id}, 0])t^{-1}\\ &= \mathrm{Stab}_G([t, 0]) = \mathrm{Stab}_G([\mathrm{Id}, L \otimes \gamma(x)])\\ &= \mathrm{Stab}_G(f_0'(L \otimes \gamma(x)))\end{aligned}$$

for all $x \in \mathbb{A}$.

It remains to show (2), i.e. $\rho(A_{f_0,w}) \subseteq A_{f_0',\sigma(w)}$, and (a) to conclude that the maps ϕ and ψ are injective. If $w = \mathrm{Id} \in W_{\mathrm{a}}$, $A_{f_0,w} = T(\mathcal{O}) \coloneqq T \cap \mathcal{O}^{n\times n}$ is the pointwise stabilizer of $f_0(\mathbb{A})$, see Proposition 3.6. By Equation ($\star$) in Example 3.4 and Lemma 7.12, for $t \in T(\mathcal{O})$ we have $t.[\mathrm{Id}, x] = [\mathrm{Id}, x]$ for all $x \in \mathbb{A}'$, so $t.f_0' = f_0$ and thus $A_{f_0,\mathrm{Id}} \subseteq A_{f_0',\sigma(\mathrm{Id})}$. On the other hand, if we start with $g \in A_{f_0',\mathrm{Id}}$, then $g \in H \coloneqq \{n \in \mathrm{Nor}_G(\mathbf{S}(\mathbb{F}))\colon [n, x] = [\mathrm{Id}, x]\,\forall x \in \mathbb{A}'\}$ by [BT72, (7.4.10)][HIL23, Corollary 6.9]. Thus, the spherical Weyl group action of $g \in N$ on $\mathbb{A}'$ is trivial, hence $n \in T$. By Lemma 7.12, $g \in T(\mathcal{O})$, so $g \in A_{f_0,\mathrm{Id}}$ as above. This shows $A_{f_0,\mathrm{Id}} = A_{f_0',\mathrm{Id}}$ and in particular condition (a) of Theorem 5.12.

If $w \in W_{\mathrm{a}}$ represents a translation by $x \in \mathbb{A}$ where $x = t.o \in B$ for $t \in T$, then $t.f_0'(y) = [t, y] = [\mathrm{Id}, L \otimes \gamma(x) + y] = f_0'(w(y))$ for all $y \in \mathbb{A}'$ by Equation ($\star$), so $A_{f_0,w} = tA_{f_0,\mathrm{Id}} \subseteq tA_{f_0',\mathrm{Id}} = A_{f_0',w}$ as required. If $w = r_a \in W_{\mathrm{s}} \subseteq W_{\mathrm{a}}$ represents the reflection along the hyperplane perpendicular to a root $\alpha \in \Sigma$, then there exists an element $m_\alpha \in \mathrm{Nor}_G(\mathbf{S}(\mathbb{F}))$ representing w, $m_\alpha \in A_{f_0,w}$ by [App24, Proposition 8.28][3]. By the compatibility of the various root systems and spherical Weyl groups [App24, Proposition 6.3], m_α also represents the reflection $r_\alpha = w$ in $\mathbb{A}'$. Thus, $A_{f_0,w} = m_\alpha A_{f_0,\mathrm{Id}} \subseteq m_\alpha A_{f_0',\mathrm{id}} = A_{f_0',w}$. Finally,

$$A_{f_0,w\tilde{w}} = A_{f_0,w}A_{f_0,\tilde{w}} \quad \text{and} \quad A_{f_0',w\tilde{w}} = A_{f_0',w}A_{f_0',\tilde{w}}$$

for all $w, \tilde{w} \in W_{\mathrm{a}}$, see Proposition 5.11 (2) (note that all involved sets are non-empty). Since translations and reflections as above generate W_{a} we have that $A_{f_0,w} = A_{f_0',w}$ for all $w \in W_{\mathrm{a}}$.

