Homology of Finite Covers of Graphs

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Abstract

This thesis will examine the first homology of a finite, regular cover of a finite graph as a representation of the group of deck transformations. Although we can characterize it in terms of representation theory, new questions arose with respect to subrepresentations spanned by elevations of particular elements in the free group. Recent work has shown that there are examples where primitive homology, the subrepresentation spanned by elevations of primitive elements, does not generate all of homology. The main contribution of this thesis is to broaden the correspondence and dictionary between the representation theory of the group of deck transformations on the one hand, and topological properties of homology classes on the other hand. This thesis uses known results and methods to extend them to primitive commutator homology, where we will present analogous findings.

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Contents

1 Introduction

Given a surface S, we know that the homology classes which are realized by simple closed curves on this surface are exactly the indivisible ones. Given a finite, regular cover $\tilde{S}$ of S, classifying the homology classes realized by elevations of simple closed curves in S becomes a significantly harder question. We will look at the first homology of the cover not only as a vector space but also as a representation of the group of deck transformations. The field of representation theory of finite groups is well elaborated and gives us many tools at hand. This thesis aims at developing a correspondence and dictionary between the representation theory of the groups of deck transformations on the one hand and topological properties of homology classes on the other hand. Instead of surfaces, we will mainly look at graphs, more precisely at the wedge of n circles for $n \geq 2$.

The homology of finite, regular covers of graphs as representations of the group of deck transformations has been studied for a while arising not only in a topological context but also in an algebraic one when studying relation modules. In particular, a main result by Gaschütz gives a decomposition of the homology with coefficients in $\mathbb{C}$ into the irreducible representations of the group of deck transformations.

Theorem 1.1 (Gaschütz)**.** *Let X be the wedge of n circles, $n \geq 2$, and $p\colon Y \to X$ a finite, regular cover with group of deck transformations G. Then $H_1(Y;\mathbb{C}) \cong \mathbb{C}_{\text{triv}} \oplus \mathbb{C}[G]^{\oplus n-1}$, which means that the homology decomposes into the trivial representation and $n-1$ copies of the regular representation of G.*

A proof of this theorem will be presented in Section 3; following Grunewald et al. in [GLLM15]. In Subsection 3.2 we will give an example of an infinite cover to see that the theorem does not hold in the infinite case.

In Section 4 we will define primitive elements in free (abelian) groups and collect a list of their important properties. We will present the work of Osborne and Zieschang [OZ81] in which they give an explicit way of listing all primitive elements in a free group of rank two. We are also interested in studying how primitive elements lift to a given finite, regular cover. Here we consider free groups as fundamental groups of graphs. We will define primitive homology $H_1^{\text{prim}}(Y;\mathbb{C})$ in Subsection 4.4 as the subrepresentation of the homology of the cover that comes from primitive elements in the original space. A natural question to ask in this context is whether primitive homology generates the whole homology group. We will sketch a proof of the following theorem by Malestein and Putman [MP17] that answers the question in the negative.

Theorem 1.2 (Malestein-Putman)**.** *For all $n \geq 2$ there exists a finite index, normal subgroup $R \trianglelefteq F_n$ with $H_1^{\text{prim}}(Y;\mathbb{C}) \neq H_1(Y;\mathbb{C})$, where $p\colon Y \to X$ is the unique finite, regular cover of X with $p_*(\pi_1(Y)) = R$.*

To prove the above theorem, we construct a finite group and a representation of the former that violates the following representation-theoretic obstruction formulated by Farb and Hensel in [FH16, Theorem 1.4].

Theorem 1.3 (Farb-Hensel)**.** *Let X be the wedge of n circles, $n \geq 2$, and let $p\colon Y \to X$ be a finite regular G-cover defined by $\phi\colon F_n \to G$ for $n \geq 2$. Then*

$$H_1^{\text{prim}}(Y;\mathbb{C}) \leq \mathbb{C}_{\text{triv}} \oplus \bigoplus_{V_i \in \text{Irr}^{\text{prim}}(\phi,G)} V_i^{(n-1)\dim(V_i)},$$

where $\text{Irr}^{\text{prim}}(\phi, G) \subseteq \text{Irr}(G)$ is the subset of those irreducible representations of G that have the property that there is a primitive element in F_n whose image in G has a nonzero fixed point.

At the end of Section 4 we will study the orbit span of the elevation of a single primitive element in homology. We will show that this span is an induced representation. A main result of this thesis is to show that we cannot characterize homology classes of elevations of primitive elements by this property.

In Section 5 we will transfer the above results to commutators of primitive elements. We will define primitive commutator homology $H_1^{\text{comm}}(Y;\mathbb{C})$, the subrepresentation coming from elevations of commutators of primitive elements. It is possible to formulate an analogue to the representation-theoretic obstruction as in [FH16, Theorem 1.4]. The proof uses surface topology. In Subsection 5.1 we try to gain more insight into primitive commutator homology using the results for primitive homology. For this purpose, we will look at iterated covers, namely such where commutators of primitive elements lift to primitive elements. A main result is the following proposition.

Proposition 1.4. *For $n = 2$ there exists a finite, regular cover $p\colon Y \to X$ such that $H_1^{\text{comm}}(Y;\mathbb{C}) \neq \ker(p_*)$.*

We will also show that we cannot characterize primitive commutators by the span of their elevations in homology.

The last section is detached from the rest of the thesis. We will compute the set of homology classes of elevations of primitive elements in rank two for cyclic covers using the theory of Osborne and Zieschang. This is important when trying to construct covers that have the property that primitive homology does not coincide with homology.

2 Preliminaries

In this section we explain the notation used in this thesis and introduce important algebraic and topological concepts.

2.1 Notations and Basic Concepts

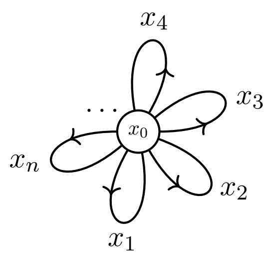
We start by defining graphs, and follow the definition by Hatcher as in [Hat02, Section 1.A].

Definition 2.1. A *graph* Γ is a one-dimensional CW complex. We call the 0-cells *vertices* and the 1-cells *edges*. Two vertices v, u form an edge if there exists a 1-cell in Γ that is attached to v and u. An edge is *oriented* if we choose a parameterisation of the edge. If all its edges are oriented, then we call Γ a *directed graph*. The graph Γ is said to be *finite* if it has only finitely many vertices and edges.

This definition naturally endows a graph with a topology, and we can study its fundamental group and its homology groups. There is a completely combinatorial definition given by Serre in [Ser80]. We can endow the latter with a topology such that the two definitions are equivalent for finite graphs. We allow multiple edges and self-loops in our graphs. Throughout this thesis all graphs are supposed to be finite if not otherwise stated.

Definition 2.2. Let Γ be a graph. Then a map $\varphi\colon \Gamma \to \Gamma$ is called a *graph automorphism*, if φ is a self-CW-homeomorphism of Γ. In other words, φ permutes the set of vertices and preserves edges and non-edges, i.e. two vertices (u, v) form an edge if and only if the pair $(\varphi(u), \varphi(v))$ also forms an edge.

For $n \in \mathbb{N}$, $n \geq 2$ we denote by X the wedge of n copies of one-dimensional spheres, denoted by S^1, i.e. the graph with one vertex x_0 and n self-loops. We chose an orientation of the edges and label the oriented edges by $x_1, \ldots, x_n$ as illustrated in the following.



The theorem of Seifert-van Kampen implies that the fundamental group of X is

$$\pi_1(X) = \pi_1\left(\bigvee_{i=1}^{n} S^1\right) \cong *_{i=1}^{n}\pi_1(S^1) = *_{i=1}^{n}\mathbb{Z} \cong F_n,$$

where by $*$ we denote the free product of the corresponding groups and by $F_n = F\langle x_1, \ldots, x_n\rangle$ the free group on the generators $x_1, \ldots, x_n$.

Let $p\colon Y \to X$ be a cover. Then Y is a graph with vertex set $p^{-1}(x_0)$. Choose $y_0 \in p^{-1}(x_0)$ as base point for Y. The edge set of Y are the lifts of the edges of X, i.e. the CW structure on Y is obtained by pulling back the CW structure of X via the map p. For further details refer to [Hat02, Lemma 1A.3]. Note that Y is a $2n$ regular graph with indegree and outdegree being n for all vertices of Y. This follows as p is a local homeomorphism and X satisfies this property.

Since X and Y are path-connected, the base points are not important when computing their fundamental groups. Nevertheless, in some definitions it is crucial to consider the base points to get well-defined objects. Thus we sometimes write $\pi_1(Y)$ for $\pi_1(Y, y_0)$ and $\pi_1(X)$ for $\pi_1(X, x_0)$ if we do not want to emphasize the base points. Denote by

$$p_*\colon \pi_1(Y) \to \pi_1(X),\ [f] \mapsto [p \circ f]$$

the induced group homomorphism on fundamental groups. Note that the homomorphism p_* is injective. The map p induces a map on chain complexes $p_*\colon C_\bullet(Y) \to C_\bullet(X)$ which commutes with the differential operators. This implies that we get a map on homology $p_*\colon H_1(Y;\mathbb{Z}) \to H_1(X;\mathbb{Z})$, which we will also denote by p_*. It will be clear from the context whether we consider the map on fundamental groups or on homology. If we assume the cover to be finite, then Y is a finite graph and we have the transfer map

$$p_\#\colon C_\bullet(X) \to C_\bullet(Y),\ x \mapsto \sum_{\tilde{x} \in p^{-1}(x)} \tilde{x}.$$

The map $p_\#$ commutes with the differential operators and induces thus a map on homology, which we will also denote by $p_\#$.

We denote by $G := \mathrm{Deck}(Y, p)$ the group of deck transformations of the cover $p\colon Y \to X$. If $p_*(\pi_1(Y))$ is normal in $\pi_1(X)$, the cover is called regular. If p is regular, then G is isomorphic to $\pi_1(X)/p_*(\pi_1(Y))$. If the cover is finite, then so is this quotient, which means that the group of deck transformations is a finite group. Note that being a regular cover means that G acts by graph automorphisms transitively on the edges and on the vertices of Y. This action descends to an action on homology, so G acts linearly on $H_1(Y;\mathbb{Z})$. We need the following definition to make use of the cellular structure of Y.

Definition 2.3. Let Z be a CW complex with k-skeleton Z^k. Denote by $H_k(Z^k)$ the singular homology of Z^k and by $H_k(Z^k, Z^{k-1})$ the singular homology of Z^k relative to Z^{k-1}. We have natural maps

$$d_k\colon H_k(Z^k, Z^{k-1}) \to H_{k-1}(Z^{k-1}),\ j_k\colon H_k(Z^k) \to H_k(Z^k, Z^{k-1})$$

coming from the long exact sequences of the pairs (Z^k, Z^{k-1}). Set

$$C_k(Z) \coloneqq H_k(Z^k, Z^{k-1})$$

for $k \geq 0$ with $Z_{-1} = \varnothing$. Set $\partial_k \coloneqq j_{k-1} \circ d_k$. We obtain the cellular chain complex

$$\begin{array}{ccccccc} \ldots \longrightarrow C_{k+1}(Z) & \xrightarrow{\partial_{k+1}} & C_k(Z) & \xrightarrow{\qquad\qquad \partial_k \qquad\qquad} & & C_{k-1}(Z) & \longrightarrow \ldots \\ & & & {\scriptstyle d_k} \searrow & & \nearrow {\scriptstyle j_{k-1}} & \\ & & & & H_{k-1}(Z^{k-1}) & & \end{array}$$

The homology groups of this chain complex are called *cellular homology groups*.

Then [Hat02, Theorem 2.35] shows that the cellular homology groups are isomorphic to the singular ones. Note that for a CW-complex Z with k-skeleton Z^k, the relative homology group $H_k(Z^j, Z^{j-1})$ is zero for $j \neq k$ and free abelian for $j = k$, with a basis the k-cells of Z, cf. [Hat02, Lemma 2.34]. Thus we can think of the relative homology groups as formal linear combinations of the k-cells of Z with coefficients in $\mathbb{Z}$, i.e.

$$C_k(Z) = H_k(Z^k, Z^{k-1}) = \bigoplus_{k\text{-cells in } Z} \mathbb{Z}.$$

For an abelian group A, we define the homology of Z with coefficients in A as the homology of the chain complex

$$C_k(Z; A) \coloneqq C_k(Z) \otimes_{\mathbb{Z}} A \cong \bigoplus_{k\text{-cells in } Z} A.$$

Note that $C_k(Z; \mathbb{Z}) = C_k(Z)$, and we use both notations interchangeably.

In our setting, we can choose an orientation of Y by pulling back the orientation of X. By the above consideration and since Y is a graph, i.e. a one-dimensional CW complex, we can identify

$$H_1(Y; \mathbb{Z}) \cong \ker(\partial_1),$$

where

$$\partial_1 \colon C_1(Y; \mathbb{Z}) \to C_0(Y; \mathbb{Z}),\ e \mapsto t(e) - o(e)$$

is the map that assigns to an edge its terminal minus its original vertex. Therefore, we have for A any abelian group that

$$H_1(Y; \mathbb{Z}) \otimes_{\mathbb{Z}} A \cong H_1(Y; A).$$

Thus we can view $H_1(Y; \mathbb{Z}) \subseteq H_1(Y; \mathbb{Q}) \subseteq H_1(Y; \mathbb{C})$ by the natural inclusions. For $A = \mathbb{C}$, we obtain a linear representation of G on $H_1(Y; \mathbb{C})$, i.e.

we have a linear representation of the group of deck transformations on the first homology of the covering space with coefficients in $\mathbb{C}$.

Recall that we can also define a finite, regular cover of X in the following way. Let G be some finite group and $\phi\colon \pi_1(X) = F_n \to G$ a surjective group homomorphism. Then $\ker(\phi) \leq F_n$ is a finite index, normal subgroup of the free group. By covering space theory, there exists a finite, regular, path-connected cover $p\colon Y \to X$ with $p_*(\pi_1(Y)) = \ker(\phi)$ and group of deck transformations G. Throughout this thesis we will use both definitions depending on which one fits better in the respective context.

There is the following theorem by Hurewicz relating fundamental groups and first homology. It holds for higher homotopy and homology groups as well.

Theorem 2.4. *Let X be a topological space. By regarding loops as singular 1-cycles, we obtain a homomorphism $h\colon \pi_1(X, x_0) \to H_1(X; \mathbb{Z})$. If X is path-connected, then h is surjective and has kernel the commutator subgroup of $\pi_1(X)$, so h induces an isomorphism from the abelianization of $\pi_1(X)$ onto $H_1(X; \mathbb{Z})$.*

A proof can be found in [Hat02, Theorem 2A.1]. Set $H := \ker(\phi)$. An immediate consequence of the above is that we can identify $H_1(Y; \mathbb{Z})$ with $H/[H, H]$. The action of G is identified with the natural action of $G = F_n/H$ on $H/[H, H]$, which is induced by the conjugation action of F_n on H.

We are particularly interested in topological spaces with free fundamental groups. An important property of the automorphism group of a free group is presented in the following.

Theorem 2.5. *Let F_n be the free group on the generators $x_1, \dots, x_n$, $n \geq 2$. Then the automorphism group $\mathrm{Aut}(F_n)$ of F_n is generated by the following four elements, called elementary Nielsen moves or transformations.*

1. *(NT1) $x_1 \mapsto x_2$, $x_2 \mapsto x_1$, $x_i \mapsto x_i$ for $3 \leq i \leq n$;*
2. *(NT2) $x_i \mapsto x_{i+1}$ for $1 \leq i \leq n-1$, $x_n \mapsto x_1$;*
3. *(NT3) $x_1 \mapsto x_1^{-1}$, $x_i \mapsto x_i$ for $2 \leq i \leq n$;*
4. *(NT4) $x_1 \mapsto x_1 x_2$, $x_i \mapsto x_i$ for $2 \leq i \leq n$.*

This result is due to Nielsen and a proof can be found in [Nie24, §1].

Proposition 2.6. *Let X be the wedge of n circles, $n \geq 2$, with base point x_0 the unique vertex in X. Then the group homomorphism*

$$_*\colon \mathrm{HE}(X) \to \mathrm{Aut}(F_n)$$

is bijective, where $\mathrm{HE}(X)$ denotes the group of homotopy equivalences of X up to homotopy.

Proof. This follows from [Hat02, Proposition 1B.9], since X is connected and an Eilenberg-MacLane space for dimension one and the group F_n, so $\pi_i(X) = \{1\}$ for all $i \geq 2$ and $i = 0$ and $\pi_1(X) = F_n$. Thus every homomorphism from $\pi_1(X, x_0)$ to itself is induced by a map from (X, x_0) to itself, that is unique up to homotopy fixing x_0. But this implies that for every automorphism $\varphi \in \mathrm{Aut}(\pi_1(X, x_0)) = \mathrm{Aut}(F_n)$ we find a map $f\colon (X, x_0) \to (X, x_0)$ with $f_* = \varphi$ and a map $g\colon (X, x_0) \to (X, x_0)$ with $g_* = \varphi^{-1}$. Thus fg and gf induce the identity on F_n which implies that fg and gf are homotopic to the identity maps. But this means nothing else than that f is a homotopy equivalence with $f_* = \varphi$. Also by the cited proposition, we know that f is unique up to homotopy fixing x_0, which implies injectivity. □

A more basic way to prove the above is to explicitly write down four homotopy equivalences whose induced maps on the fundamental group are the four elementary Nielsen moves defined in Theorem 2.5. The proposition also holds for finite graphs, as they are homotopy-equivalent to a wedge of n circles for some $n \in \mathbb{N}$.

If not otherwise stated, we denote by X the wedge of n copies of S^1 for $n \geq 2$ and by $p\colon Y \to X$ a finite regular path-connected cover with group of deck transformations G. We write 1 or 1_G for the neutral element in an abstract group G. When working with concrete abelian groups we write 0 for the neutral element.

All computations are done in [GAP18], which is also the source for the character tables presented in this thesis.

2.2 Cayley Graphs as Covering Spaces

In the following we give a more generalized definition of Cayley graph as is common in literature.

Definition 2.7. Let G be a finitely generated group and let $S \subseteq G$ be a finite generating multiset, i.e. a set that may contain multiple instances of the same elements. Denote by k_s the multiplicity of the element $s \in S$. The *Cayley graph* $\mathrm{Cay}(G, S)$ *of* G *with respect to the generating multiset* S is the graph that has the elements of G as vertices and k_s edges from g to h if and only if there exists $s \in S$ with $g = hs$.

Note that since S is a generating set, $\mathrm{Cay}(G, S)$ is connected. Extending the definition to multisets allows our Cayley graphs to have multiple edges. The group G acts on $\mathrm{Cay}(G, S)$ by left multiplication. This action is free on vertices and edges, which means that all vertex and edge stabilizers are trivial.

Proposition 2.8. *Let X be a wedge of n copies of S^1 with $n \geq 2$, G a finite group and $\phi\colon \pi_1(X) \cong F_n = F\langle x_1, \ldots, x_n\rangle \to G$ a surjective group* *homomorphism. Then the cover defined by* ϕ *is the Cayley graph* $\mathrm{Cay}(G,S)$ *of* G *with respect to the generating multiset* $S = \{\phi(x_1),\ldots,\phi(x_n)\}$ *based at the vertex* $1_G \in G$. *Furthermore, the action of* G *on* $\mathrm{Cay}(G,S)$ *by left multiplication is the action of* G *as group of deck transformations.*

PROOF. We have to show that $\mathrm{Cay}(G,S)$ is a covering space of X with covering map $p\colon \mathrm{Cay}(G,S) \to X$ such that $p_*(\pi_1(\mathrm{Cay}(G,S),1_G)) = \ker(\phi)$ and that the action of G on $\mathrm{Cay}(G,S)$ is the action of the group of deck transformations.

The edges in $\mathrm{Cay}(G,S)$ are labeled by the elements in S. If there are multiple edges between two vertices, all labeled by some $s \in S$, then the multiplicity of s in S is exactly the number of $1 \leq j \leq n$ with $\phi(x_j) = s$. Thus we can label all of these edges by a unique x_j. If the multiplicity of $s \in S$ is one, then we label all edges, which are labeled by s, by the single element x_i that has the property that $\phi(x_i) = s$. We define a map $p\colon \mathrm{Cay}(G,S) \to X$ which sends all vertices of $\mathrm{Cay}(G,S)$ to x_0 and an edge labeled by x_i to the edge of X corresponding to the free generator x_i. It is thus enough to show that every vertex is incident to $2n$ half-edges, where a half-edge is the barycentric subdivision of an edge in the graph. Because we consider S as a multiset, we have $|S| = n$ and thus every $s \in S$ defines two half-edges incident to the vertex $1_G \in G$. Namely, for $s \in S$, the vertices s and s^{-1} are adjacent to 1_G since $s = 1_G\, s$ and $1_G = s^{-1}s$. Note that $1_G = s = s^{-1}$ is possible. Thus 1_G is incident to $2n$ half-edges and therefore also every other vertex, because the action of G on $\mathrm{Cay}(G,S)$ sends 1_G to every vertex $g \in G$ by an automorphism of graphs, which preserves the number of incident half-edges.

Let $w \in \ker(\phi)$. This means that the word w, considered as an element in G, is trivial. Going along the edges labeled by the letters of w defines a closed curve in $\mathrm{Cay}(G,S)$. Thus $w \in p_*(\pi_1(\mathrm{Cay}(G,S)),1_G)$.

On the other hand, given a closed loop in $\mathrm{Cay}(G,S)$ at the identity vertex, going along the edges defines a word $w \in F_n$ that starts and ends in the same vertex. Thus it represents a trivial word in G, which is equivalent to $w \in \ker(\phi)$.

We claim that left multiplication with $g \in G$ defines a deck transformation of $\mathrm{Cay}(G,S)$. We have to check that we obtain an automorphism of graphs and that the covering map p is preserved. The first claim is clear. For the second property, note that if an edge from $h \to h'$ is labeled by s, which is equivalent to $h' = hs$, left multiplication with g sends h to gh, h' to gh', and the edge between these two vertices is still labeled by s, since $gh' = ghs$. This proves that left multiplication by g defines a deck transformation.

On the other hand, since $\mathrm{Cay}(G,S)$ is connected, a deck transformation of $\mathrm{Cay}(G,S)$ is defined by what it does on one point. Thus take $f \in \mathrm{Deck}(\mathrm{Cay}(G,S),p)$ a deck transformation and set $g \coloneqq f(1_G) \in G$. Then left multiplication with g defines a deck transformation that sends the vertex 1_G to g. By uniqueness, the latter agrees with f. □

2.3 Results from Representation Theory

In this subsection we present some basics from representation theory. All of the following results can be found in [Isa94, Chapters 1, 2 and 5], if not otherwise stated.

2.3.1 Definition and Basic Properties

Definition 2.9. Let G be a group, $\mathbb{K}$ a field and V a $\mathbb{K}$-vector space. A *representation of* G *on* V is an action of G on V by linear automorphisms, or equivalently, a homomorphism $\mathfrak{X}\colon G \to \mathrm{Aut}_{\mathbb{K}}(V)$. If V is finite-dimensional, the dimension of V over $\mathbb{K}$ is called the degree of the representation.

Example 2.10. The trivial and the regular representation are important examples that will play an essential role throughout this thesis.

1. Let G be a finite group, $V = \mathbb{K}$ the one-dimensional $\mathbb{K}$-vector space. Define the action of G on V to be trivial, i.e. $gv = v$ for all $g \in G$, $v \in V$. This representation is the trivial representation and written as $\mathbb{K}_{\text{triv}}$.

2. Let G be a group. The group algebra $\mathbb{K}[G]$ of G is the $\mathbb{K}$-vector space with basis elements the elements of G and finite formal linear combinations, i.e.

$$\mathbb{K}[G] = \left\{ \sum_{g \in G} a_g g \mid a_g \in \mathbb{K},\ \text{only finitely many } a_g \neq 0 \right\}.$$

   The group G acts on itself by left multiplication. We can extend this action to an action of G on $\mathbb{K}[G]$. This defines a representation. Indeed, it is a permutation representation, i.e. a representation where each group element acts as a permutation on a basis, as the action of an element $g \in G$ permutes the basis vectors of $\mathbb{K}[G]$. Note that if G is finite, then $\mathbb{K}[G]$ is finite-dimensional and the degree of the representation is $|G|$.

Given a representation of a group G we can construct a $\mathbb{K}[G]$-module structure on V by $\mathbb{K}[G]$-linear extension. Conversely, given a finitely generated $\mathbb{K}[G]$-module, we obtain a representation of G by restriction of the action to G. We will thus use both descriptions interchangeably.

Definition 2.11. Let G be a group and V a representation of G. A vector subspace $W \leq V$ is called a *subrepresentation* if it is invariant under the action of G. The representation $V \neq \{0\}$ is called *irreducible* if its only subrepresentations are the trivial one and V itself. The representation V is called *semisimple* if every subrepresentation W of V has a complement, i.e. there exists a subrepresentation U of V such that $V = W \oplus U$.

In the language of modules, subrepresentations correspond to submodules. Note that not every representation is semisimple. Nonetheless, there is the following important result by Maschke.

Theorem 2.12 (Maschke)**.** *Let G be a finite group and V a representation of G over $\mathbb{K}$. Then V is semisimple if the characteristic of $\mathbb{K}$ does not divide $|G|$.*

An easy consequence is that every representation of a finite group over $\mathbb{C}$ is a direct sum of irreducible representations. Note that one-dimensional representations are always irreducible. In particular, the trivial representation $\mathbb{C}_{\text{triv}}$ is irreducible. In fact, for a finite abelian group, all irreducible representations over $\mathbb{C}$ are one-dimensional.

Definition 2.13. Let G be a finite group and let V, W be representations of G over $\mathbb{K}$. A *homomorphism of representations* is a linear map $\varphi\colon V \to W$ that is G-equivariant, i.e. for all $g \in G$ and $v \in V$ we have $\varphi(gv) = g\varphi(v)$. The representations V and W are *isomorphic as representations* if there exists a homomorphism of G-representations that is an isomorphism of $\mathbb{K}$-vector spaces.

2.3.2 Character Theory

As we are interested mainly in representations of finite groups of finite degree over the complex numbers, in the following all representations are finite-dimensional and over $\mathbb{C}$. Recall that the theorem of Maschke implies that all such representations are semisimple. The following results hold in more generality for other fields as well.

Definition 2.14. Let G be a finite group and V a representation of G given by the homomorphism $\mathfrak{X}\colon G \to \mathrm{Aut}_{\mathbb{C}}(V)$. The map

$$\chi_V\colon G \to \mathbb{K},\ \chi_V \coloneqq \mathrm{Tr} \circ \mathfrak{X},$$

where $\mathrm{Tr}\colon \mathrm{Aut}_{\mathbb{C}}(V) \to \mathbb{C}$ is the trace map, is called the *character of the representation V*.

Note that characters are constant on conjugacy classes. It is easy to see that isomorphic representations have the same character. The study of the characters of G is called character theory. A list of results from this field of study is presented in the following.

Proposition 2.15. *Let G be a finite group and let $\mathrm{Irr}(G)$ be the set of isomorphism classes of the irreducible G-representations. Let $k(G)$ be the number of conjugacy classes in G. Then*

$$|\mathrm{Irr}(G)| = k(G).$$

In particular, $k(G) < \infty$.

Note that if G is abelian, then $k(G) = |G|$.

