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On the real spectrum compactification of Hitchin components

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Abstract

In this thesis, we give a geometric characterization of $\mathbb{F}$-Hitchin representations, where $\mathbb{F}$ is a real closed field extension of $\mathbb{R}$. A representation from the fundamental group of a closed surface into $\mathrm{PSL}(n, \mathbb{F})$ is called $\mathbb{F}$-*Hitchin* if it satisfies the same polynomial equalities and inequalities as Hitchin representations in $\mathrm{PSL}(n, \mathbb{R})$. More precisely, we show that the set of $\mathbb{F}$-Hitchin representations coincides with the set of $\mathbb{F}$-*positive* and weakly dynamics preserving representations, i.e. representations that admit an equivariant limit map from a dense subset of the boundary of the universal cover of the surface into the set of full flags in $\mathbb{F}^n$ satisfying specific positivity properties, that mimic Fock-Goncharov positivity [FG06].

In the first part, we give the necessary background on real algebraic geometry. This is needed to define the *real spectrum compactification* of the Hitchin component, whose study was initiated by [Bru88a] and [BIPP21b]. The points of this compactification are represented by $\mathbb{F}$-Hitchin representations for various real closed fields extensions $\mathbb{F}$ of $\mathbb{R}$.

In the second part, we introduce a variant of the Bonahon-Dreyer coordinates [BD14], that allows to define coordinates for $\mathbb{F}$-Hitchin representations. For this, we carefully study the configuration spaces of tuples of flags over general fields. For ordered fields, we introduce the notion of positivity of tuples of flags, as well as total positivity of matrices. Even though most results of this part are known to the experts for $\mathbb{R}$, we study all objects in question with the goal of generalizing to real closed fields different from $\mathbb{R}$. To keep the thesis self-contained and for lack of a good reference for the proof, some important results of this part are proven in the appendices.

The third part concerns itself with properties of representations in the real spectrum compactification of the Hitchin component. One of our main results is that both $\mathbb{F}$-Hitchin and $\mathbb{F}$-positive representations are positively hyperbolic. The proof in the first case relies on the Tarski-Seidenberg transfer principle. It allows to transfer semi-algebraic properties of representations in the Hitchin component to the boundary. In the second case, we use the positivity of the equivariant limit map to deduce the result, following [FG06]. We then show that the $\mathrm{PGL}(n, \mathbb{F})$-equivalence classes of $\mathbb{F}$-positive weakly dynamics preserving representations are described by the Bonahon-Dreyer coordinates over $\mathbb{F}$. In the proof, we use that an $\mathbb{F}$-positive representation can be reconstructed by finitely many data points from its limit map. This allows us to conclude the equivalence of $\mathbb{F}$-Hitchin and $\mathbb{F}$-positive weakly dynamics preserving representations. We finish by constructing intersection geodesic currents for $\mathbb{F}$-Hitchin representations. For this we transfer the result for the case $\mathbb{F} = \mathbb{R}$ in [MZ19] to the boundary using the Tarski-Seidenberg transfer principle. The length functions of the representation can thus be computed as intersections with the geodesic currents associated to the representation.

Zusammenfassung

In dieser Arbeit geben wir eine geometrische Charakterisierung von $\mathbb{F}$-Hitchin-Darstellungen, wobei $\mathbb{F}$ eine reell abgeschlossene Körpererweiterung von $\mathbb{R}$ ist. Eine Darstellung von der Fundamentalgruppe einer geschlossenen Fläche nach $\mathrm{PSL}(n, \mathbb{F})$ heißt $\mathbb{F}$-*Hitchin*, wenn sie die gleichen polynomiellen Gleichungen und Ungleichungen wie eine Hitchin-Darstellung nach $\mathrm{PSL}(n, \mathbb{R})$ erfüllt. Genauer gesagt zeigen wir, dass die Menge der $\mathbb{F}$-Hitchin-Darstellungen mit der Menge der $\mathbb{F}$-*positiven* und schwach dynamikerhaltenden Darstellungen übereinstimmt, d.h. Darstellungen, die eine äquivariante Randabbildung von einer dichten Teilmenge des Randes der universellen Überlagerung der Fläche in die Menge der vollständigen Fahnen in $\mathbb{F}^n$ zulassen, die bestimmte Positivitätseigenschaften im Sinne von Fock-Goncharov [FG06] erfüllen.

Im ersten Teil wird der notwendige Hintergrund zur reell algebraischen Geometrie vermittelt. Mithilfe dessen können wir die *Kompaktifizierung mithilfe des reellen Spektrums* der Hitchin-Komponente definieren, deren Studium durch [Bru88a] und [BIPP21b] initiiert wurde. Repräsentanten der Punkte dieser Kompaktifizierung sind $\mathbb{F}$-Hitchin-Darstellungen für verschiedene reell abgeschlossene Körpererweiterungen $\mathbb{F}$ von $\mathbb{R}$.

Im zweiten Teil führen wir eine Variante der Bonahon-Dreyer-Koordinaten [BD14] ein. Diese erlaubt es, Koordinaten für $\mathbb{F}$-Hitchin-Darstellungen zu definieren. Dazu untersuchen wir die Konfigurationsräume von Tupeln von Fahnen über allgemeinen Körpern. Für geordnete Körper führen wir den Begriff der Positivität von Tupeln von Fahnen sowie die totale Positivität von Matrizen ein. Obwohl die meisten Ergebnisse aus diesem Teil den Experten im Fall $\mathbb{R}$ bekannt sein sollten, untersuchen wir alle in Frage kommenden Objekte mit dem Ziel der Verallgemeinerung auf andere reell abgeschlossene Körper. Um die Arbeit in sich geschlossen zu halten und aus Ermangelung einer guten Referenz für die Beweise, werden einige wichtige Ergebnisse dieses Teils in den Anhängen bewiesen.

Der dritte Teil befasst sich mit Eigenschaften von Darstellungen in der Kompaktifizierung mithilfe des reellen Spektrums der Hitchin-Komponente. Eines unserer Hauptergebnisse ist, dass sowohl $\mathbb{F}$-Hitchin- als auch $\mathbb{F}$-positive Darstellungen positiv hyperbolisch sind. Der Beweis im ersten Fall beruht auf dem Tarski-Seidenberg-Transferprinzip. Dieses erlaubt, semi-algebraische Eigenschaften von Darstellungen in der Hitchin-Komponente auf den Rand zu übertragen. Im zweiten Fall verwenden wir die Positivität der äquivarianten Randabbildung, in Anlehnung an [FG06]. Anschließend zeigen wir, dass die $\mathrm{PGL}(n, \mathbb{F})$-Äquivalenzklassen von $\mathbb{F}$-positiven, schwach dynamikerhaltenden Darstellungen durch die Bonahon-Dreyer-Koordinaten über $\mathbb{F}$ beschrieben werden. Im Beweis verwenden wir, dass eine $\mathbb{F}$-positive Darstellung durch endliche viele Datenpunkte aus ihrer Randabbildung rekonstruiert werden kann. Daraus schließen wir die Äquivalenz von $\mathbb{F}$-Hitchin- und $\mathbb{F}$-positiven, schwach dynamikerhaltenden Darstellungen. Zum Schluss konstruieren wir geodätische Schnittpunktströme für $\mathbb{F}$-Hitchin-Darstellungen. Dafür übertragen wir das Resultat für den Fall $\mathbb{F} = \mathbb{R}$ in [MZ19] mithilfe des Tarski-Seidenberg-Transferprinzips auf den Rand. Die Längenfunktionen der Darstellung können so als Schnittzahl mit den der Darstellung zugeordneten geodätischen Strömen berechnet werden.

Contents

1. Introduction

1.1. Teichmüller space and Thurston's compactification

The *Teichmüller space* $\mathrm{Teich}(S)$ of a closed, connected, orientable surface S of genus $g \geq 2$ is a central object in the study of surfaces and their properties. It is defined as the space of equivalence classes of pairs (X, f), where X is a hyperbolic surface and $f\colon S \to X$ is a homeomorphism, referred to as a *marking*. Two pairs (X, f) and (Y, g) are *equivalent* if there exists an isometry $m\colon X \to Y$ such that $m \circ f$ is isotopic to g. If X is a hyperbolic surface and $f\colon S \to X$ a homeomorphism, then f induces an isomorphism on fundamental groups $f_*\colon \pi_1(S) \to \pi_1(X)$. The fundamental group of X acts on its universal cover $\widetilde{X}$ by orientation-preserving isometries. The latter can be identified with the hyperbolic plane $\mathbb{H}^2$, and its group of orientation-preserving isometries is isomorphic to $\mathrm{PSL}(2, \mathbb{R})$. Thus f gives rise to a homomorphism $\rho_f\colon \pi_1(S) \to \mathrm{PSL}(2, \mathbb{R})$. If (X, f) and (Y, g) represent the same point in $\mathrm{Teich}(S)$, then ρ_f and ρ_g are conjugated by an element of $\mathrm{PSL}(2, \mathbb{R})$. Thus $\mathrm{Teich}(S)$ can be naturally identified as a subset of the *character variety* $\chi(S, \mathrm{PSL}(2, \mathbb{R}))$—the space of reductive representations from $\pi_1(S)$ to $\mathrm{PSL}(2, \mathbb{R})$ up to $\mathrm{PSL}(2, \mathbb{R})$-conjugation. Compare Section 3.1 for a precise definition of the character variety. In fact, the Teichmüller space forms a whole connected component of $\chi(S, \mathrm{PSL}(2, \mathbb{R}))$ that is homeomorphic to $\mathbb{R}^{6g-6}$ and consists only of equivalence classes of faithful representations with discrete image; see [Gol80, Theorem A].

Thurston's compactification of $\mathrm{Teich}(S)$, first introduced in 1976 [Thu88], has numerous important applications in geometric topology. Informally speaking, a compactification turns a topological space into a compact topological space by adding "points at infinity" to control points from "going off to infinity". A *compactification* of a topological space X is a compact topological space $\overline{X}$ such that X embeds (homeomorphism onto its image) in $\overline{X}$ with dense image. The *boundary points* of the compactification are the elements of $\partial\overline{X} \coloneqq \overline{X} \setminus X$. There are various different compactifications, e.g. a topological space can always be compactified by adding a point. However if X is a parameter space of geometric structures on a manifold, we would like to compactify it in a geometrically meaningful way, which can be a challenging problem.

The definition of Thurston's compactification uses the data of the marked lengths of all closed geodesics on S. This data determines the marked hyperbolic structure up to isotopy. Thurston considered a point in Teichmüller space as a projectivized collection of lengths of curves. He showed that the closure of $\mathrm{Teich}(S)$ in projective space is compact and homeomorphic to a closed ball of dimension $6g - 6$. The boundary of the compactification is homeomorphic to a sphere of dimension of $6g - 7$. Compactifications that use the data of "lengths" of curves on S will be referred to as *length compactifications*. Thurston's compactification is used in the classification of elements in the mapping class group $\mathrm{MCG}(S)$ of S [Thu88] as well as in the proof of the Tits alternative for $\mathrm{MCG}(S)$ [Iva84, McC85] and various other results about its subgroup structure [Iva92]. The interpretation of its boundary points as measured laminations is essential in Thurston's definition of the so-called ending laminations of a hyperbolic 3-manifold [Thu02, Section 9.3]. It was extensively studied in [FLP12], see also [Mar22b].

1.2. The real spectrum compactification

Classical Teichmüller theory concerns itself with the study of Teichmüller space, in particular its geometric and dynamical properties. Replacing $\mathrm{PSL}(2,\mathbb{R})$ by a semisimple Lie group G of higher rank gives rise to *higher Teichmüller theory*. It tries to understand exceptional connected components of the character variety $\chi(S,G)$, consisting entirely of faithful representations with discrete image, which we call *higher Teichmüller spaces* [Wie18]. It is surprising that such components exist. For instance, they exist only for a restricted list of Lie groups, that are in particular neither compact nor complex. Higher Teichmüller theory is a very active field of research, that builds on various methods from different areas of mathematics, including geometry, analysis and dynamics. Often questions in higher Teichmüller theory are motivated by properties of the classical Teichmüller space, however there are new features that arise in higher rank. We are interested in compactifying higher Teichmüller spaces, or character varieties in general. In this thesis we will concentrate on the real spectrum compactification. Before we examine it further let us give a short non-exhaustive overview of other existing compactifications.

Thurston's compactification has been extended to $\chi(S,\mathrm{PSL}(2,\mathbb{R}))$ and to other character varieties $\chi(S,G)$ for G of rank one by various authors [Mor86, MS84, MS85, Bes88, Pau88, Sko90]. In higher rank, Parreau [Par12] generalized it to the so-called *Weyl chamber length compactification*. All of the mentioned compactifications are length compactifications and agree in the case when $G=\mathrm{PSL}(2,\mathbb{R})$. They recover Thurston's compactification when restricted to $\mathrm{Teich}(S)$. It turns out that the above compactification of $\chi(S,\mathrm{PSL}(2,\mathbb{R}))$ has the property that connected components can meet at the boundary, see Wolff [Wol11, Theorem 1.1]. There are also compactifications that only compactify higher Teichmüller components of the character variety. For example, in rank two there has been various work of Martone, Ouyang and Tamburelli using geometric interpretations of representations in higher Teichmüller spaces to compactify them, see [Ouy19, MOT21, OT21, OT23b, OT23a].

The idea to use the real spectrum to compactify Teichmüller space was first pointed out by Brumfiel in [Bru88a]. It will be introduced in detail in Chapter 2 and Chapter 3. We quickly summarize its most important properties here. For this let Γ be a finitely generated group and G the real points of a connected semisimple linear algebraic group defined over $\mathbb{R}$. The real algebraic structure of the character variety $\chi(\Gamma,G)$ [RS90] allows us to use real algebraic geometry to define its *real spectrum compactification* $\mathrm{RSp}(\chi(\Gamma,G))$ [BCR98]. It provides a formalization for the following idea: If $(\rho_n)_{n\in\mathbb{N}}$ is a sequence of representations that is not contained in any compact subset of $\chi(\Gamma,G)$, then some of the matrix coefficients of its image tend to infinity while others stay bounded. Informally speaking, in the limit the unbounded coefficients will be replaced by indeterminates X_i that are larger than any real number. In other words, this sequence converges in the real spectrum compactification to a representation that has values in a matrix group $G_{\mathbb{F}}$ with coefficients in the ordered field $\mathbb{F}=\mathbb{R}(X_i)$, compare Example 3.3.5. The order on $\mathbb{F}$ even distinguishes the different "speeds" at which matrix coefficients tend to infinity. Arranging all these ways of "going off to infinity" in a compact topological space is formalized by the real spectrum compactification.

This compactification has well-behaved topological properties in contrast to the classical length compactification. Indeed, the inclusion of the character variety into its real spectrum compactification induces a bijection on the level of connected components (Proposition 2.3.13). Brumfiel showed that the $\mathrm{MCG}(S)$-action on $\mathrm{Teich}(S)$ extends to $\mathrm{RSp}(\mathrm{Teich}(S))$ and that there is a $\mathrm{MCG}(S)$-equivariant continuous surjection from $\mathrm{RSp}(\mathrm{Teich}(S))$ on Thurston's compactification, compare [Bru88a, §7]. Intuitively, $\mathrm{RSp}(\mathrm{Teich}(S))$ distinguishes more ways in which points can "go off to infinity". For example, the sequence of marked hyperbolic structures obtained by pinching or twisting along a simple closed geodesic on S lead to the same point in Thurston's compactification [Mar22b, Section 8.2.18]—however these define different points in $\mathrm{RSp}(\mathrm{Teich}(S))$. This can for example be deduced using Fenchel-Nielsen coordinates together with Example 2.3.3. In [Bru88a, §8] Brumfiel shows how points in the compactification give rise to actions on $\mathbb{R}$-trees using non-Archimedean hyperbolic planes; see also [Bru88c]. Furthermore, he showed that the extension of a continuous semi-algebraic self map to the real spectrum compactification has a fixed point—even though the compactification is not necessarily a closed ball [Bru88b, Bru92] (Theorem 2.3.15). The price to pay is that we no longer have a complete topological description of the compact space $\mathrm{RSp}(\mathrm{Teich}(S))$, in contrast to Thurston's compactification.

In a series of papers [BIPP21b, BIPP23], Burger-Iozzi-Parreau-Pozzetti generalize Brumfiel's idea to compactify character varieties $\chi(\Gamma, G)$ in all generality. Before we can describe boundary points of $\mathrm{RSp}(\chi(\Gamma, G))$ let us introduce some notation. An ordered field $\mathbb{F}$ is *real closed* if $\mathbb{F}[\sqrt{-1}]$ is algebraically closed (Definition 2.1.1). Let $\mathbb{F}$ be a real closed field containing $\mathbb{R}$. If X is a semi-algebraic subset of some $\mathbb{R}^N$, i.e. a finite union of sets cut out by finitely many polynomial equalities and inequalities (Definition 2.1.8), then we can define its $\mathbb{F}$-*extension* $X_{\mathbb{F}}$ as the finite union of subsets of $\mathbb{F}^n$ cut out by the same equalities and inequalities (Definition 2.1.12). In particular, since G is semi-algebraic we can consider the group $G_{\mathbb{F}}$. We have the following description of $\mathrm{RSp}(\chi(\Gamma, G))$ due to Burger-Iozzi-Parreau-Pozzetti, which can be taken as a definition in this introduction.

Theorem 1.2.1 ([BIPP21b, Theorem 2])**.** Let Γ and G be as above. Then

$$\mathrm{RSp}(\chi(\Gamma, G)) \cong \left\{ (\rho, \mathbb{F}) \;\middle|\; \begin{array}{l} \rho\colon \Gamma \to G_{\mathbb{F}} \text{ reductive homomorphism,} \\ \mathbb{F} \supseteq \mathbb{R} \text{ real closed field} \end{array} \right\} \Big/ \sim,$$

where $\sim$ is the equivalence relation generated by proclaiming $\rho_1\colon \Gamma \to G_{\mathbb{F}_1}$ and $\rho_2\colon \Gamma \to G_{\mathbb{F}_2}$ equivalent if there exists an order-preserving field homomorphism $\alpha\colon \mathbb{F}_1 \to \mathbb{F}_2$ such that $\alpha \circ \rho_1$ is conjugate to ρ_2 in $G_{\mathbb{F}_2}$. Furthermore, $(\rho, \mathbb{F})$ represents a point in the boundary if and only if $\mathbb{R}(\mathrm{tr}(\rho))$ is a non-Archimedean field.

In particular if $\mathbb{F}$ is a real closed field containing $\mathbb{R}$, then any reductive homomorphism $\rho\colon \Gamma \to G_{\mathbb{F}}$ represents a point in $\mathrm{RSp}(\chi(\Gamma, G))$. We denote by $[(\rho, \mathbb{F})]$ its equivalence class in $\mathrm{RSp}(\chi(\Gamma, G))$. The equivalence relation makes sure that the representations $\rho\colon \Gamma \to G_{\mathbb{F}}$ and $\rho\colon \Gamma \to G_{\mathbb{F}} \hookrightarrow G_{\mathbb{F}'}$, for $\mathbb{F}'$ a real closed field extension of $\mathbb{F}$, define the same point. From this result it follows that boundary points can be studied in terms of representations. For higher Teichmüller spaces $\mathcal{C}$ of $\chi(\Gamma, G)$ a natural question in this context is then the following.

Question 1.2.2. Given a representation from Γ to $G_{\mathbb{F}}$ for $\mathbb{F} \supseteq \mathbb{R}$ a real closed field, can we determine "geometrically" whether or not it is in the boundary of $\mathcal{C}$?

We answer this question in the case when $\mathcal{C}$ is the Hitchin component—a higher Teichmüller space of the $\mathrm{PSL}(n, \mathbb{R})$-character variety. For maximal components in $\chi(S, \mathrm{Sp}(2n, \mathbb{R}))$ this question was studied in [BIPP21b] by Burger-Iozzi-Parreau-Pozzetti.

1.3. The Hitchin component

One of the first instances of a higher Teichmüller space is the Hitchin component. In his seminal paper [Hit92], Hitchin used Higgs bundles to show that $\chi(S, \mathrm{PSL}(n, \mathbb{R}))$ for $n \geq 3$ contains three connected components, if n is odd, and six connected components, if n is even. In the odd case, one of the three components, and in the even case, two of the six components, are homeomorphic to $\mathbb{R}^{(2g-2)(n^2-1)}$. They can be characterized as follows. We denote by ι_n the irreducible n-dimensional representation from $\mathrm{SL}(2, \mathbb{R})$ to $\mathrm{SL}(n, \mathbb{R})$, where an element of $\mathrm{SL}(2, \mathbb{R})$ acts on the n-dimensional vector space of polynomials with real coefficients in two variables X and Y of degree $n-1$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} . X^{n-i}Y^{i-1} := (aX + cY)^{n-1}(bX + dY)^{i-1}.$$

This representation is unique up to $\mathrm{PGL}(n, \mathbb{R})$-conjugation. We also denote by ι_n the induced representation from $\mathrm{PSL}(2, \mathbb{R})$ to $\mathrm{PSL}(n, \mathbb{R})$.

Definition 1.3.1. Fix $j \colon \pi_1(S) \to \mathrm{PSL}(2, \mathbb{R})$ a holonomy representation of a marked hyperbolic structure on S, i.e. j is faithful with discrete image. The *Hitchin component* $\mathrm{Hit}(S, n)$ is the connected component of $\chi(S, \mathrm{PSL}(n, \mathbb{R}))$ containing $\iota_n \circ j$. A representation $\pi_1(S) \to \mathrm{PSL}(n, \mathbb{R})$ whose $\mathrm{PSL}(n, \mathbb{R})$-conjugacy class lies in the Hitchin component will be called a *Hitchin representation*.

Denote by $\mathrm{Hit}(\bar{S}, n)$ the connected component of $\chi(S, \mathrm{PSL}(n, \mathbb{R}))$ containing $\iota_n \circ j'$, where j' is any conjugate of j under an element of $\mathrm{PGL}(2, \mathbb{R}) \setminus \mathrm{PSL}(2, \mathbb{R})$. If n is odd, $\mathrm{Hit}(\bar{S}, n)$ is the same as $\mathrm{Hit}(S, n)$.

Hitchin asked about the geometric significance of components in higher rank. In [CG93], Choi-Goldman proved that for $n = 3$ the Hitchin component parametrizes marked convex real projective structures on S, from which they concluded that $\mathrm{Hit}(S, 3)$ consists only of faithful representations with discrete image. Using the Klein model of the hyperbolic plane, one can see that hyperbolic structures are particular examples of convex real projective structures, which reflects the fact that, by definition, Teichmüller space embeds in $\mathrm{Hit}(S, 3)$.

More than a decade later, Labourie [Lab06] and, in independent work, Fock-Goncharov [FG06] showed that the Hitchin component consists only of faithful representations with discrete image, in contrast to the other components. This implies that $\mathrm{Hit}(S, n)$ constitutes an example of a higher Teichmüller space for all $n \geq 3$. Their result follows from the existence of a limit map with strong dynamical properties. To explain this we first need to introduce a definition. For more details refer to Section 4.1 and Section 5.1. Fix an auxiliary hyperbolic structure on S. Its universal cover $\tilde{S}$ is identified with $\mathbb{H}^2$, and let $\partial\tilde{S} := \partial\mathbb{H}^2 \cong \mathbb{S}^1$ denote its circle at infinity. The action of $\pi_1(S)$ on $\tilde{S}$ extends to a continuous action on $\partial\tilde{S}$. The latter is identified with $\mathbb{S}^1$ and inherits a cyclic order. Let $\mathrm{Fix}(S) \subseteq \partial\tilde{S}$ be the $\pi_1(S)$-invariant subset of $\partial\tilde{S}$ of $\pi_1(S)$-fixed points, i.e. $\mathrm{Fix}(S) := \{x \in \partial\tilde{S} \mid \mathrm{Stab}_{\pi_1(S)}(x) \neq e\}$. A flag in $\mathbb{R}^n$ is a nested sequence of $n+1$ subspaces of $\mathbb{R}^n$ of strictly increasing dimension, see Definition 4.0.1. Denote by $\mathrm{Flag}(\mathbb{R}^n)$ the set of full flags in $\mathbb{R}^n$.