By Example 5.6, G also acts transitively on $\mathcal{A}'$, τ and ρ are injective and $\rho\left(\mathrm{Stab}_G(f_0(x))\right) = \mathrm{Stab}_G(f_0'(L \otimes \gamma(x)))$, so the morphism (ψ, φ, τ) is injective by Theorem 5.12(a). If $\Lambda = \mathbb{R}$, then γ is an isomorphism, so τ is an isomorphism of apartments. We verified the other conditions of Theorem 5.12(c) above, so when $\Lambda = \mathbb{R}$, then the above morphism is an isomorphism $B_H \cong B_{\mathrm{BT}}$. □

7.3. **Bruhat–Tits and norm buildings.** In this section, we we use Theorem 5.12 to show that there is a morphism from the Bruhat–Tits building B_{BT} for GL_n to the norms building B_N. For the definitions of the buildings we refer to Example 3.4 and Example 3.2. Note that the morphism we construct in the proof consists of a bijective map $\psi\colon B_{\mathrm{BT}} \to B_N$ on the buildings themselves, but when $\Lambda \neq \mathbb{R}$, the atlas map φ is not bijective. Therefore the morphism is not an isomorphism. However, as remarked in [Par23, Remark in Section 3B4], B_N could also be equipped with a different atlas by restricting the translation part of the affine Weyl group. For the so-defined atlas, the morphism we construct is an isomorphism.

[3]In [App24], we used that $\mathrm{Nor}_G(A) = \mathrm{Nor}_G(\mathbf{S}(\mathbb{F}))$, where A is the semi-algebraically connected component of the identity of $\mathbf{S}(\mathbb{F})$.

Let now $G = \mathrm{GL}_n(\mathbb{F})$, where $\mathbb{F}$ is a field with a non-Archimedean rank one valuation $v\colon \mathbb{F}^\times \to \Lambda \subseteq \mathbb{R}$.

Theorem 7.15. *There is an injective morphism from the Bruhat–Tits building B_{BT} of type $\mathbb{A} = \mathbb{A}(A_{n-1}, \mathbb{R}, \Lambda^n/\langle(1,\dots,1)\rangle)$ to the norm building B_N of type $\mathbb{A}' = \mathbb{A}'(A_{n-1}, \mathbb{R}, \mathbb{R}^n/\langle(1,\dots,1)\rangle)$. When $\Lambda = \mathbb{R}$, then this morphism is an isomorphism $B_{\mathrm{BT}} \cong B_N$.*

Proof. We note that the apartments $\mathbb{A}$, $\mathbb{A}'$ are the same as sets. In general, the affine Weyl group of the Bruhat–Tits building only includes in the Weyl group of the norms building. Together with the identity map $\mathbb{A} \to \mathbb{A}'$ this inclusion gives a morphism τ of apartments.

Both buildings are G-buildings for $G = \mathrm{GL}_n(\mathbb{F})$, so we take $\rho = \mathrm{Id}_G$. Let $f\colon \mathbb{A} \to B_{\mathrm{BT}}$ be the standard chart given by $x \mapsto [\mathrm{Id}, x]$ and let $f'\colon \mathbb{A} \to B_N$, $\lambda \mapsto \eta_\lambda$ be the chart associated to the standard basis defined by

$$\eta_\lambda \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \max\left\{ e^{-\lambda_1}|a_1|, \dots, e^{-\lambda_n}|a_n| \right\},$$

where $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\sum_i \lambda_i = 0$ and $|a| = \exp(-v(a))$.

We apply Theorem 5.12. since G acts transitively on the atlas of the Bruhat–Tits building B_{BT}, see Example 5.6, it remains to show that the following conditions in Theorem 5.12 hold:

(1) $\mathrm{Stab}_G(f(a)) \subseteq \mathrm{Stab}_G(f'(a))$ for all $a \in \mathbb{A}$, and
(2) $A_{f,w} \subseteq A_{f',w}$ for all $w \in W_{\mathrm{a}}$.