Definition 2.16. Let M be a G-representation and V' an irreducible G-representation. We define

$$M(V') := \sum_{V \leq M, V \cong V'} V.$$

Let $V_1, \ldots, V_{k(G)}$ be representatives of the isomorphism classes of the irreducible G-representations with $V_1 = \mathbb{C}_{\mathrm{triv}}$. Then the $M(V_i)$ for $1 \leq i \leq k(G)$ are called the *homogeneous components of M*.

It follows that $M = \bigoplus_{i=1}^{k(G)} M(V_i)$. For the regular representation we can explicitly compute the homogeneous components.

Proposition 2.17. *Let G be a finite group. Then for $M = \mathbb{C}[G]$ the regular representation, we have $M(V_i) \cong V_i^{\dim(V_i)}$ for all $1 \leq i \leq k(G)$. Thus*

$$\mathbb{C}[G] \cong \bigoplus_{i=1}^{k(G)} V_i^{\dim(V_i)}.$$

If G is abelian, then $\mathbb{C}[G] \cong \bigoplus_{i=1}^{|G|} V_i$.

Definition 2.18. Given a group G, let $\mathcal{C}$ be a set of representatives of the conjugacy classes of G. The square matrix

$$[\chi_V(g)]_{V \in \mathrm{Irr}(G),\, g \in \mathcal{C}}$$

is the *character table of the group G*.

Example 2.19. For $G = S_3$ we have $\mathcal{C} = \{(), (23), (123)\}$, as conjugacy classes in symmetric groups are defined by their cycle type. We obtain the following character table, where χ_1 corresponds to the trivial character.

	$1_{S_3} = ()$	$(2,3)$	$(1,2,3)$
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Note that the entries in the first column specify the degrees of the respective representations, as the identity element in G gets mapped to the identity matrix, whose trace is precisely the degree of the representation.

Definition 2.20. Let G be a finite group. A *class function on* G is a function $f : G \to \mathbb{C}$ which is constant on conjugacy classes. We define an inner product on the set of class functions in the following way: for class functions f, f' on G we put

$$\langle f, f' \rangle \coloneqq \frac{1}{|G|} \sum_{g \in G} f(g)\overline{f'(g)}.$$

Note that all characters are class functions.

Proposition 2.21. *Let G be a finite group. We have the following identity, known as the orthogonality relations,*

$$\langle \chi_i, \chi_j \rangle = \delta_{ij},$$

for χ_i, χ_j irreducible characters of G.

This means that the irreducible characters are orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle$ defined on the set of class functions.

Definition 2.22. If V and W are two representations of G, then we set

$$\langle V, W \rangle \coloneqq \langle \chi_V, \chi_W \rangle,$$

where χ_V and χ_W are the characters of V and W, respectively.

Proposition 2.23. *Let G be a finite group, V and W representations of G. Then $V \cong W$ if and only if $\chi_V = \chi_W$.*

This proposition makes the field of character theory so powerful, as representations are already determined by their traces.

2.3.3 Induced and Restricted Representations

Definition 2.24. Let G be a finite group and $H \leq G$ a subgroup. Given a representation V of G, we can define a representation $\mathrm{Res}_H^G(V)$ of H by restricting the representation of G to H. On the other hand, if we start with a representation W of H, we can build the representation $\mathrm{Ind}_H^G(W)$, a representation of G, in the following way: Let $g_1 = 1, g_2, \ldots, g_k$ be representatives of the cosets of H in G, i.e. $g_i \in G$ with the property that $G = \bigsqcup_{i=1}^k g_i H$, where $k \coloneqq [G : H]$. Set

$$\mathrm{Ind}_H^G(W) \coloneqq \bigoplus_{i=1}^k g_i W,$$

where $g_i W$ is a copy of W as $\mathbb{C}$-vector space for all $1 \leq i \leq k$. Then for $g \in G$, we have $g g_i = g_{j(i)} h_i$ with $g_{j(i)} \in \{g_1, \ldots, g_k\}$ and $h_i \in H$ for all $1 \leq i \leq k$. The action of G on $\mathrm{Ind}_H^G(W)$ is defined as

$$g\left(\sum_{i=1}^k g_i w_i\right) \coloneqq \sum_{i=1}^k g g_i w_i = \sum_{i=1}^k g_{j(i)} \underbrace{h_i w_i}_{\in W}$$

for all $g \in G$ and $\sum_{i=1}^{k} g_i w_i \in \operatorname{Ind}_H^G(W)$. This representation is called the *induced representation* and it is of degree $[G : H] \dim(W)$. It is unique up to isomorphisms of representations.

It is easy to see from the construction that inducing representations is transitive. This means that if K, H are subgroups of G with $K \leq H$ and W a representation of K, then

$$\operatorname{Ind}_H^G(\operatorname{Ind}_K^H(W)) \cong \operatorname{Ind}_K^G(W).$$

It is also clear that the trivial representation on a subgroup H of G induces the permutation representation of G/H. Thus inducing the trivial representation from the trivial subgroup gives the regular representation,

$$\operatorname{Ind}_{\{1\}}^G(\mathbb{C}_{\text{triv}}) \cong \mathbb{C}[G].$$

Induced representations will play an important role in the course of this thesis, which is why we will collect some relevant results about induced and restricted representations in the following.

Proposition 2.25. *Let G be a finite group, $H \leq G$ a subgroup of G and U, V, W representations of H that fit into the following exact sequence*

$$0 \to U \to V \to W \to 0.$$

Then the sequence

$$0 \to \operatorname{Ind}_H^G(U) \to \operatorname{Ind}_H^G(V) \to \operatorname{Ind}_H^G(W) \to 0$$

is exact, i.e. inducing representations preserves exact sequences.

Proof. This can be easily seen when considering an equivalent description of the induced representation; see for example [Ser77, Chapter 7.1]. Namely, we know

$$\operatorname{Ind}_H^G(W) \cong \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$$

as $\mathbb{C}[G]$-modules. Now $\mathbb{C}[G]$ is a free $\mathbb{C}[H]$-module and thus flat, which by definition means that tensoring with $\mathbb{C}[G]$ preserves exact sequences. □

Proposition 2.26. *Let G be a finite group, $H \leq G$ a subgroup of G and W a representation of H. Let $V \coloneqq \operatorname{Ind}_H^G(W)$ be the induced representation. Then its character χ_V satisfies*

$$\chi_V(g) = \frac{1}{|H|} \sum_{\substack{x \in G \\ x^{-1}gx \in H}} \chi_W(x^{-1}gx)$$

for all $g \in G$.

Theorem 2.27 (Frobenius Reciprocity)**.** *Let G be a finite group, V a representation of a subgroup $H \leq G$ and W some representation of G. Then the following identity holds, which is known as Frobenius reciprocity:*

$$\langle \mathrm{Ind}_H^G(V), W\rangle_G = \langle V, \mathrm{Res}_H^G(W)\rangle_H.$$

The following lemmas will be important later on in this thesis.

Lemma 2.28. *Let G be a finite group and V a representation of G over $\mathbb{C}$. Assume there exists $0 \neq v \in V$ such that Gv is linearly independent. Let $H \coloneqq \mathrm{Stab}_G(v)$. Then*

$$\mathrm{Span}_{\mathbb{C}}\{Gv\} \cong \mathrm{Ind}_H^G(\mathbb{C}_{\mathrm{triv}}).$$

Proof. Choose representatives $g_1 = 1, g_2, \dots, g_{[G:H]}$ for the cosets of H in G. For all $g \in G$, g can be uniquely written as $g = g_i h$ for some $1 \leq i \leq [G:H]$ and $h \in H$. We then have

$$gv = g_i hv = g_i v,$$

since $h \in \mathrm{Stab}_G(v)$. Also note that if $g_i v = g_j v$, then $g_j^{-1} g_i \in H$, so $g_i \in g_j H$ which implies $i = j$. Thus $\{g_1 v, \dots, g_{[G:H]} v\}$ is a basis for $\mathrm{Span}_{\mathbb{C}}\{Gv\}$.

By construction, $\mathrm{Ind}_H^G(\mathbb{C}_{\mathrm{triv}}) = U_1 \oplus \ldots \oplus U_{[G:H]}$ as $\mathbb{C}[H]$-module with $U_1 = \mathbb{C}_{\mathrm{triv}}$ as $\mathbb{C}[H]$-module, $U_i = g_i U_1$. Let $u_1 \in U_1 \setminus \{0\}$. Then $U_1 = \langle u_1\rangle$ and thus $U_i = \langle g_i u_1\rangle$ for all $1 \leq i \leq [G:H]$. Define

$$\varphi\colon \mathrm{Span}_{\mathbb{C}}\{Gv\} \to \mathrm{Ind}_H^G(\mathbb{C}_{\mathrm{triv}}),\ g_i v \mapsto g_i u_1,$$

and $\mathbb{C}$-linear extension. This is an isomorphism of $\mathbb{C}[G]$-modules. ☐

Lemma 2.29. *Let M be a $\mathbb{C}[G]$-module and $m \in M$ be such that Gm is $\mathbb{C}$-linearly independent, i.e. $\mathrm{Span}_{\mathbb{C}[G]}\{m\} \cong \mathrm{Ind}_H^G(\mathbb{C}_{\mathrm{triv}})$ with $H = \mathrm{Stab}_G(m)$ by Lemma 2.28. Let $g_1 = 1, g_2, \dots, g_{[G:H]} \in G$ be a set of representatives of the cosets of H in G. Write $m = m_1 + \ldots + m_{k(G)}$, $m_i \in M(V_i)$, for the decomposition of m into the homogeneous components. For an element $b \in M(V_1) \setminus \{-m_1\}$ set*

$$m' \coloneqq m + b = (m_1 + b) + m_2 + \ldots + m_{k(G)}.$$

Then $Gm' = \{g_1 m', \dots, g_{[G:H]} m'\}$, Gm' is $\mathbb{C}$-linearly independent and we have $\mathrm{Span}_{\mathbb{C}[G]}\{m'\} = \mathrm{Span}_{\mathbb{C}[G]}\{m\} \cong \mathrm{Ind}_H^G(\mathbb{C}_{\mathrm{triv}})$.

Proof. We first prove that $\mathrm{Stab}_G(m') = H$. Let $g \in \mathrm{Stab}_G(m')$. Then $m' = gm'$ and $gm = g(m' - b) = gm' - b = m' - b = m$, thus $g \in \mathrm{Stab}_G(m) = H$. Let now $g \in H$, so that $gm = m$. This yields $m' = m + b = gm + b = gm + gb = g(m + b) = gm'$. Thus $g \in \mathrm{Stab}_G(m')$.

We now show that Gm' is linearly independent. Let $r_1, \ldots, r_{[G:H]} \in \mathbb{C}$ such that

$$\sum_{i=1}^{[G:H]} r_i g_i m' = 0.$$

We need to show that $r_i = 0$ for all $1 \leq i \leq [G : H]$. We have

$$\begin{aligned} 0 &= \sum_{i=1}^{[G:H]} r_i g_i m' = \sum_{i=1}^{[G:H]} r_i g_i (m + b) \\ &= \sum_{i=1}^{[G:H]} r_i g_i (m_1 + b) + \sum_{i=1}^{[G:H]} \sum_{j=2}^{k(G)} r_i g_i m_j \\ &= \underbrace{\left(\sum_{i=1}^{[G:H]} r_i \right) (m_1 + b)}_{\in M(V_1)} + \sum_{j=2}^{k(G)} \underbrace{\sum_{i=1}^{[G:H]} r_i g_i m_j}_{\in M(V_j)} . \end{aligned}$$

This sum is equal to 0 if and only if

$$\left(\sum_{i=1}^{[G:H]} r_i \right) (m_1 + b) = 0 \text{ and } \sum_{i=1}^{[G:H]} r_i g_i m_j = 0 \tag{1}$$

for all $2 \leq j \leq k(G)$.

Since $b \neq -m_1$, $\left(\sum_{i=1}^{[G:H]} r_i \right) (m_1 + b) = 0$ if and only if $\sum_{i=1}^{[G:H]} r_i = 0$. Thus we obtain

$$0 = \sum_{i=1}^{[G:H]} r_i = \left(\sum_{i=1}^{[G:H]} r_i \right) m_1 = \sum_{i=1}^{[G:H]} r_i g_i m_1.$$

Together with Equation (1) this implies

$$\sum_{i=1}^{[G:H]} r_i g_i m_j = 0 \text{ for all } 1 \leq j \leq k(G).$$

Now $\sum_{i=1}^{[G:H]} r_i g_i m = \sum_{i=1}^{[G:H]} r_i g_i \left(\sum_{j=1}^{k(G)} m_j \right) = \sum_{j=1}^{k(G)} \sum_{i=1}^{[G:H]} r_i g_i m_j = 0$. But Gm is linearly independent by assumption and thus $r_i = 0$ for all $1 \leq i \leq [G : H]$, which proves the first claim. The second claim follows directly from Lemma 2.28. □

Lemma 2.30. *Let G be abelian and $H, K \leq G$ subgroups. Then*

$$\mathrm{Ind}_H^G(\mathbb{C}_{\mathrm{triv}}) \cong \mathrm{Ind}_K^G(\mathbb{C}_{\mathrm{triv}})$$

if and only if $H = K$.

Proof. By character theory we know that two representations over $\mathbb{C}$ are isomorphic if and only if their characters agree; refer to Proposition 2.23. Let us compute the character 1_H^G of $\mathrm{Ind}_H^G(\mathbb{C}_{\mathrm{triv}})$. For $g \in G$, we have

$$1_H^G(g) = \frac{1}{|H|} \sum_{\substack{x \in G \\ x^{-1}gx \in H}} \chi_{\mathbb{C}_{\mathrm{triv}}}(x^{-1}gx);$$

refer to Proposition 2.26. Since $\chi_{\mathbb{C}_{\mathrm{triv}}}(h) = 1$ for all $h \in H$ and G is abelian, we obtain

$$1_H^G(g) = \frac{1}{|H|} \sum_{\substack{x \in G \\ g \in H}} \chi_{\mathbb{C}_{\mathrm{triv}}}(g) = [G:H]\, \mathbb{1}_H(g),$$

where by $\mathbb{1}_H$ we denote the characteristic function on H. Now it follows that $\mathrm{Ind}_H^G(\mathbb{C}_{\mathrm{triv}}) \cong \mathrm{Ind}_K^G(\mathbb{C}_{\mathrm{triv}})$ if and only if $[G:H]\, \mathbb{1}_H = [G:K]\, \mathbb{1}_K$, which is equivalent to $H = K$. □

3 First Homology as Representation of the Group of Decktransformations

In this section we want to understand, given a covering space of a graph, its first homology as a representation of the group of deck transformations. The first subsection will deal with finite regular covers, whereas in the second subsection we will see an example of an infinite cover.

3.1 Finite Covers and the Theorem of Gaschütz

We will now prove the theorem of Gaschütz and follow [GLLM15, Section 2.1, Theorem 2.1]. It gives an understanding of the first homology of a finite, regular cover Y of the wedge of n copies of S^1, $n \geq 2$, with coefficients in $\mathbb{C}$ as a representation of the group of deck transformations.

Theorem 3.1 (Gaschütz)**.** *Let X be the wedge of n copies of S^1, $n \geq 2$, and G a finite group. Let $\phi\colon F_n = \pi_1(X) \to G$ be a surjective group homomorphism and $p\colon Y \to X$ its associated finite, regular cover with $p_*(\pi_1(Y)) = \ker(\phi)$ and group of deck transformations G. Then, there exists an isomorphism of $\mathbb{C}[G]$-modules*

$$H_1(Y;\mathbb{C}) \cong \mathbb{C}_{\mathrm{triv}} \oplus \mathbb{C}[G]^{\oplus n-1},$$

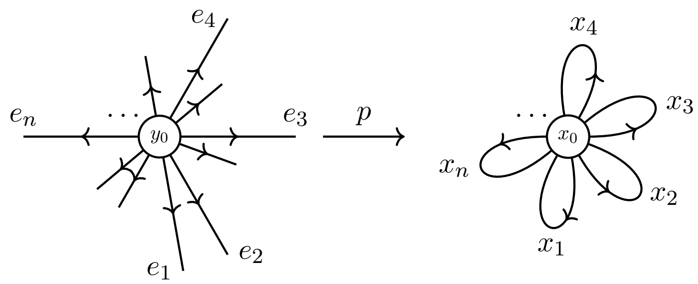
where G acts trivially on $\mathbb{C}_{\mathrm{triv}}$ and $\mathbb{C}[G]$ denotes the regular representation of G.

Proof. We first want to compute $H_1(Y;\mathbb{C})$. To do so, we can use cellular homology since Y is a graph, hence by definition a CW complex. Let $C_i(Y;\mathbb{C})$ be the set of formal linear combinations of i-cells in Y with coefficients in $\mathbb{C}$, $i \in \mathbb{N}$. Since Y is a graph and there are no 2-cells, the image of the map $\partial_2\colon C_2(Y;\mathbb{C}) \to C_1(Y;\mathbb{C})$ is zero. Hence we identify

$$H_1(Y;\mathbb{C}) = \ker(\partial_1\colon C_1(Y;\mathbb{C}) \to C_0(Y;\mathbb{C}))/\mathrm{im}(\partial_2) = \ker(\partial_1).$$

We first want to understand $C_1(Y;\mathbb{C})$ and $C_0(Y;\mathbb{C})$ as G-representations and then compute $\ker(\partial_1)$.

Let y_0 be some vertex in Y, i.e. $y_0 \in p^{-1}(x_0)$, where x_0 is the unique vertex of X. Let $e_1, \ldots, e_n$ be the oriented edges going out of y_0, where each $e_i \in p^{-1}(x_i)$ as illustrated below.



The group G acts freely on the orbit of e_i for all $1 \leq i \leq n$. Furthermore, the G-orbits of e_i and e_j are disjoint if $i \neq j$, because G acts by automorphisms of covering spaces, so fibres are preserved. Being a regular cover is equivalent to the fact that G acts transitively on the fibres of Y. Thus we have an isomorphism of $\mathbb{C}[G]$-modules given by $\mathbb{C}[G]$-linear extension of

$$\alpha_1 \colon C_1(Y;\mathbb{C}) \to \bigoplus_{i=1}^{n} \mathbb{C}[G],\ e_j \mapsto (0,\dots,0,1_G,0,\dots,0) =: f_j \tag{2}$$

with $1_G \in G$ at the j-th position for $1 \leq j \leq n$. This is well-defined, since the e_j form a $\mathbb{C}[G]$-basis for $C_1(Y;\mathbb{C})$. We also have a bijective $\mathbb{C}[G]$-module homomorphism given by $\mathbb{C}[G]$-linear extension of

$$\alpha_0 \colon C_0(Y;\mathbb{C}) \to \mathbb{C}[G],\ y_0 \mapsto 1_G.$$

This is well-defined, since y_0 is a $\mathbb{C}[G]$-basis for $C_0(Y;\mathbb{C})$. Namely, G acts transitively on the fibres and all vertices of Y are preimages of x_0. The boundary map ∂_1 is a homomorphism of $\mathbb{C}[G]$-modules. By Equation (2), it is enough to compute $\partial_1(e_j)$ for all $1 \leq j \leq n$ in order to understand ∂_1 and $\ker(\partial_1)$. By choice of e_j, we have that

$$\partial_1(e_j) = t(e_j) - o(e_j) = g_j\, y_0 - y_0 = (g_j - 1_G)\, y_0 \in C_0(Y;\mathbb{C}), \tag{3}$$

with $g_j := \phi(x_j) \in G$. We obtain the following diagram

$$\begin{array}{ccccccc} 0 & \xrightarrow{\partial_2} & C_1(Y;\mathbb{C}) & \xrightarrow{\partial_1} & C_0(Y;\mathbb{C}) & \xrightarrow{\partial_0} & 0 \\ & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} & & \\ & & \bigoplus_{i=1}^{n} \mathbb{C}[G] & \xrightarrow{\rho_1} & \mathbb{C}[G] & \xrightarrow{\varepsilon} & \mathbb{C} \end{array}$$

where $\rho_1 \colon \bigoplus_{i=1}^{n} \mathbb{C}[G] \to \mathbb{C}[G]$ is defined by $f_j \mapsto g_j - 1_G$ and $\mathbb{C}[G]$-linear extension to make the above square commute.

Consider the augmentation homomorphism

$$\varepsilon \colon \mathbb{C}[G] \to \mathbb{C},\ \sum_{g \in G} \alpha_g g \mapsto \sum_{g \in G} \alpha_g.$$

We want to show that $\alpha_0(\mathrm{im}(\partial_1)) = \ker(\varepsilon)$. By Equation (3), it is clear that $\alpha_0(\mathrm{im}(\partial_1)) \subseteq \ker(\varepsilon)$, as $\varepsilon(g_j - 1_G) = 1 - 1 = 0$ for $1 \leq j \leq n$. We want to show that $\alpha_0(\mathrm{im}(\partial_1))$ is already the whole kernel. Since $\partial_0 = 0$, we know that $H_0(Y;\mathbb{C}) = \ker(\partial_0)/\mathrm{im}(\partial_1) = C_0(Y;\mathbb{C})/\mathrm{im}(\partial_1)$. On the other hand, we have $\mathbb{C}[G]/\ker(\varepsilon) \cong \mathbb{C}$ and, since Y is connected, $\dim_{\mathbb{C}}(H_0(Y;\mathbb{C})) = 1$. Together this implies that

$$\mathbb{C} \cong \mathbb{C}[G]/\ker(\varepsilon) \hookrightarrow \mathbb{C}[G]/\alpha_0(\mathrm{im}(\partial_1)) = \alpha_0(H_0(Y;\mathbb{C})), \tag{4}$$

using that $\alpha_0(\mathrm{im}(\partial_1)) \subseteq \ker(\varepsilon)$. Since the homology in degree zero is isomorphic to $\mathbb{C}$ and α_0 is an isomorphism, we obtain an equality in Equation (4) and thus $\alpha_0(\mathrm{im}(\partial_1)) = \ker(\varepsilon)$.

By the theorem of Maschke, see Theorem 2.12, every $\mathbb{C}[G]$-module is semisimple, that means that every submodule has a complement. Using the definition of $H_1(Y;\mathbb{C})$ and the fact $\alpha_0(\mathrm{im}(\partial_1)) = \ker(\varepsilon)$ proven above, we obtain

$$\mathbb{C}[G]^{\oplus n} \cong C_1(Y;\mathbb{C}) = H_1(Y;\mathbb{C}) \oplus \mathrm{im}(\partial_1) \cong H_1(Y;\mathbb{C}) \oplus \ker(\varepsilon).$$

Additionally, we have that

$$\mathbb{C}[G] \cong \mathbb{C}_{\mathrm{triv}} \oplus \ker(\varepsilon).$$

Using the last two identities we write

$$H_1(Y;\mathbb{C}) \oplus \ker(\varepsilon) \cong \mathbb{C}[G]^{\oplus n} \cong \mathbb{C}[G]^{\oplus n-1} \oplus \mathbb{C}_{\mathrm{triv}} \oplus \ker(\varepsilon).$$

Then semisimplicity of the group algebra implies that

$$H_1(Y;\mathbb{C}) \cong \mathbb{C}_{\mathrm{triv}} \oplus \mathbb{C}[G]^{\oplus n-1},$$

which proves the theorem. □

Notice that the special case $Y = X$ with trivial group of deck transformations yields $H_1(X;\mathbb{C}) \cong \mathbb{C}^n$. The proof shows that the finiteness of the group of deck transformations is crucial to obtain semisimplicity. In the infinite case, Maschke's theorem cannot be applied and we do not obtain semisimplicity of the group algebra. In fact, the theorem is not true in the infinite case.

Using the results from Section 2.3, Proposition 2.17, we can rewrite the result of Gaschütz in the following way:

$$H_1(Y;\mathbb{C}) \cong \mathbb{C}_{\mathrm{triv}} \oplus \bigoplus_{V_i \in \mathrm{Irr}(G)} V_i^{(n-1)\dim(V_i)}.$$

We will see examples of finite, abelian covers in which we explicitly verify the above identification.

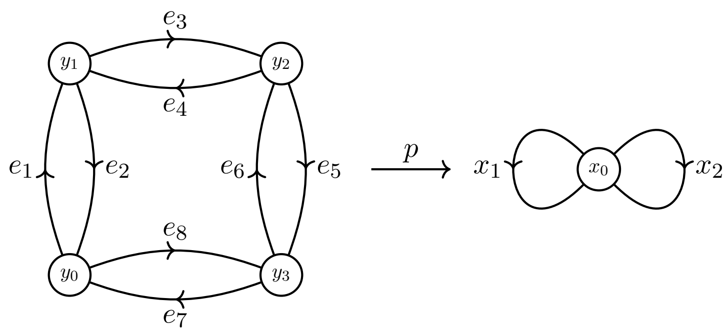
Example 3.2. Consider for $n = 2$ the mod 2-homology cover. Recall that for $m \geq 2$ the mod m-homology cover is given by the surjective group homomorphism

$$\phi\colon F_n \xrightarrow{-} F_n/[F_n, F_n] \cong \mathbb{Z}^n \xrightarrow{\bmod m} (\mathbb{Z}/m\mathbb{Z})^n = H_1(X; \mathbb{Z}/m\mathbb{Z}).$$

Now let $G := (\mathbb{Z}/2\mathbb{Z})^2 = \langle A \rangle \times \langle B \rangle$ and

$$\phi\colon F_2 = F\langle x_1, x_2 \rangle \to G,\ x_1 \mapsto A,\ x_2 \mapsto B$$

be the above group homomorphism. This defines the following covering map $p\colon Y \to X$ with $p^{-1}(x_1) = \{e_1, e_2, e_5, e_6\}$ and $p^{-1}(x_2) = \{e_3, e_4, e_7, e_8\}$ as illustrated below.



By Gaschütz, refer to Theorem 3.1, we know that

$$\dim_{\mathbb{C}}(H_1(Y; \mathbb{C})) = |G|(n-1) + 1 = 5.$$

We now want to find five linearly independent vectors $v_1, \ldots, v_5 \in H_1(Y; \mathbb{C})$ such that $\langle v_1 \rangle \cong \mathbb{C}_{\text{triv}}$ and $\langle v_2, v_3, v_4, v_5 \rangle \cong \mathbb{C}[G]$ as G-representations.

Consider the vectors

$$\begin{aligned}
w_1 &:= e_1 + e_2 + e_5 + e_6, \\
w_2 &:= e_3 + e_4 + e_7 + e_8, \\
w_3 &:= e_1 + e_2 - (e_5 + e_6), \\
w_4 &:= e_3 + e_4 - (e_7 + e_8), \\
w_5 &:= e_1 - e_2 + e_3 - e_4 + e_5 - e_6 + e_7 - e_8.
\end{aligned}$$

It is easy to verify that these are linearly independent and in $H_1(Y; \mathbb{C})$, as $\partial_1(e) = t(e) - o(e)$ for an edge e in Y. They thus form a basis and we denote it by $\mathcal{W} = \{w_1, \ldots, w_5\}$.