Definition 1.3.2. A map $\xi\colon \mathrm{Fix}(S) \to \mathrm{Flag}(\mathbb{R}^n)$ is *positive* if it maps any triple respectively quadruple of cyclically ordered points in $\mathrm{Fix}(S)$ to a positive triple respectively quadruple of flags (Definition 5.1.1).

A representation $\rho\colon \pi_1(S) \to \mathrm{PSL}(n,\mathbb{R})$ is *positive* if there exists a (not necessarily continuous) positive map

$$\xi_\rho\colon \mathrm{Fix}(S) \to \mathrm{Flag}(\mathbb{R}^n),$$

that is ρ-equivariant, i.e. for all $x \in \mathrm{Fix}(S)$ and $\gamma \in \pi_1(S)$ we have $\xi_\rho(\gamma x) = \rho(\gamma)\xi_\rho(x)$. The map ξ_ρ is called the *limit map* of ρ.

If such a limit map exists, then it is unique. With this definition we can formulate the following characterization of Hitchin representations.

Theorem 1.3.3 ([FG06, Theorem 1.15])**.** Let $\rho\colon \pi_1(S) \to \mathrm{PSL}(n,\mathbb{R})$ be a representation. Then ρ is $\mathrm{PGL}(n,\mathbb{R})$-conjugate to a Hitchin representation if and only if ρ is a positive representation.

Fock-Goncharov showed in [FG06, Theorem 1.9 and 1.10] that positive representations are injective, have discrete image and are positively hyperbolic, see Definition 1.4.4. Furthermore, if $x \in \mathrm{Fix}(S)$ is the attracting fixed point of $\gamma \in \pi_1(S)$, then $\xi_\rho(x)$ is the *stable flag* of $\rho(\gamma)$, see Definition 5.3.1. Thus the same holds true for Hitchin representations.

The notion of Hitchin representations has been generalized in many ways. Hitchin representations now fit in the broader context of so called Θ-*positive representations*. Extending Lusztig's total positivity [Lus94], Guichard-Wienhard define Θ-positivity in [GW18] for Lie groups that are not necessarily real split. All known higher Teichmüller spaces, e.g. maximal components introduced by Burger-Iozzi-Wienhard in [BIW03], consist of Θ-positive representations. Maximal representations are defined when G is a non-compact simple Lie group of Hermitian type. In this case one can associate to a representation a Toledo number, which is constant on connected components and which satisfies a Milnor-Wood type inequality. A maximal representation is a representation for which the Toledo number attains the maximal possible value. In the case when $G = \mathrm{PSL}(2,\mathbb{R})$, the Hitchin component and the maximal component agree with $\mathrm{Teich}(S)$. Guichard-Labourie-Wienhard proved in [GLW21, Theorem A] that Θ-positive representations are discrete and faithful. It is conjectured that Θ-positive representations form unions of connected components of character varieties and that all higher Teichmüller spaces are of this form [Wie18, BCGP+21]. A complete list of Lie groups that admit such representations has been found in [GW18] and in independent work using Higgs bundle techniques in [BCGP+21].

To study the properties of Hitchin representations, Labourie [Lab06] introduced the notion of *Anosov representations* using methods from dynamical systems. Hitchin representations are Anosov, however there exist Anosov representations that are not in higher Teichmüller components. Using this approach we can, additionally to changing the target Lie group, change the group of definition, i.e. we can replace $\pi_1(S)$ by any finitely generated hyperbolic group Γ, and look for injective representations with discrete image. This is an active field of research, see [Can21] for an introduction to Anosov representations.

1.4. Results

In the following we summarize our results, which partially appear in [Fla22]. The main result of this thesis is the classification of boundary points in the real spectrum compactification of the Hitchin component in terms of positivity for real closed fields.

We saw in Section 1.2 that representations of $\pi_1(S)$ into $\mathrm{PSL}(n, \mathbb{F})$, where $\mathbb{F}$ is a real closed field extension of $\mathbb{R}$, naturally occur in the real spectrum compactification of the character variety $\chi(S, \mathrm{PSL}(n, \mathbb{R}))$ and the Hitchin component $\mathrm{Hit}(S, n)$. As a connected component of a semi-algebraic set, $\mathrm{Hit}(S, n)$ is semi-algebraic (Theorem 2.1.11). The closure of $\mathrm{Hit}(S, n)$ in $\mathrm{RSp}(\chi(S, \mathrm{PSL}(n, \mathbb{R})))$ agrees with its real spectrum compactification $\mathrm{RSp}(\mathrm{Hit}(S, n))$. We define the following.

Definition 1.4.1. Let $\mathrm{Hit}(S, n)_{\mathbb{F}}$ be the $\mathbb{F}$-extension of $\mathrm{Hit}(S, n)$, called the $\mathbb{F}$-*Hitchin component*. A representation $\pi_1(S) \to \mathrm{PSL}(n, \mathbb{F})$ is $\mathbb{F}$-*Hitchin* if its $\mathrm{PSL}(n, \mathbb{F})$-conjugacy class lies in $\mathrm{Hit}(S, n)_{\mathbb{F}}$.

Remark 1.4.2. It follows from Theorem 1.2.1 and the properties of the real spectrum that if $\rho \colon \pi_1(S) \to \mathrm{PSL}(n, \mathbb{F})$ is a representation, then ρ is $\mathbb{F}$-Hitchin if and only if $(\rho, \mathbb{F})$ represents a point in $\mathrm{RSp}(\mathrm{Hit}(S, n))$, compare Theorem 2.3.4 and Proposition 2.3.13.

We will show that $\mathbb{F}$-Hitchin representations admit a positive limit map and are thus $\mathbb{F}$-positive, which we define now.

Given any ordered field $\mathbb{F}$, the concept of positive tuples of full flags in $\mathbb{F}^n$ can be defined in the same way as for $\mathbb{R}$, since it only involves positivity conditions on triple and double ratios (Definition 4.1.2 and Definition 4.2.1). Similarly, denote by $\mathrm{Flag}(\mathbb{F}^n)$ the set of full flags in $\mathbb{F}^n$. With these notations we can define an $\mathbb{F}$-*positive map* the same way as in the real case by replacing every occurrence of $\mathbb{R}$ in Definition 1.3.2 by $\mathbb{F}$. This allows us to define a generalization of positive representation for general real closed fields.

Definition 1.4.3. Let $\mathbb{F}$ be a real closed field extension of $\mathbb{R}$. A representation $\rho \colon \pi_1(S) \to \mathrm{PSL}(n, \mathbb{F})$ is $\mathbb{F}$-*positive* if there exists a ρ-equivariant (not necessarily continuous) $\mathbb{F}$-positive map $\xi_\rho \colon \mathrm{Fix}(S) \to \mathrm{Flag}(\mathbb{F}^n)$, called a *limit map* of ρ.

If ρ is an $\mathbb{F}$-positive representation with limit map ξ_ρ, then for all $e \neq \gamma \in \pi_1(S)$ the subspaces $V_a := \xi_\rho(\gamma^+)^{(a)} \cap \xi_\rho(\gamma^-)^{(n-a+1)}$, where γ^+ and $\gamma^- \in \mathrm{Fix}(S)$ are the attracting respectively repelling fixed point of γ, are one-dimensional and $\rho(\gamma)$-invariant. The eigenvalue associated to V_a is denoted by λ_a. We say that ρ is *weakly dynamics preserving* if for all $e \neq \gamma \in \pi_1(S)$ we have $|\lambda_1| \geq \ldots \geq |\lambda_n|$.

In the case $\mathbb{F} = \mathbb{R}$ we recover Definition 1.3.2, and Fock-Goncharov proved that $\mathbb{R}$-positive representations are weakly dynamics preserving [FG06, Section 7]. We expect the same to hold when $\mathbb{F}$ is a general real closed field. We saw in Theorem 1.3.3, that a representation is $\mathrm{PGL}(n, \mathbb{R})$-conjugate to a Hitchin representation if and only if it is $\mathbb{R}$-positive. In analogy to the real case, we have the following theorem.

Theorem A. *Let $\mathbb{F}$ be a real closed extension of $\mathbb{R}$. A representation $\rho\colon \pi_1(S) \to \mathrm{PSL}(n, \mathbb{F})$ is $\mathrm{PGL}(n, \mathbb{F})$-conjugate to an $\mathbb{F}$-Hitchin representation if and only if it is $\mathbb{F}$-positive and weakly dynamics preserving.*

Then Theorem A together with Remark 1.4.2 answers Question 1.2.2 for $\mathcal{C} = \mathrm{Hit}(S, n)$.

Corollary B. *If $(\rho, \mathbb{F})$ represents a point in $\mathrm{RSp}(\chi(S, \mathrm{PSL}(n, \mathbb{R})))$, then it represents a point in $\mathrm{RSp}(\mathrm{Hit}(S, n)) \cup \mathrm{RSp}(\mathrm{Hit}(\bar{S}, n))$ if and only if ρ is $\mathbb{F}$-positive and weakly dynamics preserving.*

This corollary completes the proof of a result announced in [BIPP21b, Theorem 46], compare Theorem 3.3.4. For the backward direction in the proof of Theorem A, we introduce a multiplicative variant of the Bonahon-Dreyer coordinates for the Hitchin component [BD14]. This variant can be extended to give coordinates for $\mathrm{Hit}(S, n)_{\mathbb{F}}$ for every real closed field $\mathbb{F} \supseteq \mathbb{R}$ (Corollary 6.4.3). We then prove the following.

Theorem C. *The set of $\mathrm{PGL}(n, \mathbb{F})$-equivalence classes of $\mathbb{F}$-positive weakly dynamics preserving representations is described by the Bonahon-Dreyer coordinates over $\mathbb{F}$ and hence homeomorphic to a closed semi-algebraic subset of some $\mathbb{F}^N$.*

For the proof of Theorem C as well the forward direction of Theorem A, a key result is the following proposition, which holds true over $\mathbb{R}$ ([FG06, Theorem 1.13 (i)], and which we believe to be of independent interest.

Definition 1.4.4. An element of $\mathrm{PSL}(n, \mathbb{F})$ is *positively hyperbolic* if one of its lifts to $\mathrm{SL}(n, \mathbb{F})$ has distinct and only positive eigenvalues. A representation $\rho\colon \pi_1(S) \to \mathrm{PSL}(n, \mathbb{F})$ is *positively hyperbolic* if $\rho(\gamma)$ is positively hyperbolic for all $e \neq \gamma \in \pi_1(S)$.

There is no ambiguity for odd n, since $\mathrm{PSL}(n, \mathbb{F}) = \mathrm{SL}(n, \mathbb{F})$ for every real closed field $\mathbb{F}$. If n is even, an element in $\mathrm{PSL}(n, \mathbb{F})$ admits two lifts that differ just by sign.

Proposition D (see Proposition 7.2.1 and Proposition 7.3.1)**.** *Let $\rho\colon \pi_1(S) \to \mathrm{PSL}(n, \mathbb{F})$ be $\mathbb{F}$-Hitchin or $\mathbb{F}$-positive. Then ρ is positively hyperbolic.*

The rest of the thesis concerns itself with assigning geodesic currents to $\mathbb{F}$-positive weakly dynamics preserving representations. Geodesic currents were introduced by Bonahon in [Bon86], providing a unifying framework for Thurston's compactification. They are now an active field of research by themselves and have applications in various fields in the study of surfaces; see e.g. [ES22]. To simplify we choose a hyperbolic structure on S, even though the following concept can be defined in a purely topological way, see e.g. [Bon88, Fact 1].

Definition 1.4.5. A *geodesic current* on S is a locally finite, $\pi_1(S)$-invariant, regular Borel measure on the space of (unoriented and unparametrized) geodesics in the universal cover $\tilde{S}$ of S.

Denote by $\mathscr{C}(S)$ the space of geodesic currents on S endowed with the weak*-topology, and by $\mathbb{P}\mathscr{C}(S) := \mathscr{C}(S)/\mathbb{R}_{>0}$ the space of projectivized geodesic currents, the latter being compact [Bon88, Proposition 4, Corollary 5]. Many seemingly different objects are in fact geodesic currents, e.g. homotopy classes of closed curves on S or isotopy classes of marked hyperbolic structures on S [Bon88, Lemma 9]. Martone-Zhang proved in [MZ19, Theorem 3.4], that we can associate to a Hitchin representation a geodesic current such that k-length functions of the representation can be computed as intersections (Definition 8.1.1) with this current. More precisely, let $\mathbb{F} \supseteq \mathbb{R}$ be a real closed field that admits an order-compatible valuation v, see Definition 8.2.1 (for $\mathbb{F} = \mathbb{R}$ we can take $v = -\log$). For $g \in \mathrm{PSL}(n, \mathbb{F})$ with distinct eigenvalues of the same sign, let $\lambda_1(g) > \ldots > \lambda_n(g) > 0$ be the eigenvalues of a lift of g to $\mathrm{SL}(n, \mathbb{F})$. For every $k = 1, \ldots, n-1$ we define the *k-length of g* as

$$L_k(g) \coloneqq -\sum_{j=1}^{k} v(\lambda_j(g)) + \sum_{j=n-k+1}^{n} v(\lambda_j(g)).$$

These length functions can be interpreted as the translation length of an element $g \in \mathrm{PSL}(n, \mathbb{F})$ acting on the metric space $\mathscr{B}_{\mathrm{PSL}(n,\mathbb{F})}$ as defined in [BIPP21b, Section 3.4 and 4]—a higher rank analogue of the non-Archimedean hyperbolic plane defined by Brumfiel [Bru88c].

We establish the following result announced by Burger-Iozzi-Parreau-Pozzetti [BIPP21b, Theorem 47]. An equivalent version was proven for maximal representations by Burger-Iozzi-Parreau-Pozzetti [BIPP21a, Theorem 1.2].

Theorem E. *Let $\mathbb{F} \supseteq \mathbb{R}$ be a non-Archimedean real closed field with an order-compatible valuation v (assumed to be $-\log$ if $\mathbb{F} = \mathbb{R}$) and let $\rho \colon \pi_1(S) \to \mathrm{PSL}(n, \mathbb{F})$ be $\mathbb{F}$-positive and weakly dynamics preserving. Then for every $k = 1, \ldots, n-1$, there exists a geodesic current μ_ρ^k such that for any $e \neq \gamma \in \pi_1(S)$ we have*

$$i(\mu_\rho^k, \gamma) = L_k(\rho(\gamma)).$$

The current μ_ρ^k is non-zero if and only if there exists $\gamma \in \pi_1(S)$ with $v(|\mathrm{tr}(\rho(\gamma))|) < 0$.

Considering the subset $\mathrm{RSp}^{\mathrm{cl}}(\mathrm{Hit}(S, n))$ of closed points in $\mathrm{RSp}(\mathrm{Hit}(S, n))$, this implies the following.

Corollary F. *For all $k = 1, \ldots, n-1$ the map*

$$\begin{aligned} \mathrm{RSp}^{\mathrm{cl}}(\mathrm{Hit}(S, n)) &\to \mathbb{P}\mathscr{C}(S), \\ [(\rho, \mathbb{F})] &\mapsto \left[\mu_\rho^k\right] \end{aligned}$$

is well-defined and continuous.

1.5. Method

The key tool to establish the forward direction of Theorem A, as well as the proofs of Proposition D (for $\mathbb{F}$-Hitchin representations) and Theorem E, is the *Tarski-Seidenberg transfer principle* (Theorem 2.1.13), a real closed analogue of the Lefschetz principle for algebraically closed fields. It enables to deduce results for real closed extensions of $\mathbb{R}$, that are stated in first-order logic in the language of ordered fields, from results that hold true over $\mathbb{R}$.

To prove Theorem A, we first establish Proposition D for $\mathbb{F}$-Hitchin representations (Proposition 7.2.1). Equipped with this we can construct a limit map for an $\mathbb{F}$-Hitchin representation ρ. Namely to the fixed point of a hyperbolic element γ we associate the stable flag of the positively hyperbolic element $\rho(\gamma)$. To prove its positivity properties, we use again the Tarski-Seidenberg transfer principle and the positivity properties of Hitchin representations (Theorem 1.3.3).

The backward direction of Theorem A is more involved. Here we cannot apply the transfer principle directly, as the notion of $\mathbb{F}$-positive weakly dynamics preserving representation involves infinitely many conditions. We will show that, actually, a finite set of data points suffices to determine an $\mathbb{F}$-positive weakly dynamics preserving representation up to conjugation. The equivariant limit map that comes together with an $\mathbb{F}$-positive representation allows us to associate to the representation the (adapted) Bonahon-Dreyer coordinates over $\mathbb{F}$ [BD14]. We then verify that these coordinates satisfy the same polynomial equalities and inequalities as the ones for $\mathbb{F}$-Hitchin representations, which proves Theorem C. The difficulty lies in the proof of the closed leaf inequality (iv). Together with the assumption of weakly dynamics preserving, this amounts to proving Proposition D for $\mathbb{F}$-positive representations (Proposition 7.3.1). Following Fock-Goncharov [FG06], we instead prove Proposition 7.3.2, which asserts that the image of every non-trivial element of $\pi_1(S)$ under an $\mathbb{F}$-positive representation is conjugate to a totally positive matrix with coefficients in $\mathbb{F}$. We use the Tarski-Seidenberg transfer principle to establish the theorem of Gantmacher-Krein for real closed fields, see Theorem 5.3.3. To prove Proposition 7.3.2 we explicitly describe the image of a non-trivial element of $\pi_1(S)$ under an $\mathbb{F}$-positive representation in terms of matrices whose entries involve the triple and double ratios of the representation, compare Theorem 4.3.4.

The proof of Theorem E follows closely that of [BIPP21a, Theorem 1.2] in the case of maximal representations. It is based on their result [BIPP21a, Theorem 1.6], where they show how to associate a geodesic current to a positive cross-ratio (Definition 8.1.2). To show that to an $\mathbb{F}$-Hitchin representation we can associate a positive cross-ratio we use that the same statement holds true for $\mathbb{R}$ [MZ19, Lemma 3.6]. Left to prove is thus the continuity in Corollary F which is a standard argument similar to the one in [Bru88a, Proposition 5.3].

1.6. Organization

The thesis is organized as follows. Part I provides an introduction to real algebraic geometry, character varieties and their real spectrum compactification. In Chapter 2 we define the spectrum of a ring and the real spectrum compactification of a semi-algebraic set. Chapter 3 presents a semi-algebraic model for character varieties. Finally, we apply the general theory from real algebraic geometry to character varieties and collect results on their real spectrum compactification.

Part II provides the basics on flags and positivity in order to introduce the Bonahon-Dreyer coordinates for Hitchin components. Even though most results are known to the experts for $\mathbb{R}$, we carefully study all objects in question with the goal of generalizing to real closed fields different from $\mathbb{R}$. In Chapter 4 we give the necessary preliminaries on flags, including the definition of the triple and double ratios. This leads to the notions of positive k-tuples of full flags in Chapter 5, and we explain the connection between positivity of flags and total positivity of matrices. To keep the thesis self-contained and for lack of a good reference for the proof, some important results in Chapter 4 and Chapter 5 are proven in Appendix A respectively Appendix B.

We prove our results in Part III. In Chapter 7, we prove Proposition D and collect properties of $\mathbb{F}$-Hitchin representations. Section 7.4 finishes the proof of Theorem A and Theorem C. We introduce positive cross-ratios, intersection currents and valuations in Chapter 8 and finish the chapter with the proofs of Theorem E and Corollary F.

Part I.

Preliminaries

2. Real algebraic geometry

In the first section we recall general definitions and results from real algebraic geometry and set up notation. The second section describes the semi-algebraic structure of character varieties following [RS90]. We finish this chapter by focusing on several semi-algebraic descriptions of the Hitchin component.

2.1. Background on real algebraic geometry

We refer the reader to [BCR98], in particular Chapters 1, 2 and 5, for more details and proofs.

Definition 2.1.1. An *order* on a field $\mathbb{F}$ is a total order relation $\leq$ compatible with the field operations, i.e.

$$x \leq y \implies x+z \leq y+z, \quad 0 \leq x, y \implies 0 \leq xy \quad \text{for all } x, y, z \in \mathbb{F}.$$

A field $\mathbb{F}$ is *orderable* if $\mathbb{F}$ admits an order. An order $\leq$ on $\mathbb{F}$ is *Archimedean* if for any $x \in \mathbb{F}$ there exists $n \in \mathbb{N}$ such that $x < n$. An ordered field is *real closed* if every positive element is a square and every odd degree polynomial has a root.

Note that ordered fields are infinite and of characteristic zero. Thus every ordered field contains $\mathbb{Q}$.

Example 2.1.2. The fields $\mathbb{Q}$ and $\mathbb{R}$ have a unique order, whereas $\mathbb{C}$ cannot be ordered.

The field $\mathbb{Q}(\sqrt{2})$ has exactly two distinct orderings given by the two embeddings of $\mathbb{Q}(\sqrt{2}) \hookrightarrow \mathbb{R}$ (either we send a preferred root of X^2-2 to $\sqrt{2}$ or to $-\sqrt{2}$). In general, the orderings on any number field $\mathbb{K}$ (a finite field extension of $\mathbb{Q}$) correspond bijectively to the embeddings $\mathbb{K} \hookrightarrow \mathbb{R}$ ("the real places"). They are always Archimedean, but never real closed.

The field of rational functions $\mathbb{R}(X)$ can be ordered: Indeed we can define an order by proclaiming $X > 0$ but $X < \lambda$ for all $\lambda \in \mathbb{R}_{\leq 0}$. Then $\mathbb{R}(X)$ is not real closed, since X is positive but does not have a square root in $\mathbb{R}(X)$. With this order $\mathbb{R}(X)$ is Archimedean. However, if we endow $\mathbb{R}(X)$ with a different order, say $X > \lambda$ for all $\lambda \in \mathbb{R}$, then $\mathbb{R}(X)$ is non-Archimedean.

Examples of real closed fields include the real numbers, as well as the *real algebraic numbers*, i.e. the real numbers that are algebraic over $\mathbb{Q}$.

Remark 2.1.3. A real closed field has a unique order. Indeed, the non-negative elements are exactly the squares.

Example 2.1.4. The field of *real Puiseux series* is the set of expressions

$$\mathbb{R}(X)^\wedge := \Big\{ \sum_{k=k_0}^{\infty} c_k X^{k/m} \;\Big|\; k_0 \in \mathbb{Z},\, m \in \mathbb{N} \setminus \{0\},\, c_k \in \mathbb{R} \Big\},$$

together with formal addition and multiplication. An element $\sum_{k=k_0}^{\infty} c_k X^{k/m}$ is positive if $c_{k_0} > 0$. With this order $\mathbb{R}(X)^\wedge$ is real closed, see e.g. [BPR06, Theorem 2.91].

Definition 2.1.5. Let $\mathbb{F}$ be an ordered field. A *real closure* for $\mathbb{F}$ is an algebraic field extension $\mathbb{K}$ that is real closed and such that the inclusion $\mathbb{F} \hookrightarrow \mathbb{K}$ is order-preserving.