In the special case $a = 0$, we claim that $\mathrm{Stab}_G(f(a)) = \mathrm{Stab}_G(f'(a)) = \mathrm{GL}_n(\mathcal{O})$, where

$$\begin{aligned} \mathrm{GL}_n(\mathcal{O}) &:= \left\{ g \in \mathrm{GL}_n(\mathbb{F}) \colon g_{ij} \in \mathcal{O},\ \det(g) \in \mathcal{O}^\times \right\} \\ &= \{ g \in \mathrm{GL}_n(\mathbb{F}) \colon v(g_{ij}) \geq 0,\ v(\det(g)) = 0 \}. \end{aligned}$$

Indeed, for $g \in \mathrm{Stab}_G(f(0))$ we use the fact that the determinant is a polynomial with coefficients in $\mathbb{Z}$ and the description of g in [BT72, Corollaire (10.2.9)] to see that

$$v(\det(g)) \geq \min_{i,j} \{v(g_{ij})\} \geq \frac{v(\det(g))}{n}$$

from which it follows that $v(\det(g)) \geq 0$. Similarly, since also $g^{-1} \in \mathrm{Stab}_G(f(0))$, $v(\det(g^{-1})) \geq 0$, so $v(\det(g)) = 0$ and by the description in [BT72, Corollaire (10.2.9)] $v(g_{ij}) \geq 0$ and thus $g \in \mathrm{GL}_n(\mathcal{O})$. The other inclusion follows directly from the description in Bruhat–Tits. For the norms building, [Par00, Corollaire 3.4] states that $g \in \mathrm{Stab}_G(f'(0))$ if and only if $\exp(-v(\det(g))) = 1$ and

$$\exp(-v(g_{ij})) \leq \frac{\eta_0(e_j)}{\eta_0(e_i)} = 1,$$

which is equivalent to $g \in \mathrm{GL}_n(\mathcal{O})$. This shows the claim and statement (1) in the special case $a = 0$.

For a general $a \in \mathbb{A}$, there exists a diagonal matrix $t = \mathrm{Diag}(t_1, \dots, t_n) \in \mathrm{SL}_n(\mathbb{F})$ with $(-v)(t_i) = a_i$. For the roots α_{ij} this means $(-v)(\alpha_{ij}(t)) = (-v)(t_i/t_j) = \alpha_{ij}(a)$, so $t.[\mathrm{id}, 0] = [t, 0] = [\mathrm{id}, a]$ by the compatibility condition ($\star$) on the valuations for the Bruhat–Tits building, see Example 3.4. Thus $\mathrm{Stab}_G(f(a)) = t\,\mathrm{GL}_n(\mathcal{O})t^{-1}$. In the norm building we have for all

$x = (x_1, \dots, x_n) \in \mathbb{F}^n$

$$(t.\eta_0)(x) = (\eta_0 \circ t)(x) = \eta_0\left(\frac{x_1}{t_1}, \dots, \frac{x_n}{t_n}\right) = \max_i \left\{\left|\frac{x_i}{t_i}\right|\right\}$$
$$= \max_i \left\{e^{v(t_i)}|x_i|\right\} = \max_i \left\{e^{-a_i}|x_i|\right\} = \eta_a(x).$$

Therefore also $\mathrm{Stab}_G(f'(a)) = t\,\mathrm{GL}_n(\mathcal{O})t^{-1}$, concluding the proof of condition (1) using the case $a = 0$. We actually showed $\mathrm{Stab}_G(f(a)) = \mathrm{Stab}_G(f'(a))$ for all $a \in \mathbb{A}$.