The action of A and B on $H_1(Y;\mathbb{C})$ can be computed as in Proposition 2.8. With respect to this basis it is given by the following matrices

$$M_{A,\mathcal{W}} := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \; M_{B,\mathcal{W}} := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

We have thus found a decomposition of $H_1(Y;\mathbb{C})$ into the irreducible one-dimensional representations of G. Setting

$$\begin{aligned} v_1 &:= w_1, \\ v_2 &:= w_2 + w_3 + w_4 + w_5, \\ v_3 &:= w_2 + w_3 - w_4 - w_5, \\ v_4 &:= w_2 - w_3 + w_4 - w_5, \\ v_5 &:= w_2 - w_3 - w_4 + w_5, \end{aligned}$$

these vectors form again a basis $\mathcal{V} = \{v_1, \ldots, v_5\}$ with respect to which the action of A and B on $H_1(Y;\mathbb{C})$ is given by the matrices

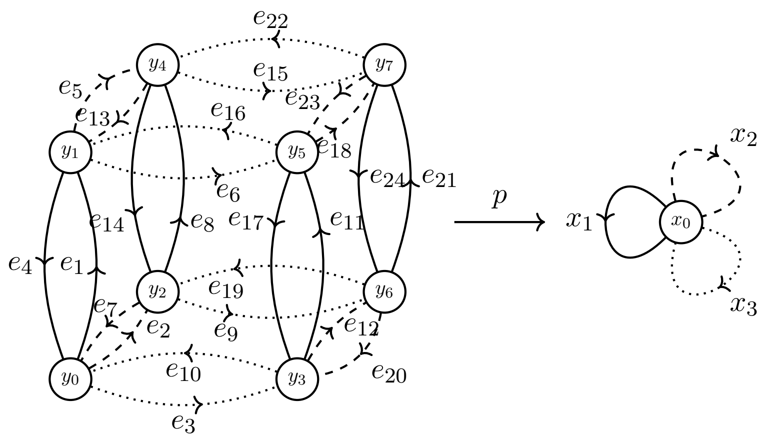
$$M_{A,\mathcal{V}} := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \; M_{B,\mathcal{V}} := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

It follows that $H_1(Y;\mathbb{C}) \cong \mathbb{C}_{\text{triv}} \oplus \mathbb{C}[(\mathbb{Z}/2\mathbb{Z})^2]$, as stated in the theorem of Gaschütz.

Example 3.3. Analogously, for $n = 3$ we can consider the mod 2-homology cover. Let $G := (\mathbb{Z}/2\mathbb{Z})^3 = \langle A \rangle \times \langle B \rangle \times \langle C \rangle$ and

$$\phi\colon F_3 = F\langle x_1, x_2, x_3 \rangle \to G, \; x_1 \mapsto A, \, x_2 \mapsto B, \, x_3 \mapsto C.$$

This corresponds to the following covering map $p\colon Y \to X$ where the preimages of x_1, x_2 and x_3 are represented by the different styles of arrows.



By the theorem of Gaschütz, we know that $H_1(Y;\mathbb{C})$ is $8\,(3-1)+1 = 17$-dimensional. A basis in terms of the 24 edges for this decomposition has been computed with [GAP18], but is not presented here because of dimension reasons.

Let G be a finite group and let $V_1 = \mathbb{C}_{\text{triv}}, \dots, V_{k(G)}$ be representatives of the isomorphism classes of the irreducible G-representations. We have the following technical lemma that follows directly from the theorem of Gaschütz.

Lemma 3.4. *In the setting of the theorem of Gaschütz, the following holds:*

1. $p_* \circ p_\# = [|G|]_{H_1(X;\mathbb{Z})}$ *and* $p_* \circ p_\# = [|G|]_{H_1(X;\mathbb{C})}$, *where by* $[|G|]_{H_1(X;\mathbb{Z})}$ *we denote the map that multiplies an element of* $H_1(X;\mathbb{Z})$ *by* $|G|$, *and equivalently for* $H_1(X;\mathbb{C})$,
2. $p_\#(H_1(X;\mathbb{C})) = M(V_1)$, *with* $M = H_1(Y;\mathbb{C})$,
3. $p_*\colon H_1(Y;\mathbb{C}) \to H_1(X;\mathbb{C})$ *is surjective,*
4. $\ker(p_*) = \bigoplus_{i=2}^{k(G)} M(V_i)$ *with* $M = H_1(Y;\mathbb{C})$.

Proof.

1. Consider the map $p_* \circ p_\#\colon C_1(X;\mathbb{Z}) \to C_1(X;\mathbb{Z})$. For e an edge in $C_1(X;\mathbb{Z})$, we have

$$p_*(p_\#(e)) = p_*\left(\sum_{\tilde{e}\in p^{-1}(e)} \tilde{e}\right) = \sum_{\tilde{e}\in p^{-1}(e)} p_*(\tilde{e}) = |G|\; e.$$

Thus $p_* \circ p_\# = [|G|]_{C_1(X;\mathbb{Z})}$ and $(p_* \circ p_\#) \otimes_{\mathbb{Z}} \mathrm{id}_{\mathbb{C}} = [|G|]_{C_1(X;\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{C}}$. These maps factor through homology, which proves 1.

2. We first claim that G acts trivially on $p_\#(C_1(X;\mathbb{Z}))$. Let $g \in G$ and $e \in C_1(X;\mathbb{Z})$. Then

$$gp_\#(e) = g \sum_{\tilde{e} \in p^{-1}(e)} \tilde{e} = \sum_{\tilde{e} \in p^{-1}(e)} g\tilde{e} = \sum_{\tilde{e} \in p^{-1}(e)} \tilde{e} = p_\#(e),$$

since an element of the group of deck transformations permutes the preimages. This identity also holds on homology. Hence the G-action on homology is trivial and thus $p_\#(H_1(X;\mathbb{C})) \leq M(V_1)$. By 1., we know that $p_\#$ is injective as map $H_1(Y;\mathbb{C}) \to H_1(X;\mathbb{C})$. For dimensionality reasons we obtain equality.

3. Follows directly from 1.

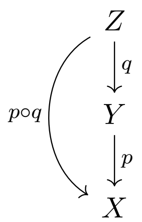
4. This follows from 3. and Gaschütz, refer to Theorem 3.1, because

$$M(V_1) \cong \mathbb{C}^n \cong H_1(X;\mathbb{C}) = \operatorname{im}(p_*) \cong M/\ker(p_*),$$

as $\mathbb{C}[G]$-modules, where G acts trivially on $H_1(X;\mathbb{C})$. This implies that $\ker(p_*)$ does not have simple submodules isomorphic to V_1. As $M = M(V_1) \oplus \bigoplus_{j=2}^{k(G)} M(V_j)$, we obtain $\ker(p_*) \leq \sum_{j=2}^{k(G)} M(V_j)$. For dimensionality reasons we obtain equality.

□

Remark 3.5. The theorem of Gaschütz is natural in the following sense: Consider two iterated finite, regular, path-connected covers $q\colon Z \to Y$ and $p\colon Y \to X$. Then $p \circ q\colon Z \to X$ defines a covering map.



Set $H := \operatorname{Deck}(Z, q)$, $G := \operatorname{Deck}(Z, p \circ q)$ and $K := \operatorname{Deck}(Y, p)$. Then they fit in the following exact sequence

$$1 \to H \to G \xrightarrow{\psi} K \to 1.$$

Assuming that Z is a regular cover of X, we can apply the theorem of Gaschütz to Y and Z as covers of X and we obtain

$$H_1(Z;\mathbb{C}) \cong \mathbb{C}_{\mathrm{triv}} \oplus \mathbb{C}[G]^{\oplus n-1}$$
$$H_1(Y;\mathbb{C}) \cong \mathbb{C}_{\mathrm{triv}} \oplus \mathbb{C}[K]^{\oplus n-1}.$$

Then the induced maps on homology $H_1(Z;\mathbb{C}) \to H_1(Y;\mathbb{C})$ coming from q and from the natural surjection $\psi\colon G \to K$ coincide.

This follows directly from the proof of Theorem 3.1. In the case of iterated covers, choose first a $\mathbb{C}[K]$-basis $\overline{e_1}, \dots, \overline{e_n}$ for $C_1(Y;\mathbb{C})$ as before. Let $z_0 \in q_*^{-1}(y_0)$ be the base point of Z, and choose a $\mathbb{C}[G]$-basis $e_1, \dots, e_n$ for $C_1(Z;\mathbb{C})$ such that $q_*(e_i) = \overline{e_i}$ for all $1 \leq i \leq n$. We also have surjections $\phi_G\colon F_n \to G$ and $\phi_K\colon F_n \to K$ with the property that this diagram commutes.

$$\begin{array}{ccc} F_n & \xrightarrow{\phi_G} & G \\ & \searrow_{\phi_K} & \downarrow \psi \\ & & K \end{array}$$

Set $g_i \coloneqq \phi_G(x_i)$ and $\overline{g_i} \coloneqq \phi_K(x_i)$ for all $1 \leq i \leq n$. Then the above diagram implies that $\overline{g_i} = \psi(g_i)$.

We have the following diagram

$$\begin{array}{ccccc}
C_1(Z;\mathbb{C}) & & \xrightarrow{\partial_{Z,1}} & & C_0(Z;\mathbb{C}) \\
& \searrow^{q_*} & & \swarrow^{q_*} & \\
\downarrow \alpha_{Z,1} & C_1(Y;\mathbb{C}) & \xrightarrow{\partial_{Y,1}} & C_0(Y;\mathbb{C}) & \downarrow \alpha_{Z,0} \\
& \downarrow \alpha_{Y,1} & & \downarrow \alpha_{Y,0} & \\
& \bigoplus_{i=1}^n \mathbb{C}[K] & \xrightarrow{\rho_{Y,1}} & \mathbb{C}[K] & \\
& \nearrow^{\psi^{\times n}} & & \nwarrow^{\psi} & \\
\bigoplus_{i=1}^n \mathbb{C}[G] & & \xrightarrow{\rho_{Z,1}} & & \mathbb{C}[G]
\end{array}$$

where the maps $\alpha_{Y,1}, \alpha_{Y,0}, \alpha_{Z,1}, \alpha_{Z,0}$ and $\rho_{Y,1}, \rho_{Z,1}$ are defined as in the proof of Gaschütz with respect to the chosen bases.

To prove the claim, we need to show two things. Firstly, we need that the left square commutes, i.e.

$$\psi^{\times n} \circ \alpha_{Z,1} = \alpha_{Y,1} \circ q_*.$$

Secondly, we need

$$\psi^{\times n}(\ker(\rho_{Z,1})) = \ker(\rho_{Y,1}),$$

since we identify the homology groups with the kernel of the maps $\rho_{Y,1}$ and $\rho_{Z,1}$, respectively.

To prove the first identity, it is enough to verify it on a $\mathbb{C}[G]$-basis, since all maps are $\mathbb{C}[G]$-module homomorphisms. Thus we compute

$$\psi^{\times n} \circ \alpha_{Z,1}(e_i) = \psi^{\times n}(f_{G,i}) = f_{H,i},\ \alpha_{Y,1} \circ q_*(e_i) = \alpha_{Y,1}(\overline{e_i}) = f_{H,i},$$

where $f_{G,i}$ and $f_{H,i}$ are defined as in the proof of Gaschütz for all $1 \leq i \leq n$.

For the second identity, let $x \in \ker(\rho_{Z,1})$. Then $\psi(\rho_{Z,1}(x)) = 0 \in \mathbb{C}[K]$. Since the bottom square commutes by construction, we obtain

$$\rho_{Y,1}(\psi^{\times n}(x)) = \psi(\rho_{Z,1}(x)) = 0,$$

thus $\psi^{\times n}(x) \in \ker(\rho_{Y,1})$. This shows the first inclusion. For the other direction, let $x \in \ker(\rho_{Y,1})$. Consider $\alpha_{Y,1}^{-1}(x) \in \ker(\partial_{Y,1}) \leq C_1(Y;\mathbb{C})$. We know by Lemma 3.4, 3. that $\ker(\partial_{Y,1}) = q_*(\ker(\partial_{Z,1}))$, since $q\colon Z \to Y$ is a finite, regular cover and we identify the kernels with the respective homology. Thus we can find $z \in \ker(\partial_{Z,1})$ with $q_*(z) = \alpha_{Y,1}^{-1}(x)$. Now $\alpha_{Z,1}(z) \in \ker(\rho_{Z,1})$ is the desired element, since the left square commutes and thus

$$\psi^{\times n}(\alpha_{Z,1}(z)) = \alpha_{Y,1}(q_*(z)) = \alpha_{Y,1}(\alpha_{Y,1}^{-1}(x)) = x.$$

Note that this does not hold in the infinite case, even when given nice covers. Let the notation be as above. For $n = 2$ and Z the universal abelian cover with group of deck transformations $G = \mathbb{Z}^2$ we will see in the following subsection (Proposition 3.12) that

$$H_1(Z;\mathbb{C}) \cong \mathbb{C}[G].$$

Let Y be any finite, abelian cover of X with group of deck transformations K. Then Z is a regular cover of Y and of X via composition, and G surjects onto K via a homomorphism ψ. We know by the theorem of Gaschütz that

$$H_1(Y;\mathbb{C}) \cong \mathbb{C}_{\text{triv}} \oplus \mathbb{C}[K].$$

The diagram as in the proof above also holds in the infinite case. Where the proof breaks down is that we cannot show that $\psi^{\times 2}(\ker(\rho_{Z,1})) = \ker(\rho_{Y,1})$, since $\ker(\rho_{Z,1}) = \mathbb{C}[G]$ and $\ker(\rho_{Y,1}) = \mathbb{C}_{\text{triv}} \oplus \mathbb{C}[K]$, but $\psi^{\times 2}(\mathbb{C}[G]) = \mathbb{C}[K]$. We only have $\psi^{\times 2}(\ker(\rho_{Z,1})) \subseteq \ker(\rho_{Y,1})$, which in particular implies that $q_*\colon H_1(Z;\mathbb{C}) \to H_1(Y;\mathbb{C})$ is not surjective. Intuitively, this makes sense, as infinite covers miss one G-invariant summand coming from the image of the transfer map.

In the following section, we will compute the first homology of the universal abelian cover.

3.2 The Universal Abelian Cover

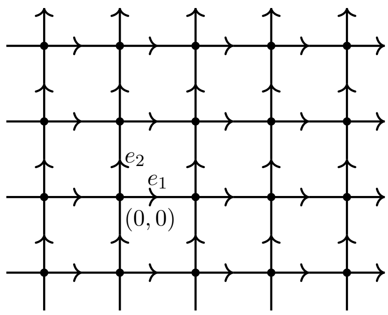
The theorem of Gaschütz fails if the cover is infinite. In this subsection we want to understand the first homology of the universal abelian cover. Additionally, we want to show that the Gaschütz decomposition does not hold in this case. Already for $n = 2$ we obtain a different result.

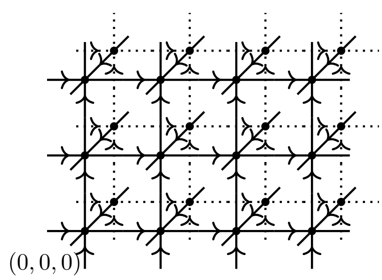
Definition 3.6. Let X be the wedge of n copies of S^1, $n \geq 2$. The *universal abelian cover Y of X* is the regular, path-connected cover of X defined by the natural surjection

$$\phi\colon F_n \to F_n/[F_n, F_n] \cong \mathbb{Z}^n$$

with base point $y_0 \in p^{-1}(x_0)$ and $p_*(\pi_1(Y)) = \ker(\phi)$.

Proposition 2.8 implies that $Y = \mathrm{Cay}(\mathbb{Z}^n, \{e_i, 1 \leq i \leq n\})$ based at $(0, \dots, 0)$, where e_i denotes the i-th standard generator of $\mathbb{Z}^n$. The group $\mathbb{Z}^n$ acts on the standard lattice graph Γ in $\mathbb{R}^n$, i.e. the graph with vertex set $\mathbb{Z}^n$ and edge set the edges from z to $z + e_i$ for $1 \leq i \leq n$. Therefore Γ is the Cayley graph of $\mathbb{Z}^n$ with respect to the standard basis. We can pull-back the orientation of the edges of X to obtain an orientation of the edges of Y. For $n = 2, 3$ this is illustrated in the following pictures.





We set $G \coloneqq \mathbb{Z}^n$ to simplify notation.

Proposition 3.7. *There is a surjective homomorphism of $\mathbb{Z}[G]$-modules*

$$\Phi\colon \mathbb{Z}[G]^{\oplus\binom{n}{2}} \to H_1(Y; \mathbb{Z}).$$

In particular, $H_1(Y; \mathbb{Z})$ is isomorphic to a quotient of $\binom{n}{2}$ copies of the regular representation of G as $\mathbb{Z}[G]$-modules.

In order to prove this proposition, we need to understand $\pi_1(Y)$ algebraically. For this we need the following lemma.

Lemma 3.8. *Let G be a group and $S \subseteq G$ a subset of G. We define $\langle S \rangle$ to be the smallest subgroup in G containing S and $\langle\langle S \rangle\rangle$ the smallest normal subgroup in G containing S. Then*

1. $\langle\langle S \rangle\rangle = \big\langle gsg^{-1} \mid s \in S, g \in G \big\rangle$, *and*
2. *if $G = \langle S \rangle$, then $[G, G] = \big\langle g[s_1, s_2]g^{-1} \mid s_1, s_2 \in S, g \in G \big\rangle$.*

Proof. We first prove

1. "$\subseteq$": Note that the subgroup on the right hand side is normal, and setting $g = 1$, it contains S.

   "$\supseteq$": For all $s \in S$, $g \in G$ we have $gsg^{-1} \in \langle\langle S \rangle\rangle$, because $S \subseteq \langle\langle S \rangle\rangle$ and $\langle\langle S \rangle\rangle$ is normal in G. As $\langle\langle S \rangle\rangle$ is a subgroup, we obtain that the subgroup generated by $\{gsg^{-1} \mid s \in S, g \in G\}$ is contained in $\langle\langle S \rangle\rangle$.

2. "$\subseteq$": Using 1., it suffices to show that

$$[G, G] \subseteq \langle\langle [s_1, s_2] \mid s_1, s_2 \in S \rangle\rangle =: K.$$

   We want to show that G/K is abelian; then we are done because the commutator subgroup is the smallest normal subgroup such that the quotient group is abelian. Note that G/K is generated by $\{sK \mid s \in S\}$ as $G = \langle S \rangle$. But now two elements in G/K commute because K consists of all pairwise commutators of elements in S.

   "$\supseteq$": For all $s_1, s_2 \in S, g \in G$ we have $g[s_1, s_2]g^{-1} \in [G, G]$ because $[G, G]$ is normal and $[s_1, s_2] \in [G, G]$.

□

Additionally, we need the following tools.

Definition 3.9. Let S, T be two partially ordered sets. The *lexicographic order on the product space* $S \times T$ is defined as follows

$$(s, t) \preccurlyeq (s', t') :\Leftrightarrow [s < s' \text{ or } (s = s' \text{ and } t \leq t')].$$

Remark 3.10. The order defined in Definition 3.9 is indeed a partial order. If both A and B are totally ordered, then also the lexicographic order on the product space is total.

Lemma 3.11. *The lexicographic order on* $G = \mathbb{Z}^n$ *defines a total order.*

Proof. This follows immediately from the preceding remark and the fact that $\mathbb{Z}$ is totally ordered with the standard order. □

We are now ready to prove Proposition 3.7.

Proof [Proposition 3.7]. Applying the last lemma to $G = F_n$ and $S = \{x_1, \ldots, x_n\}$, we obtain that

$$[F_n, F_n] = \left\langle w[x_i, x_j]w^{-1} \mid 1 \leq i < j \leq n, w \in F_n \right\rangle. \tag{5}$$

In other words, the commutator subgroup of the free group is normally generated by the pairwise commutators of its generators $x_1, \ldots, x_n$. Hence its abelianization is normally generated by $[\overline{x_i}, \overline{x_j}]$, $1 \leq i < j \leq n$, where by $\overline{x_i}$ we denote the image of x_i in $\mathbb{Z}^n$ under the natural surjection.

Note that for all $1 \leq i < j \leq n$, the commutator $x_{ij} := [x_i, x_j] \in \ker(\phi)$. Hence the loop defined by going along the edges $x_i x_j x_i^{-1} x_j^{-1}$, where by x_i^{-1}

we mean going along x_i in the opposite direction, lifts to a closed loop $\widetilde{x_{ij}}$ in Y at y_0. Thus it defines an element in $H_1(Y;\mathbb{Z})$. Namely, a path in Y is interpreted as a 1-chain in Y with multiplicities of the edges given by the number of times the path runs over a certain edge. We denote the element in $H_1(Y;\mathbb{Z})$ defined by x_{ij} as $[\widetilde{x_{ij}}]$, the equivalence class of the lifted loop under the homomorphism $\pi_1(Y, y_0) \to H_1(Y;\mathbb{Z})$. Every $1 \leq i < j \leq n$ defines an element $\widetilde{x_{ij}} \in \ker(\phi) \cong [F_n, F_n]$. By Equation (5), these $\binom{n}{2}$ elements normally generate $[F_n, F_n]$. We fix a bijection

$$\lambda\colon \left\{(i,j) \in \mathbb{N}^2 \mid 1 \leq i < j \leq n\right\} \to \left\{k \in \mathbb{N} \mid 1 \leq k \leq \binom{n}{2}\right\}$$

given by the lexicographic order of the tuples (i,j), and write $\delta_{\lambda(i,j)}$ for $\widetilde{x_{ij}}$. This leads us to define a map of $\mathbb{Z}[G]$-modules by

$$\Phi\colon \mathbb{Z}[G]^{\oplus\binom{n}{2}} \to H_1(Y;\mathbb{Z}),\ (g_1, \ldots, g_{\binom{n}{2}}) \mapsto g_1[\delta_1] + \ldots + g_{\binom{n}{2}}[\delta_{\binom{n}{2}}] \tag{6}$$

and $\mathbb{Z}$-linear continuation. Because of Equation (5) and since

$$H_1(Y;\mathbb{Z}) \cong [F_n, F_n]/\left[[F_n, F_n], [F_n, F_n]\right],$$

refer to Theorem 2.4, the map Φ is surjective. □

This already implies that $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}[G]^{\oplus\binom{n}{2}}/R$, where $R := \ker(\Phi)$ as $\mathbb{Z}[G]$-modules. We will now try to understand the relations R, and we will show that they come from relations in three-dimensional cubes. Furthermore, we will give a bound on the minimal number of relations.

Before we start proving the general case, let us first consider the cases $n = 2$ and $n = 3$. In order to do so, we need to introduce some tools.

Proposition 3.12. *For $n = 2$, the map*

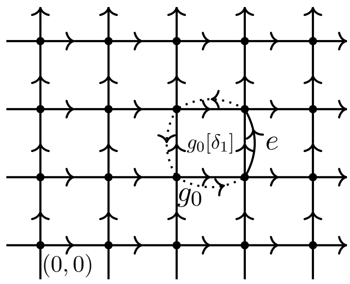
$$\Phi\colon \mathbb{Z}[G] \to H_1(Y;\mathbb{Z}),\ g \mapsto g[\delta_1],$$

defined as in Equation (6), is injective. In particular, $H_1(Y;\mathbb{C})$ is isomorphic to the regular representation.

Proof. Let $x \in \ker(\Phi) \setminus \{0\}$. Then we can write $x = \sum_{i=1}^{k} z_i g_i$ with $z_i \in \mathbb{Z} \setminus \{0\}$ and suitable $g_i \in G$ for $1 \leq i \leq k$. We have

$$0 = \Phi(x) = \sum_{i=1}^{k} z_i g_i[\delta_1].$$

Consider the loop starting at $g_0 := \max\{g_1, \ldots, g_k\}$ with respect to the lexicographic order. Consider the edge e, from $g_0 + (1,0)$ to $g_0 + (1,1)$, as illustrated in the following picture.



The edge e appears in $\Phi(x)$ non-trivially at the vertex g_0 with non-trivial multiplicity. On the other hand, $\Phi(x) = 0$. This implies, that there exists a $g_0^* \in \{g_1, \dots, g_k\}$ such that e appears in $g_0^*[\delta_1]$. But this can only be at $g_0^* = g_0 + (1,0)$, which is of higher lexicographic order. This yields a contradiction. Hence $x = 0$ and Φ is injective. $\square$

This lemma shows that already for $n = 2$ we obtain a different result than what the theorem of Gaschütz states. We will now show the case $n = 3$. Two auxiliary lemmas will be proven after the main proposition.

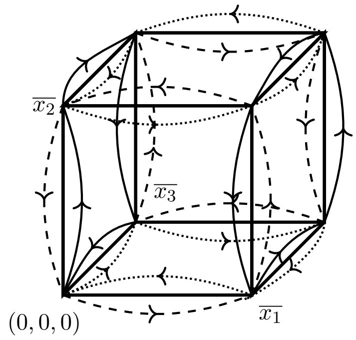
Proposition 3.13. *For $n = 3$, the kernel of the map Φ defined in Equation (6) is generated by the element $k := (1_G - \overline{x_3}, 1_G - \overline{x_1}, 1_G - \overline{x_2})$ as $\mathbb{Z}[G]$-module, i.e.*

$$\ker(\Phi) = \mathrm{Span}_{\mathbb{Z}[G]}\{(1_G - \overline{x_3}, 1_G - \overline{x_1}, 1_G - \overline{x_2})\} =: K.$$

Proof. It is easy to see that $(1_G - \overline{x_3}, 1_G - \overline{x_1}, 1_G - \overline{x_2}) \in \ker(\Phi)$. Namely, we have

$$\Phi((1_G - \overline{x_3}, 1_G - \overline{x_1}, 1_G - \overline{x_2})) = \\ 1_G[\delta_1] + 1_G[\delta_2] + 1_G[\delta_3] - \overline{x_3}[\delta_1] - \overline{x_1}[\delta_2] - \overline{x_2}[\delta_3],$$

where $[\delta_1] = [\overline{x_1}, \overline{x_2}]$, $[\delta_2] = [\overline{x_2}, \overline{x_3}]$ and $[\delta_3] = [\overline{x_3}, \overline{x_1}]$. Writing out the edges for each commutator shows that $\Phi(k) = 0$, as illustrated in the following.



There are several ways to see that this relation already generates the whole kernel. The general proof as in Proposition 3.16 works for $n = 3$ as well.

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}[G]^{\oplus 3}$ with $\Phi(\alpha) = 0$. Then we can write

$$\alpha_j = \sum_{i=1}^{m} z_{ij} g_i,\ z_{ij} \in \mathbb{Z},\ g_i \in G,$$

with m minimal such that $z_{ij} \neq 0$ for at least one j in $1 \leq j \leq 3$. We have

$$\begin{aligned} 0 = \Phi(\alpha) &= \Phi\left(\sum_{i=1}^{m} z_{i1} g_i, \sum_{i=1}^{m} z_{i2} g_i, \sum_{i=1}^{m} z_{i3} g_i\right) \\ &= \sum_{i=1}^{m} z_{i1} g_i[\delta_1] + \sum_{i=1}^{m} z_{i2} g_i[\delta_2] + \sum_{i=1}^{m} z_{i3} g_i[\delta_3] \\ &= \sum_{i=1}^{m} g_i \left(z_{i1}[\delta_1] + z_{i2}[\delta_2] + z_{i3}[\delta_3]\right). \end{aligned} \tag{7}$$

We set $M \coloneqq \{g_1, \ldots, g_m\} \subseteq \mathbb{Z}^3$ and

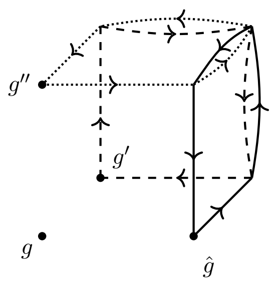
$$L(M) \coloneqq \{(x, y, z) \in M \mid \exists\, z' \in \mathbb{Z} \text{ with } (x, y, z') = \max(M)\}.$$

We will proceed by induction on $(|M|, |L(M)|) \in \mathbb{N} \times \mathbb{N}$.