Theorem 2.1.6 ([BCR98, Theorem 1.3.2])**.** Every ordered field $(\mathbb{F}, \leq)$ has a real closure. If $\mathbb{K}$ and $\mathbb{K}'$ are two real closures of $(\mathbb{F}, \leq)$, then there exists a unique order-preserving $\mathbb{F}$-isomorphism $\mathbb{K}$ and $\mathbb{K}'$.

Thus in the following we can speak of *the* real closure of an ordered field $\mathbb{F}$, and we write $\overline{\mathbb{F}}^r$.

Example 2.1.7. The real closure of $\mathbb{Q}$ are the real algebraic numbers $\overline{\mathbb{Q}}^r = \overline{\mathbb{Q}} \cap \mathbb{R}$. The real closure of $\mathbb{R}(X)$ (together with the order $X > 0$ but $X < \lambda$ for all $\lambda \in \mathbb{R}$) is $\overline{\mathbb{R}(X)}^r = \mathbb{R}(X)^\wedge_{\text{alg}}$, i.e. the field of real Puiseux series that are algebraic over $\mathbb{R}(X)$.

The main objects of study in real algebraic geometry are semi-algebraic sets. From now on let $\mathbb{F}$ be a real closed field.

Definition 2.1.8. A subset $\mathcal{B} \subseteq \mathbb{F}^n$ is a *basic semi-algebraic set*, if there exists a polynomial $f \in \mathbb{F}[X_1, \ldots, X_n]$ such that

$$\mathcal{B} = \mathcal{B}(f) = \{x \in \mathbb{F}^n \mid f(x) > 0\}.$$

A subset $X \subseteq \mathbb{F}^n$ is *semi-algebraic* if it is a Boolean combination of basic semi-algebraic sets, i.e. X is obtained by taking finite unions and intersections of basic semi-algebraic sets and their complements.

A subset $X \subseteq \mathbb{F}^n$ is *algebraic* if it is the zero set of a set of polynomials in $\mathbb{F}[X_1, \ldots, X_n]$. By Hilbert's basis theorem algebraic sets are semi-algebraic, see e.g. [Lan02, Theorem 4.1].

Let $X \subseteq \mathbb{F}^n$ and $Y \subseteq \mathbb{F}^m$ be two semi-algebraic sets. A map $f \colon X \to Y$ is called *semi-algebraic* if its graph $\operatorname{Graph}(f) \subseteq X \times Y$ is semi-algebraic in $\mathbb{F}^{n+m}$.

Proposition 2.1.9 ([BCR98, Proposition 2.2.7])**.** Let $f \colon X \to Y$ be a semi-algebraic map. If $S \subseteq X$ is semi-algebraic, then so is its image $f(S)$. If $T \subseteq Y$ is semi-algebraic, then so is its preimage $f^{-1}(T)$.

Using the order on $\mathbb{F}$ we can define a topology on $\mathbb{F}$, where a basis of open sets is given by the open intervals. Note that if $\mathbb{F} \neq \mathbb{R}$ then $\mathbb{F}$ is totally-disconnected. However we have the following notion of connectedness for semi-algebraic sets.

Definition 2.1.10. Let $\mathbb{F}$ be a real closed field. A semi-algebraic set $X \subseteq \mathbb{F}^n$ is *semi-algebraically connected* if it cannot be written as the disjoint union of two non-empty semi-algebraic subsets of $\mathbb{F}^n$ both of which are closed in X.

Theorem 2.1.11 ([BCR98, Theorem 2.4.5])**.** A semi-algebraic set of $\mathbb{R}^n$ is connected if and only if it is semi-algebraically connected. Every semi-algebraic set of $\mathbb{R}^n$ has a finite number of connected components, which are semi-algebraic.

From now denote by $\mathbb{F} \subseteq \mathbb{K}$ a real closed extension of $\mathbb{F}$.

Definition 2.1.12. Let $X \subseteq \mathbb{F}^n$ be a semi-algebraic set given as

$$X = \bigcup_{i=1}^{s} \bigcap_{j=1}^{r_i} \{x \in \mathbb{F}^n \mid f_{ij}(x) *_{ij} 0\},$$

with $f_{ij} \in \mathbb{F}[X_1, \ldots, X_n]$ and $*_{ij}$ is either $<$ or $=$ for $i = 1, \ldots, s$ and $j = 1, \ldots, r_i$. The $\mathbb{K}$-*extension* $X_{\mathbb{K}}$ of X is the set given by the same Boolean combination of sign conditions as X, more precisely

$$X_{\mathbb{K}} = \bigcup_{i=1}^{s} \bigcap_{j=1}^{r_i} \{x \in \mathbb{K}^n \mid f_{ij}(x) *_{ij} 0\}.$$

Note that $X_{\mathbb{K}}$ is semi-algebraic and depends only on the set X, and not on the Boolean combination describing it, see [BCR98, Proposition 5.1.1]. The proof of this is based on the Tarski-Seidenberg transfer principle.

Theorem 2.1.13 (Tarski-Seidenberg transfer principle, [BCR98, Theorem 5.2.1])**.** Let $X \subseteq \mathbb{F}^{n+1}$ be a semi-algebraic set. Denote the projection $\mathrm{pr}\colon \mathbb{F}^{n+1} \to \mathbb{F}^n$ onto the first n coordinates by pr. Then $\mathrm{pr}(X) \subseteq \mathbb{F}^n$ is semi-algebraic. Furthermore, if $\mathbb{K}$ is a real closed extension of $\mathbb{F}$, and $\mathrm{pr}_{\mathbb{K}}\colon \mathbb{K}^{n+1} \to \mathbb{K}$ is the projection on the first n coordinates, then

$$\mathrm{pr}_{\mathbb{K}}(X_{\mathbb{K}}) = (\mathrm{pr}(X))_{\mathbb{K}}.$$

Using this one can prove an extension theorem for semi-algebraic maps.

Theorem 2.1.14 ([BCR98, Propositions 5.3.1, 5.3.3, 5.3.5])**.** Let $X \subseteq \mathbb{F}^n$ and $Y \subseteq \mathbb{F}^m$ be two semi-algebraic sets, and $f\colon X \to Y$ a semi-algebraic map. Then $(\mathrm{Graph}(f))_{\mathbb{K}}$ is the graph of a semi-algebraic map $f_{\mathbb{K}}\colon X_{\mathbb{K}} \to Y_{\mathbb{K}}$, that is called the $\mathbb{K}$-*extension* of f. Furthermore, f is injective (respectively surjective, respectively bijective) if and only if $f_{\mathbb{K}}$ is injective (respectively surjective, respectively bijective), and f is continuous if and only if $f_{\mathbb{K}}$ is continuous.

Finally, we have the following relation between extension of semi-algebraic sets and semi-algebraically connected components.

Theorem 2.1.15 ([BCR98, Proposition 5.3.6 (ii)])**.** Let $X \subseteq \mathbb{F}^n$ be semi-algebraic. Then X is semi-algebraically connected if and only if $X_{\mathbb{K}}$ is semi-algebraically connected. More generally, if $C_1, \ldots, C_m$ are the semi-algebraically connected components of X, then $(C_1)_{\mathbb{K}}, \ldots, (C_m)_{\mathbb{K}}$ are the semi-algebraically connected components of $X_{\mathbb{K}}$.

2.2. The real spectrum of a ring

In this section we introduce the notion of the real spectrum of a ring. We follow [BCR98, Chapters 7.1 and 7.2]. Let A be a commutative ring with 1. In our examples, A will often be a polynomial ring or the coordinate ring of an algebraic set, compare Section 2.3.

Definition 2.2.1. The *real spectrum* $\mathrm{RSp}(A)$ is the topological space

$$\mathrm{RSp}(A) = \{(\mathfrak{p}, \leq) \mid \mathfrak{p} \subseteq A \text{ prime ideal }, \leq \text{ is an order on the fraction field } \mathrm{Frac}(A/\mathfrak{p})\}$$

together with the following subbasis of open sets: For $a \in A$ let

$$\mathcal{U}(a) := \{(\mathfrak{p}, \leq) \in \mathrm{RSp}(A) \mid \overline{a}^{\mathfrak{p}} > 0\},$$

where $\overline{a}^{\mathfrak{p}}$ is the image of a in $\mathrm{Frac}(A/\mathfrak{p})$ under the homomorphism

$$A \to A/\mathfrak{p} \to \mathrm{Frac}(A/\mathfrak{p}).$$

Remark 2.2.2. This topology is called the *spectral topology* on $\mathrm{RSp}(A)$.

Remark 2.2.3. Contrary to the spectrum of a ring (where we consider all prime ideals), we restrict our attention here to so called *real ideals*, which motivates the name of the real spectrum. An ideal is called real if, whenever $a_1^2 + \ldots + a_k^2 \in I$ for some $a_1, \ldots, a_k \in A$, we have $a_i \in I$ for all $i = 1, \ldots k$, see [BCR98, Definition 4.1.3]. By [BCR98, Lemma 4.1.6] we see that a prime ideal $I \subseteq A$ is real if and only if the fraction field of A/I is orderable.

Example 2.2.4. (1) Let k be a field. The real spectrum $\mathrm{RSp}(k)$ of k is homeomorphic to the set of orders on k together with the Harrison topology [BCR98, Example 7.1.4 a)]. It is non-empty if and only if k is orderable.

(2) The real spectrum of $\mathbb{Z}$ is one point corresponding to the zero prime ideal (0) and the unique order on the fraction field $\mathbb{Q}$ of $\mathbb{Z}$.

Example 2.2.5 ([BCR98, Example 7.1.4 (b), 7.5.2])**.** Let $\mathbb{R}[X]$ be the polynomial ring in one variable. To describe $\mathrm{RSp}(\mathbb{R}[X])$ we need to understand the prime ideals of $\mathbb{R}[X]$. Since $\mathbb{R}$ is real closed and $\mathbb{R}[X]$ is a principal ideal domain, all non-zero prime ideals are generated by an irreducible polynomial which is either of degree one or two. Irreducible polynomials of degree one correspond to maximal ideals, hence to elements of $\mathbb{R}$, with residue field $\mathbb{R}$. Since $\mathbb{R}$ is real closed it has a unique order. Irreducible polynomials of degree two correspond to algebraic extensions of $\mathbb{R}$ of degree two, which are algebraically closed, and hence not orderable. We also need to describe the total orders on the field of rational function $\mathbb{R}(X)$, which is the fraction field of $\mathbb{R}[X]$ corresponding to the prime ideal (0). It suffices to order the variable X with respect to $\mathbb{R}$. More precisely, we define the following orders. For $\lambda \in \mathbb{R}$ we set

- λ^+: We have $\lambda <_{\lambda^+} X$, but $X <_{\lambda^+} \mu$ for every $\mu \in \mathbb{R}$ with $\lambda < \mu$.
- λ^-: We have $X <_{\lambda^-} \lambda$, but $\mu <_{\lambda^-} X$ for every $\mu \in \mathbb{R}$ with $\mu < \lambda$.
- $+\infty$: We have $\mu <_{+\infty} X$ for all $\mu \in \mathbb{R}$.
- $-\infty$: We have $X <_{-\infty} \mu$ for all $\mu \in \mathbb{R}$.

It turns out that this list is the complete set of total orders on $\mathbb{R}[X]$. Putting everything together we obtain

$$\mathrm{RSp}(\mathbb{R}[X]) = \{(\langle X - \lambda \rangle, \leq_{\mathbb{R}}) \mid \lambda \in \mathbb{R}\} \cup \{((0), \leq_o) \mid o \in \{\lambda^{\pm}, \pm\infty\}\}.$$

The description of prime ideals in $\mathbb{F}[X]$ for $\mathbb{F}$ any real closed field is similar. The set of orders of $\mathbb{F}(X)$ is in one-to-one correspondence with the set of *cuts* (I, J) of $\mathbb{F}$, with $I = \{\mu \in \mathbb{F} \mid \mu < X\}$ and $J = \{\mu \in \mathbb{F} \mid X < \mu\}$. However, in general these need not correspond to elements in $\mathbb{F}$. For example in the case $\mathbb{F} = \overline{\mathbb{Q}} \cap \mathbb{R}$ a cut can be given by a transcendental number in $\mathbb{R} \setminus (\overline{\mathbb{Q}} \cap \mathbb{R})$.

We have an equivalent way of describing points in the real spectrum using homomorphisms.

Proposition 2.2.6 ([BCR98, Proposition 7.1.2])**.** Points $(\mathfrak{p}, \leq) \in \mathrm{RSp}(A)$ are in bijection with equivalence classes of ring homomorphisms $[\varphi\colon A \to \mathbb{F}]$ for a real closed field $\mathbb{F}$, where we consider the equivalence relation generated by proclaiming two homomorphisms $\varphi\colon A \to \mathbb{F}$ and $\varphi'\colon A \to \mathbb{F}'$ to be *equivalent* if there exists an order-preserving field homomorphism $\mathbb{F} \to \mathbb{F}'$ such that the diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[dr, "\varphi'"'] & \mathbb{F} \arrow[d] \\ & \mathbb{F}' \end{tikzcd}$$

More precisely, given $(\mathfrak{p}, \leq)$ we get a homomorphism from A into the real closure of the residue field $\mathrm{Frac}(A/\mathfrak{p})$ with the order $\leq$ by composing the following maps

$$\varphi\colon A \to A/\mathfrak{p} \hookrightarrow \mathrm{Frac}(A/\mathfrak{p}) \hookrightarrow \overline{(\mathrm{Frac}(A/\mathfrak{p}), \leq)}^r.$$

On the other hand, given $[\varphi\colon A \to \mathbb{F}]$ for a real closed field $\mathbb{F}$, take $(\ker(\varphi), \leq)$ with the restriction of the order of $\mathbb{F}$ to $\mathrm{Frac}(A/\ker(\varphi))$. Under this identification we have that for $a \in A$

$$\mathcal{U}(a) = \{[\varphi\colon A \to \mathbb{F}] \mid \mathbb{F} \text{ real closed}, \varphi(a) > 0\}.$$

We now define a second topology on the real spectrum that we only use to prove that the real spectrum is compact. If not otherwise stated, the real spectrum is always considered with the spectral topology.

Definition 2.2.7. A subset of $\mathrm{RSp}(A)$ is called *constructible* if it can be obtained as a Boolean combination, i.e. finite unions, finite intersections and complements, from the sets $\mathcal{U}(a)$ defined above. The *constructible topology* is the topology on $\mathrm{RSp}(A)$ which is generated by the constructible subsets, or equivalently, for which the constructible subsets form a subbasis of the topology.

The constructible topology has more open sets than the spectral topology, since we also define $\mathcal{U}(a)^c$ to be open in the constructible topology.

Example 2.2.8 ([BCR98, Remark 7.1.11])**.** For $A = k$ the constructible and the spectral topology on $\mathrm{RSp}(k)$ agree. Indeed, for $a \in k$ we have

$$\begin{aligned} \mathrm{RSp}(k) \setminus \mathcal{U}(a) &= \{\text{orders on } k \text{ for which } a < 0\} \\ &= \{\text{orders on } k \text{ for which } -a > 0\} = \mathcal{U}(-a). \end{aligned}$$

Theorem 2.2.9 ([Lam84, Theorem 4.1], [BCR98, Proposition 7.1.12])**.** The topological space $\mathrm{RSp}(A)$ together with the constructible topology is a compact, totally disconnected Hausdorff space. In particular, $\mathrm{RSp}(A)$ with its spectral topology is compact (but not necessarily Hausdorff nor totally disconnected).

Proposition 2.2.10 ([BCR98, Proposition 7.1.25 (ii)]). Let $C \subseteq \mathrm{RSp}(A)$ be a constructible subset endowed with the subspace topology of the spectral topology. The topological space C^{cl} of closed points of C is a compact Hausdorff space. In particular $\mathrm{RSp}^{\mathrm{cl}}(A)$, the set of closed points of $\mathrm{RSp}(A)$, is a compact Hausdorff space.

Example 2.2.11 (continuation of Example 2.2.5). The points $((0), \pm\infty) \in \mathrm{RSp}(\mathbb{R}[X])$ are closed. The points $((0), \lambda^{\pm}) \in \mathrm{RSp}(\mathbb{R}[X])$ for $\lambda \in \mathbb{R}$ are not closed. We have $\overline{\{\lambda^{\pm}\}} = \{\lambda^{\pm}, \lambda\}$. The space $\mathrm{RSp}^{\mathrm{cl}}(\mathbb{R}[X])$ is homeomorphic to the two point compactification of $\mathbb{R}$, i.e. to the closed interval $[0,1]$.

We have the following characterization of closed points of $\mathrm{RSp}(A)$. For this we generalize the notion of Archimedean order from Definition 2.1.1.

Definition 2.2.12. Let $A \subseteq A' \subseteq \mathbb{F}$ be two subsets of an ordered field $\mathbb{F}$. We say that A' is *Archimedean over* A if for all $a' \in A$ there exists $a \in A$ with $a' < a$.

With this new definition, an order on $\mathbb{F}$ is Archimedean if $\mathbb{F}$ is Archimedean over $\mathbb{N}$.

Proposition 2.2.13 ([Bru88a, Proposition 2.2 (e)]). A point $(\mathfrak{p}, \leq) \in \mathrm{RSp}(A)$ is closed if and only if $\mathrm{Frac}(A/\mathfrak{p})$ is Archimedean over $\varphi(A)$, where

$$\varphi \colon A \to A/\mathfrak{p} \hookrightarrow \mathrm{Frac}(A/\mathfrak{p}) \hookrightarrow \overline{(\mathrm{Frac}(A/\mathfrak{p}), \leq)}^{r}$$

is defined as in Proposition 2.2.6.

2.3. The real spectrum compactification of semi-algebraic sets

We begin by describing the real spectrum compactification for algebraic sets. Let $V \subset \mathbb{R}^n$ be an algebraic set, i.e. V is the zero set of a family of polynomials with coefficients in $\mathbb{R}$. Let $A(V) := \mathbb{R}[X_1, \ldots, X_n]/I$ be the *coordinate ring* of V, where I is the ideal of all polynomials vanishing on V, i.e. $I = \{f \in \mathbb{R}[X_1, \ldots, X_n] \mid f(v) = 0 \text{ for all } v \in V\}$. Note that $A(V)$ is naturally an $\mathbb{R}$-algebra, hence contains $\mathbb{R}$.

Lemma 2.3.1. Let $\varphi \colon A(V) \to \mathbb{F}$, for $\mathbb{F}$ a real closed field, represent a point in $\mathrm{RSp}(A(V))$. Then $\mathbb{R} \subseteq \mathbb{F}$ and φ is $\mathbb{R}$-linear.

Proof. The ring homomorphism φ restricted to $\mathbb{R}$ is injective (since the only ideals of $\mathbb{R}$ are trivial and φ sends one to one), hence $\mathbb{R} \subseteq \mathbb{F}$ via φ. Furthermore, for $\lambda \in \mathbb{R}$ and $f \in A(V)$, we have $\varphi(\lambda f) = \varphi(\lambda)\varphi(f)$, since φ is a ring homomorphism. $\square$

Proposition 2.3.2 ([BCR98, Proposition 7.1.5]). The map

$$\Psi \colon V \to \mathrm{RSp}(A(V)), \quad v \mapsto (\langle X_1 - v_1, \ldots, X_n - v_n \rangle, \leq_{\mathbb{R}}),$$

where $\leq_{\mathbb{R}}$ is the unique order on $\mathbb{R}$ (refer to Remark 2.1.3), is injective and induces a homeomorphism from V, with its Euclidean topology, onto its image in $\mathrm{RSp}(A(V))$, with its spectral topology.

Proof. The injectivity of Ψ is clear. We write V for its image in $\mathrm{RSp}(A(V))$. To show the second part we consider for every $\varepsilon > 0$ and $v = (v_1, \ldots, v_n) \in V$ the element

$$f_{\varepsilon,v}(X_1, \ldots, X_n) := \varepsilon - \sum_{i=1}^{n} (X_i - v_i)^2 \in A(V).$$

Then

$$\begin{aligned} V \cap \mathcal{U}(f_{\varepsilon,v}) &= \{(\langle X - w\rangle, \leq_{\mathbb{R}}) \in \mathrm{RSp}(A(V)) \mid w \in V,\ \overline{f_{\varepsilon,v}}^{\langle X-w\rangle} > 0\} \\ &= \Big\{ w \in V \;\Big|\; \sum_{i=1}^{n} (w_i - v_i)^2 < \varepsilon \Big\}, \end{aligned}$$

which is the usual basis of the Euclidean topology of V. □

Example 2.3.3. We have already seen an example of the above proposition in Example 2.2.5 for $V = \mathbb{R}$.

We now consider $\mathbb{R}^2$ with its standard basis (e_1, e_2). Let $\mathbb{R}[X, Y]$ be the polynomial ring in two variables. The sequences $s_n := \Psi(ne_1)$ and $s'_n := \Psi(ne_2)$ converge (up to subsequence) in $\mathrm{RSp}(\mathbb{R}[X, Y])$, since the latter is compact. We claim that $\lim_{n\to\infty} s_n \neq \lim_{n\to\infty} s'_n$. We first verify that $\lim_{n\to\infty} s_n = (\langle Y\rangle, \leq_{+\infty})$, where $\leq_{+\infty}$ denotes the order as in Example 2.2.5 on the fraction field of $\mathbb{R}[X, Y]/\langle Y\rangle \cong \mathbb{R}[X]$. Then we show that $\lim_{n\to\infty} s'_n \neq (\langle Y\rangle, \leq_{+\infty})$.

Let us describe the basic open sets $\mathcal{U}(f)$ with $(\langle Y\rangle, \leq_{+\infty}) \in \mathcal{U}(f)$. By definition, this mean that $\overline{f}^{\langle Y\rangle} >_{+\infty} 0$. If we write

$$f(X, Y) = \sum_{i,j} a_{ij} X^i Y^j = \sum_i a_{i0} X^i + \sum_{j\neq 0, i} a_{ij} X^i Y^j,$$

we see that $f(X, Y) + \langle Y\rangle = \sum_i a_{i0} X^i + \langle Y\rangle$. Thus $\overline{f}^{\langle Y\rangle} >_{+\infty} 0$ is equivalent to asking that $p(X) := f(X, 0) = \sum_i a_{i0} X^i >_{+\infty} 0$. The latter means that the coefficient $a_{\deg p, 0}$ of $X^{\deg p}$ is positive.

Choose any $f \in \mathbb{R}[X, Y]$ with $(\langle Y\rangle, \leq_{+\infty}) \in \mathcal{U}(f)$. Recall that for $(x, y) \in \mathbb{R}^2$ we have $\Psi(x, y) \in \mathcal{U}(f)$ if and only if $f(x, y) > 0$. Thus $s_n \in \mathcal{U}(f)$ for all $n \geq n_0 \in \mathbb{N}$, since $f(ne_1) = f(n, 0)$, which is positive for all n large enough by the assumption on f. Thus for all open neighborhoods $\mathcal{U}$ of $(\langle Y\rangle, \leq_{+\infty})$, there exists $n_0 \in \mathbb{N}$ such that $s_n \in \mathcal{U}$ for all $n \geq n_0$, in other words $\lim_{n\to\infty} s_n = (\langle Y\rangle, \leq_{+\infty})$.