For $w \in W_{\mathrm{s}} \cong S_n$ there exists a permutation matrix $k \in \mathrm{SO}_n(\mathbb{F}) \subseteq \mathrm{GL}_n(\mathbb{F})$ such that for all $t = \mathrm{Diag}(t_1, \dots, t_n) \in \mathrm{SL}_n(\mathbb{F})$, $ktk^{-1} = \mathrm{Diag}(t_{w^{-1}(1)}, \dots, t_{w^{-1}(n)})$ as in the proof of Lemma 7.9. If $a_i = (-v)(t_i)$, we use $k^{-1} \in \mathrm{SO}_n(\mathbb{F}) \subseteq \mathrm{GL}_n(\mathcal{O})$ to obtain

$$k.f(a) = [k, a] = k.[t, 0] = ktk^{-1}.[\mathrm{Id}, 0] = \left[\mathrm{Diag}(t_{w^{-1}(1)}, \dots, t_{w^{-1}(n)}), 0\right]$$
$$= \left[\mathrm{Id}, (a_{w^{-1}(1)}, \dots, a_{w^{-1}(n)})\right] = [\mathrm{Id}, w(a)] = f(w(a))$$

and similarly

$$k.f'(a) = k.\eta_a = k.t.\eta_0 = ktk^{-1}.\eta_0 = \eta_{w(a)} = f'(w(a)).$$

Finally, we know from [BT72, Corollaire (10.2.9)] that the pointwise stabilizer of the apartment $\Omega \coloneqq f(\mathbb{A})$ is described by

$$\hat{P}_\Omega \coloneqq \{g \in \mathrm{GL}_n(\mathcal{O})\colon g.f(a) = f(a) \text{ for all } a \in \mathbb{A}\}$$
$$= \{\mathrm{Diag}(t_1, \dots, t_n)\colon v(t_i) = 0\},$$

since there are $x \in \mathbb{A}$ with arbitrarily large $x_i - x_j$ when $i \neq j$. Similarly, we know that if $g \in G$ satisfies $g.f' = f'$, then $t^{-1}gt.\eta_0 = \eta_0$, so $t^{-1}gt \in \mathrm{GL}_n(\mathcal{O})$, for all $t = \mathrm{Diag}(t_1, \dots, t_n)$ with $(-v)(t_i) = a_i$ for some $a \in \mathbb{A}$. Since there are $a \in \mathbb{A}$ with $a_i - a_j$ arbitrarily large (when $i \neq j$), the condition

$$(t^{-1}gt)_{ij} = g_{ij}\frac{t_j}{t_i} \in \mathcal{O}$$

forces $t^{-1}gt$ and hence g to be diagonal. Thus also

$$\left\{g \in G\colon g.f'(a) = f'(a) \text{ for all } a \in \mathbb{A}\right\} = \{\mathrm{Diag}(t_1, \dots, t_n)\colon v(t_i) = 0\} = \hat{P}_\Omega.$$

If now $w = (a, w_s) \in W_{\mathrm{a}} = \Lambda^n/\Lambda(1, \dots, 1) \rtimes W_{\mathrm{s}}$, let $t = \mathrm{Diag}(t_1, \dots, t_n) \in \mathrm{SL}_n(\mathbb{F})$ with $a_i = (-v)(t_i)$ and let $k \in \mathrm{SO}_n(\mathbb{F})$ as above. Now for $g \in A_{f,w} = \{h \in \mathrm{GL}_n(\mathbb{F})\colon h.f = f \circ w\}$ we have $g.f = f \circ w = t.f \circ w_s = tk.f$, so $g^{-1}tk \in \hat{P}_{f(\mathbb{A})}$, which is equivalent to $g.f' = tk.f' = t.f' \circ w_s = f' \circ (a, w_s) = f' \circ w$, so $g \in A_{f',w}$ as required for the condition (2). In fact, we have shown $A_{f,w} = A_{f',w}$ for all $w \in W_{\mathrm{a}}$.