For $|M| = 0$ or $|L(M)| = 0$, we have that $\alpha_j = 0$ for all $1 \leq j \leq 3$ and thus $\alpha = 0$, so $\alpha \in K$.

Let now $|M|, |L(M)| > 0$. Choose $\hat{g} \in M$ of highest lexicographic order. Without loss of generality we can assume that it is g_m; otherwise reorder the sum. Then it follows by Lemma 3.14 below that $z_1 = z_3 = 0$ and $z_2 \neq 0$, where $z_j \coloneqq z_{mj} \in \mathbb{Z}$ for $1 \leq j \leq 3$. By Lemma 3.15 below, we have $g' \coloneqq \hat{g} + (-1, 0, 1)$ and $g'' \coloneqq \hat{g} + (-1, 1, 0) \in M$ with $z_1' = z_2 = z_3''$, where $z_1', z_3'' \in \mathbb{Z}$ are the coefficients of $g'[\delta_1]$ and $g''[\delta_3]$ in Equation (7), respectively. Set $g \coloneqq \hat{g} + (-1, 0, 0)$. Then we have

$$\hat{g} = \overline{x_1} g,\ g' = \overline{x_3} g,\ g'' = \overline{x_2} g.$$



Consider the element $-z_2 g(1 - \overline{x_3}, 1 - \overline{x_1}, 1 - \overline{x_2}) \in K \leq \ker(\Phi)$. Then also

$$\alpha' \coloneqq (\alpha_1, \alpha_2, \alpha_3) - (-z_2 g(1 - \overline{x_3}, 1 - \overline{x_1}, 1 - \overline{x_2})) \in \ker(\Phi)$$

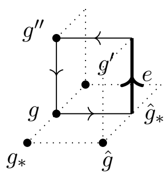
and from Equation (7),

$$
\begin{aligned}
\Phi(\alpha') &= \sum_{i=1}^{m-1} \left(g_i \sum_{j=1}^{3} z_{ij}[\delta_j] \right) + \hat{g} z_2[\delta_2] \\
&\quad + g \left(\sum_{j=1}^{3} z_2[\delta_j] \right) - g' z_2[\delta_1] - \hat{g} z_2[\delta_2] - g'' z_2[\delta_3] \\
&= \sum_{i=1}^{m-1} \left(g_i \sum_{j=1}^{3} \widehat{z_{ij}}[\delta_j] \right) + g \sum_{j=1}^{3} z_2[\delta_j], \qquad (8)
\end{aligned}
$$

with $\widehat{z_{ij}} \in \mathbb{Z}$ since $g', g'' \in M \setminus \{\hat{g}\}$. We have to distinguish two cases.

1. If $g \in M \setminus \{\hat{g}\}$, then we have found an element $\alpha' \in \ker(\Phi)$ with support $M \setminus \{\hat{g}\}$, and thus $\alpha' \in K$ by induction and therefore also $\alpha \in K$.

2. If $g \notin M \setminus \{\hat{g}\}$, set $M' := (M \setminus \{\hat{g}\}) \cup \{g\}$. Notice that $g \prec \hat{g}$. Consider the edge $e\colon g + (1,0,0) \to g + (1,1,0)$. Since $z_2 \neq 0$ in Equation (8), the edge e appears non-trivially in $\Phi(\alpha')$.



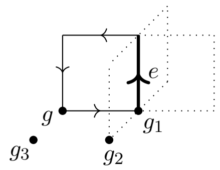
Because $\alpha' \in \ker(\Phi)$ and $\hat{g} \notin M'$, the edge e has to be eliminated by the commutator $[\delta_2]$ at the element $\hat{g}_* := \hat{g} + (0,0,-1) = g + (1,0,-1)$, i.e. $\hat{g}_* \in M'$ and the multiplicity of the commutator $[\delta_2]$ is $z_2 \neq 0$. Note that now $\hat{g}_* \in M'$ is the element of highest lexicographic order, since $\hat{g} \notin M'$ and for $h \in G$ with $\hat{g}_* \preccurlyeq h \preccurlyeq \hat{g}$, it follows $h = \hat{g}_*$ or $h = \hat{g}$. But this implies that $L(M') = L(M \setminus \{\hat{g}\}) = L(M) \setminus \{\hat{g}\}$ since $\hat{g}$ and $\hat{g}_*$ have the same first two coordinates. Thus we are done by induction.

□

Lemma 3.14. *Let $g \in M$ be the element of highest lexicographic order. Then $z_1 = z_3 = 0$ and $z_2 \neq 0$, where $z_j \in \mathbb{Z}$ is the coefficient of $g[\delta_j]$ in Equation (7) for $1 \leq j \leq 3$.*

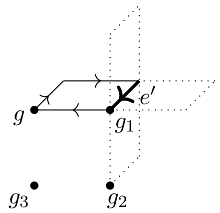
Proof. We show $z_1 = z_3 = 0$. Then it follows that $z_2 \neq 0$ because otherwise $g \notin M$.

Assume $z_1 \neq 0$. The edge $e\colon g + (1,0,0) \to g + (1,1,0)$ appears non-trivially in $gz_1[\delta_1]$.



Because $\Phi(\alpha) = 0$, the edge e has to be eliminated by some commutator starting at another vertex, say g'. The edge e belongs to four different cubes with left lower corners at g, $g_1 := g + (1,0,0)$, $g_2 := g + (1,0,-1)$ and $g_3 := g + (-1,0,-1)$. Note that $g \prec g_1, g_2$ and that none of the $[\delta_j]$, $1 \leq j \leq 3$, at g_3 can eliminate the edge e. Thus we obtain a contradiction.

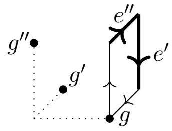
If we assume $z_3 \neq 0$, then we can use the same argument for the edge $e'\colon g + (1,0,1) \to g + (1,0,0)$ which appears non-trivially in $gz_3[\delta_3]$.



Now the edge e' is contained in exactly one of the commutators at either g, $g_1 := g + (1,0,0)$, $g_2 := g + (1,-1,0)$ or $g_3 := g + (0,-1,0)$. Again, we have $g \prec g_1, g_2$, and $[\delta_j]$ at g_3 cannot eliminate e' for any $1 \leq j \leq 3$. $\square$

Lemma 3.15. *Let $g \in M$ be of highest lexicographic order. Then the elements $g' := g + (-1,0,1)$ and $g'' := g + (-1,1,0)$ are in M. Furthermore, $z_1' = z_2 = -z_3''$, where z_1', z_2, $z_3'' \in \mathbb{Z}$ and z_1', z_2, z_3'' are the coefficients of $g'[\delta_1]$, $g[\delta_2]$, $g''[\delta_3]$, respectively.*

Proof. The proof works similary to the proof of Lemma 3.14, using that $\Phi(\alpha) = 0$, $z_2 \neq 0$ by Lemma 3.14 and the fact that g is maximal. Use the edges $e'\colon g + (0,0,1) \to g + (0,1,1)$ and $e''\colon g + (0,1,1) \to g + (0,1,0)$ for g' and g'' respectively.



$\square$

Recall that we want to analyze the universal abelian cover Y of the wedge of n circles. In order to understand $H_1(Y;\mathbb{Z})$, we use our knowledge about the homology groups of $\mathbb{R}^n$. The CW structure on $\mathbb{R}^n$ has $\mathbb{Z}^n$ as 0-skeleton. The 1-cells in $\mathbb{R}^n$ are intervals $[0,1]$ attached to $\mathbb{Z}^n$ by the characteristic maps $f^1_{(g,i)}\colon [0,1] \to \mathbb{R}^n,\ x \mapsto g + xe_i$ for $g \in \mathbb{Z}^n$ and $1 \le i \le n$. Now the boundary map takes an edge to its terminal vertex minus its initial vertex, where we orient the edges from 0 to 1. To glue in the 2-cells take $1 \le i < j \le n$ and $g \in \mathbb{Z}^n$ and the characteristic maps

$$f^2_{(g,i,j)}\colon [0,1]^2 \to \mathbb{R}^n,\ (x_1,x_2) \mapsto g + x_1e_i + x_2e_j.$$

Then we chose an orientation of the 2-cells such that for a 2-cell $Q_{(g,i,j)}$ defined by the triple (g,i,j), we have

$$\partial_{2,\mathbb{R}^n}(Q_{(g,i,j)}) = g\,[\widetilde{x_{ij}}].$$

Recall that $[\widetilde{x_{ij}}]$ is the homology class of the lift of the loop given by the commutator $[x_i,x_j]$ for x_i,x_j elements of a free basis of F_n. To define the 3-skeleton, take $1 \le i < j < k \le n$ and $g \in \mathbb{Z}^n$ and set

$$f^3_{(g,i,j,k)}\colon [0,1]^3 \to \mathbb{R}^n,\ (x_1,x_2,x_3) \mapsto g + x_1e_i + x_2e_j + x_3e_k.$$

Now if $W_{(g,i,j,k)}$ is a 3-cell in $\mathbb{R}^n$ defined by the quadruple (g,i,j,k), then we chose an orientation such that

$$\begin{aligned}\partial_{3,\mathbb{R}^n}(W_{(g,i,j,k)}) = {} & Q_{(g,i,j)} - Q_{(g,i,k)} + Q_{(g,j,k)} \\ & - Q_{(g+e_k,i,j)} + Q_{(g+e_j,i,k)} - Q_{(g+e_i,j,k)}.\end{aligned}$$

This is well-defined, as opposite squares appear once with positive and once with negative sign. We continue this process for all cubes up to dimension n. In this way, we define a CW structure on $\mathbb{R}^n$. This is the standard cubic lattice cell structure on $\mathbb{R}^n$.

We know that $Y = \mathrm{Cay}(G, \{e_i, 1 \le i \le n\})$ based at the vertex $(0,\dots,0)$. Thus we can view it as a subcomplex of $\mathbb{R}^n$ with compatible G-actions. Note that by construction

$$\partial_{1,Y} = \partial_{1,\mathbb{R}^n},$$

where $\partial_{1,Y}\colon C_1(Y;\mathbb{Z}) \to C_0(Y;\mathbb{Z})$ and $\partial_{1,\mathbb{R}^n}\colon C_1(\mathbb{R}^n;\mathbb{Z}) \to C_0(\mathbb{R}^n;\mathbb{Z})$ with the above defined cell structure on $\mathbb{R}^n$. We also know that the map $\partial_{2,\mathbb{R}^n}$ has the property that for each 2-cell Q in $\mathbb{R}^n$, we can find $g \in G$ and $1 \le i < j \le n$ such that $\partial_{2,\mathbb{R}^n}(Q) = g\,[\widetilde{x_{ij}}]$. We label the element $\widetilde{x_{ij}}$ by $\delta_{\lambda(i,j)}$, where λ is the ordering of the tuples (i,j).

Proposition 3.16. *For all $n \ge 3$, the kernel of the map*

$$\Phi\colon \mathbb{Z}[G]^{\oplus\binom{n}{2}} \to H_1(Y;\mathbb{Z}),$$

as defined in the proof of Proposition 3.7, is non-trivial. In fact, let

$$k_{ijl} \coloneqq (0,\ldots,0,\underbrace{1_G-\overline{x_l}}_{\lambda(i,j)},0,\ldots,0,\underbrace{-(1_G-\overline{x_j})}_{\lambda(i,l)},0,\ldots,0,\underbrace{1_G-\overline{x_i}}_{\lambda(j,l)},0,\ldots,0)$$

for $1 \le i < j < k \le n$. *Then*

$$\ker(\Phi) = \operatorname{Span}_{\mathbb{Z}[G]}\{k_{ijl} \mid 1 \le i < j < l \le n\} =: K,$$

i.e. the kernel is generated by the $\mathbb{Z}[G]$*-span of the projections onto the three-dimensional cubes in the unit cube* I^n *at the vertex* $(0,\ldots,0)$ *analogously to the case* $n = 3$.

Proof. It is easy to see that $k_{ijl} \in \ker(\Phi)$ for all $1 \le i < j < l \le n$; refer to the three-dimensional case in Proposition 3.13.

To show that $\ker(\Phi) \subseteq K$, we will make use of the cell structure on $\mathbb{R}^n$ defined as above. We have a chain complex of the form

$$\ldots \xrightarrow{\partial_{i+2,\mathbb{R}^n}} C_{i+1}(\mathbb{R}^n;\mathbb{Z}) \xrightarrow{\partial_{i+1,\mathbb{R}^n}} C_i(\mathbb{R}^n;\mathbb{Z}) \xrightarrow{\partial_{i,\mathbb{R}^n}} C_{i-1}(\mathbb{R}^n;\mathbb{Z}) \xrightarrow{\partial_{i-1,\mathbb{R}^n}} \ldots,$$

where $C_i(\mathbb{R}^n;\mathbb{Z}) = \bigoplus_{i\text{-dim. cells}} \mathbb{Z}$, and the i-dimensional cells are exactly the i-dimensional cubes. The homology of this chain complex equals $H_\bullet(\mathbb{R}^n;\mathbb{Z})$; refer to [Hat02, Theorem 2.35]. Since $\mathbb{R}^n$ is contractible, we know that $H_i(\mathbb{R}^n;\mathbb{Z}) = 0$ for all $i > 0$. It follows that

$$0 = H_2(\mathbb{R}^n;\mathbb{Z}) = \ker(\partial_{2,\mathbb{R}^n})/\operatorname{im}(\partial_{3,\mathbb{R}^n}),$$

and thus $\ker(\partial_{2,\mathbb{R}^n}) = \operatorname{im}(\partial_{3,\mathbb{R}^n})$.

Let x_i, x_j be two different elements of the chosen free basis of F_n. Then the loop $[x_i, x_j]$ lifts to a loop $\widetilde{x_{ij}}$ in Y. Since $Y \subseteq \mathbb{R}^n$, we can find a square Q_{ij}, i.e. a 2-cell in $\mathbb{R}^n$, such that $\partial_{2,\mathbb{R}^n}(Q_{ij}) = [\widetilde{x_{ij}}]$.

Take an element $\alpha = (\alpha_1,\ldots,\alpha_{\binom{n}{2}}) \in \ker(\Phi)$ with $\alpha_i \in \mathbb{Z}[G]$. Then

$$0 = \Phi(\alpha) = \sum_{i=1}^{\binom{n}{2}} \alpha_i[\delta_i].$$

Write $\alpha_i = \sum_{j=1}^{r} \beta_{ji} g_j$ for all $1 \le i \le \binom{n}{2}$ with r minimal such that for all $1 \le j \le r$, $\beta_{ji} \ne 0$ for at least one i. Let Q_i be 2-cells in $\mathbb{R}^n$ with the property that $\partial_{2,\mathbb{R}^n}(Q_i) = [\delta_i]$. We can reorder the right hand side of the above equation to obtain

$$\begin{aligned}
0 = \Phi(\alpha) &= \sum_{i=1}^{\binom{n}{2}} \alpha_i [\delta_i] \\
&= \sum_{j=1}^{r} g_j \left(\sum_{i=1}^{\binom{n}{2}} \beta_{ji} [\delta_i] \right) \\
&= \sum_{j=1}^{r} g_j \left(\sum_{i=1}^{\binom{n}{2}} \beta_{ji} \partial_{2,\mathbb{R}^n}(Q_i) \right) \\
&= \partial_{2,\mathbb{R}^n} \left(\sum_{j=1}^{r} g_j \left(\sum_{i=1}^{\binom{n}{2}} \beta_{ji} Q_i \right) \right) \qquad (9)
\end{aligned}$$

But now this means nothing else than that

$$\sum_{j=1}^{r} g_j \left(\sum_{i=1}^{\binom{n}{2}} \beta_{ji} Q_i \right) \in \ker(\partial_{2,\mathbb{R}^n}) = \operatorname{im}(\partial_{3,\mathbb{R}^n}).$$

We find a finite number of 3-cells $W_l \in I^n$, elements $g_l \in G$ and scalars $c_l \in \mathbb{Z}$ such that

$$\sum_{j=1}^{r} g_j \left(\sum_{i=1}^{\binom{n}{2}} \beta_{ji} Q_i \right) = \partial_{3,\mathbb{R}^n} \left(\sum_{l} c_l g_l W_l \right).$$

The map $\partial_{3,\mathbb{R}^n}$ is G-equivariant, so $\partial_{3,\mathbb{R}^n}(\sum_l c_l g_l W_l) = \sum_l c_l g_l \partial_{3,\mathbb{R}^n}(W_l)$. Plugging this in in the above Equation (9) yields

$$\Phi(\alpha) = \sum_{l} c_l g_l \partial_{2,\mathbb{R}^n} \left(\partial_{3,\mathbb{R}^n}(W_l) \right),$$

which also holds in the free chain group $C_1(\mathbb{R}^n)$. Furthermore, in $C_1(\mathbb{R}^n)$ we have

$$\partial_{2,\mathbb{R}^n} \left(\partial_{3,\mathbb{R}^n}(W_l) \right) = \Phi(k_{ijm})$$

for suitable directions $1 \le i < j < m \le n$ by construction. But now we can substract for all l exactly these relations from α to obtain that $\alpha \in K$, which proves the claim.

□

Remark 3.17. Since the maps $\partial_{2,\mathbb{R}^n}$ and $\partial_{3,\mathbb{R}^n}$ are $\mathbb{Z}[G]$-module homomorphism, it is enough to study the 3-dimensional subcubes in the n-dimensional unit cube I^n. This leads us to give a bound on the maximal number of generators of the kernel of the map Φ. We need to compute the dimension of $\mathrm{im}(\partial_{3,\mathbb{R}^n}) = \ker(\partial_{2,\mathbb{R}^n})$. For the unit cube, we obtain a cell structure with the following chain groups

$$C_i(I^n;\mathbb{Z}) = \bigoplus_{\substack{i\text{-dim. cubes in} \\ n\text{-dim. unit cube}}} \mathbb{Z} = \mathbb{Z}^{\binom{n}{i}2^{n-i}}.$$

Since the unit cube is contractible, we still have $H_i(I^n;\mathbb{Z}) = 0$ for all $i > 0$ and $H_0(I^n;\mathbb{Z}) = \mathbb{Z}$. Thus $\ker(\partial_{i,\mathbb{R}^n}) = \mathrm{im}(\partial_{i+1,\mathbb{R}^n})$ for all $i > 0$. We obtain

$$\dim(\ker \partial_{2,\mathbb{R}^n}) = \sum_{i=3}^{n} (-1)^{i+1} \dim(C_i(I^n;\mathbb{Z})) = \sum_{i=3}^{n} (-1)^{i+1} \binom{n}{i} 2^{n-i}.$$

For $n = 3$ this formula yields $\dim(\ker \partial_{2,\mathbb{R}^n}) = 1$, which corresponds to the result in Proposition 3.13. For $n = 4$, we have 8 three-dimensional subcubes in I^4, but $\dim(\ker \partial_{2,\mathbb{R}^n}) = 7$. In fact, it has been verified by [GAP18] that the eight elements k_{ijl}, $1 \leq i < j < l \leq 8$, are of rank seven.

4 Primitive Elements and Primitive Homology

4.1 Primitive Elements

Definition 4.1. Let $n \geq 2$ be an integer and F_n the free group on n generators. An element $w \in F_n$ is called *primitive* if there exist $w_2, \dots, w_n \in F_n$ such that $\{w, w_2, \dots, w_n\}$ is a free basis of F_n.

In an analogous manner, an element $z \in \mathbb{Z}^n$ is called *primitive* if there exist $z_2, \dots, z_n \in \mathbb{Z}^n$ such that $\{z, z_2, \dots, z_n\}$ is a free basis of $\mathbb{Z}^n$.

Lemma 4.2. *Let F_n be the free group on the generators $x_1, \dots, x_n$, and let $\overline{}\colon F_n \to F_n/[F_n, F_n] \cong \mathbb{Z}^n$ be the projection map. Then we have*

1. *for $w \in F_n$ primitive, $\overline{w} \in \mathbb{Z}^n$ is primitive, and*
2. *for $z \in \mathbb{Z}^n$ primitive, there exists a primitive element $w \in F_n$ with $\overline{w} = z$.*

Proof. The subgroup $[F_n, F_n]$ is invariant under every automorphism of F_n. This implies that an automorphism of F_n induces an automorphism of $\mathbb{Z}^n$, so we get a map

$$\Phi\colon \mathrm{Aut}(F_n) \to \mathrm{Aut}(\mathbb{Z}^n) = \mathrm{GL}(n; \mathbb{Z}),\ \alpha \mapsto \Phi(\alpha)$$

with $\Phi(\alpha)(\overline{x_i}) \coloneqq \overline{\alpha(x_i)}$. We know that Φ is surjective, since every matrix in $\mathrm{GL}(n; \mathbb{Z})$ can be obtained by a finite number of row transformations from the identity matrix. The row transformations correspond exactly to the Nielsen transformations which generate $\mathrm{Aut}(F_n)$; refer to Theorem 2.5.

To prove the first claim, let $w \in F_n$ be primitive. Then there exists an $\alpha \in \mathrm{Aut}(F_n)$ with $\alpha(w) = x_1$. We obtain

$$\Phi(\alpha)(\overline{w}) = \overline{\alpha(w)} = \overline{x_1}$$

by definition of Φ. Since $\overline{x_1}$ is primitive in $\mathbb{Z}^n$, we get that $\overline{w}$ is primitive, which proves the first part of the lemma.

For the second claim, take $z \in \mathbb{Z}^n$ primitive. There exists an automorphism ψ of $\mathbb{Z}^n$ with $\psi(\overline{x_1}) = z$. Since Φ is surjective, we can find $\alpha \in \mathrm{Aut}(F_n)$ with $\Phi(\alpha) = \psi$. By definition, $z = \psi(\overline{x_1}) = \Phi(\alpha)(\overline{x_1})$. Now $\overline{\alpha(x_1)} = z$, so $w \coloneqq \alpha(x_1)$ is primitive in F_n and has the property that $\overline{w} = z$. □

Using this lemma, it is easy to see that not all elements of F_n are primitive.

Remark 4.3. Let $x_1, \dots, x_n$ be a free basis of F_n. Then $w_1 \coloneqq x_1^2$ is not primitive. Assuming it were, we would find words $w_2, \dots, w_n$ that complete to a basis of F_n. Then, by Lemma 4.2, their images in $\mathbb{Z}^n$ would form a basis. We could thus find a bijective group homomorphism $\varphi\colon \mathbb{Z}^n \to \mathbb{Z}^n$

that sends $\overline{x_i}$ to $\overline{w_i}$. Because of the definition of w_1, the homomorphism φ could be represented by the matrix

$$M = \begin{pmatrix} 2 & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{pmatrix}.$$

But M is not invertible over $\mathbb{Z}$, hence we would obtain a contradiction.

Commutators of primitive elements are not primitive because their image in $\mathbb{Z}^n$ is trivial.

It is not easy to list all primitive elements in the free group for $n \geq 3$. There is an algorithmic way to test for primitivity of words based on a result by Whitehead; see [HL74]. Depending on the length of the word, this algorithm has poor theoretical upper bounds on its run time. For $n = 2$ there is a nice way of listing primitive elements. This will be elaborated in the following subsection.

Remark 4.4. Another important property is that the automorphism group of a free group acts transitively on the set of primitive elements. This follows from the universal property of the free group.

Primitive elements in $\mathbb{Z}^n$ however are better understood.

Definition 4.5. An element $z \in \mathbb{Z}^n$ is *indivisible* if for every $z' \in \mathbb{Z}^n$ and $m \in \mathbb{Z}$ with $z = mz'$, we have $m = \pm 1$.

The following lemma gives a constructive way to see if elements are indivisible.

Lemma 4.6. *Let $z_1, \ldots, z_n$ be a basis of $\mathbb{Z}^n$, $z := \sum_{i=1}^n \alpha_i z_i \in \mathbb{Z}^n$ with $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}$. Then z is indivisible if and only if* $\gcd(\alpha_1, \ldots, \alpha_n) = 1$.

Proof. We first show that an indivisible element has coprime coefficients. Let $m := \gcd(\alpha_1, \ldots, \alpha_n)$. Then we have

$$z = \sum \alpha_i z_i = m \sum \alpha_i' z_i,$$

with $\alpha_i' \in \mathbb{Z}$ for $1 \leq i \leq n$. Thus $\gcd(\alpha_1, \ldots, \alpha_n) = m = \pm 1$, since z is indivisible.

For the other direction let $z = mz'$ for some $z' \in \mathbb{Z}^n$ and $m \in \mathbb{Z}$. Then $mz' = z = \sum \alpha_i z_i$ and $z' = \sum (\alpha_i / m) z_i$. This implies that $m \mid \alpha_i$ for all $1 \leq i \leq n$. Thus $m \mid \gcd(\alpha_1, \ldots, \alpha_n)$. Since $\gcd(\alpha_1, \ldots, \alpha_n) = 1$, we obtain $m = \pm 1$ and thus z is indivisible. □

For primitive elements in $\mathbb{Z}^n$ we have the following complete characterization.

Proposition 4.7. *An element in $\mathbb{Z}^n$ is primitive if and only if it is indivisible.*

PROOF. The easy direction is to show that primitive elements are indivisible. Let thus $z_1 \in \mathbb{Z}^n$ be primitive, $z' \in \mathbb{Z}^n$ and $m \in \mathbb{Z}$ with $z_1 = mz'$. We extend z_1 to a basis $z_1, z_2, \ldots, z_n$ of $\mathbb{Z}^n$ and write $z' = \sum_{i=1}^{n} \alpha_i z_i$ with $\alpha_i \in \mathbb{Z}$ for $1 \leq i \leq n$. Then

$$0 = mz' - z_1 = m\left(\sum_{i=1}^{n} \alpha_i z_i\right) - z_1 = (m\alpha_1 - 1)\, z_1 + m \sum_{i=2}^{n} \alpha_i z_i.$$

Since $z_1, \ldots, z_n$ is a basis, we obtain $m\alpha_1 - 1 = 0$. Thus $m\alpha_1 = 1$, and since $\alpha_1 \in \mathbb{Z}$, it follows that $m = \pm 1$, which proves that z_1 is indivisible.

To show that indivisible elements are primitive, we adapt a proof by [Rad51, Lemma] which uses Lemma 4.6 and an induction on the sum of the coefficients. Thus let $z_1, \ldots, z_n$ be a basis of $\mathbb{Z}^n$. Let $z = \sum_{i=1}^{n} \alpha_i z_i$ with $\alpha_i \in \mathbb{Z}$ for $1 \leq i \leq n$ and $\gcd(\alpha_1, \ldots, \alpha_n) = 1$. Without loss of generality we can assume $\alpha_i \in \mathbb{N}$, because otherwise we can replace z_i by $-z_i$. We induct on $s := \sum_{i=1}^{n} \alpha_i$ and we claim that z is primitive.

For $s = 1$, there is only one $\alpha_i = 1$, and thus $z = z_i$ for some i, hence z is primitive. If $s > 1$, there are at least two $\alpha_i > 0$. Up to reordering we can assume $\alpha_1 \geq \alpha_2 > 0$. With respect to the basis $z_1, z_2 + z_1, z_3, \ldots, z_n$, the coefficients of z are $(\alpha_1 - \alpha_2, \alpha_2, \ldots, \alpha_n)$. Then

$$\gcd(\alpha_1 - \alpha_2, \alpha_2, \ldots, \alpha_n) = \gcd(\alpha_1, \alpha_2, \ldots, \alpha_n) = 1$$

and

$$s' = \alpha_1 - \alpha_2 + \alpha_2 + \ldots + \alpha_n = \alpha_1 + \alpha_3 + \ldots + \alpha_n < s.$$

By induction, we have that $z = (\alpha_1 - \alpha_2) z_1 + \alpha_2 (z_2 + z_1) + \sum_{i=3}^{n} \alpha_i z_i$ is primitive.