On the other hand, if we choose $f \in \mathbb{R}[X] \subseteq \mathbb{R}[X, Y]$ with $(\langle Y\rangle, \leq_{+\infty}) \in \mathcal{U}(f)$ and constant term $c \leq 0$ (which exists by the above considerations), then $f(ne_2) = f(0, n) = c \leq 0$, so $s'_n \notin \mathcal{U}(f)$ for all $n \in \mathbb{N}$. Thus $\lim_{n\to\infty} s'_n \neq (\langle Y\rangle, \leq_{+\infty})$.

Theorem 2.3.4 ([BCR98, Theorem 7.2.3])**.** Let $V \subseteq \mathbb{F}^n$ be an algebraic set. If $X \subseteq V$ is a semi-algebraic set given by a Boolean combination of the basic semi-algebraic sets $\mathcal{B}(f_i)$ for some $f_i \in A(V)$ (Definition 2.1.8), then we define $\widetilde{X}$ to be the constructible subset of $\mathrm{RSp}(A(V))$ given by the same Boolean combination of the open sets $\mathcal{U}(f_i)$ (Definition 2.2.1).

(1) There is an isomorphism of Boolean algebras

$$\begin{aligned}\{\text{semi-algebraic subsets of } V\} &\leftrightarrow \{\text{constructible subsets of } \mathrm{RSp}(A(V))\}\\ X &\mapsto \widetilde{X},\\ \widetilde{X}\cap V &\mapsfrom \widetilde{X}.\end{aligned}$$

(2) X is closed (respectively open) if and only if $\widetilde{X}$ is closed (respectively open).

Remark 2.3.5 ([BCR98, Corollary 7.2.4, Remark 7.2.5])**.** It turns out that $\widetilde{X}$ is intrinsically defined by the semi-algebraic set X (up to homeomorphism) and does not depend on the algebraic set V in which X is embedded.

Proposition 2.3.6 ([BCR98, Proposition 7.2.7])**.** Let X be a closed (respectively open) semi-algebraic subset of an algebraic set V. Then $\widetilde{X}$ is the smallest (respectively largest) closed (respectively open) subset of $\mathrm{RSp}(A(V))$ whose intersection with $\Psi(V)$ is $\Psi(X)$.

With this at hand we are now ready to define and study the properties of the real spectrum compactification of semi-algebraic sets.

Definition 2.3.7. Let $X \subseteq V \subseteq \mathbb{R}^n$ be a semi-algebraic subset of some algebraic set V. Its *real spectrum compactification* $\mathrm{RSp}(X)$ is the closure of its image

$$X \subseteq V \overset{\Psi}{\hookrightarrow} \mathrm{RSp}(A(V)).$$

The definition of the compactification depends on an embedding $X \subseteq V$. This is not the case if we restrict ourselves to closed semi-algebraic sets. The following result, together with Remark 2.3.5, implies that in this case the real spectrum compactification $\mathrm{RSp}(X)$ is intrinsic to X.

Lemma 2.3.8. Let $X \subseteq V \subseteq \mathbb{R}^n$ be a closed semi-algebraic subset of some algebraic set V. Then $\mathrm{RSp}(X) = \widetilde{X}$.

Proof. We use Proposition 2.3.6. Clearly $\Psi(X) \subseteq \mathrm{RSp}(X) \cap \Psi(V)$. But since Ψ is a homeomorphism onto its image (Proposition 2.3.2) we have equality. If Y is now any closed subset of $\mathrm{RSp}(A(V))$ with $Y \cap \Psi(V) = \Psi(X)$, then $\Psi(X) \subseteq Y$, and since Y is closed, also the closure of $\Psi(X)$ is in Y. Thus $\mathrm{RSp}(X) = \widetilde{X}$ by Proposition 2.3.6. □

From now on let $X \subseteq V$ be a closed semi-algebraic set. We saw in Example 2.2.5 and Example 2.2.11 that in general X is not open in $\mathrm{RSp}(X)$ (not even for algebraic sets). Moreover, $\mathrm{RSp}(X)$ with its spectral topology is in general not Hausdorff (Theorem 2.2.9), which is something one often desires when compactifying a Hausdorff topological space. Both of these issues can be resolved when considering the subset of closed points, see Theorem 2.3.11.

Definition 2.3.9. Let $\mathrm{RSp}^{\mathrm{cl}}(X) := \widetilde{X}^{\mathrm{cl}}$ be the subset of closed points in $\mathrm{RSp}(X)$.

Remark 2.3.10. If we view $X \subseteq V$ as a closed semi-algebraic subset of some algebraic set V, then $\mathrm{RSp}^{\mathrm{cl}}(X) = \widetilde{X} \cap \mathrm{RSp}^{\mathrm{cl}}(A(V))$, since $\widetilde{X}$ is closed by Theorem 2.3.4 (2).

Theorem 2.3.11. The closed semi-algebraic set X is dense in $\operatorname{RSp}(X)$, and open and dense in $\operatorname{RSp}^{\mathrm{cl}}(X)$. In particular, $\operatorname{RSp}^{\mathrm{cl}}(X)$ provides a compactification of X.

Before proving this we need the following lemma. Recall that all points in $\Psi(X)$ are closed. On the other hand for boundary points we have the following implication using the point of view of homomorphisms as in Proposition 2.2.6.

Lemma 2.3.12. Let $X \subseteq V \subseteq \mathbb{R}^n$ be a closed semi-algebraic subset of some algebraic set V, and $A(V) = \mathbb{R}[X_1, \ldots, X_n]/I(V)$ the coordinate ring of V. Let $[\varphi\colon A(V) \to \mathbb{F}] \in \partial\operatorname{RSp}^{\mathrm{cl}}(X)$. Then $\varphi(A(V))$ is non-Archimedean over $\mathbb{R}$; in other words, there exists $i \in \{1, \ldots, n\}$ such that $|\varphi(X_i)| > \mu$ for all $\mu \in \mathbb{R}$.

Proof. Assume that $\varphi\colon A(V) \to \mathbb{F}$ represents a point in $\partial\operatorname{RSp}^{\mathrm{cl}}(X)$ and is such that $\varphi(A(V))$ is Archimedean over $\mathbb{R}$. We know that $\operatorname{Frac}(A(V)/\ker(\varphi))$ is Archimedean over $\varphi(A(V))$, since the homomorphism represents a closed point, see Proposition 2.2.13. Thus $\operatorname{Frac}(A(V)/\ker(\varphi))$ is Archimedean over $\mathbb{R}$ and hence a subfield of $\mathbb{R}$, compare e.g. [Pla13]. Since it contains $\mathbb{R}$ we obtain $\operatorname{Frac}(A(V)/\ker(\varphi)) = \mathbb{R}$. Thus $\varphi\colon A(V) \to \mathbb{R}$ is surjective, since it is also $\mathbb{R}$-linear by Lemma 2.3.1. Hence $\ker(\varphi)$ corresponds to a maximal ideal in $A(V)$, i.e. a point in $\Psi(X)$. This is a contradiction, since we started with a point in the boundary. □

Proof of Theorem 2.3.11. By definition X is dense in $\operatorname{RSp}(X)$. For the second claim we simplify the exposition and assume that $X = V$. Similar arguments then also prove the general case.

We need to prove that the image of V is open and dense in $\operatorname{RSp}^{\mathrm{cl}}(A(V))$. We already know that V is dense in $\operatorname{RSp}(A(V))$, hence also in $\operatorname{RSp}^{\mathrm{cl}}(A(V))$. For $v \in V$ let $\Psi(v) = (\langle X - v\rangle, \leq_{\mathbb{R}})$ be its image in $\operatorname{RSp}^{\mathrm{cl}}(A(V))$. We construct an open neighbourhood of $\Psi(v)$ contained in $\Psi(V)$, which implies that the latter is open in $\operatorname{RSp}^{\mathrm{cl}}(A(V))$. Consider, as in the proof of Proposition 2.3.2, the element $f_{\varepsilon,v} \in A(V)$. Then clearly $\Psi(v) \in \mathcal{U}(f_{\varepsilon,v})$ and we claim that

$$\mathcal{U}(f_{\varepsilon,v}) \cap \operatorname{RSp}^{\mathrm{cl}}(A(V)) \subseteq \Psi(V).$$

We take the point of view of homomorphisms as in Proposition 2.2.6. We immediately see that if $\varphi\colon A(V) \to \mathbb{F}$ is in $\mathcal{U}(f_{\varepsilon,v})$, then

$$|\varphi(X_i) - v_i| \leq \varepsilon \text{ for all } i = 1, \ldots, n,$$

which implies that $\varphi(A(V))$ is Archimedean over $\mathbb{R}$. However the only closed points Archimedean over $\mathbb{R}$ are points in $\Psi(V)$, see Lemma 2.3.12, which proves the claim. □

$$\begin{tikzcd}
 & \operatorname{RSp}^{\mathrm{cl}}(X) \arrow[dd, hook] \\
X \arrow[ur, hook, "\text{open and dense}"] \arrow[dr, hook, "\text{dense}"'] & \\
 & \operatorname{RSp}(X)
\end{tikzcd}$$

Let us collect some important properties of the real spectrum compactification of closed semi-algebraic sets. Recall that in this case we have $\operatorname{RSp}(X) = \widetilde{X}$.

Proposition 2.3.13 ([BCR98, Proposition 7.5.1])**.** Let X be a semi-algebraic set. Then X is semi-algebraically connected if and only if $\widetilde{X}$ is connected. Furthermore, if $X_1, \dots, X_m$ are the semi-algebraically connected components of X, then $\widetilde{X}_1, \dots, \widetilde{X}_m$ are the connected components of $\widetilde{X}$.

From this remark we immediately obtain that the real spectrum compactification of a closed connected semi-algebraic set is connected. The following proposition tells us that semi-algebraic maps extend continuously to the compactification.

Proposition 2.3.14 ([BCR98, Proposition 7.2.8])**.** Let X and Y be two semi-algebraic sets and $f\colon X \to Y$ a semi-algebraic map. Then there exists a unique map $\tilde{f}\colon \widetilde{X} \to \widetilde{Y}$ such that for all semi-algebraic subsets $Y' \subseteq Y$ we have

$$\tilde{f}^{-1}(\widetilde{Y'}) = \widetilde{f^{-1}(Y')}.$$

If additionally f is a homeomorphism then so if $\tilde{f}$.

In 2.2.5, we saw that $\mathrm{RSp}^{\mathrm{cl}}(\mathbb{R})$ is homeomorphic to the closed interval. In higher dimension, this compactification is no longer a closed ball. However, there is still a fixed-point theorem for this compactification.

Theorem 2.3.15 ([Bru88b], Hopf fixed point theorem for semi-algebraic maps [Bru92])**.** Let $X \subseteq \mathbb{R}^n$ be a semi-algebraic set and $f\colon X \to X$ a continuous semi-algebraic map with $\mathrm{tr}(f_*) \neq 0$, where

$$\mathrm{tr}(f_*) = \sum_{i=0}^{n} (-1)^i \,\mathrm{tr}(f_*\colon H_i(X;\mathbb{Q}) \to H_i(X;\mathbb{Q}))$$

is the Lefschetz number of f. Then either f has a fixed point in X or $\tilde{f}$ has a closed fixed point in $\widetilde{X}$.

Remark 2.3.16. If $X \subseteq \mathbb{R}^n$ is contractible semi-algebraic, then $\mathrm{tr}(f_*) = 1$ for any continuous semi-algebraic map $f\colon X \to X$, so the Hopf fixed point theorem implies a Brouwer fixed point theorem for semi-algebraic maps, compare [Bru88b].

3. Character varieties and compactifications

3.1. Character varieties

Let Γ be a finitely generated group. Let $\mathbf{G} \leq \mathrm{GL}_n$ be a connected semisimple algebraic group defined over $\mathbb{R}$. Its real points $\mathbf{G}(\mathbb{R})$ are a real Lie group. Let G be a subgroup of $\mathbf{G}(\mathbb{R})$ which contains the identity component of $\mathbf{G}(\mathbb{R})$, i.e. $\mathbf{G}(\mathbb{R})^\circ \subseteq G \subseteq \mathbf{G}(\mathbb{R})$. Note that G is not necessarily a real algebraic group. However, since G has finite index in $\mathbf{G}(\mathbb{R})$, it is open and closed, hence made out of components of $\mathbf{G}(\mathbb{R})$ and therefore semi-algebraic.

Example 3.1.1. If we take $\mathbf{G} = \mathrm{PGL}_m \leq \mathrm{GL}_{m^2}$, then $\mathrm{PGL}_m(\mathbb{R})^\circ = \mathrm{PSL}_m(\mathbb{R})$, which has index two in $\mathrm{PGL}_m(\mathbb{R})$, and G can thus be either $\mathrm{PSL}_m(\mathbb{R})$ or $\mathrm{PGL}_m(\mathbb{R})$.

The topology of G endows the space $\mathrm{Hom}(\Gamma, G)$ of group homomorphisms from Γ to G with the topology of point-wise convergence. The group G acts on $\mathrm{Hom}(\Gamma, G)$ by conjugation, i.e. for all $\rho \in \mathrm{Hom}(\Gamma, G)$ and $g \in G$ we have

$$(g.\rho)(\gamma) \coloneqq g\rho(\gamma)g^{-1} \text{ for all } \gamma \in \Gamma.$$

Definition 3.1.2. A representation $\rho\colon \Gamma \to G$ is *reductive* if, seen as a linear representation on $\mathbb{R}^n$, it is completely reducible, i.e. a direct sum of irreducible subspaces. Similarly, if $\mathbb{F}$ is a real closed field extension and $G_{\mathbb{F}}$ the $\mathbb{F}$-extension of G (Definition 2.1.12), we say that $\rho\colon \Gamma \to G_{\mathbb{F}}$ is reductive if, seen as a linear representation on $\mathbb{F}^n$, it is completely reducible.

Denote the set of reductive homomorphisms from Γ to G by $\mathrm{Hom}_{\mathrm{red}}(\Gamma, G)$. Any irreducible representation is reductive.

Theorem 3.1.3 ([Bou12, §20, p. 376 Corollaire a)])**.** Let $\rho, \rho'\colon \Gamma \to \mathrm{GL}(n, \mathbb{R})$ be two reductive linear representations of Γ with $\mathrm{tr}(\rho(\gamma)) = \mathrm{tr}(\rho'(\gamma))$ for all $\gamma \in \Gamma$. Then ρ and ρ' are conjugate, i.e. there exists $g \in \mathrm{GL}(n, \mathbb{R})$ with $\rho(\gamma) = g\rho'(\gamma)g^{-1}$ for all $\gamma \in \Gamma$. In other words, reductive representations are up to isomorphism determined by their trace function. The same holds true for reductive representations into $\mathrm{GL}(n, \mathbb{F})$.

The subset of reductive homomorphisms is invariant under the action of G on $\mathrm{Hom}(\Gamma, G)$ by conjugation, which allows us to define the following.

Definition 3.1.4. The *character variety* is the topological quotient

$$\chi(\Gamma, G) \coloneqq \mathrm{Hom}_{\mathrm{red}}(\Gamma, G)/G,$$

where G acts on $\mathrm{Hom}_{\mathrm{red}}(\Gamma, G)$ by conjugation.

The following theorem explains why we restrict our attention to the set of reductive representations.

Theorem 3.1.5. Let Γ and G be as above. A representation $\rho\colon \Gamma \to G$ is reductive if and only if the orbit $G \cdot \rho$ is closed in $\operatorname{Hom}(\Gamma, G)$.

For a proof of this result see e.g. [Sik12, Theorem 30] based on an argument by [JM87, Theorem 1.1]. From this we see that the quotient $\operatorname{Hom}(\Gamma, G)/G$ has non closed points in general. This is the reason why we replace $\operatorname{Hom}(\Gamma, G)$ by $\operatorname{Hom}_{\mathrm{red}}(\Gamma, G)$ in the above definition. In fact more is true.

Theorem 3.1.6. The character variety $\chi(\Gamma, G)$ is a Hausdorff topological space.

This follows from Theorem 3.2.8. For more details on the subtleties in the various definitions of character varieties and how they relate, we refer to [Mar22a, Sections 2 and 3].

3.2. Semi-algebraic models for character varieties and their components

Let Γ and G be as in Section 3.1. The real semi-algebraic structure of G endows the space $\operatorname{Hom}(\Gamma, G)$ with a real semi-algebraic structure by choosing a finite set of generators for Γ. More precisely, if $F = \{\gamma_1, \ldots, \gamma_k\}$ is a finite generating set for Γ, the map

$$\mathrm{ev}\colon \operatorname{Hom}(\Gamma, G) \to G^F, \quad \rho \mapsto (\rho(\gamma_1), \ldots, \rho(\gamma_k))$$

induces a homeomorphism between $\operatorname{Hom}(\Gamma, G)$ and its image $X_F(\Gamma, G)$. Note that $X_F(\Gamma, G)$ is a semi-algebraic subset of $\mathbb{R}^N$ for some $N \in \mathbb{N}$. If F' is a different choice of a generating set for Γ, then $X_F(\Gamma, G)$ and $X_{F'}(\Gamma, G)$ are semi-algebraically homeomorphic. From now on we drop the choice of a generating set F from the notation. Denote by $X^{\mathrm{red}}(\Gamma, G)$ the image under the map ev of the space of reductive homomorphisms $\operatorname{Hom}_{\mathrm{red}}(\Gamma, G)$. We show in Lemma 3.2.9 that $X^{\mathrm{red}}(\Gamma, G)$ is real semi-algebraic as a subset of $X(\Gamma, G)$. From now on we identify $\operatorname{Hom}(\Gamma, G)$ and $\operatorname{Hom}_{\mathrm{red}}(\Gamma, G)$ with their respective images under the map ev.

Definition 3.2.1. A *semi-algebraic model for* $\chi(\Gamma, G)$ is a semi-algebraic, continuous map

$$p\colon \operatorname{Hom}_{\mathrm{red}}(\Gamma, G) \to \mathbb{R}^d$$

for some $d \in \mathbb{N}$, such that the fibres over the image of p are exactly the G-orbits, and p induces a homeomorphism $\operatorname{Hom}_{\mathrm{red}}(\Gamma, G)/G \xrightarrow{\sim} \operatorname{Im}(p) \subseteq \mathbb{R}^d$.

Note that $\operatorname{Im}(p)$ as the image of a semi-algebraic map is semi-algebraic, see Proposition 2.1.9.

Lemma 3.2.2. A semi-algebraic model for $\chi(\Gamma, G)$ is unique up to semi-algebraic homeomorphism, that means that if $p\colon \operatorname{Hom}_{\mathrm{red}}(\Gamma, G) \to \mathbb{R}^d$ and $p'\colon \operatorname{Hom}_{\mathrm{red}}(\Gamma, G) \to \mathbb{R}^{d'}$ are two such models, then there exists a unique semi-algebraic homeomorphism $f\colon \operatorname{Im}(p) \to \operatorname{Im}(p')$ with $f \circ p = p'$.

Proof. Assume we have two semi-algebraic models p and p' for $\chi(\Gamma, G)$. The map p factors through $\mathrm{Im}(p')$, since both p and p' have the same fibres, and induces a continuous bijective map $\bar{p}\colon \mathrm{Im}(p') \to \mathrm{Im}(p)$. We claim that $\bar{p}$ is also semi-algebraic. We have

$$\begin{aligned}\mathrm{Graph}(\bar{p}) &= \{(x, y) \in \mathrm{Im}(p') \times \mathrm{Im}(p) \mid \bar{p}(x) = y\} \\ &= \{(x, y) \in \mathrm{Im}(p') \times \mathrm{Im}(p) \mid \exists\, c \in \mathrm{Hom}_{\mathrm{red}}(\Gamma, G) \text{ s.t. } p'(c) = x,\, p(c) = y\},\end{aligned}$$

which by Tarski-Seidenberg, see Theorem 2.1.13, is a semi-algebraic set since both p and p' are semi-algebraic. Reversing the roles of p and p' in the above argument proves the claim. $\square$

If $\mathcal{C} \subseteq \mathrm{Hom}(\Gamma, G)$ is a G-invariant connected component–hence semi-algebraic by Theorem 2.1.11–consisting of reductive homomorphisms, we can define in an analogous way a semi-algebraic model for the quotient $\mathcal{C}/G$. Indeed, the restriction to $\mathcal{C}$ of the semi-algebraic model for the whole character variety coming from [RS90, Theorem 7.6] provides a semi-algebraic model for $\mathcal{C}/G$. In general we can find different such models, which are related by semi-algebraic homeomorphisms, see Lemma 3.2.2. In fact, for connected components of geometric significance, there are often other semi-algebraic models that exploit their geometric interpretation. For example in Chapter 6 we introduce a variant of the Bonahon-Dreyer coordinates for the Hitchin component, which generalize the shear coordinates for Teichmüller space (the case of $\mathrm{PSL}(2, \mathbb{R})$) developed by Thurston [Thu22, Section 9] and [Bon96, Theorem A].

Lemma 3.2.3. Let $p\colon \mathcal{C} \to \mathbb{R}^d$ be a semi-algebraic model for $\mathcal{C}/G$, with image $p(\mathcal{C})$. For a real closed field extension $\mathbb{R} \subseteq \mathbb{F}$, we can consider the $\mathbb{F}$-extension of this model, i.e.

$$p_{\mathbb{F}}\colon \mathcal{C}_{\mathbb{F}} \to \mathbb{F}^d$$

with image $p(\mathcal{C})_{\mathbb{F}} = p_{\mathbb{F}}(\mathcal{C}_{\mathbb{F}})$. Then the fibres over $p(\mathcal{C})_{\mathbb{F}}$ are exactly the $G_{\mathbb{F}}$-orbits.

Proof. For $\rho \in \mathcal{C}$ the fibre over $p(\rho) \coloneqq [\rho]$ is exactly its G-orbit, since p is a semi-algebraic model. Hence

$$\begin{aligned}p^{-1}([\rho]) &= \{\rho' \in \mathcal{C} \mid p(\rho) = p(\rho')\} \\ &= \{\rho' \in \mathcal{C} \mid \exists\, g \in G : \rho = g\rho' g^{-1}\} \\ &= \mathrm{pr}_{\mathcal{C}}\left(\{(\rho', g) \in \mathcal{C} \times G \mid \rho = g\rho' g^{-1}\}\right),\end{aligned}$$

which is semi-algebraic as a projection of a semi-algebraic set by the Tarski-Seidenberg principle (Theorem 2.1.13). Thus the $p_{\mathbb{F}}$-fibre over $p(\mathcal{C})_{\mathbb{F}}$ is a $G_{\mathbb{F}}$-orbit; see also Theorem 2.1.14. $\square$

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook, "\mathbb{F}\text{-extension}"'] \arrow[d, two heads, "p"'] & \mathcal{C}_{\mathbb{F}} \arrow[d, two heads, "p_{\mathbb{F}}"] \\ p(\mathcal{C}) \arrow[r, hook, "\mathbb{F}\text{-extension}"'] & p(\mathcal{C})_{\mathbb{F}} \end{tikzcd}$$

The aim for the rest of this section is to show the existence of a semi-algebraic model for $\chi(\Gamma, G)$. For this we follow Richardson-Slodowy [RS90]. We first state the general theory developed by the authors and later apply it to character varieties. Let G be as before. Choose a Cartan involution $\theta\colon G \to G$ and let $K = G^\theta$ be the subgroup of fixed points of θ. Then K is compact. Furthermore the Lie algebra $\mathfrak{g}$ of G splits as a direct sum in the $+1$ and -1 eigenspaces of θ, denoted $\mathfrak{k}$ respectively $\mathfrak{p}$. Let $\mathbf{V}$ be a finite dimensional complex vector space and $\mathbf{G} \to \mathrm{GL}(\mathbf{V})$ a rational representation of $\mathbf{G}$ defined over $\mathbb{R}$. The real structure of $\mathbf{V}$ is a real vector space V and we are interested in the orbit spaces of the induced G-action on V, i.e. $G \to \mathrm{GL}(V)$.