When $\Lambda = \mathbb{R}$ (meaning $v\colon \mathbb{F}^\times \to \Lambda$ is surjective), then we claim that for every ultra-norm η' adapted to the standard basis $\mathcal{E}_0 = \{e_1, \dots, e_n\}$ there is some $g \in \mathrm{GL}_n(\mathbb{F})$ such that $\eta' = g.\eta_0$. Indeed, take $g = \mathrm{Diag}(g_1, \dots, g_n) \in \mathrm{GL}_n(\mathbb{F})$ such that $\exp((-v)(g_i)) = \eta'(e_i)$ (use the surjectivity of v). Then since η' is adapted to $\mathcal{E}_0$,

$$g.\eta_0\left(\sum_{i=1}^n a_i e_i\right) = \eta_0\left(\sum_{i=1}^n g_i a_i e_i\right) = \max_i \left\{e^{-v(g_i a_i)}\right\}$$
$$= \max_i \left\{|a_i|\eta'(e_i)\right\} = \eta'\left(\sum_{i=1}^n a_i e_i\right)$$

for all $a_i \in \mathbb{F}$, where we used the fact that η' is adapted to $\mathcal{E}_0$. Note that $\mathrm{GL}_n(\mathbb{F})$ acts transitively on the bases of $\mathbb{F}^n$ (and the action preserves adaptedness), so together with the claim, we obtain that $\mathrm{GL}_n(\mathbb{F})$ acts transitively on the atlas $\mathcal{A}'$. Moreover, ρ is an isomorphism and the inclusions in conditions (1) and (2) are equalities. If $\Lambda = \mathbb{R}$, then τ is an isomorphism, all the conditions of (c) are satisfied and we can conclude that $B_{\mathrm{BT}} \cong B_N$. □

Appendix A. Basics from real algebraic geometry

We summarize general definitions and results from real algebraic geometry and set up notation. We refer the reader to [BCR98], in particular Chapters 1, 2 and 5, for more details and proofs. The main objects of study in real algebraic geometry are semi-algebraic sets. From now on let $\mathbb{F}$ be a real closed field.

Definition A.1. A subset $\mathcal{B} \subseteq \mathbb{F}^n$ is a *basic semi-algebraic set*, if there exists a polynomial $f \in \mathbb{F}[X_1, \ldots, X_n]$ such that

$$\mathcal{B} = \mathcal{B}(f) = \{x \in \mathbb{F}^n \mid f(x) > 0\}.$$

A subset $X \subseteq \mathbb{F}^n$ is *semi-algebraic* if it is a Boolean combination of basic semi-algebraic sets, i.e. X is obtained by taking finite unions and intersections of basic semi-algebraic sets and their complements.

Let $X \subseteq \mathbb{F}^n$ and $Y \subseteq \mathbb{F}^m$ be two semi-algebraic sets. A map $f \colon X \to Y$ is called *semi-algebraic* if its graph $\mathrm{Graph}(f) \subseteq X \times Y$ is semi-algebraic in $\mathbb{F}^{n+m}$.

Algebraic sets are semi-algebraic and any polynomial or rational map is semi-algebraic.

Proposition A.2 ([BCR98, Proposition 2.2.7])**.** *Let $f \colon X \to Y$ be a semi-algebraic map. If $S \subseteq X$ is semi-algebraic, then so is its image $f(S)$. If $T \subseteq Y$ is semi-algebraic, then so is its preimage $f^{-1}(T)$.*

Note that if $\mathbb{F} \neq \mathbb{R}$, then $\mathbb{F}$ is totally-disconnected in the order topology on $\mathbb{F}$. However we have the following notion of connectedness for semi-algebraic sets.

Definition A.3. A semi-algebraic set $X \subseteq \mathbb{F}^n$ is *semi-algebraically connected* if it cannot be written as the disjoint union of two non-empty semi-algebraic subsets of $\mathbb{F}^n$ both of which are closed in X.

Theorem A.4 ([BCR98, Theorem 2.4.5])**.** *A semi-algebraic set of $\mathbb{R}^n$ is connected if and only if it is semi-algebraically connected. Every semi-algebraic set of $\mathbb{R}^n$ has a finite number of connected components, which are semi-algebraic.*

We record the following proposition which justifies why closed and bounded semi-algebraic sets are the right analogue of compact sets in real algebraic geometry.