A constructive proof in which the basis is given explicitly can be found in [New72, Theorem II.1.]. For a topological proof refer to [FM12, Proposition 6.2]. □

In the following, we will see an important property of primitive elements in free groups. Given a surjective homomorphism onto a finite, abelian group, we can choose a basis for the free group in which this homomorphism has a particularly nice form, i.e. redundant generators get mapped to the trivial element.

Lemma 4.8. *Let F_n be the free group on the generators $x_1, \ldots, x_n$ and G a non-trivial, finite, abelian group. Given a surjective group homomorphism $\phi\colon F_n \to G$, we can find two primitive elements $w, w' \in F_n$ such that*

1. $\langle \phi(w) \rangle \neq \{1\}$, *and*

2. $\langle \phi(w) \rangle \cap \langle \phi(w') \rangle = \{1\}$.

PROOF. Since G is abelian, the map ϕ factors through $\mathbb{Z}^n$, so we obtain a map $\tilde{\phi}\colon \mathbb{Z}^n \to G$. Note that $\tilde{\phi}$ is a surjective homomorphism of abelian groups. Consider $M \coloneqq \ker(\tilde{\phi}) \leq \mathbb{Z}^n$. Now M is a $\mathbb{Z}$-submodule of rank n as the quotient $\mathbb{Z}^n/M \cong G$ is finite. Since $\mathbb{Z}^n$ is a finitely generated free $\mathbb{Z}$-module, the invariant factor decomposition, refer to [Bos09, Chapter 2.9, Theorem 2], tells us that we can find a basis $z_1, \dots, z_n$ of $\mathbb{Z}^n$ and coefficients $\alpha_1, \dots, \alpha_n \in \mathbb{N} \setminus \{0\}$ such that

1. $\alpha_1 z_1, \dots, \alpha_n z_n$ is a basis of M, and
2. $\alpha_i \mid \alpha_{i+1}$ for $1 \leq i \leq n-1$.

These coefficients are uniquely determined by the submodule M. We obtain

$$\mathbb{Z}^n = \bigoplus_{i=1}^{n} \mathbb{Z}\, z_i, \text{ and } M = \bigoplus_{i=1}^{n} \mathbb{Z}\, \alpha_i\, z_i.$$

We denote by $\alpha\colon \mathbb{Z}^n/M \to \bigoplus_{i=1}^n \mathbb{Z}/\alpha_i\mathbb{Z}$ the isomorphism that sends

$$z_i + M \mapsto (0, \dots, 0, \underbrace{1 + \alpha_i\mathbb{Z}}_{i}, 0, \dots, 0)$$

with the non-zero entry at the i-th position. Since $M = \ker(\tilde{\phi})$, we have an isomorphism $\psi\colon G \to \bigoplus_{i=1}^n \mathbb{Z}/\alpha_i\mathbb{Z}$ such that the following diagram commutes, i.e. we have $\psi \circ \tilde{\phi} = \alpha \circ \mathrm{pr}$.

$$\begin{array}{ccc} F_n & & \\ {\scriptstyle -}\downarrow & \searrow^{\phi} & \\ \mathbb{Z}^n & \xrightarrow[\tilde{\phi}]{} & G \\ {\scriptstyle \mathrm{pr}}\downarrow & & {\scriptstyle \psi}\downarrow \\ \mathbb{Z}^n/M & \xrightarrow{\alpha} & \bigoplus_{i=1}^n \mathbb{Z}/\alpha_i\mathbb{Z} \end{array}$$

From the commutativity of the diagram and the definition of α we obtain

$$\langle \tilde{\phi}(z_i) \rangle \cap \langle \tilde{\phi}(z_j) \rangle = \{1\}$$

for all $1 \leq i \neq j \leq n$.

The group G is non-trivial and $G \cong \mathbb{Z}^n/M \cong \bigoplus_{i=1}^n \mathbb{Z}/\alpha_i\mathbb{Z}$, which implies that at least one of the $\alpha_i \neq 1$. Furthermore $\alpha_i \mid \alpha_{i+1}$, so we know that $\alpha_n \neq 1$. Thus

$$\langle \tilde{\phi}(z_n) \rangle \neq \{1\}. \tag{10}$$

By the previous claim, we additionally obtain

$$\langle \tilde{\phi}(z_{n-1}) \rangle \cap \langle \tilde{\phi}(z_n) \rangle = \{1\} \tag{11}$$

with $z_{n-1}, z_n \in \mathbb{Z}^n$ primitive.

Lemma 4.2 tells us that we can find primitive elements $w, w' \in F_n$ with

$$\overline{w} = z_{n-1}, \text{ and } \overline{w'} = z_n.$$

Furthermore, we have $\phi(w) = \tilde{\phi}(\overline{w}) = \tilde{\phi}(z_{n-1})$ and $\phi(w') = \tilde{\phi}(\overline{w'}) = \tilde{\phi}(z_n)$. This, together with Equations (10) and (11), proves the lemma. □

4.2 Primitive Elements in Rank 2

For $n = 2$, there is a nice way of listing all primitive elements. This is done in the paper [OZ81] by Osborne and Zieschang.

Theorem 4.9 (Osborne-Zieschang). *Up to conjugation, the primitive elements in the free group on two generators x_1, x_2 are*

$$\{W_{m,k}(x_1, x_2) \mid m, k \in \mathbb{Z}, \gcd(m, k) = 1\},$$

where $W_{m,k}$ is defined as follows.

For $m, k \geq 0$, $\gcd(m, k) = 1$ we define a function $f_{m,k} \colon \mathbb{Z} \to \{1, 2\}$ by $f_{m,k}(l) = f_{m,k}(l')$ if $l \equiv l' \operatorname{mod}(m + k)$ and

$$f_{m,k}(l) = \begin{cases} 1 & \text{if } 0 \leq l < m, \\ 2 & \text{if } m \leq l < m + k. \end{cases}$$

We set

$$W_{m,k}(x_1, x_2) = \prod_{i=0}^{m+k-1} x_{f_{m,k}(1+im)}.$$

For $m < 0$ define $W_{m,k}(x_1, x_2) = W_{-m,k}(x_1^{-1}, x_2)$, and for $k < 0$ define $W_{m,k}(x_1, x_2) = W_{m,-k}(x_1, x_2^{-1})$.

The proof of the above can be found in [OZ81, Theorem 1.2]. An important consequence of this theorem is summarized in the following remark.

Remark 4.10. Let $w \in F_2 = F\langle x_1, x_2 \rangle$ be a primitive word in the letters x_1 and x_2. Then, up to conjugation, all exponents of x_1 are either all positive or all negative. The same is true for the exponents of x_2.

This will be useful in testing if certain homology classes in the covering space appear as lifts of powers of primitives. The result allows us to give a bound on the length of a primitive that can elevate to a given homology class. Note that conjugation of a primitive element corresponds to elevations at different preimages of the base point in the cover.

Another important result is the following geometric interpretation of primitive words in rank two. Let L be the set of all lines in $\mathbb{R}^2$ that are parallel to one of the coordinate axes of $\mathbb{R}^2$ and passing through $\mathbb{Z}^2 \subset \mathbb{R}^2$.

Any directed line segment in $\mathbb{R}^2$ that does not contain a point of $\mathbb{Z}^2$ defines a word in F_2 in the following way: Write x_1 when traveling along the line segment and crossing a vertical line of L from left to right and write x_2 when crossing a horizontal line from below. Write x_1^{-1} respectively x_2^{-1} if the lines are crossed in the opposite direction. For $(m,k) \in \mathbb{N}^2$ with $\gcd(m,k) = 1$ the open segment from $(0,0)$ to (m,k) does not contain any point from $\mathbb{Z}^2$, so the above process defines a word $V'_{m,k}(x_1,x_2)$, and we set

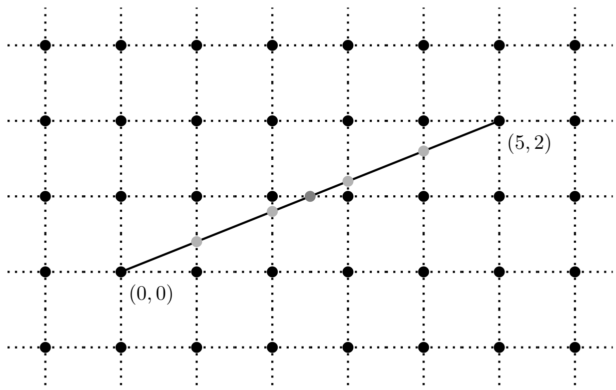
$$V_{m,k}(x_1,x_2) \coloneqq x_1x_2V'_{m,k}(x_1,x_2).$$

We also define

$$V_{0,1}(x_1,x_2) \coloneqq x_2,\ V_{1,0}(x_1,x_2) \coloneqq x_1$$

and

$$V_{-m,k}(x_1,x_2) \coloneqq V_{m,k}(x_1^{-1},x_2),\ V_{m,-k}(x_1,x_2) \coloneqq V_{m,k}(x_1,x_2^{-1}).$$



For $(m,k) = (5,2)$, as illustrated above, we obtain $V'_{5,2}(x_1,x_2) = x_1^2x_2x_1^2$, and thus the associated word is

$$V_{5,2}(x_1,x_2) \coloneqq x_1x_2V'_{5,2}(x_1,x_2) = (x_1x_2)(x_1^2x_2x_1^2).$$

Proposition 4.11. *For $(m,k) \in \mathbb{Z}^2$ with $\gcd(m,k) = 1$ we have*

$$W_{m,k}(x_1,x_2) = V_{m,k}(x_1,x_2).$$

A proof is given in [OZ81, Proposition 2.3]. From this geometric interpretation it is clear that $m-1$ is equal to the number of x_1's in $V'_{m,k}(x_1,x_2)$ and $k-1$ is equal to the number of x_2's. But now m is equal to the number of x_1's in $W_{m,k}(x_1,x_2)$ and k is equal to the number of x_2's by definition of $V_{m,k}(x_1,x_2)$. In the following we will list some important properties of the words $V'_{m,k}(x_1,x_2)$.

Lemma 4.12. *Let $(m,k) \in \mathbb{N}^2$. Denote by a_i the number of x_1's between the $(i-1)$-st and the i-th x_2 in $V'_{m,k}(x_1,x_2)$. Analogously, let b_i be the number of x_2's between the $(i-1)$-st and the i-th x_1 in $V'_{m,k}(x_1,x_2)$.*

1. *If $k \leq m$, then the line segment from $(0,0)$ to (m,k) has slope $k/m < 1$ and $b_i = 1$ for all $1 \leq i \leq m$. Additionally,*

$$\left\lfloor \frac{m-1}{k} \right\rfloor \leq a_i \leq \left\lfloor \frac{m-1}{k} \right\rfloor + 1$$

for all $1 \leq i \leq k$, and

$$a_i = a_{k-i+1}$$

for all $1 \leq i \leq k/2$.

2. *If $k \geq m$, then the line segment from $(0,0)$ to (m,k) has slope $k/m > 1$ and $a_i = 1$ for all $1 \leq i \leq k$. We have*

$$\left\lfloor \frac{k-1}{m} \right\rfloor \leq b_i \leq \left\lfloor \frac{k-1}{m} \right\rfloor + 1$$

for all $1 \leq i \leq m$, and

$$b_i = b_{m-i+1}$$

for all $1 \leq i \leq m/2$.

PROOF. This just follows when analyzing the line segments in $\mathbb{R}^2$ not going through points of $\mathbb{Z}^2$. For the first part in the case $k \leq m$, note that the line segment from $(0,0)$ to (m,k) crosses $m-1$ vertical and $k-1$ horizontal lines. Thus we have to evenly distribute $m-1$ points on k intervals, so $\lfloor (m-1)/k \rfloor \leq a_i \leq \lfloor (m-1)/k \rfloor + 1$.

For the second part, we compute using the functional equation for the line segment that

$$a_i = \left| \left\{ n \in \mathbb{N} \,\middle|\, (i-1)\frac{m}{k} \leq n \leq i\frac{m}{k} \right\} \right|$$

for all $1 \leq i \leq k$. But this we can compute, since for $x, y \in \mathbb{R}$, $0 \leq x \leq y$ we have

$$|\{n \in \mathbb{N} \mid x \leq n \leq y\}| = \max(0, \lceil y \rceil - \lfloor x \rfloor - 1).$$

These two observations together imply that

$$\begin{aligned}
a_{k-i+1} &= \max\left(0, \left\lceil (k-i+1)\frac{m}{k} \right\rceil - \left\lfloor (k-i)\frac{m}{k} \right\rfloor - 1\right) \\
&= \max\left(0, \left\lceil m-(i-1)\frac{m}{k} \right\rceil - \left\lfloor m - i\frac{m}{k} \right\rfloor - 1\right) \\
&= \max\left(0, \left\lceil -(i-1)\frac{m}{k} \right\rceil - \left\lfloor -i\frac{m}{k} \right\rfloor - 1\right) \\
&= \max\left(0, -\left\lfloor (i-1)\frac{m}{k} \right\rfloor + \left\lceil i\frac{m}{k} \right\rceil - 1\right) \\
&= a_i
\end{aligned}$$

The case $k \geq m$ works similarly by interchanging the roles of m and k, and a_i and b_i. □

4.3 Primitive Elements in the Covering Space

By the theorem of Nielsen-Schreier, refer to [Rob93, Theorem 6.1.1], we know that a subgroup of a free group is again free. We are also interested in finding the free generators of this subgroup. For this we identify the free group on n generators F_n with the fundamental group of the wedge of n circles X based at the single vertex x_0 of X, with free generators the oriented edges $x_1, \ldots, x_n$ of X. Now the above question translates into finding primitive elements in a given covering space, since fundamental groups of covering spaces correspond to subgroups of the fundamental group of X. Covering spaces of graphs are graphs and thus their fundamental groups are free.

The following proposition can be used to explicitly describe the free generators of the fundamental group of the covering space in terms of those of X by pulling back the labels of the edges of X via the covering map p. It can be found in [Hat02, Proposition 1A.2].

Proposition 4.13. *Let Y be a connected graph with base vertex y_0 and T a spanning tree in Y with $y_0 \in T$. Each oriented edge e_α of $Y \setminus T$ determines a loop f_α in Y in the following way: First go from y_0 to one endpoint of e_α by a path in T, then across e_α, then back to y_0 by a path in T. Then $\pi_1(Y, y_0)$ is a free group with basis the classes $[f_\alpha]$ corresponding to the edges e_α of $Y \setminus T$.*

We are interested in the following question: given a finite-index subgroup of a free group, are powers of primitive elements that lie in the subgroup also primitive in the subgroup? For this we need a more formal definition, but before we will setup a notation for the rest of this section.

Setup 4.14. Fix $n \in \mathbb{N}$, $n \geq 2$, a finite group G and a surjective homomorphism $\phi\colon F_n \to G$, where $F_n = F\langle x_1, \ldots, x_n \rangle$ is the free group on n generators. Let X be the wedge of n copies of S^1. Then we can associate to ϕ a finite regular path-connected cover $p\colon Y \to X$ with base point $y_0 \in p^{-1}(x_0)$ and $p_*(\pi_1(Y, y_0)) = \ker(\phi)$. Since p_* is injective, we sometimes view $\pi_1(Y, y_0) \subseteq \pi_1(X, x_0)$. Then

$$G \cong F_n/\ker(\phi) \cong \pi_1(X, x_0)/p_*(\pi_1(Y, y_0)) \cong \mathrm{Deck}(Y, p)$$

and G acts on Y by graph automorphisms.

Definition 4.15. Let the situation be as in Setup 4.14. For an element $w \in F_n$, let $k(w)$ be the minimal number such that $w^{k(w)} \in \ker(\phi)$. The *preferred elevation* $\widetilde{w}$ *of* w is the lift of $w^{k(w)}$ to Y at the base point y_0. If we want to emphasize the cover to which we elevate, we write $\widetilde{w}^{Y \to X}$.

Note that $k(w)$ is finite for all $w \in F_n$, since G is finite. The elevation $\widetilde{w}$ is a closed loop in Y, and it makes sense to consider its homology class, which is denoted by square brackets $[\widetilde{w}]$. Lifts of $w^{k(w)}$ at other preimages of

x_0 are just called elevations. Note that we obtain all elevations by applying all elements of the group of deck transformations G to $\widetilde{w}$.

The following lemma shows that it is often sufficient to only prove certain properties for representatives in an $\mathrm{Aut}(F_n)$-orbit that are of a particularly nice form. As a consequence of Proposition 2.6, it is sufficient to verify assumptions about primitive elements on standard primitive elements, i.e. primitive elements that correspond to a single loop in X. We will refer to it as the Relabeling lemma.

Lemma 4.16 (Relabeling Lemma)**.** *In the situation of Setup 4.14, let w, v be two elements in the same $\mathrm{Aut}(F_n)$-orbit.*

1. *If $\tilde{v}$ is primitive in H, then $\widetilde{w}$ is also primitive in H.*
2. *If $G\,[\tilde{v}]$ is linearly independent in $H_1(Y;\mathbb{Z})$, then $G\,[\widetilde{w}]$ is linearly independent in $H_1(Y;\mathbb{Z})$.*

Proof. Set $H \coloneqq p_*(\pi_1(Y, y_0)) = \ker(\phi)$. Take $k(w)$ minimal such that $w^{k(w)} \in H$. We know there exists an automorphism $\alpha \in \mathrm{Aut}(F_n)$ with $\alpha(v) = w$, since w, v lie in the same orbit. Note that

$$\alpha(v^{k(w)}) = \alpha(v)^{k(w)} = w^{k(w)} \in H.$$

Thus $v^{k(w)} \in \alpha^{-1}(H)$ with $k(w)$ minimal with this property. Now $\alpha^{-1}(H)$ is again a finite index normal subgroup of the fundamental group of X. Then $v^{k(w)}$ is primitive in $\alpha^{-1}(H)$, since we can apply the assumption to $v, k(w)$ and $\alpha^{-1}(H)$. Applying α implies that $w^{k(w)} = \alpha(v^{k(w)})$ is primitive in H, which proves 1.

$$\begin{array}{ccc} F_n & \xrightarrow{\ \alpha\ } & F_n \\ \big\downarrow & & \big\downarrow \\ F_n/H & \underset{\alpha}{\dashrightarrow} & F_n/\alpha^{-1}(H) \end{array}$$

To prove 2., we have to understand $g'\,[v^{k(w)}]$, where $[v^{k(w)}]$ is the image of $v^{k(w)}$ in $\alpha^{-1}(H)/[\alpha^{-1}(H), \alpha^{-1}(H)]$ and $g' \in G' \coloneqq F_n/\alpha^{-1}(H)$. Recall that the action of G respectively G' is induced by the conjugation action of F_n on H respectively $\alpha^{-1}(H)$ as described in Subsection 2.1. Note that $[w^{k(w)}]$ is the image of $w^{k(w)}$ in $H/[H, H]$. Set $\gamma \coloneqq v^{k(w)}$ and $g' = z\,\alpha^{-1}(H)$ for some $z \in F_n$. Then

$$g'\,[\gamma] = (z\gamma z^{-1})[\alpha^{-1}(H), \alpha^{-1}(H)].$$

Now we can compute

$$\alpha(g'\,[\gamma]) = \alpha(z)\alpha(\gamma)\alpha(z)^{-1}[H, H] = (\alpha(z)H)\,[w^{k(w)}] = \alpha(g')\,[w^{k(w)}],$$

where the last equality comes from the fact that $\alpha(g') = \alpha(z)h$ for some $h \in H$. We want to show that $G\,[w^{k(w)}]$ is linearly independent. Let $a_i \in \mathbb{Z}$, $g_i \in G$ with

$$0 = \sum_{i=1}^{|G|} a_i g_i [w^{k(w)}].$$

Take $g_i' \in G'$ with $\alpha(g_i') = g_i$ for all $1 \leq i \leq |G|$. Then

$$0 = \sum_{i=1}^{|G|} a_i \alpha(g_i')[w^{k(w)}] = \sum_{i=1}^{|G|} a_i \alpha\left(g_i'\,[v^{k(w)}]\right) = \alpha\left(\sum_{i=1}^{|G|} a_i g_i'\,[v^{k(w)}]\right).$$

Since α is bijective and $G[\tilde{v}]$ is linearly independent in $H_1(Y;\mathbb{Z})$, it follows that $a_i = 0$ for all $1 \leq i \leq |G|$, which proves 2. □

Note that the second conclusion of the above lemma also holds when considering coefficients in $\mathbb{C}$.

Corollary 4.17. *In Setup 4.14, elevations of primitive elements are primitive.*

Proof. It is enough to check the claim for the preferred elevation, since the other elevations are conjugates of the preferred one and thus also primitive.

Let $l = x_i \in F_n$ for some $1 \leq i \leq n$ be one of the standard primitive elements. Set $k := k(l)$, the minimal number such that $l^{k(l)} \in \ker(\phi)$. The preferred elevation $\tilde{l}$ of l is a simple closed curve in Y. Namely, for every vertex in the covering space, there is exactly one incoming and outgoing edge that is the preimage of x_i under p. Say we come back to the vertex $y_1 \neq y_0$ before we return to y_0. Since $\tilde{l}$ is the lift of a power of l, we only go along edges labeled by x_i. Thus we can never come back to y_0 which is a contradiction to the fact that $\tilde{l}$ is a closed curve at y_0 by construction.

Denote by e_k the last edge in $\tilde{l}$. Then $\tilde{l} \setminus e_k$ is a tree T_l in Y. By [Hat02, Proposition 1A.1], T_l is contained in a maximal tree T with base point y_0 in Y. Note that $e_k \in Y \setminus T$, because if e_k were in T, then T could not be a tree, since $\tilde{l}$ is a loop. Additionally, we have $f_k = \tilde{l}$ with f_k as defined in Proposition 4.13. But this immediately implies that f_k is primitive, since its class is part of a free basis of $\pi_1(Y, y_0)$; see Proposition 4.13.

Now for $l \in F_n$ some primitive element, not necessarily a standard generator, the first part of the Relabeling lemma, Lemma 4.16, 1., and Proposition 2.6 immediately imply that $\tilde{l}$ is primitive. □

4.4 Primitive Homology

In Setup 4.14, the group G acts on Y and this action descends to an action of G on $H_1(Y;\mathbb{C})$ by linear maps. We can thus study $H_1(Y;\mathbb{C})$ as a G-representation.

Definition 4.18. Let $S \subseteq F_n$ be a subset of F_n. We set

$$H_1^S(Y;\mathbb{C}) := \operatorname{Span}_{\mathbb{C}[G]} \{[\tilde{s}] \mid s \in S\} \leq H_1(Y;\mathbb{C}).$$

We write $H_1^S(Y \to X;\mathbb{C})$ if we want to emphasize the cover to which we elevate the primitive elements. If S is the set of primitive elements, we write $H_1^{\text{prim}}(Y;\mathbb{C})$ and call this subrepresentation the *primitive homology of the covering space* Y.

Note that if $S' \subseteq S$, then $H_1^{S'}(Y;\mathbb{C}) \leq H_1^S(Y;\mathbb{C})$. One question that arises naturally is whether $H_1^{\text{prim}}(Y;\mathbb{C}) = H_1(Y;\mathbb{C})$. In general, the answer is no. There is the following theorem by Malestein and Putman that answers the question in the negative; refer to [MP17, Theorem B, Example 1.2].

Theorem 4.19. *For all $n \geq 2$ there exists a finite index, normal subgroup $R \trianglelefteq F_n$ with $H_1^{\text{prim}}(Y;\mathbb{C}) \neq H_1(Y;\mathbb{C})$, where $p\colon Y \to X$ is the unique finite, regular cover of X with $p_*(\pi_1(Y,y_0)) = R$.*

We will give a sketch of the proof in the following. For more details refer to Sections 1 to 3 of [MP17].

Proof. We begin by constructing a group G with the following property: For all $n \geq 2$, p a prime, there exists a finite p-group G, i.e. a group of order a power of p, a central subgroup C of G and a homomorphism $\Psi\colon C \to \mathbb{Z}/p\mathbb{Z}$ such that $H_1(G;\mathbb{F}_p) = \mathbb{F}_p^n$, and for all $g \in G$ that have nonzero image in $H_1(G;\mathbb{F}_p)$, there is a power of g that lies in $C \setminus \ker(\Psi)$. Here we mean $H_1(G;\mathbb{F}_p) := H_1(K(G,1);\mathbb{F}_p) \cong G/[G,G] \otimes_{\mathbb{Z}} \mathbb{F}_p$, with $K(G,1)$ the Eilenberg-MacLane space with fundamental group G. The construction of such groups is the hard part of this proof and uses restricted Lie algebras and Lie theory. More details can be found in [MP17, Subsections 2.2 ff.].

A small example for $n = 2$ and $p = 2$ is the quaternion group

$$Q_8 = \langle x, y \mid x^4 = 1, x^2 = y^2, yxy^{-1} = x^{-1} \rangle.$$

Then C can be chosen as the center of Q_8, i.e. $C = Z(Q_8) = \langle x^2 \rangle$, which is cyclic of order two, and Ψ the identity map. Note that Q_8 is a 2-group and $H_1(Q_8;\mathbb{F}_2) \cong Q_8/[Q_8,Q_8] \otimes_{\mathbb{Z}} \mathbb{F}_2 \cong (\mathbb{Z}/2\mathbb{Z})^2$, so the first condition is satisfied. Furthermore, every $g \in Q_8 \setminus \{1\}$ has the property that some power is equal to x^2, since $y^2 = x^2$.

For p a prime, we denote by $\mathfrak{D}_p$ the set of all p-primitive elements, i.e. all $w \in F_n$ with the property that $0 \neq [w] \in H_1(X;\mathbb{F}_p)$. Note that a primitive element is p-primitive for all primes p. This follows immediately from Lemma 4.2, 1. and Proposition 4.7. Recalling

$$H_1^{\mathfrak{D}_p}(Y;\mathbb{C}) = \operatorname{Span}_{\mathbb{C}[G]} \left\{ [\tilde{l}] \mid l \in F_n \ p\text{-primitive} \right\},$$

with the help of the above group G, we construct a finite index, normal subgroup $R \trianglelefteq F_n$ such that $H_1^{\mathfrak{D}_p}(Y;\mathbb{C}) \neq H_1(Y;\mathbb{C})$ and such that F_n/R is a p-group.

This is done by using the first property of the group G to define a homomorphism $\rho\colon F_n \to G$ in order for the induced map on homology $\rho_*\colon H_1(F_n;\mathbb{F}_p) \to H_1(G;\mathbb{F}_p)$ to be an isomorphism. Define $R := \ker(\rho)$. This is the desired normal subgroup. Next, we construct a $\mathbb{C}$-representation V of G such that for all $g \in G$ that have nonzero image in $H_1(G;\mathbb{F}_p)$, the action of g on V does not fix any nonzero vector. Indeed, V can be taken to be the induced representation of $W = \mathbb{C}$, a representation of C, where C acts on W by multiplication with p-th roots of unity using the map Ψ. This implies that $H_1^{\mathfrak{D}_p}(Y;\mathbb{C}) \neq H_1(Y;\mathbb{C})$. But now we are done because $H_1^{\mathrm{prim}}(Y;\mathbb{C}) \leq H_1^{\mathfrak{D}_p}(Y;\mathbb{C}) \neq H_1(Y;\mathbb{C})$. □

In particular, we can choose $p = 2$. The size of the group G constructed in the above proof depends on both p and n. Although we can choose $p = 2$, the group G grows exponentially fast in n.