Remark 3.2.4. For all $v \in V$ there exists a unique closed G-orbit in the closure of $G \cdot v$, see [RS90, Section 7.3.1]. This allows us to define the following, compare [RS90, Section 7.2].

Definition 3.2.5. Let $V/\!\!/G$ be the set of closed G-orbits on V. We define

$$\pi\colon V \to V/\!\!/G,\ v \mapsto \text{unique closed } G\text{-orbit in the closure of } G \cdot v.$$

We endow $V/\!\!/G$ with the quotient topology under the map π and call it the *Richardson-Slodowy quotient* of V by G.

In the following we describe how $V/\!\!/G$ is homeomorphic to a closed semi-algebraic set in some $\mathbb{R}^d$, which implies that $V/\!\!/G$ is Hausdorff.

The action of G on V induces a homomorphism $\mathfrak{g} \to \mathfrak{gl}(V)$, the Lie algebra of $\mathrm{GL}(V)$. There is a K-invariant scalar product $\langle\,,\,\rangle$ on V such that $\mathfrak{p}$ acts on V by self-adjoint operators, compare [RS90, Section 2].

Definition 3.2.6. We say that $v \in V$ is a *minimal vector* if $\langle v, v\rangle \le \langle gv, gv\rangle$ for all $g \in G$. The set of minimal vectors is denoted by $\mathcal{M}$.

The main results about minimal vectors are the following.

Theorem 3.2.7 ([RS90, Theorem 4.3, 4.4])**.** Let $v \in V$. Then the following hold.

(1) $v \in \mathcal{M} \iff \langle X \cdot v, v\rangle = 0$ for all $X \in \mathfrak{p}$. Furthermore, if $v \in \mathcal{M}$ then $G \cdot v \cap \mathcal{M} = K \cdot v$.

(2) $G \cdot v \cap \mathcal{M} \neq \emptyset$ if and only if $G \cdot v$ is closed.

Theorem 3.2.8 ([RS90, Theorem 7.6, 7.7])**.** The inclusion $\mathcal{M} \subseteq V$ induces a homeomorphism $\mathcal{M}/K \to V/\!\!/G$. In particular, $V/\!\!/G$ is homeomorphic to a closed semi-algebraic set in some $\mathbb{R}^d$.

More generally, if X is a closed G-invariant subset of V, set $\mathcal{M}_X = \mathcal{M} \cap X$. Then the continuous map $\mathcal{M}_X/K \to X/\!\!/G$ determined by the inclusion map $\mathcal{M}_X \hookrightarrow X$ is a homeomorphism. If, in addition, X is a real semi-algebraic subset of V, then $X/\!\!/G$ is homeomorphic to a closed semi-algebraic set in some $\mathbb{R}^d$.

By Theorem 3.2.7 (1) we see that $\mathcal{M}$ is an algebraic subset of V, compare [RS90, Remark 4.5 (e)]. Thus the fact that $\mathcal{M}/K$ is homeomorphic to a closed semi-algebraic subset of some $\mathbb{R}^d$ follows from classical results in the study of quotients by compact

Lie groups and go back to Schwarz [Sch75]. We refer to [RS90, Section 7.1] or for more details to [NGSdS03, Section 11].

We apply the general theory to character varieties in the following way: The map which associates to a reductive homomorphism the K-equivalence class of the minimal vector of its conjugacy class is a semi-algebraic model for the character variety, refer to Theorem 3.2.11. Let us describe it in more detail. As in Section 3.1 let Γ be a finitely generated group on k generators and $G \subseteq \mathrm{Mat}(m, \mathbb{R}) = \mathbb{R}^{m \times m}$ a closed semi-algebraic subset. We consider the vector space $V = \bigoplus_{i=1}^{k} \mathrm{Mat}(m, \mathbb{R})$, on which G acts diagonally by conjugation. We saw in Section 3.2 how $\mathrm{Hom}(\Gamma, G)$ can be naturally identified with a closed semi-algebraic subset $X := X(\Gamma, G) \subseteq V$, which is invariant under G-conjugation. Thanks to Theorem 3.1.5 and Theorem 3.2.7 (2) minimal homomorphisms are reductive, i.e. $\mathcal{M}_X \subseteq X^{\mathrm{red}} := X^{\mathrm{red}}(\Gamma, G)$. Let us take this moment to explain something we announced at the beginning of this section.

Lemma 3.2.9. The set X^{red} is semi-algebraic.

Proof. Consider the semi-algebraic set $\mathcal{M}' := \{(x, g) \in X \times G \mid g \cdot x \in \mathcal{M}_X\} \subseteq X \times G$. By the Tarski-Seidenberg transfer principle, see Theorem 2.1.13, the image of the projection map

$$\mathrm{pr} \colon \mathcal{M}' \to X,\ (x, g) \mapsto x$$

is semi-algebraic. But $\mathrm{pr}(\mathcal{M}')$ is exactly X^{red} by Theorem 3.2.7 (2). □

Remark 3.2.10. By Theorem 3.1.5, the reductive homomorphisms are exactly those with closed G-orbit, hence their Richardson-Slodowy quotients agree, i.e. $X /\!\!/ G = X^{\mathrm{red}} /\!\!/ G$. Since minimal homomorphisms are reductive, we also have $\mathcal{M}_{X^{\mathrm{red}}} = \mathcal{M}_X$.

Theorem 3.2.11. The composition of maps

$$p \colon \mathrm{Hom}_{\mathrm{red}}(\Gamma, G) \xrightarrow{\mathrm{ev}} X^{\mathrm{red}} \xrightarrow{\pi} X^{\mathrm{red}} /\!\!/ G \cong \mathcal{M}_{X^{\mathrm{red}}}/K \subseteq \mathbb{R}^d$$

is a semi-algebraic model for $\chi(\Gamma, G)$.

Proof. We apply Theorem 3.2.8 to X^{red}, and obtain that p is continuous, surjective onto a semi-algebraic set in $\mathbb{R}^d$ such that the fibres over the image of p are exactly the G-orbits, and such that p induces a homeomorphism of $\mathrm{Hom}_{\mathrm{red}}(\Gamma, G)/G$ onto its image. □

3.3. The real spectrum compactification of character varieties

We finish the preliminaries by combining the results of Section 2.3 and Section 3.2. In the last subsection we saw that $\chi(\Gamma, G)$ is homeomorphic to a closed semi-algebraic set $X(\Gamma, G) := \mathcal{M}_{X^{\mathrm{red}}}/K$ of some $\mathbb{R}^d$. Using the results from Section 2.3 we can embed $\chi(\Gamma, G)$ in the compact space

$$\mathrm{RSp}(\chi(\Gamma, G)) := \widetilde{X(\Gamma, G)} \subseteq \mathrm{RSp}(\mathbb{R}[X_1, \ldots, X_d])$$

with dense image, and hence the latter provides a compactification of $\chi(\Gamma, G)$.

We saw in Theorem 1.2.1 that a point in the compactification can be represented by a reductive representation $\rho\colon \Gamma \to G_{\mathbb{F}}$ for $\mathbb{F} \supseteq \mathbb{R}$ a real closed field. Even more is true: In the equivalence class of $(\rho, \mathbb{F})$ there is a "smallest" field of definition for ρ. Let us make this precise.

Definition 3.3.1. Let $\mathbb{F}$ be a real closed field and $\rho\colon \mathbb{F} \to G_{\mathbb{F}}$ a homomorphism. Then $\mathbb{F}$ is *ρ-minimal* if there is no proper real closed subfield $\mathbb{K} \subseteq \mathbb{F}$ such that ρ is conjugate into $G_{\mathbb{K}}$ by an element of $G_{\mathbb{F}}$.

Proposition 3.3.2 ([BIPP23])**.** If $\rho\colon \Gamma \to G_{\mathbb{F}}$ is reductive, then there exists a unique real closed subfield $\mathbb{F}_\rho \subseteq \mathbb{F}$ such that ρ is $G_{\mathbb{F}}$-conjugate to a representation $\rho'\colon \Gamma \to G_{\mathbb{F}_\rho}$ and $\mathbb{F}_\rho$ is ρ'-minimal.

We can now state a refined version of Theorem 1.2.1 from the introduction.

Theorem 3.3.3 ([BIPP21b, Theorem 2])**.** Let Γ and G be as above. Then

$$\mathrm{RSp}(\chi(\Gamma, G)) \cong \left\{ (\rho, \mathbb{F}_\rho) \;\middle|\; \begin{array}{l} \rho\colon \Gamma \to G_{\mathbb{F}_\rho} \text{ reductive homomorphism,} \\ \mathbb{F}_\rho \supseteq \mathbb{R} \text{ real closed, } \rho\text{-minimal} \end{array} \right\} \Big/ \sim,$$

for the same equivalence relation as in Theorem 1.2.1. Points in the boundary correspond to representations into $G_{\mathbb{F}_\rho}$ for $\mathbb{F}_\rho$ a non-Archimedean field. Moreover $\mathbb{F}_\rho$ is of finite transcendence degree over $\mathbb{R}$.

Let us now focus on the case when $\Gamma = \pi_1(S)$ is a surface group and $G = \mathrm{PSL}(n, \mathbb{R})$. The Hitchin component $\mathrm{Hit}(S, n) \subseteq \chi(S, \mathrm{PSL}(n, \mathbb{R}))$ is again closed semi-algebraic as it is a connected component of a closed semi-algebraic set. We denote by $\mathrm{RSp}(\mathrm{Hit}(S, n))$ its real spectrum compactification. By Theorem 2.3.4 and Proposition 2.3.13 it agrees with the closure of $\mathrm{Hit}(S, n)$ in $\mathrm{RSp}(\chi(S, \mathrm{PSL}(n, \mathbb{R})))$. Recall from Definition 1.4.1 that a representation $\rho\colon \pi_1(S) \to \mathrm{PSL}(n, \mathbb{F})$ is $\mathbb{F}$-Hitchin if its $\mathrm{PSL}(n, \mathbb{F})$-equivalence class lies in the $\mathbb{F}$-extension of $\mathrm{Hit}(S, n)$.

Theorem 3.3.4 ([BIPP21b, Theorem 46])**.**

$$\mathrm{RSp}(\mathrm{Hit}(S, n)) \cong \left\{ (\rho, \mathbb{F}_\rho) \;\middle|\; \begin{array}{l} \rho\colon \pi_1(S) \to \mathrm{PSL}(n, \mathbb{F}_\rho) \text{ is } \mathbb{F}_\rho\text{-Hitchin,} \\ \mathbb{F}_\rho \supseteq \mathbb{R} \text{ real closed, } \rho\text{-minimal} \end{array} \right\} \Big/ \sim.$$

Furthermore, $\mathbb{F}_\rho = \overline{\mathbb{R}(\mathrm{tr}(\mathrm{Ad}(\rho)))}^r$, where Ad is the adjoint representation of $\mathrm{PSL}(n, \mathbb{F}_\rho)$. In addition, $(\rho, \mathbb{F}_\rho)$ represents a closed point if and only if $\mathbb{F}_\rho$ is Archimedean over the ring of traces $\mathbb{R}[\mathrm{tr}(\mathrm{Ad}(\rho))]$ of $\mathrm{Ad} \circ \rho$.

Loosely speaking, the goal of the following chapters is to replace the word "$\mathbb{F}_\rho$-Hitchin" by the word "$\mathbb{F}_\rho$-positive weakly dynamics preserving" in this theorem. Let us end the preliminaries by illustrating in an example how one can think of a sequence of representations to converge in the real spectrum compactification.

Example 3.3.5. Let $\Delta := \Delta(3, 3, 4) := \langle a, b \mid a^3 = b^3 = (ab)^4 = 1 \rangle$ be the $(3, 3, 4)$-triangle rotation group. It contains a surface subgroup $\pi_1(S)$ of genus two of finite index. Long-Reid-Thistlethwaite [LRT11] prove that for all $t \in \mathbb{R}$ the restriction of

$$\begin{aligned} &\rho_t \colon \Delta \to \mathrm{PSL}(3,\mathbb{R}), \\ &a \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 1 & 2-t+t^2 & 3+t^2 \\ 0 & -2+2t-t^2 & -1+t-t^2 \\ 0 & 3-3t+t^2 & (t-1)^2 \end{pmatrix} \end{aligned}$$

to $\pi_1(S)$ are Hitchin representations. Denote by $\rho := \lim_{t\to\infty} \rho_t$ the limit of this sequence of representations as $t \to \infty$ in $\mathrm{RSp}(\chi(\Delta, \mathrm{PSL}(3,\mathbb{R})))$. By similar argument as in Example 2.2.5 and Example 2.3.3, ρ can be represented by the homomorphism

$$\begin{aligned} &\rho \colon \Delta \to \mathrm{PSL}(3, \overline{\mathbb{R}(X)}^r), \\ &a \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 1 & 2-X+X^2 & 3+X^2 \\ 0 & -2+2X-X^2 & -1+X-X^2 \\ 0 & 3-3X+X^2 & (X-1)^2 \end{pmatrix}, \end{aligned}$$

where $\overline{\mathbb{R}(X)}^r$ denotes the real closure of $\mathbb{R}(X)$ together with the order $+\infty$, compare Example 2.2.5. The restriction of ρ to $\pi_1(S)$ then describes a point in $\mathrm{RSp}(\mathrm{Hit}(S,3))$ and is $\overline{\mathbb{R}(X)}^r$-Hitchin. Note that $t \to -\infty$ gives a different limit point in the compactification.

Part II.

Coordinates for the Hitchin component

4. Configuration spaces of tuples of flags

In this part we describe an explicit semi-algebraic model for the Hitchin component. It is based on the seminal work of Fock-Goncharov [FG06]. For an introduction to their work in low-dimensional cases we recommend [CTT20], [FG07], and more generally [Pal13]. Originally defined for non-closed surfaces, Bonahon-Dreyer [BD14] adapt their coordinates to closed surfaces. To define them we make use of the positive limit map into the flag variety associated to Hitchin representations, see Theorem 1.3.3. Thus before we introduce them, we need to study the configuration space of tuples of flags and the notion of positivity.

For now let $\mathbb{F}$ be any field.

Definition 4.0.1. A *(full) flag E in $\mathbb{F}^n$* is an increasing sequence of subspaces in the finite-dimensional $\mathbb{F}$-vector space $\mathbb{F}^n$, i.e.

$$E = \left(\{0\} = E^{(0)} \subset E^{(1)} \subset \ldots \subset E^{(n-1)} \subset E^{(n)} = \mathbb{F}^n \right),$$

such that $\dim \left(E^{(a)}\right) = a$ for all $a = 0, \ldots, n$.

Given a flag E we use the notation $E^{(a)}$ to denote the a-dimensional subspace of $\mathbb{F}^n$ defined by E. In this thesis we are only concerned with full flags, and we omit the word *full* in the following when referring to full flags in $\mathbb{F}^n$. The natural action of the general linear group $\mathrm{GL}(\mathbb{F}^n)$ on $\mathbb{F}^n$ induces an action on the space of flags $\mathrm{Flag}(\mathbb{F}^n)$. The action descends to an action of the projective linear group $\mathrm{PGL}(\mathbb{F}^n)$ on $\mathrm{Flag}(\mathbb{F}^n)$, which is transitive.

Definition 4.0.2. A k-tuple $(E_1, \ldots, E_k)$ of flags in $\mathbb{F}^n$ is called *transverse* if for every $a_1, \ldots, a_k \in \{0, \ldots, n\}$ with $\sum_{i=1}^k a_i = n$

$$E_1^{(a_1)} + \ldots + E_k^{(a_k)} = \mathbb{F}^n.$$

The space of k-tuples of transverse flags is denoted by $\mathrm{Flag}(\mathbb{F}^n)^{(k)}$.

Definition 4.0.3. For $k \geq 1$ consider the diagonal action of $\mathrm{PGL}(\mathbb{F}^n)$ on $\mathrm{Flag}(\mathbb{F}^n)^{(k)}$. The space

$$\mathrm{Conf}^{(k)}(\mathbb{F}) := \mathrm{Flag}(\mathbb{F}^n)^{(k)}/\mathrm{PGL}(\mathbb{F}^n)$$

is the *configuration space* of k-tuples of transverse flags in $\mathbb{F}^n$.

We observe that $\mathrm{PGL}(\mathbb{F}^n)$ acts transitively on flags. For a pair of transverse flags (E, F) the subspaces $E^{(a)} \cap F^{(n-a+1)}$ for $a = 1, \ldots, n$ are one-dimensional and in direct sum. By considering a basis adapted to this line decomposition of $\mathbb{F}^n$ one sees that $\mathrm{PGL}(\mathbb{F}^n)$ acts transitively on pairs of transverse flags. Hence $\mathrm{Conf}^{(1)}(\mathbb{F}) = \mathrm{Conf}^{(2)}(\mathbb{F}) = \{\cdot\}$. As soon as $k > 2$ we have $|\mathrm{Conf}^{(k)}(\mathbb{F})| > 1$. We will now parametrize $\mathrm{Conf}^{((3)}(\mathbb{F})$ and $\mathrm{Conf}^{(4)}(\mathbb{F})$.

4.1. Triple ratios and configuration spaces of triples of flags

The goal of this section is to find coordinates to describe the configuration space $\mathrm{Conf}^{(3)}(\mathbb{F})$. We begin with the following observation about the stabilizer in $\mathrm{PGL}(\mathbb{F}^n)$ of a triple of transverse flags.

Proposition 4.1.1. Let $(E, F, G) \in \mathrm{Flag}(\mathbb{F}^n)^{(3)}$ be a transverse triple of flags. Then $\mathrm{Stab}_{\mathrm{PGL}(\mathbb{F}^n)}(E, F, G) = \{\mathrm{Id}_{\mathrm{PGL}(\mathbb{F}^n)}\}$.

Proof. By the transversality of the triple, we can choose a basis $e'_1, \ldots, e'_n$ of $\mathbb{F}^n$ such that for all $a = 0, \ldots, n$

$$E^{(a)} = \langle e'_1, \ldots, e'_a \rangle, \text{ and } F^{(a)} = \langle e'_n, \ldots, e'_{n-a+1} \rangle.$$

Let $0 \neq g \in G^{(1)}$ be a generator, and write $g = \sum_{i=1}^n g_i e'_i$ for some $g_i \in \mathbb{F}$. Again by transversality, $g_i \neq 0$ for all $i = 1, \ldots, n$, and we set $e_i := \frac{1}{g_i} e'_i$. Then $e_1, \ldots, e_n$ is a basis of $\mathbb{F}^n$ such that $g = e_1 + \ldots + e_n$. Let now $\varphi \in \mathrm{GL}(\mathbb{F}^n)$ be in the stabilizer of (E, F, G). The matrix M representing φ in the basis $e_1, \ldots, e_n$ is diagonal. Since M maps the vector $e_1 + \ldots + e_n$ to a non-trivial multiple of itself, it follows that there exists $0 \neq \alpha \in \mathbb{F}$ such that $M = \mathrm{diag}(\alpha, \ldots, \alpha)$, and thus M lies in the center of $\mathrm{GL}(n, \mathbb{F})$. $\square$

To parametrize the configuration space of triples of transverse flags, we introduce so-called triple ratios, which are rational maps from $\mathrm{Flag}(\mathbb{F}^n)^{(3)}$ to $\mathbb{F}$. The triple ratios are expressed in terms of the exterior algebra $\bigwedge^n \mathbb{F}^n$ of $\mathbb{F}^n$. If E is a full flag, then for every a between 0 and n the space $\bigwedge^a E^{(a)}$ is isomorphic to $\mathbb{F}$. Choose a non-zero element $e^{(a)} \in \bigwedge^a E^{(a)}$. We use the same notation to denote its image in $\bigwedge^a \mathbb{F}^n$. The following definition is independent of the choices of $e^{(a)} \in \bigwedge^a E^{(a)}$.

Definition 4.1.2. Let (E, F, G) be a transverse triple of flags in $\mathbb{F}^n$. For $a, b, c \in \{1, \ldots, n-2\}$ with $a + b + c = n$, we define the *(abc)-triple ratio* T_{abc} of (E, F, G) by

$$T_{abc}(E, F, G) = \frac{e^{(a+1)} \wedge f^{(b)} \wedge g^{(c-1)}}{e^{(a-1)} \wedge f^{(b)} \wedge g^{(c+1)}} \cdot \frac{e^{(a)} \wedge f^{(b-1)} \wedge g^{(c+1)}}{e^{(a)} \wedge f^{(b+1)} \wedge g^{(c-1)}} \cdot \frac{e^{(a-1)} \wedge f^{(b+1)} \wedge g^{(c)}}{e^{(a+1)} \wedge f^{(b-1)} \wedge g^{(c)}} \in \mathbb{F}.$$

Since the ratios involve elements of $\Lambda^n \mathbb{F}^n$, $T_{abc}(E, F, G)$ is an element in $\mathbb{F}$. Note that all of the involved expressions are non-zero by the transversality of the triple. The triple ratios are invariant under the action of $\mathrm{PGL}(\mathbb{F}^n)$. In fact the following theorem relates the triple ratios and the action of $\mathrm{PGL}(\mathbb{F}^n)$ on $\mathrm{Flag}(\mathbb{F}^n)^{(3)}$.

Theorem 4.1.3 ([FG06, Section 9], [Bon23, Theorem 4.1])**.** *Let $\mathbb{F}$ be a field, and (E, F, G), $(E', F', G') \in \mathrm{Flag}(\mathbb{F}^n)^{(3)}$ two triples of transverse flags. Then there exists $\varphi \in \mathrm{PGL}(\mathbb{F}^n)$ with $\varphi(E, F, G) = (E', F', G')$ (which is unique by Proposition 4.1.1) if and only if*

$$T_{abc}(E, F, G) = T_{abc}(E', F', G')$$

for all $a + b + c = n$, $a, b, c \in \mathbb{N}_{>0}$.

Furthermore, for all $(a,b,c) \in \mathbb{N}^3_{>0}$ *such that* $a+b+c=n$*, pick* $x_{abc} \in \mathbb{F} \setminus \{0\}$*. Then there exists a triple of transverse flags* (E,F,G) *such that* $T_{abc}(E,F,G) = x_{abc}$ *for all such* $(a,b,c) \in \mathbb{N}^3_{>0}$*. Thus there is a one-to-one correspondence between*

$$\mathrm{Conf}^{(3)}(\mathbb{F}) \longleftrightarrow (\mathbb{F} \setminus \{0\})^{\frac{(n-1)(n-2)}{2}}.$$

We will give a proof of the first statement in the above theorem in Appendix A, which follows [Bon23].

Let us draw our attention to the low-dimensional cases. For $n=2$ a flag in $\mathbb{F}^2$ is nothing but a line in $\mathbb{F}^2$, hence a point in $\mathbb{FP}^1$, the one-dimensional projective space over $\mathbb{F}$. Two lines in $\mathbb{F}^2$ are transverse if they are distinct, hence define different points in $\mathbb{FP}^1$. Basic linear algebra shows that we can send any triple of distinct lines in $\mathbb{F}^2$ to any other, hence $\mathrm{Flag}(\mathbb{F}^2)^{(3)}/\mathrm{PGL}(\mathbb{F}^2)$ is just one point, which is consistent with the fact that for $n=2$ there are no triple ratios.