Proposition A.5 ([BCR98, Theorem 2.5.7])**.** *Let X be a closed and bounded semi-algebraic subset of $\mathbb{F}^n$ and $\mathrm{pr} \colon \mathbb{F}^n \to \mathbb{F}^{n-1}$ the projection on the space of the first $n-1$ coordinates. Then $\mathrm{pr}(X)$ is a closed and bounded semi-algebraic set.*

From now on, denote by $\mathbb{K}$ a real closed extension of $\mathbb{F}$.

Definition A.6. Let $X \subseteq \mathbb{F}^n$ be a semi-algebraic set given as

$$X = \bigcup_{i=1}^{s} \bigcap_{j=1}^{r_i} \{x \in \mathbb{F}^n \mid f_{ij}(x) *_{ij} 0\},$$

with $f_{ij} \in \mathbb{F}[X_1, \ldots, X_n]$ and $*_{ij}$ is either $<$ or $=$ for $i = 1, \ldots, s$ and $j = 1, \ldots, r_i$. The *$\mathbb{K}$-extension* $X_{\mathbb{K}}$ of X is the set given by the same Boolean combination of sign conditions as X, more precisely

$$X_{\mathbb{K}} = \bigcup_{i=1}^{s} \bigcap_{j=1}^{r_i} \{x \in \mathbb{K}^n \mid f_{ij}(x) *_{ij} 0\}.$$

Note that $X_{\mathbb{K}}$ is semi-algebraic and depends only on the set X, and not on the Boolean combination describing it, see [BCR98, Proposition 5.1.1]. The proof of this is based on the Tarski–Seidenberg transfer principle.

Theorem A.7 (Tarski–Seidenberg transfer principle, [BCR98, Theorem 5.2.1])**.** *Let $X \subseteq \mathbb{F}^{n+1}$ be a semi-algebraic set. Denote the projection $\mathrm{pr}\colon \mathbb{F}^{n+1} \to \mathbb{F}^n$ onto the first n coordinates by pr. Then $\mathrm{pr}(X) \subseteq \mathbb{F}^n$ is semi-algebraic. Furthermore, if $\mathbb{K}$ is a real closed extension of $\mathbb{F}$, and $\mathrm{pr}_{\mathbb{K}}\colon \mathbb{K}^{n+1} \to \mathbb{K}$ is the projection on the first n coordinates, then*

$$\mathrm{pr}_{\mathbb{K}}(X_{\mathbb{K}}) = \mathrm{pr}(X)_{\mathbb{K}}.$$

Using this one can prove an extension theorem for semi-algebraic maps.

Theorem A.8 ([BCR98, Propositions 5.3.1, 5.3.3, 5.3.5])**.** *Let $X \subseteq \mathbb{F}^n$ and $Y \subseteq \mathbb{F}^m$ be two semi-algebraic sets, and $f\colon X \to Y$ a semi-algebraic map. Then $(\mathrm{Graph}(f))_{\mathbb{K}}$ is the graph of a semi-algebraic map $f_{\mathbb{K}}\colon X_{\mathbb{K}} \to Y_{\mathbb{K}}$, that is called the* $\mathbb{K}$-extension *of f. Furthermore, f is injective (respectively surjective, respectively bijective) if and only if $f_{\mathbb{K}}$ is injective (respectively surjective, respectively bijective), and f is continuous if and only if $f_{\mathbb{K}}$ is continuous.*

Finally, we have the following relation between extension of semi-algebraic sets and semi-algebraically connected components.

Theorem A.9 ([BCR98, Proposition 5.3.6 (ii)])**.** *Let $X \subseteq \mathbb{F}^n$ be semi-algebraic. Then X is semi-algebraically connected if and only if $X_{\mathbb{K}}$ is semi-algebraically connected. More generally, if $C_1, \dots, C_m$ are the semi-algebraically connected components of X, then $(C_1)_{\mathbb{K}}, \dots, (C_m)_{\mathbb{K}}$ are the semi-algebraically connected components of $X_{\mathbb{K}}$.*

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