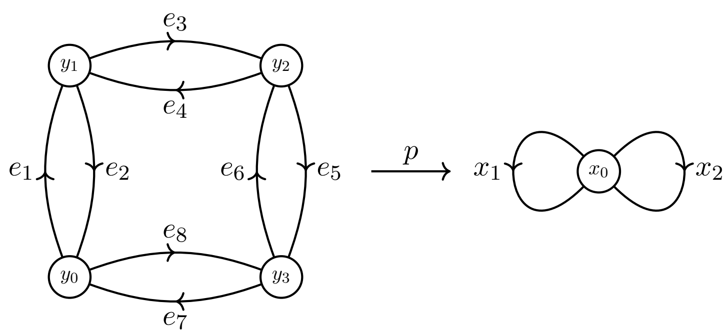
If our data is given by a surjection $\phi\colon F_n \to G$ for some finite group G, the construction in the above theorem will not tell us whether primitive homology is all of homology. The following result gives an answer for finite abelian and 2-step nilpotent groups, and can be found in [FH16, Proposition 3.2, Proposition 3.3].

Theorem 4.20 (Farb-Hensel). *Let $n \geq 2$ and $\phi\colon F_n \to G$ a surjective homomorphism onto a finite abelian group G. Then $H_1^{\mathrm{prim}}(Y;\mathbb{C}) = H_1(Y;\mathbb{C})$. For $n \geq 3$, the same is true for G a finite 2-step nilpotent group.*

Note that for $n = 2$, there are examples of 2-step nilpotent groups where primitive homology does not coincide with homology. See for example [FH16, Section 7.3].

In the following we want to explicitly see the primitive elements that generate homology for the mod 2-homology cover.

Example 4.21. Let the setup be as in Example 3.2.



Consider the primitive elements x_1, x_2, $x_1x_2 \in F_2$. Set

$$v_1 := [\widetilde{x_1}] = e_1 + e_2,\ v_2 := B.[\widetilde{x_1}] = e_5 + e_6,$$
$$v_3 := [\widetilde{x_2}] = e_7 + e_8,\ v_4 := A.[\widetilde{x_2}] = e_3 + e_4.$$

Furthermore, define $v_5 := [\widetilde{x_1 x_2}] = e_1 + e_3 + e_5 + e_7$. It is reasonably obvious that the vectors $v_1, \dots, v_5$ are linearly independent and in $H_1(Y;\mathbb{C})$. We know that

$$\dim_{\mathbb{C}}(H_1(Y;\mathbb{C})) = |G|(n-1) + 1 = 5,$$

so $H_1^{\mathrm{prim}}(Y;\mathbb{C}) = H_1(Y;\mathbb{C})$.

The main result to prove both of the above theorems is the following obstruction specified by Farb and Hensel in [FH16, Theorem 1.4]. To properly understand this, we need the following purely representation-theoretic definition.

Definition 4.22. Let G be a finite group and let $\phi\colon F_n \to G$ be a surjective group homomorphism. Let $S \subseteq F_n$ be a subset of F_n. We define

$$\mathrm{Irr}^S(\phi, G) \subseteq \mathrm{Irr}(G)$$

to be the subset of those irreducible representations V of G that have the property that there is an element in S whose image has a nonzero fixed point. More precisely, $V \in \mathrm{Irr}^S(\phi, G)$ if and only if there exists an element $s \in S$ and $0 \neq v \in V$ such that $\phi(s)(v) = v$. If S is the set of primitive elements, we write $\mathrm{Irr}^S(\phi, G) = \mathrm{Irr}^{\mathrm{prim}}(\phi, G)$.

Theorem 4.23 (Farb-Hensel)**.** *In the situation of Setup 4.14, we have*

$$H_1^{\mathrm{prim}}(Y;\mathbb{C}) \leq \mathbb{C}_{\mathrm{triv}} \oplus \bigoplus_{V_i \in \mathrm{Irr}^{\mathrm{prim}}(\phi, G)} V_i^{(n-1)\dim(V_i)}.$$

The following proposition is the main ingredient to prove the above representation-theoretic criterion for primitive homology.

Proposition 4.24. *Let the situation be as in Setup 4.14. Let l be a primitive loop in X and let $\tilde{l}$ be its preferred elevation in Y. We define $z := [\tilde{l}]$ and $g := \phi(l)$. Then there is an isomorphism of G-representations*

$$\mathrm{Span}_{\mathbb{C}[G]}\{[z]\} \cong \mathrm{Ind}_{\langle g \rangle}^G(\mathbb{C}_{\mathrm{triv}}).$$

The proof presented in [FH16, Proposition 2.1] uses surface topology. In the following we will give a direct proof for graphs.

Proof. Let $l = x_i \in F_n$ for some $1 \leq i \leq n$ be one of the standard primitive elements. The preferred elevation $\tilde{l}$ of l is a simple closed curve in Y by choice of l. We claim that Gz is linearly independent in $H_1(Y;\mathbb{Z})$. Since $\tilde{l}$ is a simple closed curve, $Z := \{g\,\tilde{l} \mid g \in G\} = \bigsqcup_{i=1}^k S^1$ is a finite disjoint union of embedded circles. By [FH17, Claim 2.4], the inclusion $Z \hookrightarrow Y$ induces an injection I on homology, so

$$I\colon \bigoplus_{i=1}^k H_1(S^1;\mathbb{Z}) \hookrightarrow H_1(Y;\mathbb{Z}).$$

Note that this is still injective when we consider homology classes with coefficients in $\mathbb{C}$. Now the action of G on the left hand side is just the permutation action of $G/\langle\phi(l)\rangle$. This is clear since $\mathrm{Stab}_G(\tilde{l}) = \langle\phi(l)\rangle$. This proves the proposition for standard primitive elements using Lemma 2.28.

Now, let l be some primitive element in X, not necessarily a standard generator. By Proposition 2.6 and the Relabeling lemma, see Lemma 4.16, 2., it follows that Gz is linearly independent in $H_1(Y;\mathbb{C})$, and by Lemma 2.28 the proposition is proven. □

Now we are ready to prove Theorem 4.23.

Proof. Let V be an irreducible G-representation. For l a primitive loop in X, set $g_l \coloneqq \phi(l)$ and $G_l \coloneqq \langle g_l\rangle$. Then Proposition 4.24 and Frobenius reciprocity, refer to Theorem 2.27, imply that

$$\begin{aligned}\langle \mathrm{Span}_{\mathbb{C}[G]}\{[\tilde{l}]\}, V\rangle_G &= \langle \mathrm{Ind}_{G_l}^G(\mathbb{C}_{\mathrm{triv}}), V\rangle_G \\ &= \langle \mathbb{C}_{\mathrm{triv}}, \mathrm{Res}_{G_l}^G(V)\rangle_{G_l} \\ &= \dim(V^{G_l}),\end{aligned}$$

where by V^{G_l} we denote the fixed point set of V, i.e.

$$V^{G_l} = \{v \in V : gv = v \text{ for all } g \in G_l\}.$$

The last equality follows from the orthogonality relations.

Now an irreducible representation V appears in $H_1^{\mathrm{prim}}(Y;\mathbb{C})$ if and only if there exists a primitive element $l \in F_n$ such that

$$\langle \mathrm{Span}_{\mathbb{C}[G]}\{[\tilde{l}]\}, V\rangle_G \neq 0.$$

By the above, this is true if and only if $\dim(V^{G_l}) \neq 0$, which is equivalent to the existence of some nonzero $v \in V$ with $g_l v = v$. Thus $V \in \mathrm{Irr}^{\mathrm{prim}}(\phi, G)$, which proves the theorem. □

Lemma 4.25. *Let the setup be as in Proposition 4.24. For $z \in H_1(Y;\mathbb{Z})$, write $z = z_1 + \ldots + z_{k(G)}$ for its decomposition into the homogeneous components, i.e. $z_i \in M(V_i)$ for $M = H_1(Y;\mathbb{C})$. Then $z_1 \in H_1(Y;\mathbb{Q})$.*

Proof. By Lemma 3.4, 4., we know that $p_*(z) = p_*(z_1) \in H_1(X;\mathbb{Z})$, since $z \in H_1(Y;\mathbb{Z})$. Now since $z_1 \in M(V_1)$, using 2. in the same lemma, we know that there exists $x \in H_1(X;\mathbb{C})$ such that $z_1 = p_\#(x)$. Lemma 3.4, 1. implies that

$$|G|\, x = p_* \circ p_\#(x) = p_*(z_1) = p_*(z) \in H_1(Y;\mathbb{Z}).$$

Thus,

$$|G|\, z_1 = p_\#(|G|\, x) \in H_1(Y;\mathbb{Z}),$$

which proves the lemma. □

Lemma 4.26. *Let the setting be as in Proposition 4.24, $z = [\tilde{l}]$ for l a loop on X. Then*

$$p_*(z) = |\langle \phi(l) \rangle| \cdot [l] \in H_1(X; \mathbb{Z}),$$

where $\phi\colon F_n \to G$ and p_* is the induced map on homology.

If G is abelian, then

$$\phi(l) = \tilde{\phi}\left(\frac{1}{|\langle \phi(l) \rangle|}\, p_*(z) \right).$$

Proof. The first part is true by the definition of an elevation $\tilde{l}$ of a loop l. Namely,

$$p_*(z) = p_*([\tilde{l}]) = p_*\left([\widetilde{l^{k(l)}}] \right) = k(l)\,[l].$$

Note that $k(l) = |\phi(l)|$.

The second claim follows easily from the first, because if G is abelian, the map $\phi\colon F_n \to G$ factors through $\mathbb{Z}^n$ as illustrated in the following commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(Y, y_0) & \longrightarrow & \pi_1(X, x_0) & \xrightarrow{\phi} & G & \longrightarrow & 1 \\
 & & \big\downarrow \text{ab} & & \big\downarrow \text{ab} & \nearrow \tilde{\phi} & & & \\
 & & H_1(Y; \mathbb{Z}) & \xrightarrow[p_*]{} & H_1(X; \mathbb{Z}) & & & &
\end{array}$$

Since the first homology with coefficients in $\mathbb{Z}$ is the abelianization of the fundamental group of a path-connected space, see Theorem 2.4, the result follows. □

This lemma shows that in the case of an abelian group, the homology class of some elevation of a primitive loop already encodes information about the cover $\phi\colon F_n \to G$. This will be essential in understanding how much topological data is encoded in the representation of the first homology of the covering space.

In the following we want to show that the condition given in Proposition 4.24 from [FH16] is not sufficient to characterize elevations of primitive elements. We already know that elevations of primitive elements are themselves primitive in the covering space, refer to Corollary 4.17, and by Proposition 4.7 they are thus indivisible. We will show that there are indivisible homology classes in $H_1(Y; \mathbb{Z})$ that satisfy the condition in Proposition 4.24 but do not come from primitive elements. The main idea is to start with the preferred elevations of two different primitive elements and then exchange the trivial components in the induced representations that they span. Now the trivial and the non-trivial components do not "fit" together anymore. This is formalized in the following proposition. An explicit example will be given subsequently.

Proposition 4.27. *In the situation of Setup 4.14, there exists $z \in H_1(Y;\mathbb{Z})$ such that*

1. *z is indivisible,*
2. *$\mathrm{Span}_{\mathbb{C}[G]}\{z\} \cong \mathrm{Ind}_{\langle g \rangle}^{G}(\mathbb{C}_{\mathrm{triv}})$ as $\mathbb{C}[G]$-representations for some $g \in G$,*
3. *but z is not the homology class of any elevation of any primitive loop l on X.*

Proof. Let l', l'' be two primitive loops on X with $\langle g' \rangle \cap \langle g'' \rangle = \{1\}$ and $\langle g' \rangle \neq \{1\}$, where $g' := \phi(l')$ and $g'' := \phi(l'') \in G$. This is possible by Lemma 4.8. Consider the homology classes $z' = [\tilde{l'}]$ respectively $z'' = [\tilde{l''}] \in H_1(Y;\mathbb{Z})$ of the preferred elevations $\tilde{l'}$ and $\tilde{l''}$ of the two primitive loops l' and l''. By Proposition 4.24, we know that

$$\mathrm{Span}_{\mathbb{C}[G]}\{z'\} \cong \mathrm{Ind}_{\langle g' \rangle}^{G}(\mathbb{C}_{\mathrm{triv}}), \text{ and } \mathrm{Span}_{\mathbb{C}[G]}\{z''\} \cong \mathrm{Ind}_{\langle g'' \rangle}^{G}(\mathbb{C}_{\mathrm{triv}}).$$

Let $V_1 = \mathbb{C}_{\mathrm{triv}}, \dots, V_{|G|}$ be representatives of the isomorphism classes of the irreducible G-representations. By the theorem of Gaschütz, see Theorem 3.1, we know that

$$H_1(Y;\mathbb{C}) \cong \mathbb{C}_{\mathrm{triv}} \oplus \mathbb{C}[G]^{\oplus n-1} \cong V_1^n \oplus \bigoplus_{i=2}^{|G|} V_i^{n-1},$$

as G is abelian, and all irreducible representations over the complex numbers are one-dimensional. We write

$$z' = z_1' + z_2' + \ldots + z_{|G|}', \; z'' = z_1'' + z_2'' + \ldots + z_{|G|}''$$

for their decomposition into the homogeneous components, i.e. $z_1', z_1'' \in V_1^n$ and $z_i', z_i'' \in V_i^{n-1}$ for $2 \leq i \leq |G|$. Lemma 4.25 yields that $z_1', z_1'' \in H_1(Y;\mathbb{Q})$, so

$$z_2' + \ldots + z_{|G|}' = z' - z_1' \in H_1(Y;\mathbb{Q}).$$

The same holds for z''. Now set $b := z_1'' - z_1' \in V_1^n$.

Notice that $b \neq -z_1'$ and $b \neq 0$ and thus $b \in V_1^n \setminus \{-z_1'\}$. Indeed, if $b = -z_1'$, which is equivalent to $z_1'' = 0$, Lemma 3.4, 4. and the first part of Lemma 4.26 would imply that $[l''] = 0$ which contradicts the fact that $[l'']$ is indivisible since l'' is primitive. If we assume $b = 0$, we would have $z_1' = z_1''$. By Lemma 3.4, 4., it follows that $p_*(z') = p_*(z'')$. Then Lemma 4.26 implies that

$$\phi(l') = \tilde{\phi}\left(\frac{1}{|\langle \phi(l') \rangle|}\, p_*(z'')\right) = \tilde{\phi}\left(\frac{|\langle \phi(l'') \rangle|}{|\langle \phi(l') \rangle|}\,[l'']\right) = \frac{|\langle \phi(l'') \rangle|}{|\langle \phi(l') \rangle|}\,\phi(l''),$$

since $[l''] \in H_1(X;\mathbb{Z})$ indivisible and therefore $|\langle \phi(l'') \rangle| / |\langle \phi(l') \rangle| \in \mathbb{Z}$. Thus $\langle g' \rangle \leq \langle g'' \rangle$, which contradicts the choice of l' and l''.

Set

$$\hat{z} := z' + b = z''_1 + z'_2 + \ldots + z'_{|G|} \in H_1(Y;\mathbb{Q}).$$

If $\hat{z} \notin H_1(Y;\mathbb{Z})$, choose $m \in \mathbb{Z}$ minimal such that $z := m\hat{z} \in H_1(Y;\mathbb{Z})$. Then z is indivisible. If $\hat{z} \in H_1(Y;\mathbb{Z})$, take $k \in \mathbb{Z}$ so that $z := \hat{z}/k \in H_1(Y;\mathbb{Z})$ is indivisible.

Thus we have found $z \in H_1(Y;\mathbb{Z})$ indivisible with $z = q\hat{z}$ for some $q \in \mathbb{Q}$, $\hat{z} = z''_1 + z'_2 + \ldots + z'_{|G|}$. Setting $M = H_1(Y;\mathbb{C})$ in Lemma 2.29, we know that

$$\mathrm{Span}_{\mathbb{C}[G]}\{z\} = \mathrm{Span}_{\mathbb{C}[G]}\{\hat{z}\} = \mathrm{Span}_{\mathbb{C}[G]}\{z'\} \cong \mathrm{Ind}^G_{\langle g'\rangle}(\mathbb{C}_{\mathrm{triv}}). \tag{12}$$

By Proposition 4.24, it follows that if z were the homology class of an elevation of a primitive element l in X, then $\mathrm{Span}_{\mathbb{C}[G]}\{z\} \cong \mathrm{Ind}^G_{\langle\phi(l)\rangle}(\mathbb{C}_{\mathrm{triv}})$. By Equation (12), we would also have

$$\mathrm{Ind}^G_{\langle g\rangle}(\mathbb{C}_{\mathrm{triv}}) \cong \mathrm{Ind}^G_{\langle g'\rangle}(\mathbb{C}_{\mathrm{triv}}),$$

with $g := \phi(l)$. By Lemma 2.30, this would imply that $\langle g\rangle = \langle g'\rangle$.

On the other hand, we compute

$$p_*(z) = q\, p_*(\hat{z}) = q\, p_*(z''_1) = q\, p_*(z'') = q\,|\langle g''\rangle|\,[l''].$$

By Lemma 4.26 and since G is abelian, we obtain

$$g = \tilde{\phi}\left(\frac{1}{|\langle g\rangle|}\,p_*(z)\right) = \tilde{\phi}\left(\frac{q|\langle g''\rangle|}{|\langle g\rangle|}\,[l'']\right) = \underbrace{\frac{q|\langle g''\rangle|}{|\langle g\rangle|}}_{=:r}\,\tilde{\phi}([l'']) = (g'')^r,$$

since $[l''] \in H_1(X;\mathbb{Z})$ indivisible implies that $r \in \mathbb{Z}$. Thus $\langle g'\rangle = \langle g\rangle \leq \langle g''\rangle$. This contradicts the fact that $\langle g'\rangle \cap \langle g''\rangle = \{1\}$ and $\langle g'\rangle \neq \{1\}$. ☐

In fact, we can show an even stronger result using the same techniques as in the proof of the above proposition.

Corollary 4.28. *Let $z \in H_1(Y;\mathbb{Z})$ be the element constructed in the proof of Proposition 4.27. Then there is no element $z^* \in \mathrm{Span}_{\mathbb{C}[G]}\{z\}$ which is the homology class of an elevation of a primitive loop on X.*

PROOF. If we assume there is, let $z^* \in \mathrm{Span}_{\mathbb{C}[G]}\{z\} \cap H_1(Y;\mathbb{Z})$ be indivisible. Then there exist $\alpha_1, \ldots, \alpha_{|G|} \in \mathbb{C}$ such that

$$z^* = \sum_{i=1}^{|G|} \alpha_i g_i z$$

for $g_i \in G$. Additionally, we can assume that $\mathrm{Span}_{\mathbb{C}[G]}\{z^*\} \cong \mathrm{Ind}^G_{\langle g^*\rangle}(\mathbb{C}_{\mathrm{triv}})$ for some $g^* \in G$ by Proposition 4.24. This implies that $\langle g^*\rangle = \mathrm{Stab}_G(z^*)$.

We claim that $\mathrm{Stab}_G(z) \le \mathrm{Stab}_G(z^*)$. But this is clear since an element in the stabilizer of z also stabilizes z^*, as G is abelian. This implies that

$$\langle g' \rangle = \mathrm{Stab}_G(z) \le \mathrm{Stab}_G(z^*) = \langle g^* \rangle. \tag{13}$$

On the other hand, we have

$$p_*(z^*) = \sum_{i=1}^{|G|} \alpha_i p_*(g_i z) = \underbrace{\left(\sum_{i=1}^{|G|} \alpha_i \right)}_{:=A} p_*(z),$$

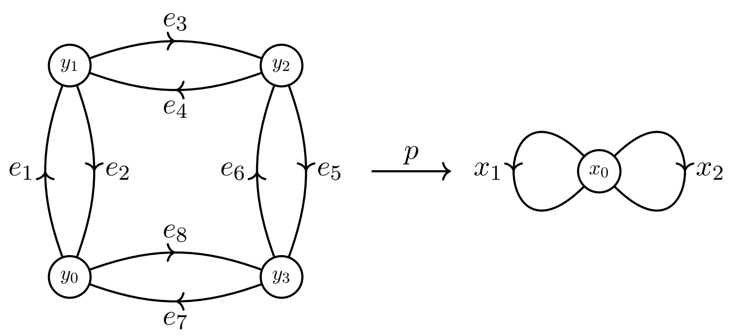
since G acts as homeomorphisms of the covering space which preserve the projection map p. Then $A \in \mathbb{Q}$ since both $p_*(z^*)$, $p_*(z)$ are in $H_1(X;\mathbb{Z})$. Assume there exists a primitive element $l^* \in F_n$ with $[\tilde{l^*}] = z^*$. Then $g^* = \phi(l^*)$ by Proposition 4.24. Together with Lemma 4.26 we have

$$\begin{aligned} g^* &= \tilde{\phi}\left(\frac{1}{|\langle g^* \rangle|} \, p_*(z^*) \right) \\ &= \tilde{\phi}\left(\frac{A}{|\langle g^* \rangle|} \, p_*(z) \right) \\ &= \tilde{\phi}\left(\frac{q \, A \, |\langle g'' \rangle|}{|\langle g^* \rangle|} \, [l''] \right) \\ &= \underbrace{\frac{q \, A \, |\langle g'' \rangle|}{|\langle g^* \rangle|}}_{=:r} \, \tilde{\phi}([l'']) \\ &= (g'')^r, \end{aligned}$$

since $[l''] \in H_1(X;\mathbb{Z})$ indivisible and therefore $r \in \mathbb{Z}$. Thus $\langle g^* \rangle \le \langle g'' \rangle$. Together with Equation (13) we obtain $\langle g' \rangle \le \langle g^* \rangle \le \langle g'' \rangle$, which is a contradiction. □

In the following example we want to explicitly compute the induced representations spanned by homology classes for rank $n = 2$ and $G = (\mathbb{Z}/2\mathbb{Z})^2$ in order to see the abstract argument in a specific case.

Example 4.29. We are given the same setup as in Examples 3.2 and 4.21 and set $G = (\mathbb{Z}/2\mathbb{Z})^2 = \langle A \rangle \times \langle B \rangle$ to simplify notation.



Let $l' := x_1$, $l'' := x_2$. Then $z' := [\tilde{l'}] = e_1 + e_2$, $z'' := [\tilde{l''}] = e_7 + e_8$ and $g' := \phi(l') = A$, $g'' := \phi(l'') = B$ and thus the primitives l' and l'' satisfy the conditions $\langle g' \rangle \cap \langle g'' \rangle = \{0\}$ and $\langle g' \rangle \neq \{0\}$.

A $\mathbb{Z}$-basis $\mathcal{W}$ for $H_1(Y;\mathbb{Z})$ written in terms of the e_i's is given by

$$\mathcal{W} := \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}^T = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{pmatrix}^T;$$

refer to Example 4.21.

A basis $\mathcal{V}$ for the decomposition of $H_1(Y;\mathbb{C})$ into the irreducible representations of G is given as in Example 3.2. We have

$$\mathcal{V} := \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{pmatrix}^T = \begin{pmatrix} v_{11} \\ v_{12} \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}^T,$$

and $\langle v_{11} \rangle_{\mathbb{C}} = \langle v_{12} \rangle_{\mathbb{C}} = V_1, \langle v_2 \rangle_{\mathbb{C}} = V_2, \langle v_3 \rangle_{\mathbb{C}} = V_3, \langle v_4 \rangle_{\mathbb{C}} = V_4$, where $V_1, \dots, V_4$ are the irreducible representations of G corresponding to the characters $\chi_1, \dots, \chi_4$ in the following character table.

	1	A	B	AB
χ_1	1	1	1	1
χ_2	1	1	−1	−1
χ_3	1	−1	1	−1
χ_4	1	−1	−1	1

With respect to the basis $\mathcal{V}$ we have $z' = (v_{11} + v_2)/2$ and $z'' = (v_{12} - v_3)/2$. We set

$$\hat{z} := \frac{1}{2}(v_{12} + v_2)$$

as in the proof of the preceding proposition. Then

$$\hat{z} = \frac{1}{2}(w_1 + w_2 - w_3 + w_4) = \frac{1}{2}(e_1 + e_2 + e_3 + e_4 - e_5 - e_6 + e_7 + e_8).$$

We define $z \coloneqq 2\hat{z}$. Then $z \in H_1(Y;\mathbb{Z})$ and it is indivisible; see Lemma 4.6.

Now it is easy to see that

$$\begin{aligned}
\mathrm{Span}_{\mathbb{C}[G]}\{z'\} &= \mathrm{Span}_{\mathbb{C}[G]}\{v_{11} + v_2\} = V_1 \oplus V_2 \cong \mathrm{Ind}_{\langle A\rangle}^{G}(\mathbb{C}_{\mathrm{triv}}),\\
\mathrm{Span}_{\mathbb{C}[G]}\{z''\} &= \mathrm{Span}_{\mathbb{C}[G]}\{v_{12} - v_3\} = V_1 \oplus V_3 \cong \mathrm{Ind}_{\langle B\rangle}^{G}(\mathbb{C}_{\mathrm{triv}}),\\
\mathrm{Span}_{\mathbb{C}[G]}\{z\} &= \mathrm{Span}_{\mathbb{C}[G]}\{v_{12} + v_2\} = V_1 \oplus V_2 \cong \mathrm{Ind}_{\langle A\rangle}^{G}(\mathbb{C}_{\mathrm{triv}}).
\end{aligned}$$

If z was the homology class of an elevation of some primitive $l \in F_n$, then we would need $g \coloneqq \phi(l) \in (\mathbb{Z}/2\mathbb{Z})^2$ to be such that $\langle g\rangle = \langle A\rangle$. On the other hand, by Lemma 4.26, we also know that $g = \tilde{\phi}(p_*(z)/|\langle g\rangle|)$. We compute

$$p_*(z) = p_*(e_1 + e_2 + e_3 + e_4 - e_5 - e_6 + e_7 + e_8) = 4[x_2].$$

Using the formula $g = \tilde{\phi}(p_*(z)/|\langle g\rangle|)$, we obtain

$$g = 2\,\tilde{\phi}([x_2]) = 2\,\phi(x_2) = B^2 = 0,$$

but this is a contradiction since $\langle g\rangle = \langle A\rangle$.

Alternatively, Remark 4.10 in Subsection 4.2 directly implies that z cannot be homology class of an elevation of some primitive. This is because the edges e_1, e_2, e_5 and e_6 appear in z once with positive and once with negative multiplicity.

We can ask if the conditions on l' and l'' as in the proof of Proposition 4.27 are necessary. It turns out that the statement holds in a weaker sense where we demand compatibility with the map ϕ. This is illustrated in the following example. We take the same situation as in Example 4.29, and additionally ask that $\langle g'\rangle = \langle g''\rangle$. We can show that there are cases where the homology class cannot be realised by an elevation of a primitive element.