For $n=3$ there is exactly one triple ratio $T_{111}(E,F,G)$ for $(E,F,G) \in \mathrm{Flag}(\mathbb{F}^3)^{(3)}$. If $\mathbb{F}$ is an ordered field, the triple ratio is positive if and only if, in an affine chart of the 2-dimensional projective space $\mathbb{FP}^2$, the points $E^{(1)}$, $F^{(1)}$ and $G^{(1)}$ are on the boundary of a convex domain bounded by the lines $E^{(2)}$, $F^{(2)}$ and $G^{(2)}$, as in Figure 4.1; see [FG06, Lemma 9.1]. Positivity of tuples of flags will be discussed in more detail in Section 5.1.

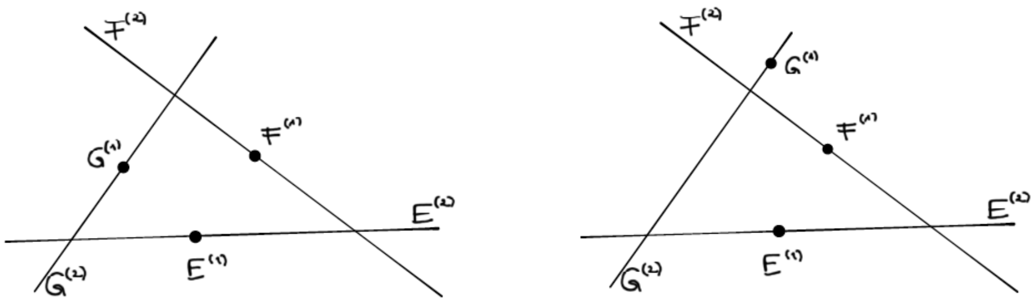

Figure 4.1.: A configuration of a positive (left) and a negative (right) triple in $\mathbb{F}^3$ visualized in an affine chart of $\mathbb{FP}^2$.

In fact, the case $n=3$ is more general than it seems, since all triple ratios of a transverse triple of flags in $\mathbb{F}^n$ for any $n>3$ arise in this way. A more precise statement can be found in Appendix A.1.

4.2. Double ratios and configuration spaces of quadruples of flags

Let us now turn our attention to the case $k=4$. We consider another $\mathrm{PGL}(n,\mathbb{F})$-invariant rational function, so-called double ratios, which are, similarly to the triple ratios, expressed in terms of the exterior algebra $\bigwedge^n \mathbb{F}^n$ of $\mathbb{F}^n$, and we keep the notations that were introduced in Section 4.1.

Definition 4.2.1. Let (E, F, G, H) be a transverse quadruple of flags in $\mathbb{F}^n$. For $a = 1, \ldots, n-1$ we define the *a-th double ratio* D_a of (E, F, G, H) by

$$D_a(E,F,G,H) = -\frac{e^{(a)} \wedge f^{(n-a-1)} \wedge g^{(1)}}{e^{(a)} \wedge f^{(n-a-1)} \wedge h^{(1)}} \cdot \frac{e^{(a-1)} \wedge f^{(n-a)} \wedge h^{(1)}}{e^{(a-1)} \wedge f^{(n-a)} \wedge g^{(1)}}.$$

For later use we also define the *a-th quadruple ratio* Q_a of (E, F, G) by

$$\begin{aligned} Q_a(E,F,G) = & \frac{e^{(a-1)} \wedge f^{(n-a)} \wedge g^{(1)}}{e^{(a)} \wedge f^{(n-a-1)} \wedge g^{(1)}} \cdot \frac{e^{(a)} \wedge f^{(1)} \wedge g^{(n-a-1)}}{e^{(a-1)} \wedge f^{(1)} \wedge g^{(n-a)}} \\ & \cdot \frac{e^{(a+1)} \wedge f^{(n-a-1)}}{e^{(a+1)} \wedge g^{(n-a-1)}} \cdot \frac{e^{(a)} \wedge f^{(n-a)}}{e^{(a)} \wedge g^{(n-a)}} \end{aligned}$$

for all $a = 1, \ldots, n-1$.

We remark that the definition of the double ratios only involves the one-dimensional subspaces of the flags G and H. We summarize Lemmas 5, 6 and 7 from [BD14], which relate the different ratios defined above and explain how they behave under permutations of the involved flags in the following lemma. The proof is a direct computation.

Lemma 4.2.2 ([BD14, Lemmas 5, 6, 7])**.** Let (E, F, G) be a transverse triple of flags in $\mathbb{F}^n$.

(1) $T_{abc}(E,F,G) = T_{bca}(F,G,E) = T_{bac}(F,E,G)^{-1}$ for all $a+b+c = n$, $a,b,c \in \mathbb{N}_{>0}$,

(2) $Q_a(E,F,G) = \prod_{b,c \geq 1, b+c=n-a} T_{abc}(E,F,G)$ for all $a = 1, \ldots, n-1$.

Let H be a fourth flag in $\mathbb{F}^n$ such that (E, F, G, H) is a transverse quadruple.

(3) $D_a(E,F,H,G) = D_a(E,F,G,H)^{-1}$ for all $a = 1, \ldots, n-1$,

(4) $D_a(F,E,G,H) = D_{n-a}(E,F,G,H)^{-1}$ for all $a = 1, \ldots, n-1$.

Even though we will not use the following theorem it illustrates why we consider double ratios, and how triple and double ratios can be used to parametrize the configuration space of quadruples of flags.

Theorem 4.2.3 ([FG06, Proposition 5.5])**.** Let $\mathbb{F}$ be a field, and (E, F, G, H), $(E', F', G', H') \in \mathrm{Flag}(\mathbb{F}^n)^{(4)}$ two quadruples of transverse flags. Then there exists $\varphi \in \mathrm{PGL}(\mathbb{F}^n)$ with $\varphi(E, F, G, H) = (E', F', G', H')$ (which is unique by Proposition 4.1.1) if and only if

$$\begin{aligned} T_{abc}(E,F,G) &= T_{abc}(E',F',G'), \\ T_{abc}(E,G,H) &= T_{abc}(E',G',H'), \\ D_a(E,G,F,H) &= D_a(E',G',F',H') \end{aligned}$$

for all $a+b+c = n$, $a,b,c \in \mathbb{N}_{>0}$ and $a = 1, \ldots, n-1$.

4.3. Triangulations of polygons and parametrizing the base change

We explain how to associate triple and double ratios to a transverse k-tuple of flags. The following definition is used to single out preferred subtriples and -quadruples. Let $y_1, \dots, y_k$ be distinct points on the unit circle $\mathbb{S}^1$ that are cyclically ordered in clockwise direction, and P the inscribed polygon that we obtain by connecting consecutive points by a straight line.

Definition 4.3.1. An *ideal triangulation* of P is a collection of *oriented diagonals* $\mathcal{E} = \{e_1, \dots, e_{k-3}\}$, i.e. straight lines in P that do not intersect and that connect two non-consecutive vertices, such that $P \setminus \mathcal{E}$ is a union of $k-2$ triangles, together with a choice of preferred vertex $x_{t_i} \in \{y_1, \dots, y_k\}$ for each triangle $t_1, \dots, t_{k-2}$. Let $\mathcal{V} = \{x_{t_1}, \dots, x_{t_{k-2}}\}$.

A choice of ideal triangulation $(\mathcal{E}, \mathcal{V})$ gives the following information, compare Figure 4.2.

- Each connected component t of $P \setminus \mathcal{E}$, together with a choice of preferred vertex $x_t \in \mathcal{V}$, singles out a triple of vertices (x_t, x'_t, x''_t) that appear in this clockwise order around the circle.
- Each oriented diagonal $e \in \mathcal{E}$ is contained in the closure of exactly two connected components of $P \setminus \mathcal{E}$ with vertices $x_{e^+}, x_{e^r}, x_{e^-}$ and $x_{e^+}, x_{e^-}, x_{e^l}$ respectively, and therefore e singles out four vertices $x_{e^+}, x_{e^r}, x_{e^-}, x_{e^l}$ which appear in this clockwise order around the circle.

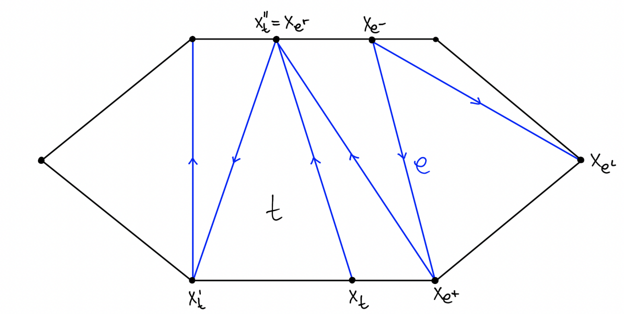

Figure 4.2.: Part of an ideal triangulation $(\mathcal{E}, \mathcal{V})$ of a polygon.

For every choice of ideal triangulation $(\mathcal{E}, \mathcal{V})$ we can define a map

$$\phi_{(\mathcal{E},\mathcal{V})} \colon \mathrm{Flag}(\mathbb{F}^n)^{(k)} \to \mathbb{F}^{\frac{(n-1)(n-2)}{2}(k-2)} \times \mathbb{F}^{(k-3)(n-1)} \tag{4.1}$$

by assigning to a k-tuple $(F_1, \dots, F_k)$ of transverse flags the following data:

- for each triangle t of $P \setminus \mathcal{E}$, together with a choice of preferred vertex $x_t \in \mathcal{V}$, we compute the $\frac{(n-1)(n-2)}{2}$ triple ratios $T_{abc}(F_{x_t}, F_{x'_t}, F_{x''_t})$, and
- for each oriented diagonal $e \in \mathcal{E}$, we compute the $n-1$ double ratios $D_a(F_{x_{e^+}}, F_{x_{e^-}}, F_{x_{e^r}}, F_{x_{e^l}})$,

where we set $F_{y_i} := F_i$ for all $i = 1, \ldots, k$. Note that the triple ratios depend on the choice of preferred vertex. Any two such choices permute the flags and are thus related by the equalities given in Lemma 4.2.2 (1).

We now explain how an ideal triangulation of P can be used to describe an element in $\mathrm{GL}(n, \mathbb{F})$ that maps one subtriple of a k-tuple of transverse flags to another. It is given as a product of elementary matrices with entries given by the triple and double ratios of subtriples and -quadruples of P singled out by the ideal triangulation. For an interpretation of the matrices defined in the following we refer to Appendix B.1 and Appendix B.2. Roughly speaking, they describe base change matrices between different bases that can be associated to a triple of transverse flags.

Definition 4.3.2. For $k = 1, \ldots, n-1$ let

$$E_k := \mathrm{Id}_n + E_{k,k+1} = \left(\begin{array}{c|cc|c} \mathrm{Id}_{k-1} & & & \\ \hline & 1 & 1 & \\ & 0 & 1 & \\ \hline & & & \mathrm{Id}_{n-k-1} \end{array}\right) \in \mathrm{GL}(n, \mathbb{F})$$

be the elementary upper triangular matrix with ones on the diagonal, a 1 at position $(k, k+1)$ and zeroes everywhere else. We also set $F_k := E_k^\top$, the transpose of E_k. For $x \in \mathbb{F} \setminus \{0\}$ and $k = 1, \ldots, n$ let

$$H_k(x) := \mathrm{diag}(\underbrace{1, \ldots, 1}_{k}, \underbrace{x, \ldots, x}_{n-k}) = \left(\begin{array}{c|c} \mathrm{Id}_k & \\ \hline & x\,\mathrm{Id}_{n-k} \end{array}\right) \in \mathrm{GL}(n, \mathbb{F})$$

be the diagonal matrix with the first k entries equal to 1, and the last $n-k$ entries equal to x. Furthermore, we set

$$S := \begin{pmatrix} & & & & (-1)^{n-1} \\ & & & \iddots & \\ & & -1 & & \\ & 1 & & & \\ -1 & & & & \\ 1 & & & & \end{pmatrix}.$$

Definition 4.3.3. Let (E, F, G) be a transverse triple of flags in $\mathbb{F}^n$. Define

$$M_{(E,F,G)} := \prod_{k=1}^{n-1} \left(\left(\prod_{i=1}^{k-1} F_{n-k+i-1} H_{n-k+i}(x_{k-i,i,n-k}) \right) F_{n-1} \right),$$

where $x_{abc} := T_{abc}(E, F, G)$ denotes the (abc)-triple ratio for all integers $a, b, c \geq 1$ with $a + b + c = n$. Let (E, F, G, H) be a transverse quadruple of flags in $\mathbb{F}^n$ and denote by $d_k := D_k(E, G, F, H)$ the k-th double ratio of (E, G, F, H) for all $k = 1, \ldots, n-1$. Define

$$D_{(E,F,G,H)} := \begin{pmatrix} 1 & & & & \\ & d_{n-1} & & & \\ & & d_{n-2}d_{n-1} & & \\ & & & \ddots & \\ & & & & d_1 \cdot \ldots \cdot d_{n-1} \end{pmatrix}.$$

In the following we describe a particular ideal triangulation of a polygon P with k vertices $x_1, \ldots, x_k$ in clockwise order around the polygon. Fix $4 \leq j \leq k-2$. Consider the following ideal triangulation $\mathcal{E} = \mathcal{E}_j$ of P as indicated in Figure 4.3: the diagonals of $\mathcal{E}$ are formed by the (directed) edges $e_1, \dots, e_{k-3}$, labelled from left to right, from the vertex x_3 to the vertices $x_1, x_k, x_{k-1}, \dots x_{j+2}$, as well as from the vertices $x_4, \dots, x_j$ to the vertex x_{j+2}. We label the triangles obtained in this way from left to right by $t_0, \dots, t_{k-3}$. For $i = 1, \dots, k-4$ the preferred vertex of t_i is where two diagonals of $\mathcal{E}$ meet, i.e. either x_3 or x_{j+2}. We define the preferred vertex of t_0 to be x_3 and the preferred vertex of t_{k-3} to be x_{j+2}. For all $i = 1, \dots, k-3$ the triangle t_i lies to the left of the (directed) edge e_i.

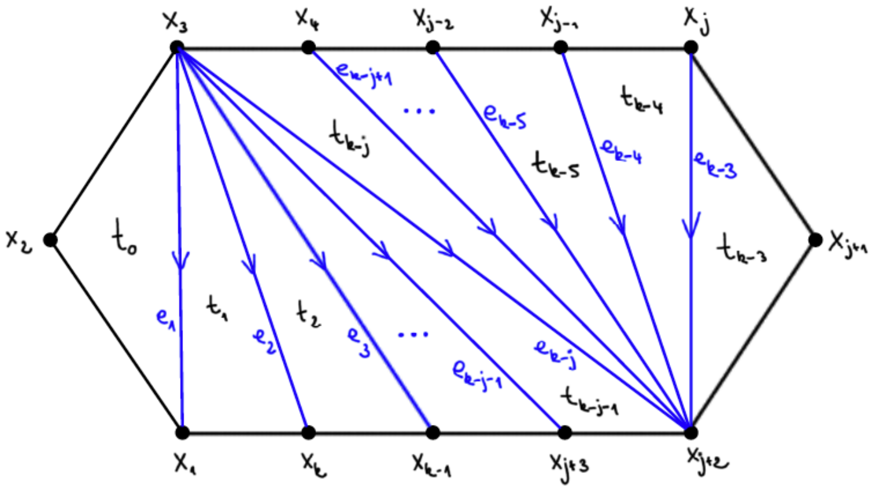

Figure 4.3.: The triangulation $\mathcal{E}_j$ of the polygon P.

Let $(F_1, \dots, F_k) \in \mathrm{Flag}(\mathbb{F}^n)^{(k)}$ be a transverse k-tuple of flags associated to P, i.e. to the vertex x_j we associate the flag F_j for all $j = 1, \dots, k$. We now compute the matrices defined in Definition 4.3.3 for the subtriples and -quadruples singled out by this triangulation. Every triangle t_i together with its preferred vertex determines a unique triple $(E_{t_i}, F_{t_i}, G_{t_i})$ of transverse flags singled out by the triple of vertices of t_i, starting at the preferred vertex, that appear in this clockwise order around P. For example for t_0 we obtain $E_{t_0} = F_3$, $F_{t_0} = F_1$ and $G_{t_0} = F_2$. We set for all $i = 0, \dots, k-3$

$$M_i := M(t_i) := M_{(E_{t_i}, F_{t_i}, G_{t_i})}.$$

Similarly every oriented diagonal $e_i \in \mathcal{E}$ is contained in exactly two adjacent triangles of $P \setminus \mathcal{E}$, and hence the four vertices of the two adjacent triangles, starting from the forward endpoint of e_i in clockwise order around P, single out four transverse flags $(E_{t_i}, F_{t_i}, G_{t_i}, H_{t_i})$. We set for all $i = 1, \dots, k-3$

$$D_i := D(e_i) := S^{-1} D_{(E_{t_i}, F_{t_i}, G_{t_i}, H_{t_i})} S.$$

With these notations we can now explain how the ideal triangulation of P can be used to describe an element of $\mathrm{GL}(n, \mathbb{F})$ that maps one subtriple to another. The following theorem will later be applied in the proof of Proposition D for $\mathbb{F}$-positive representations (see Proposition 7.3.2) to an $\mathbb{F}$-positive representation ρ with limit map ξ_ρ. More precisely, for

$\gamma \in \pi_1(S)$ we consider a transverse triple of flags $(\xi_\rho(x), \xi_\rho(y), \xi_\rho(z))$ and its γ-translate $(\xi_\rho(\gamma x), \xi_\rho(\gamma y), \xi_\rho(\gamma z))$, which lie in the same $\mathrm{PGL}(\mathbb{F}^n)$-orbit by ρ-equivariance of ξ_ρ. We can then express $\rho(\gamma)$ in terms of the triple and double ratios.

Theorem 4.3.4 ([FG06, Proposition 9.2])**.** *Let $(F_1, \dots, F_k) \in \mathrm{Flag}(\mathbb{F}^n)^{(k)}$ be a transverse k-tuple of flags associated to a polygon P with k vertices $x_1, \dots, x_k$ in clockwise order around the polygon. Assume either that there exists $4 \leq j \leq k-2$ so that (F_1, F_2, F_3) and*

(1) (F_{j+2}, F_j, F_{j+1}), *or*

(2) (F_{j+1}, F_{j+2}, F_j)

have the same triple ratios, and let $\varphi \in \mathrm{PGL}(\mathbb{F}^n)$ be the element that maps (F_1, F_2, F_3) to the corresponding triple. Then, in the respective cases, there exists a basis of $\mathbb{F}^n$ in which φ is represented by

(1) $M_\varphi := \left(\prod_{i=1}^{k-j-1} D_i M_i\right)\left(\prod_{i=k-j+1}^{k-3} D_i S M_i^{-1} S^{-1}\right)$, *or*

(2) $M_\varphi := M_0 \left(\prod_{i=1}^{k-j-1} D_i M_i\right)\left(\prod_{i=k-j+1}^{k-4} D_i S M_i^{-1} S^{-1}\right) D_{k-3}$,

where the D_i and M_i are defined as above associated to the ideal triangulation $\mathcal{E}_j$ as described before.

A self-contained proof of this theorem is presented in Appendix B.3.

5. Positivity

5.1. Positivity of tuples of flags

The definitions from the last section allow us to define positive triples and quadruples of flags in $\mathbb{F}^n$ for $\mathbb{F}$ any ordered field. It is defined analogously as in [BD14] for $\mathbb{R}$.

Definition 5.1.1. A triple (E, F, G) of flags in $\mathbb{F}^n$ is called *positive* if the triple is transverse and all triple ratios are positive. A quadruple (E, F, G, H) of flags in $\mathbb{F}^n$ is called *positive* if the quadruple is transverse, the triples (E, F, G) and (E, G, H) are positive, and all double ratios of (E, G, F, H) are positive.

Example 5.1.2. For $n = 3$, Figure 4.1 shows a positive triple of transverse flags and a negative triple (that means with a negative triple ratio) of transverse flags. A quadruple of flags in $\mathbb{F}^3$ is positive if and only if the convex quadruple in $\mathbb{FP}^2$ formed by the one dimensional subspaces, is inscribed in the convex quadruple formed by the two-dimensional subspaces [FG07, Lemma 2.4], compare Figure 5.1.

Let $\mathcal{B} = (e_1, e_2, e_3)$ be a basis of $\mathbb{F}^3$, and let $E = (\langle e_1 \rangle \subseteq \langle e_1, e_2 \rangle \subseteq \mathbb{F}^3)$ the ascending, $F = (\langle e_3 \rangle \subseteq \langle e_2, e_3 \rangle \subseteq \mathbb{F}^3)$ the descending flags associated to $\mathcal{B}$ and

$$G = \left(\langle e_1 + e_2 + e_3 \rangle \subseteq \langle e_1 + e_2 + e_3, e_2 + (t+1)e_3 \rangle \subseteq \mathbb{F}^3\right) \text{ for } t \in \mathbb{F}.$$

Then (E, F, G) is positive if and only if $t > 0$. Note that up to $\mathrm{PGL}(\mathbb{F}^3)$-action, every triple of transverse flags is of the above form.

We can also define a notion of positivity for k-tuples of flags with $k \geq 4$ by fixing the additional data of an ideal triangulation (Definition 4.3.1).

Definition 5.1.3. Let $\mathcal{E}$ be an ideal triangulation of a polygon with k vertices. A k-tuple of flags $(F_1, \ldots, F_k) \in \mathrm{Flag}(\mathbb{F}^n)^{(k)}$ is *positive* if $\phi_{(\mathcal{E},\mathcal{V})}(F_1, \ldots, F_k)$ has only positive coordinates. We denote the space of positive k-tuples of flags by $\mathrm{Flag}(\mathbb{F}^n)^{(k,+)}$.

Example 5.1.4. For $n = 3$, a k-tuple of positive flags can be visualized in $\mathbb{FP}^2$ as a k-gon defined by the one-dimensional subspaces inscribed into a k-gon defined by the two-dimensional subspaces, see Figure 5.1.

At first sight, it seems that this definition depends on the choice of triangulation $(\mathcal{E}, \mathcal{V})$. Fock-Goncharov proved in [FG06, Theorem 5.8, Theorem 9.1] that if a k-tuple is positive with respect to one triangulation it is positive with respect to any other. From this is also follows that if (E, F, G, H) is a positive quadruple, then all possible subtriples are positive. Their proof gives an explicit rational expression, which preserves positivity, for the change of coordinates under a *flip of a diagonal*, that is, for a pair of adjacent triangles we remove the edge that forms the diagonal and add an edge that forms the other diagonal of that quadrilateral. Since we can get from any ideal triangulation of a

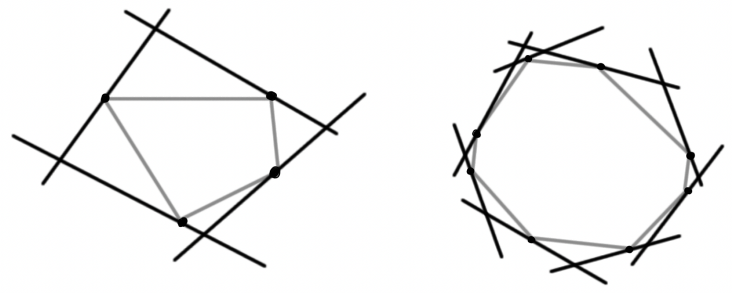

Figure 5.1.: A configuration of a positive quadruple (left) and a positive 8-tuple (right) in $\mathbb{F}^3$ visualized in an affine chart of $\mathbb{FP}^2$.

polygon to any other by a sequence of flips of diagonals [Hat91], we obtain the desired result. We also recommend [Mar19b, Section 2.2] for a detailed account of how the flip of a diagonal changes the coordinates $\phi_{(\mathcal{E},\mathcal{V})}$. Since the triple and double ratios are $\mathrm{PGL}(\mathbb{F}^n)$-invariant, a k-tuple is positive if and only if a k-tuple in the same $\mathrm{PGL}(\mathbb{F}^n)$-orbit is. We refer the reader to [FG06] for the more general definition of positivity for k-tuples of flags which clarifies the minus sign in the definition of the double ratios.