Lemma 4.30. *Using the notation in Setup 4.14, there is a finite, regular, path-connected cover $p\colon Y \to X$ with group of deck transformations G such that there exists an indivisible element $z \in H_1(Y;\mathbb{Z})$ with the property that*

1. $\mathrm{Span}_{\mathbb{C}[G]}\{z\} \cong \mathrm{Ind}_{\langle g\rangle}^{G}(\mathbb{C}_{\mathrm{triv}})$ *for some* $1 \neq g \in G$ *with*
2. $g = \tilde{\phi}(p_*(z)/2)$, *but*
3. *z is not the homology class of an elevation of a primitive element in X.*

PROOF. Consider the mod 2-homology cover as in Examples 3.2, 4.21 and 4.29. Let $G = (\mathbb{Z}/2\mathbb{Z})^2$ and take $\mathcal{W}$ and $\mathcal{V}$ as in Example 4.29 as bases for $H_1(Y;\mathbb{Z})$ and the Gaschütz decomposition of $H_1(Y;\mathbb{C})$, respectively.

We start with the primitive element $l' = x_1$. By the same considerations as in Example 4.29, we see that $z' := [\tilde{l'}] = e_1 + e_2 = (v_{11} + v_2)/2$ and $\mathrm{Span}_{\mathbb{C}[G]}\{z'\} = \mathrm{Span}_{\mathbb{C}[G]}\{v_{11} + v_2\} = V_1 \oplus V_2 \cong \mathrm{Ind}_{\langle A\rangle}^G(\mathbb{C}_{\mathrm{triv}})$.

For a general element in $H_1(Y;\mathbb{Z})$, we have

$$\begin{aligned}\sum_{i=1}^{5} \beta_i w_i = {} & \frac{\beta_1 + \beta_3 + \beta_5}{2}\, v_{11} + \frac{\beta_2 + \beta_4 + \beta_5}{2}\, v_{12} \\ & + \frac{\beta_1 - \beta_3}{2}\, v_2 + \frac{\beta_2 - \beta_4}{2}\, v_3 + \frac{\beta_5}{2}\, v_4 \end{aligned} \tag{14}$$

with $\beta = (\beta_1, \dots, \beta_5) \in \mathbb{Z}^5$. We want to find $z \in H_1(Y;\mathbb{Z})$ with

$$\mathrm{Span}_{\mathbb{C}[G]}\{z\} = \mathrm{Span}_{\mathbb{C}[G]}\{z'\}.$$

Therefore, we need $\beta_1 \neq \beta_3$, $\beta_2 = \beta_4$, $\beta_5 = 0$ and either $\beta_1 + \beta_3 + \beta_5 \neq 0$ or $\beta_2 + \beta_4 + \beta_5 \neq 0$. Then Equation (14) simplifies to

$$z = \frac{\beta_1 + \beta_3}{2}\, v_{11} + \beta_2\, v_{12} + \frac{\beta_1 - \beta_3}{2}\, v_2$$

with $p_*(z) = 2(\beta_1 + \beta_3)x_1 + 4\beta_2 x_2$. Thus

$$\tilde{\phi}\left(\frac{1}{2}\, p_*(z)\right) = A^{\beta_1+\beta_3} B^{2\beta_2} = A^{\beta_1+\beta_3}.$$

We need $\beta \in \mathbb{Z}^5$ such that $\gcd(\beta_1, \dots, \beta_5) = 1$ and $\beta_1 + \beta_3$ odd.

Set $\beta := (2, 0, -1, 0, 0)^T$. Then $\tilde{\phi}(p_*(z)/2) = A$ and

$$z = 2(e_1 + e_2) - (e_5 + e_6).$$

Note that all four edges appearing in z are preimages of x_1. By the results of Osborne and Zieschang, see Remark 4.10, z cannot be the homology class of an elevation of a primitive, because this primitive element would need to include the letter x_1 with positive as well as negative powers. $\square$

5 Primitive Commutator Homology

In this section we aim to develop an analogous theory for a different subset of elements in the free group. Namely, we are interested in the subset of commutators of primitive elements. It is clear that this subset is preserved by automorphisms of the free group. This is one of the reasons why we are interested in these specific elements. In the following we will set up the precise notation.

Definition 5.1. Let $n \geq 2$ and let F_n be the free group. A *primitive commutator in* F_n is a commutator of the form $[w, w']$ for w, w' primitive elements that can be extended to a basis of F_n.

Remark 5.2. If we fix a basis of F_n, the normal subgroup generated by all primitive commutators is the commutator subgroup $[F_n, F_n]$ of the free group; see the proof of Proposition 3.7, Equation (5), and Lemma 3.8.

We adapt the setup of Subsection 4.4, see Setup 4.14, so let X be a wedge of n circles, $n \geq 2$, and identify F_n with its fundamental group based at the single vertex x_0 of X. For G a finite group, a surjective group homomorphism $\phi\colon F_n \to G$ defines a finite, regular, path-connected cover $p\colon Y \to X$ with base point $y_0 \in p^{-1}(x_0)$ and $p_*(\pi_1(Y, y_0)) = \ker(\phi)$.

Definition 5.3. For S the set of primitive commutators, we define

$$H_1^{\mathrm{comm}}(Y; \mathbb{C}) := H_1^S(Y; \mathbb{C}) \leq H_1(Y; \mathbb{C}).$$

as in Definition 4.18. This subrepresentation is called the *primitive commutator homology.* We also set $\mathrm{Irr}^{\mathrm{comm}}(Y; \mathbb{C}) := \mathrm{Irr}^S(Y; \mathbb{C})$.

We have an analogue obstruction as in the primitive case; refer to Theorem 4.23.

Theorem 5.4. *In the situation of Setup 4.14, we have*

$$H_1^{\mathrm{comm}}(Y; \mathbb{C}) \leq \mathbb{C}_{\mathrm{triv}} \oplus \bigoplus_{V_i \in \mathrm{Irr}^{\mathrm{comm}}(\phi, G)} V_i^{(n-1)\dim(V_i)}.$$

The main ingredient to prove this theorem is the following proposition, an analogue of Proposition 4.24.

Proposition 5.5. *Let $x_1, \ldots, x_n$ be a basis of F_n, $n \geq 2$ and set*

$$x_{ij} := [x_i, x_j],$$

i.e. the primitive commutator of x_i and x_j for fixed $1 \leq i, j \leq n$. Let

$$K := \langle \phi(x_{ij}) \rangle \leq \langle \phi(x_i), \phi(x_j) \rangle := H \leq G.$$

Then we have the following isomorphism

$$\mathrm{Span}_{\mathbb{C}[G]}\{[\widetilde{x_{ij}}]\} \cong \mathrm{Ind}_K^G(\mathbb{C}_{\mathrm{triv}})/\mathrm{Ind}_H^G(\mathbb{C}_{\mathrm{triv}})$$

of G-representations.

Using this proposition we can prove Theorem 5.4, adapting the proof of Theorem 4.23.

Proof [Theorem 5.4]. For $x = [l, l']$ a primitive commutator in F_n, set $K := \langle \phi(x) \rangle \leq \langle \phi(l), \phi(l') \rangle := H \leq G$. Let V be an irreducible G-representation. Then Proposition 5.5 and Frobenius reciprocity; refer to Theorem 2.27, imply that

$$\begin{aligned}\langle \mathrm{Span}_{\mathbb{C}[G]}\{[\tilde{x}]\}, V\rangle_G &\leq \langle \mathrm{Ind}_K^G(\mathbb{C}_{\mathrm{triv}}), V\rangle_G \\ &= \langle \mathbb{C}_{\mathrm{triv}}, \mathrm{Res}_K^G(V)\rangle_K \\ &= \dim(V^K),\end{aligned}$$

where the last equality follows from the orthogonality relations.

By Definitions 4.18 and 5.3, an irreducible representation V appears in $H_1^{\mathrm{comm}}(Y;\mathbb{C})$ if and only if there exists a primitive commutator $x = [l, l']$ in F_n such that

$$\langle \mathrm{Span}_{\mathbb{C}[G]}\{[\tilde{x}]\}, V\rangle_G \neq 0.$$

By the above, this implies $\dim(V^K) \neq 0$, which is equivalent to the existence of some nonzero $v \in V$ with $\phi(x)(v) = v$. Thus, for V to appear in $\mathrm{Span}_{\mathbb{C}[G]}\{[\tilde{x}]\}$, we necessarily need $V \in \mathrm{Irr}^{\mathrm{comm}}(\phi, G)$ which proves the theorem. □

The proof of Proposition 5.5 consists of two parts. First, we reduce to the case $n = 2$ and secondly, we use surface topology to prove the claim in this special case. Let us have a look at the special case.

Remark 5.6. For $n = 2$, we have $H = G$ and the above proposition simplifies to

$$\mathrm{Span}_{\mathbb{C}[G]}\{[\widetilde{x_{12}}]\} \cong \mathrm{Ind}_K^G(\mathbb{C}_{\mathrm{triv}})/\mathbb{C}_{\mathrm{triv}}.$$

We will proceed by giving a proof of Proposition 5.5.

Proof [Proposition 5.5]. We first want to reduce to the case $n = 2$. Denote by Y_1 the subgraph of Y that corresponds to the Cayley graph of H with generating set $\{\phi(x_i), \phi(x_j)\}$ that contains the base point y_0. Now choose representatives for the cosets of H in G, say $g_1 = 1, \ldots, g_{[G:H]}$. Set $Y_i := g_i Y_1$ for $1 \leq i \leq [G : H]$. These are pairwise disjoint subgraphs of Y, since the vertices of Y correspond to the group elements, and every group element is contained in exactly one Y_i. By [FH17, Claim 2.4], we know that

$$I \colon \bigoplus_{i=1}^{[G:H]} H_1(Y_i;\mathbb{C}) \hookrightarrow H_1(Y;\mathbb{C})$$

is injective, where the map I is induced by the inclusions of the graphs into Y. Note that I is a map of G-representations, since the action of G on the direct sum is given by permutation of the summands according to the permutation of the subgraphs Y_i. Now set $v := [\widetilde{x_{ij}}]$. By definition of Y_1, we obtain that $v \in H_1(Y_1;\mathbb{C})$ and for every $h \in H$, we have $hv \in H_1(Y_1;\mathbb{C})$. For $g \in G$ write $g = g_j h$ for some $1 \leq j \leq [G:H]$ and $h \in H$. Thus $gv = g_j hv \in H_1(g_j Y_1;\mathbb{C}) = H_1(Y_j;\mathbb{C})$.

We know that

$$\operatorname{Span}_{\mathbb{C}[G]}\{v\} = \sum_{i=1}^{[G:H]} g_i \operatorname{Span}_{\mathbb{C}[H]}\{v\}$$

and we want to show that this sum is direct. Let therefore $1 \leq j \leq [G:H]$ and

$$v^* \in g_j \operatorname{Span}_{\mathbb{C}[H]}\{v\} \cap \sum_{i \neq j} g_i \operatorname{Span}_{\mathbb{C}[H]}\{v\}.$$

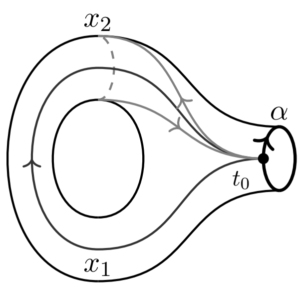
This implies $v^* \in H_1(Y_j;\mathbb{C}) \cap \sum_{i\neq j} H_1(Y_i;\mathbb{C})$ by the above considerations. But then it follows immediately that $v^* = 0$, since I is injective. Thus we obtain

$$\operatorname{Span}_{\mathbb{C}[G]}\{v\} = \bigoplus_{i=1}^{[G:H]} g_i \operatorname{Span}_{\mathbb{C}[H]}\{v\}.$$

We will show that $\operatorname{Span}_{\mathbb{C}[H]}\{v\} \cong \operatorname{Ind}_K^H(\mathbb{C}_{\mathrm{triv}})/\mathbb{C}_{\mathrm{triv}}$. By the exactness of induction, refer to Proposition 2.25, it then follows that

$$\operatorname{Span}_{\mathbb{C}[G]}\{v\} \cong \operatorname{Ind}_H^G(\mathbb{C}_{\mathrm{triv}})/\operatorname{Ind}_K^H(\mathbb{C}_{\mathrm{triv}}).$$

We can without loss of generality assume that $H = G$ and $n = 2$. To prove the proposition in this case, we need some surface topology. We want to show that $\operatorname{Span}_{\mathbb{C}[G]}\{[\widetilde{x_{12}}]\} \cong \operatorname{Ind}_K^G(\mathbb{C}_{\mathrm{triv}})/\mathbb{C}_{\mathrm{triv}}$; see Remark 5.6. We identify F_2 with the fundamental group of the torus with one boundary component T and base point $t_0 \in \partial T$, with generators x_1 and x_2 the simple closed curves as in the image below.



Then $x := x_{12} = [x_1, x_2]$ is represented by a null-homologous simple closed curve α on T that starts and ends at t_0 with the property that $\alpha = \partial T$. We have a group homomorphism $\phi\colon \pi_1(T) \cong F_2 \to G$ and we can associate to $\ker(\phi)$ a finite, regular, path-connected cover $q\colon S \to T$ with base point $s_0 \in q^{-1}(t_0)$ and $q_*(\pi_1(S, s_0)) = \ker(\phi)$. Note that S is again an orientable surface with $\partial S = q^{-1}(\partial T) = q^{-1}(\alpha) = \alpha_1 \cup \ldots \cup \alpha_k$. The surface S is homotopy-equivalent to Y. Set $A \coloneqq \alpha_1 \cup \ldots \cup \alpha_k$. We want to show that we have the following short exact sequence of G-representations

$$0 \to \mathbb{C}[\partial S] \xrightarrow{\psi} \bigoplus_{i=1}^{k} \mathbb{C}[\alpha_i] \xrightarrow{\varphi} \mathrm{Span}_{H_1(S;\mathbb{C})}\{[\alpha_1], \ldots, [\alpha_k]\} \to 0, \tag{15}$$

where the map ψ is induced by the natural identification of the boundary of S with the multicurve A, and φ sends a curve α_i to its homology class. It is clear, that φ is surjective and G-equivariant, since G acts by permutation on the α_i. The map ψ is injective, since for $c \in \mathbb{C}$, $\psi(c\,\partial S) = c\sum_{i=1}^{k} \alpha_i = 0$ if and only if $c = 0$. Let $g \in G$, then

$$g\,\psi(\partial S) = g\sum_{i=1}^{k} \alpha_i = \sum_{i=1}^{k} \alpha_i = \psi(\partial S) = \psi(g\,\partial S),$$

because the action of G is orientation-preserving and thus trivial on ∂S. It is left to show that $\mathrm{im}(\psi) = \ker(\varphi)$.

By definition of homology, the boundary of a compact subsurface is null-homologous, thus $\varphi(\psi(\partial S)) = 0$, which shows $\mathrm{im}(\psi) \subseteq \ker(\varphi)$.

For the other inclusion, we show that $\{[\alpha_2], \ldots, [\alpha_k]\}$ is linearly independent in $H_1(S; \mathbb{C})$. Then it follows immediately that $\ker(\varphi) \subseteq \mathrm{im}(\psi)$, since $\sum_{i=1}^{k} [\alpha_i] = 0$. Let now $a_i \in \mathbb{C}$ with $\sum_{i=2}^{k} a_i[\alpha_i] = 0$. Since S is path-connected, for all $1 \leq j \leq n$ we can find an arc c_j from $\partial S \cap \alpha_1$ to $\partial S \cap \alpha_j$ that intersects ∂S only in the two end points. Then the algebraic intersection number i between α_m and c_j is

$$i(\alpha_m, c_j) = \pm\delta_{mj}.$$

We know that there exists a non-degenerate bilinear form

$$\langle\ ,\ \rangle\colon H_1(S;\mathbb{Z}) \times H_1(S, \partial S; \mathbb{Z}) \to \mathbb{Z}$$

that evaluated on homology classes of a simple closed curve and an arc from ∂S to ∂S counts intersections with sign; see [FM12, Chapter 6.1.2]. For more technical details refer to [Bre93, Chapter VI.11]. Thus, for all $1 \leq j \leq k$ we have

$$0 = \langle \sum_{i=2}^{k} a_i[\alpha_i], [c_j] \rangle = \sum_{i=2}^{k} a_i \underbrace{\langle [\alpha_i], [c_j] \rangle}_{\pm\delta_{ij}} = \pm a_j.$$

Thus we obtain $a_j = 0$ for $2 \leq j \leq k$, which implies linear independence and ultimately the exactness of the sequence in Equation (15).

We are interested in understanding $\mathrm{Span}_{H_1(S;\mathbb{C})}\{[\alpha_1], \dots, [\alpha_k]\}$. Using the short exact sequence from Equation (15), it is enough to understand $\mathbb{C}[\partial S]$ and $\bigoplus_{i=1}^k \mathbb{C}[\alpha_i]$ as G-representations.

Say α_1 contains the base point $s_0 \in S$. Then

$$\mathrm{Stab}_G(\alpha_1) = \langle \phi(x) \rangle = K \leq G,$$

since α is simple and thus runs through t_0 only once. The curves $\alpha_1, \dots, \alpha_k$ are the elevations of α and G permutes these. Thus we can identify the curves $\alpha_2, \dots, \alpha_k$ with the cosets of K in G, because K is the stabilizer of α_1. Choose representatives $g_1 = 1, g_2, \dots, g_k \in G$ of the cosets of K in G. Then

$$\bigoplus_{i=1}^{k} \mathbb{C}[\alpha_i] = \bigoplus_{i=1}^{k} \mathbb{C} g_i[\alpha_1] \cong \mathrm{Ind}_K^G(\mathbb{C}_{\mathrm{triv}})$$

as G-representations by the defining property of the induced representation. Thus

$$\mathrm{Span}_{H_1(S;\mathbb{C})}\{[\alpha_1], \dots, [\alpha_k]\} = \mathrm{Span}_{\mathbb{C}[G]}\{[\alpha_1]\} \cong \mathrm{Ind}_K^G(\mathbb{C}_{\mathrm{triv}})/\mathbb{C}_{\mathrm{triv}},$$

which proves the proposition. □

Remark 5.7. It is clear, that $H_1^{\mathrm{comm}}(Y;\mathbb{C}) \neq H_1(Y;\mathbb{C})$. Namely, each primitive commutator lies in the kernel of the induced map p_* on homology, since the homology groups are the abelianizations of the respective fundamental groups; refer to Theorem 2.4. We already know that $H_1(Y;\mathbb{C}) = M(V_1) \oplus \ker(p_*)$ as G-representations by Lemma 3.4, 4. One natural question to ask is whether $H_1^{\mathrm{comm}}(Y;\mathbb{C}) = \ker(p_*)$ in general.

If $n = 2$ and G is abelian, we have $\phi(x_{12}) = 1_G$ and we obtain

$$\mathrm{Span}_{\mathbb{C}[G]}\{[\widetilde{x_{12}}]\} \cong \mathrm{Ind}_{\{1_G\}}^G(\mathbb{C}_{\mathrm{triv}})/\mathbb{C}_{\mathrm{triv}} \cong \mathbb{C}[G]/\mathbb{C}_{\mathrm{triv}} \cong \bigoplus_{i=2}^{|G|} M(V_i)$$

for $M = H_1(Y;\mathbb{C})$. This implies that $\mathrm{Span}_{\mathbb{C}[G]}\{[\widetilde{x_{12}}]\} \cong \ker(p_*)$ and therefore, $H_1^{\mathrm{comm}}(Y;\mathbb{C}) = \ker(p_*)$.

A more general answer to this question is presented in the following.

5.1 Iterated Covers

To see that there are examples where $H_1^{\mathrm{comm}}(Y;\mathbb{C}) \neq \ker(p_*)$ for certain covers $p\colon Y \to X$, we use the idea of iterated covers and the theorem of Malestein-Putman; see Theorem 4.19. We start with an important property of finite, regular covers. Namely, every such cover is itself covered by a finite, characteristic cover, which means that the fundamental group of the largest covering space is characteristic in the fundamental group of the original space. A definition is given in the following.

Definition 5.8. Let G be a group. A subgroup $H \leq G$ is *characteristic* if it is invariant under every automorphism of G, i.e. for all $\varphi \in \mathrm{Aut}(G)$ we have $\varphi(H) \subseteq H$.

Note that characteristic subgroups are normal.

Lemma 5.9. *Let G be a finitely generated group and $H \trianglelefteq G$ a normal subgroup of finite index k. Then there is a subgroup $H^* \leq H$ which is of finite index in G and characteristic in G.*

Proof. Set

$$H^* := \bigcap_{\substack{H' \trianglelefteq G \text{ normal} \\ [G:H']=k}} H'.$$

We will show that H^* is characteristic in G and of finite index.

Let $\varphi \in \mathrm{Aut}(G)$. For $H' \trianglelefteq G$ with $[G : H'] = k$, we know that $\varphi(H')$ is normal in G and $[G : \varphi(H')] = k$. This implies that

$$\varphi(H^*) = \bigcap_{\substack{H' \trianglelefteq G \text{ normal} \\ [G:H']=k}} \varphi(H') \leq \bigcap_{\substack{H' \trianglelefteq G \text{ normal} \\ [G:H']=k}} H' = H^*,$$

thus H^* is characteristic in G.

In order to show that H^* is of finite index in G, we want to bound the number of all normal subgroups in G of index k. If we can show that this number is finite, it is enough to verify that the intersection of two finite-index, normal subgroups is again a finite-index, normal subgroup. Then we can do an induction on the number of such subgroups. It is clear that the intersection of two normal subgroups is normal. Let H_1, H_2 be two normal subgroups of finite index in G. Then we know by the first isomorphism theorem for groups that $H_1/(H_1 \cap H_2) \cong H_1H_2/H_2$; refer to [Bos09, Chapter 1.3, Satz 8]. Thus $[H_1 : H_1 \cap H_2] \leq [G : H_2]$. Since the index is multiplicative, we obtain $[G : H_1 \cap H_2] = [G : H_1] \cdot [H_1 : H_1 \cap H_2] \leq [G : H_1] \cdot [G : H_2]$, and we are done.

Now let K be a finite group of order k. We claim that

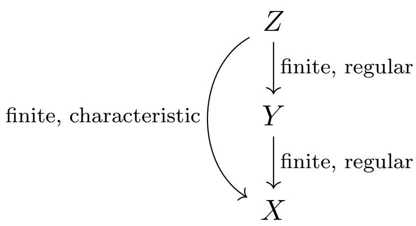
$$|\{H' \trianglelefteq G, G/H' \cong K\}| < \infty.$$

Let $H' \trianglelefteq G$ normal with $G/H' \cong K$, i.e. there exists an isomorphism $\psi_{H'} \colon G/H' \to K$. Every such isomorphism defines a surjective group homomorphism $f_{H'} \colon G \to G/H' \to K$ with $\ker(f_{H'}) = H'$. Note that $\ker(f_{H'})$ is independent of the choice of the isomorphism $\psi_{H'}$. Since G is finitely generated and K is finite, there are only finitely many surjective group homomorphism $G \to K$. The claim now follows, since every normal subgroup of G with quotient isomorphic to K is the kernel of a surjective group homomorphism $G \to K$ of which there are only finitely many.

We additionally know, that there are up to isomorphism only finitely many groups of order k. Altogether, this proves the lemma. $\square$

The following is a direct application of the above lemma.

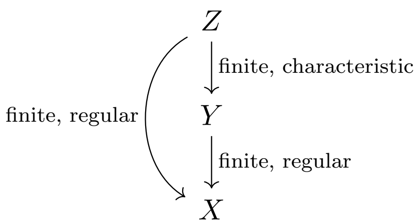
Corollary 5.10. *Let the situation be as in Setup 4.14. Then there exists a cover* $q\colon Z \to Y$ *such that* $p \circ q\colon Z \to X$ *is a finite, characteristic cover, i.e.* $\pi_1(Z)$ *is a characteristic subgroup of* $\pi_1(X)$.



Proof. We can apply the above lemma, since F_n is finitely generated, and $\ker(\phi) \trianglelefteq F_n$ is a finite-index, normal subgroup. $\square$

Given a group G and a sequence of subgroups of the form K char $H \trianglelefteq G$ with K characteristic in H and H normal in G, then K is also normal in G. This is because conjugation by an element in G induces an automorphism of H, since H is normal. Now K is characteristic in H, so it is invariant by this automorphism.

Corollary 5.11. *In the situation of Setup 4.14, let* $q\colon Z \to Y$ *be a finite, characteristic cover, i.e.* $\pi_1(Z)$ *is a characteristic subgroup of* $\pi_1(Y)$. *Then* $p \circ q\colon Z \to X$ *is a finite, regular cover.*



Proof. This follows immediately from the above consideration for the sequence $\pi_1(Z)$ char $\pi_1(Y) \trianglelefteq \pi_1(X)$. We obtain that $\pi_1(Z) \trianglelefteq \pi_1(X)$. Furthermore, it is of finite index since

$$[\pi_1(X) : \pi_1(Z)] = [\pi_1(X) : \pi_1(Y)] \cdot [\pi_1(Y) : \pi_1(Z)] < \infty.$$

$\square$

Let us look at an example of a finite, characteristic cover of the wedge of n circles.

Lemma 5.12. *For all $n, m \geq 2$ the* $\bmod m$*-homology cover is a characteristic cover of the wedge of n circles.*

Proof. Recall that the $\bmod m$-homology cover is given by the surjective group homomorphism

$$\phi\colon F_n \xrightarrow{\overline{\ \ }} F_n/[F_n, F_n] \cong \mathbb{Z}^n \xrightarrow{\bmod m} (\mathbb{Z}/m\mathbb{Z})^n.$$

The cover associated to $\ker(\phi)$ is by definition a finite cover. It suffices thus to show that $\ker(\phi)$ is characteristic in F_n.

Take $\alpha \in \mathrm{Aut}(F_n)$. We need to show that $\alpha(\ker(\phi)) \subseteq \ker(\phi)$. Notice that

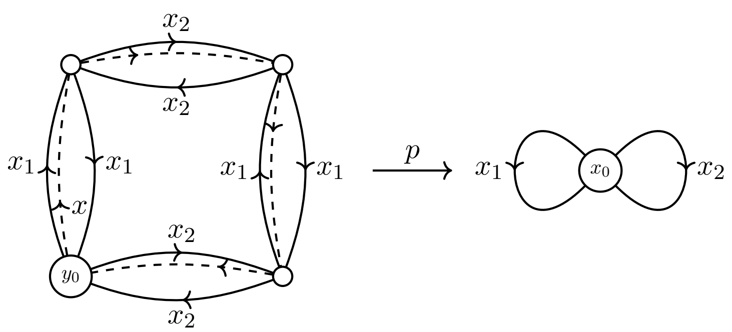
$$\ker(\phi) = \langle [x, y], z^m \mid x, y, z \in F_n \rangle =: K.$$

But now $\alpha([x, y]) = [\alpha(x), \alpha(y)] \in K$ and $\alpha(z^m) = \alpha(z)^m \in K$ for all $x, y, z \in F_n$, which proves the lemma. □

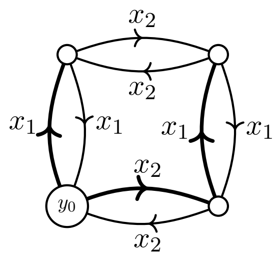
In order to prove that $H_1^{\mathrm{comm}} \neq \ker(p_*)$ in general, we want to reduce the case of primitive commutator homology to the case of primitive homology. In the latter case, we have a negative result by Malestein and Putman; see Theorem 4.19. We will look at a particular cover of the wedge of two circles that has the property that all primitive commutators lift to primitive elements.