Remark 5.1.5. In light of the above definition of positive k-tuples of flags an $\mathbb{F}$-positive map $\mathrm{Fix}(S) \to \mathrm{Flag}(\mathbb{F}^n)$ (Definition 1.3.2) can equivalently be defined as a map that sends k-tuples of distinct points in $\mathrm{Fix}(S)$, occurring in this clockwise order, to positive k-tuples of flags for any $k \geq 3$.

5.2. Total positivity

In this section we shed light on the connection between positivity of flags and total positivity of matrices. For a more conceptual approach to the latter definition we refer the reader to [Lus94].

Definition 5.2.1. Let $\mathbb{F}$ be an ordered field. An element in $\mathrm{GL}(n,\mathbb{F})$ is *totally non-negative*, respectively *totally positive*, if all its minors are non-negative, respectively positive. An upper respectively lower triangular matrix in $\mathrm{GL}(n,\mathbb{F})$ is *totally positive* if all its minors are positive, except the minors that are necessary zero because the matrix is triangular.

Example 5.2.2. Consider the matrices

$$M_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 4 & 8 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 3 & 5 \\ 0 & 4 & 8 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{pmatrix}.$$

Then M_1 is totally positive, M_2 is totally non-negative and M_3 is upper triangular totally positive.

The following theorem is due to Fock-Goncharov, see [FG06, Section 9]. We also refer to [Pal13, Section 4.4] and [Mar19b, Section 5.1] for a detailed treatment of this theorem in the case $\mathbb{F} = \mathbb{R}$. The version in which it is stated here is adapted for our setting from [Mar19b, Proposition 5.4]. A self-contained proof which relies on Theorem 4.3.4 can be found in Appendix B.4.

Theorem 5.2.3 ([FG06, Theorem 9.3]). *Under the hypotheses and conclusions of Theorem 4.3.4, if we additionally assume that $(F_1, \ldots, F_k) \in \mathrm{Flag}(\mathbb{F}^n)^{(k,+)}$ is a positive k-tuple of flags, then M_φ is a totally positive matrix.*

5.3. Positive hyperbolicity

Over real closed fields, totally positive matrices are positively hyperbolic.

Definition 5.3.1. Let $\mathbb{F}$ be a real closed field. A matrix $M \in \mathrm{GL}(n, \mathbb{F})$ is *positively hyperbolic* if all its eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$ are distinct and positive. In this case we sort them in descending order, i.e. we assume $\lambda_1 > \ldots > \lambda_n > 0$. If M is positively hyperbolic, then M is diagonalizable over $\mathbb{F}$. In this case, let $V_1, \ldots, V_n$ be the eigenspaces corresponding to the eigenvalues $\lambda_1 > \ldots > \lambda_n > 0$. We define its *stable flag* F_M^+ and its *unstable flag* F_M^- by

$$
\begin{aligned}
F_M^+ &= (V_1 \subseteq V_1 \oplus V_2 \subseteq \ldots \subseteq V_1 \oplus \ldots \oplus V_{n-1}) \,, \\
F_M^- &= (V_n \subseteq V_n \oplus V_{n-1} \subseteq \ldots \subseteq V_n \oplus \ldots \oplus V_2) \,.
\end{aligned}
$$

For positively hyperbolic elements of $\mathrm{PSL}(n, \mathbb{F})$ (Definition 1.4.4) we define analogously their *stable and unstable flags*.

Example 5.3.2. Let $M = \mathrm{diag}(2, 1, 1/2)$ be a diagonal matrix in a basis (e_1, e_2, e_3) for $\mathbb{F}^3$. Then $F_M^+ = (\langle e_1 \rangle \subseteq \langle e_1, e_2 \rangle \subseteq \mathbb{F}^3)$.

Theorem 5.3.3. Let $\mathbb{F}$ be a real closed field, and $M \in \mathrm{GL}(n, \mathbb{F})$ totally positive. Then M is positively hyperbolic.

Proof. We first assume that the theorem holds for the field $\overline{\mathbb{Q}}^r$ of real algebraic numbers, the real closure of $\mathbb{Q}$. Then we can prove the theorem using the Tarski-Seidenberg transfer principle as in the proof of Proposition 7.2.1. Indeed, every real closed field $\mathbb{F}$ contains $\overline{\mathbb{Q}}^r$. Since the set of totally positive matrices $\mathrm{Pos}(n, \overline{\mathbb{Q}}^r)$ in $\mathrm{GL}(n, \overline{\mathbb{Q}}^r)$ is a semi-algebraic subset of $(\overline{\mathbb{Q}}^r)^{n \times n}$, we can consider the projection onto the first $n \times n$ coordinates of the semi-algebraic set

$$
\begin{aligned}
&\{M, v_1, \ldots, v_n, \, \lambda_1, \ldots, \lambda_n \mid M \in \mathrm{Pos}(n, \overline{\mathbb{Q}}^r), \\
&\quad v_i \in \mathbb{F}^n \setminus \{0\}, \, \lambda_i \in \mathbb{F}, \, M v_i = \lambda_i v_i, \lambda_1 > \ldots > \lambda_n\} \to \mathrm{Pos}(n, \overline{\mathbb{Q}}^r),
\end{aligned}
$$

which is surjective by assumption. By the Tarski-Seidenberg transfer principle, see Theorem 2.1.13, we obtain that for any real closed field $\mathbb{F}$, the extension of the above map to a map between the $\mathbb{F}$-extensions is still surjective. Since $\mathrm{Pos}(n, \overline{\mathbb{Q}}^r)_{\mathbb{F}} = \mathrm{Pos}(n, \mathbb{F})$, we conclude.

Let now $M \in \mathrm{GL}(n, \overline{\mathbb{Q}}^r)$ be a totally positive matrix. Since $\overline{\mathbb{Q}}^r \subset \mathbb{R}$, we can apply the theorem of Gantmacher-Krein, which tells us that $M \in \mathrm{GL}(n, \mathbb{R})$ is diagonalizable over $\mathbb{R}$ with distinct positive eigenvalues $\lambda_1 > \ldots > \lambda_n > 0 \in \mathbb{R}$, see [GK02, Theorem

6.1]. Left to show is that the eigenvalues are algebraic over $\mathbb{Q}$. Since they are the roots of the characteristic polynomial, which has only real algebraic coefficients they are by definition algebraic over an algebraic extension of $\mathbb{Q}$, hence algebraic. □

Remark 5.3.4. The last theorem implies that a totally positive matrix in $\mathrm{GL}(n, \mathbb{F})$ is positively hyperbolic. Conversely, He-Lusztig proved in [HL22, Theorem 2.6] that a positively hyperbolic matrix can be conjugated to lie in $\mathrm{Pos}(n, \mathbb{F})$.

As a consequence we can associate to every totally positive matrix its stable flag. We now show that this defines a semi-algebraic map.

Lemma 5.3.5. The spaces $\mathrm{Flag}(\mathbb{F}^n)$, $\mathrm{Flag}(\mathbb{F}^n)^k$ and $\mathrm{Flag}(\mathbb{F}^n)^{(k)}$ are semi-algebraic.

Proof. The product of Grassmannians $\mathrm{Gr}(1,n)_{\mathbb{F}} \times \ldots \times \mathrm{Gr}(n-1,n)_{\mathbb{F}}$ is an algebraic set, see for example [BCR98, Proposition 3.4.4]. In the proof they define a bijection

$$\begin{aligned} \Psi_k \colon \mathrm{Gr}(k,n)_{\mathbb{F}} &\to \{M \in \mathrm{Mat}(n, \mathbb{F}) \mid M^\top = M,\, M^2 = M,\, \mathrm{tr}(M) = k\} =: H_k \\ V &\mapsto P_V, \end{aligned}$$

where P_V is the matrix of the orthogonal projection on V with respect to some scalar product on $\mathbb{F}^n$. If $V, W \subseteq \mathbb{F}^n$ are two subspaces, then $V \subseteq W$ if and only if $P_W P_V = P_V$. Thus the image of $\mathrm{Flag}(\mathbb{F}^n)$ under the map $\Psi_1 \times \cdots \times \Psi_{n-1}$ is the algebraic set

$$\{(M_1, \ldots, M_{n-1}) \in H_1 \times \cdots \times H_{n-1} \mid M_{i+1} M_i = M_i \text{ for all } i = 1, \ldots, n-2\}.$$

Thus $\mathrm{Flag}(\mathbb{F}^n)$ and $\mathrm{Flag}(\mathbb{F}^n)^k$ are algebraic. Since being transverse is a semi-algebraic condition, it follows that $\mathrm{Flag}(\mathbb{F}^n)^{(k)}$ is semi-algebraic. □

Lemma 5.3.6. Let $\mathbb{F}$ be a real closed field. The map

$$f \colon \mathrm{Pos}(n, \mathbb{F}) \to \mathrm{Flag}(\mathbb{F}^n), \quad M \mapsto F_M^+$$

is semi-algebraic. The same is true if we replace the stable flag by the unstable flag.

Proof. We need to prove that $\mathrm{Graph}(f)$ is semi-algebraic (Definition 2.1.8). We have

$$\begin{aligned} \mathrm{Graph}(f) &= \{(M, F) \in \mathrm{Pos}(n, \mathbb{F}) \times \mathrm{Flag}(\mathbb{F}^n) \mid F = F_M^+\} \\ &= \{(M, F) \in \mathrm{Pos}(n, \mathbb{F}) \times \mathrm{Flag}(\mathbb{F}^n) \mid \exists\, \lambda_1 > \ldots > \lambda_n \in \mathbb{F}_{>0},\, v_1, \ldots, v_n \in \mathbb{F}^n : \\ &\qquad \det(v_1 | \ldots | v_n) \neq 0,\, M v_i = \lambda_i v_i,\, v_1, \ldots, v_i \in F^{(i)} \text{ for all } i = 1, \ldots, n\}. \end{aligned}$$

As in the proof of Theorem 5.3.3, we can conclude that $\mathrm{Graph}(f)$ is semi-algebraic as the projection of a semi-algebraic set to its first coordinates (Theorem 2.1.13). Note that the condition $v_1, \ldots, v_i \in F^{(i)}$ can be expressed as $P_{F^{(i)}}(v_j) = v_j$ for all $j = 1, \ldots, i$; see the proof of Lemma 5.3.5. □

6. A variant of the Bonahon-Dreyer coordinates

6.1. The coordinates

In [BD14] and [BD17], Bonahon-Dreyer introduce coordinates to parametrize the Hitchin component, which in the case $n = 2$ agree with the shear coordinates associated to a maximal geodesic lamination (Definition 6.1.1) to parametrize Teichmüller space, see [Thu22, Section 9] and [Bon96, Theorem A]. In the following, we give a slight variant of Bonahon-Dreyer's parametrization presented in [BD14], that gives a semi-algebraic model of the Hitchin component in the sense of Definition 3.2.1. It allows thus to be generalized to real closed fields different from $\mathbb{R}$. The coordinates built on the existence of positive limit maps for Hitchin representations, see Theorem 1.3.3 due to Labourie and Fock-Goncharov. We would like to emphasize that what is presented here differs only from the original Bonahon-Dreyer coordinates by not taking logarithms and requiring an additional positivity condition. For the convenience of the reader, we recall the definition of the coordinates here. For a more detailed description and more information we recommend [BD14].

In order to define the coordinates we need to fix some topological data on the connected, closed, oriented topological surface S of genus $g \geq 2$. Choose an auxiliary hyperbolic metric on S. Denote by $\partial\tilde{S}$ the boundary of the universal cover $\tilde{S}$ of S.

Definition 6.1.1. A *lamination* λ of S is a closed subset of S that is partitioned into smooth curves, called the *leaves* of the lamination. A *geodesic lamination* λ on S is a lamination of S whose leaves are geodesics. We say λ is *finite* if λ has finitely many leaves. It is called *maximal* if it is maximal with respect to inclusion. In this case a connected component of its complement $S \setminus \lambda$ is called an *ideal triangle*.

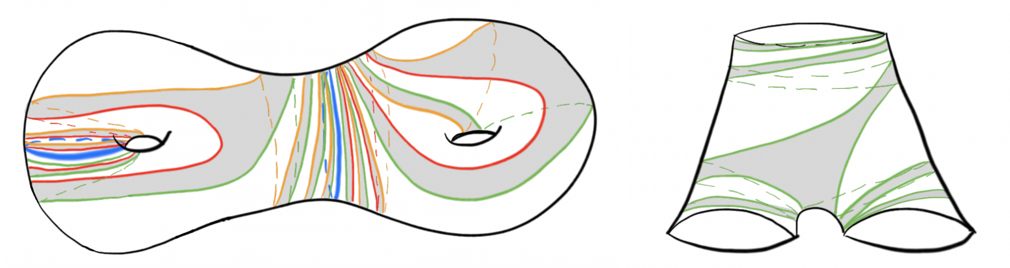

Figure 6.1.: A maximal geodesic lamination of S (left) and of a pair of pants (right). The white and shaded regions are ideal triangles.

Example 6.1.2. To obtain a finite maximal geodesic lamination on S we can first decompose S into pairs of pants. Then for each pair of pants choose three infinite geodesics that spiral around the three boundary components, see Figure 6.1.

Fix a maximal geodesic lamination λ of S with finitely many leaves, and denote by $\tilde{\lambda} \subset \tilde{S}$ its lift to $\tilde{S}$. Although geodesic laminations can be defined in a metric-independent way and, in particular, are purely topological objects, see for example [Thu02, Proposition 8.9.4] or [Bon01, Lemma 18], it is convenient here to fix a hyperbolic metric on S. Furthermore, we (arbitrarily) choose an orientation on each leaf of λ. Note that the leaves of λ come in two flavours: there are closed leaves and infinite, or so called open leaves. Finally, for every closed geodesic leaf γ in λ we choose an arc k that is transverse to λ, cuts γ in exactly one point x, and has endpoints in $S \setminus \lambda$. We write $\partial\tilde{\lambda} \subseteq \partial\tilde{S}$ for the endpoints of the leaves of $\tilde{\lambda}$. Since λ is a closed subset of S, we observe that $\partial\tilde{\lambda} \subseteq \mathrm{Fix}(S)$.

We now describe the Bonahon-Dreyer coordinates that parametrize the Hitchin component. Let $\rho\colon \pi_1(S) \to \mathrm{PSL}(n, \mathbb{R})$ be a Hitchin representation in the equivalence class of $[\rho] \in \mathrm{Hit}(S, n)$, and $\xi_\rho\colon \mathrm{Fix}(S) \to \mathrm{Flag}(\mathbb{R}^n)$ the limit map associated to ρ by Theorem 1.3.3. Before we define the invariants let us quickly explain where they come from. Since ρ is determined by ξ_ρ we would like a way to encode the limit map. However it turns out that it suffices to have the data of the *flag decoration*, i.e. the restriction of ξ_ρ to $\partial\tilde{\lambda} \subseteq \mathrm{Fix}(S)$. By ρ-equivariance of ξ_ρ this is the same as the data of a k-tuple of positive flags for some k, since λ has only finitely many leaves. We have seen in Chapter 4 that triple and double ratios are crucial in encoding the data of k-tuples of positive flags. Thus we associate to $[\rho]$ the following invariants.

(1) *Triangle invariants*: For every ideal triangle t of $S \setminus \lambda$ and every vertex v of t, we associate for $a, b, c \in \mathbb{N}_{\geq 1}$ with $a + b + c = n$ the triangle invariant

$$T^{\rho}_{abc}(t, v) := T_{abc}(\xi_\rho(\tilde{v}), \xi_\rho(\tilde{v}'), \xi_\rho(\tilde{v}'')),$$

where $\tilde{v}, \tilde{v}', \tilde{v}''$ are the vertices of a lift $\tilde{t}$ of t to $\tilde{S}$, $\tilde{v}$ is the vertex corresponding to the vertex v of t, and T_{abc} is the (abc)-triple ratio (Definition 4.1.2) of the triple of flags. The vertices $\tilde{v}, \tilde{v}', \tilde{v}'' \in \mathrm{Fix}(S)$ are labelled in clockwise order around $\tilde{t}$.

(2) *Shear invariants for infinite leaves*: For every infinite leaf $h \in \lambda$, let $\tilde{h}$ be a lift of h to $\tilde{S}$, $h^{\pm} \in \mathrm{Fix}(S)$ its positive respectively negative endpoint, and $z, z' \in \mathrm{Fix}(S)$ the third vertices of $\tilde{t}$ and $\tilde{t}'$, respectively, where $\tilde{t}$ and $\tilde{t}'$ are the ideal triangles that lie on the left respectively right side of $\tilde{h}$ (where the orientation of $\tilde{h}$ comes from the orientation of h which was part of the topological data), see Figure 6.2. We associate to h for $a = 1, \ldots, n-1$ the shear invariant

$$D^{\rho}_{a}(h) := D_a(\xi_\rho(h^+), \xi_\rho(h^-), \xi_\rho(z), \xi_\rho(z')),$$

where D_a denotes the a-th double ratio (Definition 4.2.1) of the quadruple of flags.

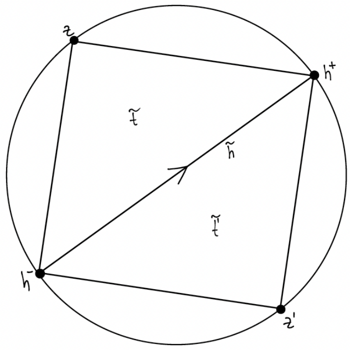

Figure 6.2.: The construction of the shear invariants for infinite leaves.

(3) *Shear invariants for closed leaves*: For every closed leaf $\gamma \in \lambda$, let $\tilde{\gamma}$ be a lift of γ to $\tilde{S}$, $\gamma^{\pm} \in \mathrm{Fix}(S)$ its positive respectively negative endpoint, and $z, z' \in \mathrm{Fix}(S)$ are vertices of the triangles $\widetilde{t}$ and $\widetilde{t'}$ defined by the arc k intersecting γ in the following way: Lift k to an arc $\tilde{k}$ that intersects $\tilde{\gamma}$ in exactly one point. The two endpoints of $\tilde{k}$ lie in two ideal triangles $\widetilde{t}$ and $\widetilde{t'}$, that each share one vertex with the endpoints of $\tilde{\gamma}$ (this is by definition of the arc k), and such that $\widetilde{t}$ lies to the left of $\tilde{\gamma}$ and $\widetilde{t'}$ to the right of $\tilde{\gamma}$. Now z, z' are the vertices of $\widetilde{t}$ respectively $\widetilde{t'}$ that are not adjacent to a component of $\tilde{S} \setminus (\widetilde{t} \cup \widetilde{t'})$ that contains $\tilde{\gamma}$, see Figure 6.3 for one of the four possible configurations. We associate to γ for $a = 1, \ldots, n-1$ the shear invariant

$$D_a^{\rho}(\gamma) := D_a(\xi_\rho(\gamma^+), \xi_\rho(\gamma^-), \xi_\rho(z), \xi_\rho(z')),$$

where D_a denotes the a-th double ratio (Definition 4.2.1) of the quadruple of flags.

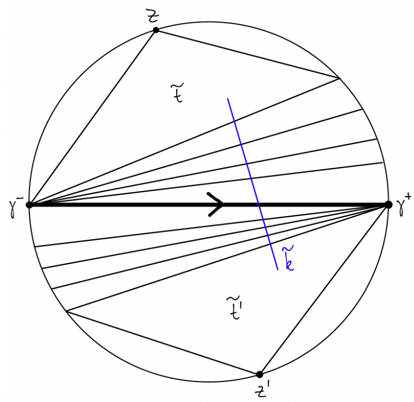

Figure 6.3.: The construction of the shear invariants for closed leaves.

Remark 6.1.3. We note that the above definitions do not depend on the choice of ρ in its equivalence class, since if ρ' and ρ are conjugate by an element $g \in \mathrm{PSL}(n, \mathbb{R})$, then $\xi_\rho = g\xi_{\rho'}$. This is however not a problem since the triple and double ratios are $\mathrm{PSL}(n, \mathbb{R})$-invariant.

6.2. The relations

The Bonahon-Dreyer coordinates are not independent but satisfy the following relations.

(i) *Positivity condition*: For every ideal triangle $t \in S \setminus \lambda$ and every vertex v of t, the triangle invariants $T^{\rho}_{abc}(t, v)$ are positive for all $a, b, c \in \mathbb{N}_{\geq 1}$ with $a + b + c = n$. Similarly for every (infinite or closed) leaf $l \in \lambda$ the shear invariants $D^{\rho}_{a}(l)$ are positive for all $a = 1, \ldots, n-1$.

(ii) *Rotation condition*: For every ideal triangle $t \in S \setminus \lambda$ with vertices v and v' such that v' immediately follows v when going in clockwise direction around the boundary of t, and for every $a, b, c \in \mathbb{N}_{\geq 1}$ with $a + b + c = n$, we have

$$T^{\rho}_{abc}(t, v) = T^{\rho}_{bca}(t, v').$$

(iii) *Closed leaf equality*: For every closed leaf $\gamma \in \lambda$ and every $a = 1, \ldots, n-1$, we have

$$L^{\text{right}}_{a}(\gamma) = L^{\text{left}}_{a}(\gamma),$$

where $L^{\text{right}}_{a}(\gamma)$ and $L^{\text{left}}_{a}(\gamma)$ will be defined in the following paragraph.

(iv) *Closed leaf inequality*: For every closed leaf $\gamma \in \lambda$ and every $a = 1, \ldots, n-1$, we have

$$L^{\text{right}}_{a}(\gamma) > 1.$$

Recall that we chose an orientation on each leaf of λ. Let us now quickly define $L^{\text{right}}_{a}(\gamma)$ and $L^{\text{left}}_{a}(\gamma)$ for γ a closed leaf of the finite maximal lamination λ. Choose a side of γ. Denote by h_l and t_l for $l = 1, \ldots, k$ the infinite leaves and the ideal triangles that spiral on the chosen side of γ. An infinite leaf appears twice in this list if its two ends spiral on the selected side of γ. The spiralling of the triangle t_l on this side of γ occurs in the direction of a vertex which we call v_l. Define

$$\overline{D}_a(h_l) := \begin{cases} D^{\rho}_{a}(h_l), & \text{if } h_l \text{ is oriented toward } \gamma, \\ D^{\rho}_{n-a}(h_l), & \text{if } h_l \text{ is oriented away from } \gamma. \end{cases}$$

If the chosen side is the right side of γ, we define

$$L^{\text{right}}_{a}(\gamma) := \prod_{l=1}^{k} \overline{D}_a(h_l) \prod_{l=1}^{k} \prod_{b+c=n-a} T^{\rho}_{abc}(t_l, v_l),$$

if the spiralling of the triangles occurs in the direction of the orientation of γ, and otherwise

$$L^{\text{right}}_{a}(\gamma) := \left(\prod_{l=1}^{k} \overline{D}_{n-a}(h_l) \prod_{l=1}^{k} \prod_{b+c=a} T^{\rho}_{(n-a)bc}(t_l, v_l) \right)^{-1}.$$

Similarly, if the chosen side is the left side of γ, we define

$$L^{\text{left}}_{a}(\gamma) := \left(\prod_{l=1}^{k} \overline{D}_a(h_l) \prod_{l=1}^{k} \prod_{b+c=n-a} T^{\rho}_{abc}(t_l, v_l) \right)^{-1},$$

if the spiralling of the triangles occurs in the direction of the orientation of γ, and otherwise

$$L_a^{\mathrm{left}}(\gamma) := \prod_{l=1}^{k} \overline{D}_{n-a}(h_l) \prod_{l=1}^{k} \prod_{b+c=a} T^{\rho}_{(n-a)bc}(t_l, v_l).$$

Note that the triangles and infinite leaves that spiral towards a side of γ differ depending on which side of γ we choose.