Lemma 5.13. *Let $\phi\colon F_2 \xrightarrow{\overline{\ \ }} F_2/[F_2, F_2] \cong \mathbb{Z}^2 \xrightarrow{\bmod 2} (\mathbb{Z}/2\mathbb{Z})^2$ be the* $\bmod 2$*-homology cover of the wedge of two circles X as in Examples 3.2, 4.21 and 4.29. We denote by $p\colon Y \to X$ the associated finite, characteristic cover. Then for $x := [l, l']$ any primitive commutator in F_2 with $l \neq l'$, its preferred elevation $\widetilde{x} \in \pi_1(Y)$ is primitive. Note that this implies that every elevation of x is primitive.*

Proof. We will show the claim for the basis primitive commutator first. Let $x := [x_1, x_2]$. Then $\phi(x) = 0$ and thus x lifts to a closed curve on Y.



Now we just need to extend x to a free basis of $\pi_1(Y)$. For example $\{x, x_1^2, x_2^2, x_2x_1^2x_2^{-1}, x_2x_1x_2x_1^{-1}\}$ is a free basis of $\pi_1(Y)$. Namely, consider the spanning tree illustrated as the bold lines in the picture below.



Then the construction as in Proposition 4.13 gives rise to the above free basis.

Let l be some primitive commutator. By Proposition 2.6, it lies in the same $\mathrm{Aut}(F_2)$-orbit as x. Thus the Relabeling lemma, see Lemma 4.16, 1., shows that $\tilde{l}$ is primitive. $\square$

We additionally need to understand how primitive homology behaves in iterated covers. Therefore, we need the following results.

Lemma 5.14. *Let G be a group and $K \trianglelefteq H \trianglelefteq G$ normal subgroups of finite index with K normal in G. Take g in G, and let r minimal such that $g^r \in H$, s minimal such that $(g^r)^s \in K$ and t minimal such that $g^t \in K$. Then $t = rs$.*

Proof. It is clear, that $t \leq rs$ since $g^{rs} = (g^r)^s \in K$ and t minimal with this property. To see that $t \geq rs$, consider $g^t \in K \leq H$. This implies that $r \mid t$, since r is the order of g in G/H. Thus $t = rs'$ for some $s' \in \mathbb{N}$. It suffices to show that $s \leq s'$. We have $(g^r)^{s'} = g^t \in K$, and s chosen minimal with the property that $(g^r)^s \in K$, so $s \leq s'$. $\square$

Corollary 5.15. *In the situation of Setup 4.14, take $q\colon Z \to Y$ a finite, regular cover of Y such that $p \circ q\colon Z \to X$ is a finite, regular cover with base point $z_0 \in q^{-1}(y_0)$. Then for l any loop on X, we have*

$$\tilde{l}^{Z\to X} = \widetilde{(\tilde{l}^{Y\to X})}^{Z\to Y}.$$

Proof. Take r minimal such that $l^r \in \pi_1(Y, y_0)$, s minimal such that $(l^r)^s \in \pi_1(Z, z_0)$ and t minimal such that $l^t \in \pi_1(Z, z_0)$. By Lemma 5.14, we know that $t = sr$. The uniqueness of lifts and the choice of base points immediately imply that

$$\tilde{l}^{Z\to X} = \widetilde{(\tilde{l}^{Y\to X})}^{Z\to Y}.$$

$\square$

Lemma 5.16. *In the situation of Setup 4.14, take* $q\colon Z \to Y$ *a finite, regular cover of* Y *such that* $p \circ q\colon Z \to X$ *is a finite, regular cover with base point* $z_0 \in q^{-1}(y_0)$. *Then*

$$q_*(H_1^{\mathrm{prim}}(Z \to X;\mathbb{C})) \subseteq H_1^{\mathrm{prim}}(Y \to X;\mathbb{C}),$$

in other words, primitive homology gets mapped into primitive homology.

Proof. For l a primitive element in $\pi_1(X, x_0)$, Corollary 5.15 implies that

$$q_*\left(\left[\tilde{l}^{Z\to X}\right]\right) = q_*\left(\left[\widetilde{(\tilde{l}^{Y\to X})}^{Z\to Y}\right]\right) = s\left[\tilde{l}^{Y\to X}\right] \in H_1^{\mathrm{prim}}(Y \to X;\mathbb{C}),$$

where s is as in the proof of Corollary 5.15. This proves the claim. □

The following is an immediate corollary of the above lemma.

Corollary 5.17. *Let the setup be as in Lemma 5.16 with the condition that* $H_1^{\mathrm{prim}}(Y \to X;\mathbb{C}) \neq H_1(Y;\mathbb{C})$ *as* K*-representations with* $K := \mathrm{Deck}(Y, p)$. *Then*

$$H_1^{\mathrm{prim}}(Z \to X;\mathbb{C}) \neq H_1(Z;\mathbb{C})$$

as G*-representations with* $G := \mathrm{Deck}(G, p \circ q)$.

Proof. Assume $H_1^{\mathrm{prim}}(Z \to X;\mathbb{C}) = H_1(Z;\mathbb{C})$ as G-representations. Consider the map $q_*\colon H_1(Z;\mathbb{C}) \to H_1(Y;\mathbb{C})$. By Lemma 3.4, 3., this map is surjective. By assumption, we have

$$q_*(H_1^{\mathrm{prim}}(Z \to X;\mathbb{C})) = H_1(Y;\mathbb{C}).$$

By Lemma 5.16, we know that $q_*(H_1^{\mathrm{prim}}(Z \to X;\mathbb{C})) \subseteq H_1^{\mathrm{prim}}(Y \to X;\mathbb{C})$. But this contradicts the fact that $H_1^{\mathrm{prim}}(Y \to X;\mathbb{C}) \neq H_1(Y;\mathbb{C})$. □

Remark 5.18. The last corollary together with Corollary 5.10 imply that we can assume without loss of generality that the cover constructed by Malestein and Putman in Theorem 4.19 is characteristic.

Now we want to bring everything together in the following proposition.

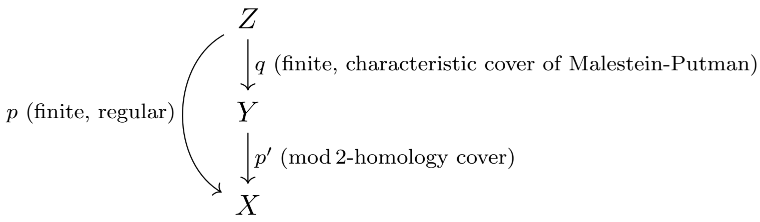
Proposition 5.19. *For* $n = 2$ *there exists a finite, regular cover* $p\colon Z \to X$ *of* X *such that* $H_1^{\mathrm{comm}}(Z;\mathbb{C}) \neq \ker(p_*)$.

Proof. Let $p'\colon Y \to X$ be the mod 2-homology cover with group of deck transformations $K = (\mathbb{Z}/2\mathbb{Z})^2$. By Lemma 5.12, we know that this cover is characteristic. Lemma 5.13 tells us that all primitive commutators lift to primitive elements in Y.

The space Y is again a graph with free fundamental group of rank five, so we can apply the result of Malestein and Putman, see Theorem 4.19, to find a finite, regular cover $q\colon Z \to Y$ with group of deck transformations H such that

$$H_1^{\text{prim}}(Z \to Y; \mathbb{C}) \neq H_1(Z; \mathbb{C})$$

as H-representations. By Remark 5.18, we can without loss of generality assume that this cover is characteristic. Therefore, $p \coloneqq p' \circ q\colon Z \to X$ is regular by Corollary 5.11. Denote its group of deck transformations by G. We want to show that $H_1^{\text{comm}}(Z \to X; \mathbb{C}) \neq \ker(p_*)$.



Let us assume $H_1^{\text{comm}}(Z \to X; \mathbb{C}) = \ker(p_*)$. Denote by $V_1 = \mathbb{C}_{\text{triv}}, \dots, V_{k(G)}$ representatives of the isomorphism classes of the irreducible representations of G. Then

$$H_1^{\text{comm}}(Z \to X; \mathbb{C}) = \bigoplus_{i=2}^{k(G)} M(V_i)$$

with $M = H_1(Z; \mathbb{C})$ as G-representation; refer to Lemma 3.4, 4. We additionally know that

$$M(V_1) = p_\#(H_1(X; \mathbb{C})) = p_\#\left(\text{Span}_{\mathbb{C}}\{[l], l \in \pi_1(X) \text{ primitive}\}\right)$$

by Lemma 3.4, 2. By definition, we have $p_\#([l]) = \sum_{g \in G} g\,[\tilde{l}]$. For l a primitive element, it follows that $p_\#([l]) \in H_1^{\text{prim}}(Z; \mathbb{C})$. Thus we obtain $M(V_1) \leq H_1^{\text{prim}}(Z \to X; \mathbb{C})$. This implies that

$$H_1(Z; \mathbb{C}) = M(V_1) \oplus \bigoplus_{i=2}^{k(G)} M(V_i) = H_1^{\text{prim}}(Z; \mathbb{C}) + H_1^{\text{comm}}(Z; \mathbb{C}). \tag{16}$$

But on the other hand, we know by Lemma 5.13 and Corollary 4.17 that primitive elements and primitive commutators in X both lift to primitive elements in Y. Define

$$S \coloneqq \{\widetilde{x} \mid x \in \pi_1(X) \text{ primitive or a primitive commutator}\}.$$

Then $S \subseteq \{l \in \pi_1(Y) \text{ primitive}\}$ and by Equation (16)

$$\begin{aligned} H_1(Z; \mathbb{C}) &= H_1^S(Z \to Y; \mathbb{C}) \\ &\leq H_1^{\text{prim}}(Z \to Y; \mathbb{C}) \end{aligned}$$

But this is a contradiction to the choice of cover $q\colon Z \to Y$. $\square$

We have seen that we can use the results of primitive homology to construct counterexamples for primitive commutator homology. One can now ask whether the questions "$H_1^{\text{prim}} \neq H_1$?" and "$H_1^{\text{comm}} \neq \ker(p_*)$?" are equivalent. From the preceding proposition we can see that given a cover with $H_1^{\text{prim}} \neq H_1$, we can construct covers with $H_1^{\text{comm}} \neq \ker(p_*)$.

We now want to find an analogous result to Proposition 4.27 for primitive commutator homology. This means that we want to show that the condition in Propositon 5.5 is not sufficient to characterize homology classes of elevations of primitive commutators.

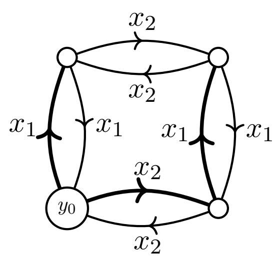
Proposition 5.20. *For $n = 2$ there exists a finite, regular cover $p\colon Z \to X$ with group of deck transformations G and $z \in H_1(Y;\mathbb{Z})$ such that*

1. *$z \in \ker(p_*)$,*
2. *$\mathrm{Span}_{\mathbb{C}[G]}\{z\} \cong \mathrm{Ind}_{\langle g\rangle}^{G}(\mathbb{C}_{\text{triv}})/\mathbb{C}_{\text{triv}}$ for some $g \in G$, but*
3. *z is not the homology class of an elevation of a primitive commutator in X.*

Proof. Let $\pi_1(X) = F_2$ be the free group on the generators x_1 and x_2. Consider the primitive commutators

$$x' := [x_1, x_2],\ x'' := [x_2^{-1}, x_1].$$

We start by considering the mod 2-homology cover $p'\colon Y \to X$ defined by $F_2 \to K = (\mathbb{Z}/2\mathbb{Z})^2 = \langle A\rangle \times \langle B\rangle,\ x_1 \mapsto A, x_2 \mapsto B$. Then $l' := \widetilde{x'}$ and $l'' := \widetilde{x''}$ are primitive in $\pi_1(Y)$ by Lemma 5.13. We additionally have that l' and l'' lie in one basis. For example, the elements x_1^2, x_2^2 and $x_2x_1^2x_2^{-1}$ complete l' and l'' to a basis of $\pi_1(Y, y_0) \cong F_5$. Namely, consider the spanning tree illustrated as the bold lines in the picture below. Then the construction as in Proposition 4.13 gives rise to the elements x_1^2, x_2^2, $x_2x_1^2x_2^{-1}$ and $x_2x_1x_2x_1^{-1}$. The last element multiplied with x_2^{-2} gives x'' and thus the set $\{x', x'', x_1^2, x_2^2, x_2x_1^2x_2^{-1}\}$ forms a basis.



We now want to use the same trick as in the result for primitive elements; refer to the proof of Proposition 4.27. We put a second mod 2-homology cover over Y, so $q\colon Z \to Y$ is defined by the surjection

$$\phi\colon F_5 \to H = (\mathbb{Z}/2\mathbb{Z})^5 = \langle C\rangle \times \langle D\rangle \times \langle E\rangle \times \langle F\rangle \times \langle G\rangle,\ l' \mapsto C,\ l'' \mapsto D$$

and extension. Then $p \coloneqq p' \circ q\colon Z \to Y \to X$ is a finite, regular cover because q is characteristic; see Corollary 5.11. We have

$$\widetilde{x'}^{Z\to X} = \left(\widetilde{\widetilde{x'}^{Y\to X}}\right)^{Z\to Y} = \widetilde{l'}^{Z\to Y}$$

by Corollary 5.15, and analogously for the elevation of x''. Denote by G the group of deck transformations of the cover $p\colon Z \to X$. We have the following exact sequence of groups

$$1 \to H \to G \to K \to 1.$$

Since $0 = [x'] = [x''] \in H_1(X;\mathbb{Z})$, it follows that

$$\left[\widetilde{x'}^{Z\to X}\right], \left[\widetilde{x''}^{Z\to X}\right] \in \ker(p_*).$$

We already know that $\ker(p_*) \cong \bigoplus_{i=2}^{k(G)} M(V_i)$ with $V_1 = \mathbb{C}_{\text{triv}}, V_2, \ldots, V_{k(G)}$ representatives of the isomorphism classes of the irreducible representations of G. We write

$$z' \coloneqq \left[\widetilde{l'}^{Z\to Y}\right] = z'_1 + \ldots + z'_{k(H)}$$

and

$$z'' \coloneqq \left[\widetilde{l''}^{Z\to Y}\right] = z''_1 + \ldots + z''_{k(H)},$$

with $z'_i, z''_i \in M(W_i)$, where $W_1 = \mathbb{C}_{\text{triv}}, W_2, \ldots, W_{k(H)}$ are representatives of the isomorphism classes of the irreducible representations of H.

Define $\hat{z} \coloneqq z' - (z'_1 - z''_1) = z''_1 + z'_2 + \ldots z'_{k(H)}$. Then $\hat{z}$ still lies in the kernel of p_* since $z'_1 - z''_1$ does. Namely, define

$$e_G \coloneqq \frac{1}{|G|}\sum_{g\in G} g,\ e_H \coloneqq \frac{1}{|H|}\sum_{h\in H} h.$$

Then e_G is a central idempotent in $\mathbb{C}[G]$ and $M(V_1) = e_G(M)$. Thus $(1 - e_G)(M) = \bigoplus_{i=2}^{k(G)} M(V_i) \cong \ker(p_*)$. Now $z' \in \ker(p_*)$ is equivalent to $e_G(z') = 0$. We also have $z'_1 = e_H(z')$ and $z''_1 = e_H(z'')$. We compute

$$e_G(z'_1 - z''_1) = e_G e_H z' - e_G e_H z'' = e_H e_G z' - e_H e_G z'' = 0,$$

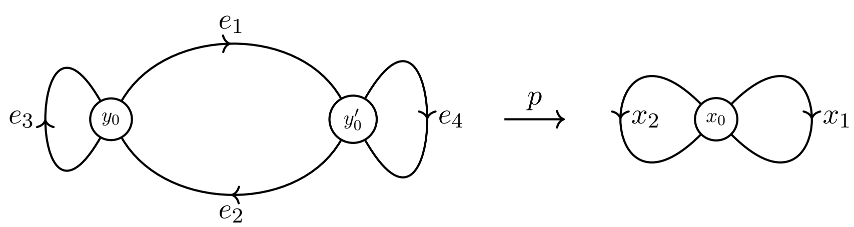
since e_G is central. Thus $e_G(\hat{z}) = e_G(z' - (z'_1 - z''_1)) = 0$, which is equivalent to $\hat{z} \in \ker(p_*)$.

Define $z \coloneqq m\hat{z} \in \ker(p_*)$ with $m \in \mathbb{Q}$ such that $z \in H_1(Y;\mathbb{Z})$. By the same argument as in Proposition 4.27 for homology classes of elevations of primitive elements, we obtain that z is not the homology class of an elevation of a primitive element in Y, and thus also not the homology class of an elevation of a primitive commutator in X. □

6 Finite Cyclic Covers

When trying to explicitly construct covers with $H_1 \neq H_1^{\text{prim}}$ respectively $H_1^{\text{comm}} \neq \ker(p_*)$, we are not only interested in the span but also in the set of all elevations of all primitives respectively primitive commutators. We hope that this set is rather small compared to the whole homology group and that we can in this way construct a cover with $H_1^{\text{prim}} \neq H_1$. This appears to be a difficult task as the set of all primitives is not even completely understood for ranks larger than or equal to three. In the following, we try to answer this question in the special case of a finite cyclic cover for $n = 2$ using the results from Subsection 4.2.

The easiest case is the cyclic cover of order two. Let X be the wedge of two circles based at the single vertex x_0 with fundamental group F_2, the free group on the generators x_1 and x_2, $G = \mathbb{Z}/2\mathbb{Z} = \langle A \rangle$ and $\phi\colon F_2 \to G$ the group homomorphism sending $x_1 \mapsto 0$, $x_2 \mapsto A$. This data defines a cover $p\colon Y \to X$ with base point $y_0 \in p^{-1}(x_0)$ and we are now interested in which homology classes can be realized by elevations of primitives. We label the edges in the cover in the following way.



An obvious basis for $H_1(Y;\mathbb{Z})$ is $\{e_1 + e_2, e_3, e_4\}$. We know by the theorem of Gaschütz that $H_1(Y;\mathbb{C})$ decomposes as G-representation as follows

$$H_1(Y;\mathbb{C}) \cong \mathbb{C}_{\text{triv}} \oplus \mathbb{C}[G] = \mathbb{C}^2_{\text{triv}} \oplus \mathbb{C}_{\text{flip}},$$

where by $\mathbb{C}_{\text{flip}}$ we denote the one-dimensional representation where the action of $A \in G$ is given by multiplication with -1. It is easy to see that a $\mathbb{C}$-basis of this decomposition is given by $\{e_1 + e_2, e_3 + e_4, e_3 - e_4\}$.

Proposition 6.1. *Let* $z = k(e_1+e_2)+le_3+me_4 \in H_1(Y;\mathbb{Z})$, $k, l, m \in \mathbb{Z}$ *and* $\gcd(k, l, m) = 1$. *Then* z *is the homology class of an elevation of a primitive element in* X *if and only if either*

1. $l = m$ *and* $\gcd(k, 2l) = 1$, *or*
2. $l = m \pm 1$ *and* $\gcd(k, 2l \pm 1) = 1$.

Proof. Let w be a primitive word in F_2. Then w lies in the conjugacy class of some $W_{i,j}(x_1, x_2)$ for some $(i, j) \in \mathbb{Z}^2$, $\gcd(i, j) = 1$ by Theorem 4.9 of Osborne and Zieschang; refer to [OZ81]. We can assume without loss of generality that $i, j \geq 0$, else replace x_1^{-1}, x_2^{-1} by x_1, x_2 respectively. We will compute the elevation of $W_{i,j}(x_1, x_2)$, and see that we either land in case one or in case two.

We have to distinguish two cases. First, let j be odd. Then $W_{i,j}(x_1, x_2)$ does not lift to a closed loop in Y since j is equal to the number of x_2's in the word. Note that $W_{i,j}(x_1, x_2)^2$ does. We begin by lifting $W_{i,j}(x_1, x_2)$ to a path in Y starting at y_0. Every time x_2 appears in the word we switch from vertex y_0 to y_0' or the other way round, either going along e_1 or e_2. Thus the number of times e_1 and e_2 appear in the lifted path differ by one and their sum is equal to j. Therefore, we obtain an element of the form

$$\beta e_3 + \gamma e_4 + \alpha e_1 + (\alpha - 1)e_2 \in C_1(Y; \mathbb{Z}),$$

with $\beta + \gamma = i$ and $2\alpha - 1 = j$, $\alpha, \beta, \gamma \in \mathbb{N}$. To obtain the preferred elevation, we have to lift the same path at the other vertex y_0'. Symmetry reasons imply that

$$\begin{aligned} z := \left[\widetilde{W_{i,j}(x_1, x_2)}\right] &= (\beta + \gamma)e_3 + (\beta + \gamma)e_4 + (2\alpha - 1)e_1 + (2\alpha - 1)e_2 \\ &= j(e_1 + e_2) + ie_3 + ie_4. \end{aligned}$$

Thus z is of the desired form with $l = m = i$ and $k = j$. Additionally, $1 = \gcd(i, j) = \gcd(2i, j) = \gcd(2l, k)$. Thus we are in Case 1.

Now let j be even. Then $W_{i,j}(x_1, x_2)$ lifts to a closed loop. We will use the geometric interpretation of $V'_{i,j}(x_1, x_2)$ as in Lemma 4.12, to better understand $W_{i,j}(x_1, x_2)$. The slope of the line segment defining $V'_{i,j}(x_1, x_2)$ is equal to j/i.

Let us first treat the case $j < i$. This means that the slope of the line segment defining $V'_{i,j}(x_1, x_2)$ is less than 1 and thus all x_2's in $V'_{i,j}(x_1, x_2)$ only appear to the first power. Our word is thus of the form

$$W_{i,j}(x_1, x_2) = (x_1 x_2)\underbrace{x_1 \dots x_1}_{a_1} x_2 \underbrace{x_1 \dots x_1}_{a_2} x_2 x_1 \dots x_1 x_2 \underbrace{x_1 \dots x_1}_{a_{j-1}} x_2 \underbrace{x_1 \dots x_1}_{a_j},$$

with $\sum_{r=1}^{j} a_r = i - 1$. Since every time x_2 appears in the word we switch from vertex y_0 to y_0' or the other way round, and the lift starts at y_0, a_r counts the multiplicity of e_3 for r odd, and the multiplicity of e_4 for r even. Note that

$$\begin{aligned} m &= 1 + a_2 + a_4 + \dots + a_k, \\ l &= a_1 + a_3 + \dots + a_{k-1} \end{aligned}$$

with $l + m = i$. We now want to show that $l = m + 1$, then we are in Case 2. Equivalently, it suffices to prove that

$$\sum_{r=1}^{j/2} a_{2r} = \sum_{r=1}^{j/2} a_{2r-1}.$$

This follows directly from Lemma 4.12, 1., since j is even and $a_r = a_{j-r+1}$ for all $1 \leq r \leq j/2$.

For the case $j > i$, the word looks like this

$$W_{i,j}(x_1, x_2) = (x_1x_2)\underbrace{x_2 \ldots x_2}_{b_1} x_1 \underbrace{x_2 \ldots x_2}_{b_2} x_1x_2 \ldots x_2x_1 \underbrace{x_2 \ldots x_2}_{b_{i-1}} x_1 \underbrace{x_2 \ldots x_2}_{b_i}.$$

Lemma 4.12, 2. implies that the b_r's only differ by 1. Since the x_1's appear in $W_{i,j}(x_1, x_2)$ only to the first power they put weight 1 on e_3 or e_4 depending on the current vertex. We also know that $b_r = b_{i-r+1}$ for all $1 \leq r \leq i/2$. Since j is even and $\gcd(i, j) = 1$, it follows that i is odd. We want to show that $b_{(i-1)/2}$ is odd. This implies that the number of x_2's in the middle of the word $V'_{i,j}(x_1, x_2)$ is odd, so the start and the end vertex do not agree. But because $b_r = b_{i-r+1}$ for all $1 \leq r \leq i/2$, symmetry reasons imply that $V'_{i,j}(x_1, x_2)$ puts exactly weight $(i - 1)/2$ on e_3 and e_4. Note that

$$b_{(i-1)/2} = \left| \left\{ n \in \mathbb{N} \,\middle|\, \frac{j(i-1)}{2i} \leq n \leq \frac{j(i+1)}{2i} \right\} \right|.$$

We compute as in Lemma 4.12

$$\begin{aligned} b_{(i-1)/2} &= \max\left(0, \left\lceil \frac{j}{i}\frac{i+1}{2} \right\rceil - \left\lfloor \frac{j}{i}\frac{i-1}{2} \right\rfloor - 1\right) \\ &= \max\left(0, \frac{j}{2} + \left\lceil \frac{j}{2i} \right\rceil - \frac{j}{2} - \left\lfloor -\frac{j}{2i} \right\rfloor - 1\right) \\ &= \max\left(0, 2\left\lceil \frac{j}{2i} \right\rceil - 1\right) \\ &= 2\left\lceil \frac{j}{2i} \right\rceil - 1, \end{aligned}$$

since j is even and $k > n$. Now multiplication with x_1x_2 to obtain the word $W_{i,j}(x_1, x_2)$ puts one more weight on either e_3 or e_4 depending on the base point of the elevation, which finishes the proof that every elevation of a primitive word is either of the form 1. or 2.

To prove the other direction, assume we are given an element of the form $z = k(e_1 + e_2) + l(e_3 + e_4) \in H_1(Y; \mathbb{Z})$ with $\gcd(k, l) = 1$ and $\gcd(k, 2l) = 1$. Consider the primitive $w := W_{l,k}(x_1, x_2)$. Since k is odd, w^2 lifts to a closed path in Y and $[\widetilde{w}] = z$, using the same considerations as in the proof of the other direction.

For the second case, assume we are given $z = k(e_1 + e_2) + le_3 + me_4$ with $\gcd(k, l, m) = 1$, $\gcd(k, 2l \pm 1) = 1$ and $l = m \pm 1$. Set $w \coloneqq W_{2l\pm1,2k}(x_1, x_2)$. Since the number of x_2's in w is $2k$ and thus even, w lifts to the first power and satisfies $[\widetilde{w}] = z$, using the same considerations as in the proof of the other direction. □

We want to extend the above to cyclic groups of arbitrary order. An equivalent result to Proposition 6.1 is the following.

Proposition 6.2. *For $k \in \mathbb{N}$, $G = \mathbb{Z}/k\mathbb{Z} = \langle A \rangle$ and*

$$\phi \colon F_2 \to G,\ x_1 \mapsto 0,\ x_2 \mapsto A,$$

a $\mathbb{Z}$-basis for $H_1(Y;\mathbb{Z})$ is given by $w_0 = e_1 + \ldots + e_k$, $w_i = e_{k+i}$ for $1 \leq i \leq n$ with $e_1, \ldots, e_k$ preimages of x_1 and $e_{k+1}, \ldots, e_{2k}$ preimages of x_2.

Then an indivisible element $z = \sum_{i=0}^{k} \alpha_i w_i \in H_1(Y;\mathbb{Z})$ is primitive if and only if $\alpha_1, \ldots, \alpha_k \in \{s, s+1\}$ for some $s \in \mathbb{Z}$ and $\gcd(\alpha_0, \sum_{i=1}^{k} \alpha_i) = 1$.

Note that for $k = 2$ this corresponds to the claim in Proposition 6.1.

Proof. This proof is an adaption of the case $G = \mathbb{Z}/2\mathbb{Z}$. The main work lies in understanding Lemma 4.12 and the primitive words in rank two. □

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