6.3. The parametrization

Suppose the geodesic lamination λ has p closed leaves and q infinite leaves, and its complement $S \setminus \lambda$ consists of r ideal triangles. There are $(n-1)(n-2)/2$ triples of integers $a, b, c \geq 1$ with $a + b + c = n$. For

$$N = 3r\frac{(n-1)(n-2)}{2} + (p+q)(n-1)$$

the functions $\{T_{abc}\}$ and $\{D_a\}$ define a map from the Hitchin component to $\mathbb{R}^N$ satisfying the following properties:

(A) For all $a, b, c \in \mathbb{N}_{\geq 1}$ with $a + b + c = n$, the function T_{abc} associates a positive real number $T_{abc}(t, v)$ to every triangle $t \in S \setminus \lambda$ and to every vertex v of t;

(B) For all $a = 1, \dots, n-1$, the function D_a associates a positive real number $D_a(l)$ to each leaf $l \in \lambda$;

(C) For every triangle $t \in S \setminus \lambda$ and all indices $a, b, c \in \mathbb{N}_{\geq 1}$ with $a + b + c = n$, the functions T_{abc} satisfy the rotation condition as in Section 6.2 (ii);

(D) For every closed leaf $\gamma \in \lambda$ and every index $a = 1, \dots, n-1$, the functions T_{abc} and D_a satisfy the closed leaf equality and the closed leaf inequality as in Section 6.2 (iii) respectively Section 6.2 (iv).

Consider now the semi-algebraic subset $\mathcal{P} \subseteq \mathbb{R}^N$ defined by the properties (A)-(D). More precisely, let

$$(x_{abc,t,v}, y_{m,l}) \in \mathbb{R}^N$$

where $a + b + c = n$, $m = 1, \dots, n-1$, v is a vertex of $t \in S \setminus \lambda$ and l is a leaf of λ. Then $\mathcal{P}$ is the set of all $(x_{abc,t,v}, y_{m,l}) \in \mathbb{R}^N$ satisfying:

- $x_{abc,t,v}$, $y_{m,l} > 0$ for all a, b, c, t, v, m, l (Positivity condition (i));
- $x_{abc,t,v} = x_{bca,t,v'}$ for all a, b, c, t, where $v' \in t$ is the vertex that follows v in clockwise direction (Rotation condition (ii));
- For every closed leaf of λ and every index $1, \dots, n-1$ the coordinates $x_{abc,t,v}$, $y_{m,l}$ satisfy the closed leaf equality and inequality as in (iii) respectively (iv).

We have seen in Section 6.1 and Section 6.2 that the map that assigns to a Hitchin representation its triangle and shear invariants has image in $\mathcal{P}$. In fact, the image of the map is exactly $\mathcal{P}$.

Theorem 6.3.1 ([BD14, Theorem 17])**.** *The semi-algebraic set $\mathcal{P} \subseteq \mathbb{R}^N$ is homeomorphic to* $\mathrm{Hit}(S, n)$.

The difficult part of the proof is the construction of a representation from a given point in $\mathcal{P}$ that has the correct Bonahon-Dreyer coordinates. From a point in $\mathcal{P}$ they construct a flag decoration $\partial\tilde{\lambda} \to \mathrm{Flag}(\mathbb{R}^n)$ and show that there is a unique representation $\rho\colon \pi_1(S) \to \mathrm{PGL}(n, \mathbb{R})$ for which the flag decoration is ρ-equivariant. They conclude by showing that this representation is valued in $\mathrm{PSL}(n, \mathbb{R})$, a Hitchin representation and has the right invariants.

Using this parametrization we can define a semi-algebraic model for the Hitchin component in the sense of Definition 3.2.1.

Definition 6.3.2. We define $\mathrm{Hom}_{\mathrm{Hit}}(\pi_1(S), \mathrm{PSL}(n, \mathbb{R}))$ as the connected component of $\mathrm{Hom}(\pi_1(S), \mathrm{PSL}(n, \mathbb{R}))$ that contains $\iota_n \circ j$ as defined in Definition 1.3.1.

With this definition $\mathrm{Hit}(S, n) \cong \mathrm{Hom}_{\mathrm{Hit}}(\pi_1(S), \mathrm{PSL}(n, \mathbb{R}))/\mathrm{PSL}(n, \mathbb{R})$. Thus a Hitchin representation is the same as an element of $\mathrm{Hom}_{\mathrm{Hit}}(\pi_1(S), \mathrm{PSL}(n, \mathbb{R}))$.

Proposition 6.3.3. *The map*

$$\mathrm{pr}^{\mathrm{BD}}\colon \mathrm{Hom}_{\mathrm{Hit}}(\pi_1(S), \mathrm{PSL}(n, \mathbb{R})) \to \mathcal{P}$$

defined by Bonahon-Dreyer in Section 6.1 (1)-(3) is a semi-algebraic model for $\mathrm{Hit}(S, n)$.

Proof. The map $\mathrm{pr}^{\mathrm{BD}}$ is continuous and its image $\mathcal{P}$ is by Theorem 6.3.1 homeomorphic to $\mathrm{Hit}(S, n)$, thus the fibres over $\mathcal{P}$ are exactly the $\mathrm{PSL}(n, \mathbb{R})$-orbits. We need to prove that $\mathrm{pr}^{\mathrm{BD}}$ is a semi-algebraic map. Let $y = (y_1, \ldots, y_k)$ be the finite set of points in $\mathrm{Fix}(S)$ that are endpoints of the lifts of the leaves of the lamination λ needed to define the Bonahon-Dreyer coordinates. The points $y_1, \ldots, y_k$ define a polygon P. Given a k-tuple of transverse flags we can assign it to the polygon P. In the same way as in the definition of the Bonahon-Dreyer coordinates we then obtain a map

$$\phi_\lambda\colon \mathrm{Flag}(\mathbb{R}^n)^{(k)} \to \mathbb{R}^N,$$

for N as in Section 6.3 by assigning the triangle and shear invariants Section 6.1 (1)-(3) according to λ. The map $\mathrm{pr}^{\mathrm{BD}}$ is then the composition of the maps $\phi_\lambda \circ \xi_y$, where the latter is defined as

$$\begin{aligned} \xi_y\colon \mathrm{Hom}_{\mathrm{Hit}}(\pi_1(S), \mathrm{PSL}(n, \mathbb{R})) &\to \mathrm{Flag}(\mathbb{R}^n)^{(k)}, \\ \rho &\mapsto (\xi_\rho(y_1), \ldots, \xi_\rho(y_k)). \end{aligned}$$

We show that ϕ_λ and ξ_y are semi-algebraic. Refer to Lemma 5.3.5 for the real semi-algebraic structure on $\mathrm{Flag}(\mathbb{R}^n)^{(k)}$. The map ϕ_λ is given by the triple and double ratios, and is hence semi-algebraic as a regular rational mapping on $\mathrm{Flag}(\mathbb{R}^n)^{(k)}$.

Let us discuss ξ_y. Let $\gamma_1, \ldots, \gamma_k \in \pi_1(S)$ with $\gamma_i^+ = y_i$. Recall from Section 1.3 that $\xi_\rho(y_i) = F^+_{\rho(\gamma_i)}$ for all $i = 1, \ldots, k$. By Lemma 5.3.6 it follows immediately that ξ_y is semi-algebraic. Putting the two maps together we obtain that $\mathrm{pr}^{\mathrm{BD}}$ is semi-algebraic. $\square$

6.4. Extension to real closed fields

Let $\mathbb{F}$ be a real closed field extension of $\mathbb{R}$. We would like to generalize the coordinates for the $\mathbb{F}$-Hitchin component.

Definition 6.4.1. Let $\mathrm{Hom}_{\mathrm{Hit}}(\pi_1(S), \mathrm{PSL}(n, \mathbb{F}))$ be the $\mathbb{F}$-extension of $\mathrm{Hom}_{\mathrm{Hit}}(\pi_1(S), \mathrm{PSL}(n, \mathbb{R}))$.

Lemma 6.4.2. The $\mathbb{F}$-extension $\mathrm{Hom}_{\mathrm{Hit}}(\pi_1(S), \mathrm{PSL}(n, \mathbb{F}))$ is the semi-algebraically connected component of $\mathrm{Hom}(\pi_1(S), \mathrm{PSL}(n, \mathbb{F}))$ that contains $\iota_n \circ j$ as defined in Definition 1.3.1.

Proof. This follows from Theorem 2.1.11 and Theorem 2.1.15. □

Corollary 6.4.3. The semi-algebraic set $\mathcal{P}_{\mathbb{F}} \subseteq \mathbb{F}^N$ is homeomorphic to $\mathrm{Hit}(S, n)_{\mathbb{F}}$.

Proof. We consider the $\mathbb{F}$-extension of the semi-algebraic model for $\mathrm{Hit}(S, n)$, i.e.

$$\mathrm{pr}_{\mathbb{F}}^{\mathrm{BD}} \colon \mathrm{Hom}_{\mathrm{Hit}}(\pi_1(S), \mathrm{PSL}(n, \mathbb{F})) \to \mathcal{P}_{\mathbb{F}}.$$

By Lemma 3.2.3, $\mathrm{pr}_{\mathbb{F}}^{\mathrm{BD}}$ induces a continuous bijection between $\mathrm{Hit}(S, n)_{\mathbb{F}}$ and $\mathcal{P}_{\mathbb{F}}$. To prove that this is a homeomorphism, we consider the inverse map of $\mathrm{pr}^{\mathrm{BD}}$ from $\mathcal{P} \to \mathrm{Hit}(S, n)$. It is continuous and semi-algebraic, thus its extension to $\mathbb{F}$ is continuous; see Theorem 2.1.14. □

Part III.

Hitchin representations over real closed fields

7. Properties of boundary representations

7.1. $\mathbb{F}$-Hitchin representations lift to $\mathrm{SL}(n, \mathbb{F})$

Let $\mathbb{F}$ be a real closed field. The group $\mathrm{SL}(n, \mathbb{F})$ is naturally an algebraic subset of the set of $n \times n$-matrices, denoted $\mathrm{Mat}(n, \mathbb{F})$, which we can identify with $\mathbb{F}^{n^2}$.

Lemma 7.1.1. The group $\mathrm{PGL}(n, \mathbb{F})$ is algebraic. The group $\mathrm{PSL}(n, \mathbb{F})$ is semi-algebraic.

Proof. By the Skolem-Noether theorem, see e.g. [GS06, Theorem 2.7.2], $\mathrm{PGL}(n, \mathbb{F})$ is isomorphic to the set of $\mathbb{F}$-algebra automorphisms $\mathrm{Aut}(\mathrm{Mat}(n, \mathbb{F}))$ of the set of $n \times n$-matrices, and the isomorphism is given by the adjoint representation

$$\mathrm{Ad} \colon \mathrm{PGL}(n, \mathbb{F}) \to \mathrm{Aut}(\mathrm{Mat}(n, \mathbb{F})),\ [A] \mapsto (M \mapsto AMA^{-1}).$$

Thus $\mathrm{PGL}(n, \mathbb{F}) \cong \mathrm{Aut}(\mathrm{Mat}(n, \mathbb{F})) \subseteq \mathrm{GL}(n^2, \mathbb{F})$ is a real algebraic subset of $\mathbb{F}^{n^4+1}$, since being an $\mathbb{F}$-algebra automorphism is an algebraic condition. Hence $\mathrm{PSL}(n, \mathbb{F})$—as a connected component of the algebraic set $\mathrm{PGL}(n, \mathbb{F})$—is real semi-algebraic, see Theorem 2.1.11. □

As in the real case, we have $\mathrm{PSL}(n, \mathbb{F}) = \mathrm{SL}(n, \mathbb{F})$ when n is odd, and $\mathrm{PSL}(n, \mathbb{F})$ is an index two subgroup of $\mathrm{SL}(n, \mathbb{F})$ when n is even. Thus the following considerations only concern the case when n is even.

Lemma 7.1.2. The natural projection map $\mathrm{SL}(n, \mathbb{F}) \to \mathrm{PSL}(n, \mathbb{F})$ is semi-algebraic.

Proof. The projection map is realized by the semi-algebraic map

$$\mathrm{Ad} \colon \mathrm{SL}(n, \mathbb{F}) \to \mathrm{Aut}(\mathrm{Mat}(n, \mathbb{F})),\ A \mapsto (M \mapsto AMA^{-1}).$$ □

Remark 7.1.3. If $\mathbb{F}$ is a real closed extension of $\mathbb{R}$, the extension to $\mathbb{F}$ of the real semi-algebraic sets $\mathrm{SL}(n, \mathbb{R})$ and $\mathrm{PSL}(n, \mathbb{R})$ correspond to the groups $\mathrm{SL}(n, \mathbb{F})$ respectively $\mathrm{PSL}(n, \mathbb{F})$.

Let S be a closed connected orientable surface of genus at least two, and $\pi_1(S)$ its fundamental group. We denote by G either $\mathrm{SL}(n, \mathbb{R})$ or $\mathrm{PSL}(n, \mathbb{R})$. We saw in Section 3.2 how we identify the space $\mathrm{Hom}(\pi_1(S), G)$ with a real semi-algebraic set by choosing a finite set of generators for $\pi_1(S)$. Assume from now on that $\mathbb{R} \subseteq \mathbb{F}$.

Remark 7.1.4. The adjoint representation Ad in the proof of Lemma 7.1.2 induces a continuous semi-algebraic map, which we also denote by Ad, between the spaces of homomorphisms

$$\mathrm{Ad} \colon \mathrm{Hom}(\pi_1(S), \mathrm{SL}(n, \mathbb{F})) \to \mathrm{Hom}(\pi_1(S), \mathrm{PSL}(n, \mathbb{F})).$$

Definition 7.1.5. We define

$$\mathrm{Hom}_{\mathrm{Hit}}(\pi_1(S), \mathrm{SL}(n,\mathbb{F})) := \mathrm{Ad}_{\mathbb{F}}^{-1}\left(\mathrm{Hom}_{\mathrm{Hit}}(\pi_1(S), \mathrm{PSL}(n,\mathbb{F}))\right)$$

as the set of $\mathbb{F}$-*Hitchin representations into* $\mathrm{SL}(n,\mathbb{F})$ (as opposed to $\mathrm{PSL}(n,\mathbb{F})$).

Proposition 7.1.6. The space $\mathrm{Hom}_{\mathrm{Hit}}(\pi_1(S), \mathrm{SL}(n,\mathbb{F}))$ is non-empty and semi-algebraic. Furthermore, every $\mathbb{F}$-Hitchin representation lifts to an $\mathbb{F}$-Hitchin representation into $\mathrm{SL}(n,\mathbb{F})$.

Proof. Goldman proved in [Gol80, Theorem A] that representations $j\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ that are discrete and faithful lift to representations into $\mathrm{SL}(2,\mathbb{R})$, and hence $\iota_n \circ j$ lifts to $\mathrm{SL}(n,\mathbb{R}) \subseteq \mathrm{SL}(n,\mathbb{F})$; see Definition 1.3.1. Thus $\mathrm{Hom}_{\mathrm{Hit}}(\pi_1(S), \mathrm{SL}(n,\mathbb{F}))$ is non-empty. The set $\mathrm{Hom}_{\mathrm{Hit}}(\pi_1(S), \mathrm{SL}(n,\mathbb{F}))$ is semi-algebraic as it is the preimage of a semi-algebraic set under a semi-algebraic map, see Proposition 2.1.9.

Since the set of Hitchin representations is connected, it follows that all Hitchin representations lift. In other words, the map Ad, when restricted to the set of Hitchin representations into $\mathrm{SL}(n,\mathbb{R})$,

$$\mathrm{Ad}\colon \mathrm{Hom}_{\mathrm{Hit}}(\pi_1(S), \mathrm{SL}(n,\mathbb{R})) \to \mathrm{Hom}_{\mathrm{Hit}}(\pi_1(S), \mathrm{PSL}(n,\mathbb{R}))$$

is surjective. If now $\mathbb{F}$ is a real closed field extension of $\mathbb{R}$, the extension of the map Ad to $\mathbb{F}$, when restricted to $\mathbb{F}$-Hitchin representations into $\mathrm{SL}(n,\mathbb{F})$,

$$\mathrm{Ad}_{\mathbb{F}}\colon \mathrm{Hom}_{\mathrm{Hit}}(\pi_1(S), \mathrm{SL}(n,\mathbb{F})) \to \mathrm{Hom}_{\mathrm{Hit}}(\pi_1(S), \mathrm{PSL}(n,\mathbb{F}))$$

is still surjective, see Theorem 2.1.14. Hence every $\mathbb{F}$-Hitchin representation is the image of an $\mathbb{F}$-Hitchin representation into $\mathrm{SL}(n,\mathbb{F})$ under the map $\mathrm{Ad}_{\mathbb{F}}$, and thus every $\mathbb{F}$-Hitchin representation lifts to an $\mathbb{F}$-Hitchin representation into $\mathrm{SL}(n,\mathbb{F})$. □

Remark 7.1.7. Note that $\mathrm{Hom}_{\mathrm{Hit}}(\pi_1(S), \mathrm{SL}(n,\mathbb{F}))$ is only semi-algebraically connected if n is odd. Otherwise it is a union of 2^{2g+1} semi-algebraically connected components if n is even, where g is the genus of S. This follows from the equivalent statement over $\mathbb{R}$ and Theorem 2.1.15.

7.2. $\mathbb{F}$-Hitchin representations are positively hyperbolic

For this section let $\mathbb{F}$ be a real closed field extension of $\mathbb{R}$. Recall that a representation $\rho\colon \pi_1(S) \to \mathrm{PSL}(n,\mathbb{F})$ is called positively hyperbolic if for every non-trivial $\gamma \in \pi_1(S)$ the element $\rho(\gamma)$ is positively hyperbolic (Definition 1.4.4).

Proposition 7.2.1. Let $\rho\colon \pi_1(S) \to \mathrm{PSL}(n,\mathbb{F})$ be an $\mathbb{F}$-Hitchin representation. Then ρ is positively hyperbolic.

Proof. Fix a non-trivial $\gamma \in \pi_1(S)$. Since $\mathrm{Hom}_{\mathrm{Hit}}(\pi_1(S), \mathrm{SL}(n,\mathbb{R}))$ is real semi-algebraic we can consider the real semi-algebraic subset $X_\gamma \subseteq \mathbb{R}^{2g(n\times n)} \times \mathbb{R}^{n\times n} \times \mathbb{R}^n$ defined by

$$\begin{aligned} X_\gamma := \big\{ &(\rho', v_1, \dots, v_n, \lambda_1, \dots, \lambda_n) \mid \rho' \in \mathrm{Hom}_{\mathrm{Hit}}(\pi_1(S), \mathrm{SL}(n,\mathbb{R})), \\ &v_i \in \mathbb{R}^n \setminus \{0\},\ \lambda_i \in \mathbb{R},\ \rho'(\gamma) v_i = \lambda_i v_i \text{ for all } i = 1, \dots, n, \\ &\det(v_1 | \dots | v_n) \neq 0,\ \lambda_1 > \dots > \lambda_n > 0 \text{ or } \lambda_1 < \dots < \lambda_n < 0 \big\}. \end{aligned}$$

Then the projection pr onto the first $2g(n \times n)$ coordinates gives a surjection of semi-algebraic sets onto the component of Hitchin representations into $\mathrm{SL}(n, \mathbb{R})$

$$\begin{aligned} \mathrm{pr}\colon X_\gamma &\twoheadrightarrow \mathrm{Hom}_{\mathrm{Hit}}(\pi_1(S), \mathrm{SL}(n, \mathbb{R})), \\ (\rho', v_i, \lambda_i) &\mapsto \rho'. \end{aligned}$$

This map is surjective since for every Hitchin representation $\rho'\colon \pi_1(S) \to \mathrm{SL}(n, \mathbb{R})$ and every non-trivial $\gamma \in \pi_1(S)$ the matrix $\rho'(\gamma)$ is diagonalizable over $\mathbb{R}$ with distinct eigenvalues all of the same sign; refer for example to [BD14, Proposition 8 and Lemma 9], since every Hitchin representation into $\mathrm{SL}(n, \mathbb{R})$ is a lift of a Hitchin representation into $\mathrm{PSL}(n, \mathbb{R})$. By the Tarski-Seidenberg transfer principle, see Theorem 2.1.13, we obtain that for any real closed extension $\mathbb{F}$ of $\mathbb{R}$,

$$\mathrm{pr}_{\mathbb{F}}((X_\gamma)_{\mathbb{F}}) = (\mathrm{pr}(X_\gamma))_{\mathbb{F}} = \mathrm{Hom}_{\mathrm{Hit}}(\pi_1(S), \mathrm{SL}(n, \mathbb{F})).$$

This means that $\mathbb{F}$-Hitchin representations into $\mathrm{SL}(n, \mathbb{F})$ are diagonalizable over $\mathbb{F}$ with distinct eigenvalues all of the same sign. Now every $\mathbb{F}$-Hitchin representation into $\mathrm{PSL}(n, \mathbb{F})$ lifts to an $\mathbb{F}$-Hitchin representation into $\mathrm{SL}(n, \mathbb{F})$ (Proposition 7.1.6), which proves the lemma. □

7.3. $\mathbb{F}$-positive representations are positively hyperbolic

We have seen that $\mathbb{F}$-Hitchin representations are positively hyperbolic in the last section. Now we deduce that $\mathbb{F}$-positive representations are positively hyperbolic. This result is crucial in the backward direction of the proof of Theorem A; namely it will imply the closed leaf inequality Section 6.2 (iv). Recall that a representation $\rho\colon \pi_1(S) \to \mathrm{PSL}(n, \mathbb{F})$ is $\mathbb{F}$-positive if there exists a ρ-equivariant $\mathbb{F}$-positive map $\xi_\rho\colon \mathrm{Fix}(S) \to \mathrm{Flag}(\mathbb{F}^n)$, i.e. ξ_ρ maps any triple respectively quadruple of cyclically ordered points in $\mathrm{Fix}(S)$ to a positive triple respectively quadruple of flags (Definition 5.1.1). In particular, Remark 5.1.5 implies that ξ_ρ maps any k cyclically ordered points in $\mathrm{Fix}(S)$ to a positive k-tuple of flags (Definition 5.1.3) for any $k \geq 3$.

Proposition 7.3.1. Let $\rho\colon \pi_1(S) \to \mathrm{PSL}(n, \mathbb{F})$ be $\mathbb{F}$-positive. Then ρ is positively hyperbolic.

The proof of this proposition follows from Theorem 5.3.3 and the following proposition. Recall that an element in $\mathrm{GL}(n, \mathbb{F})$ is totally positive, if all its minors are positive (Definition 5.2.1).

Proposition 7.3.2. Let $\mathbb{F}$ be an ordered field (not necessarily real closed) and $\rho\colon \pi_1(S) \to \mathrm{PSL}(n, \mathbb{F})$ an $\mathbb{F}$-positive representation. Then for all $\gamma \in \pi_1(S)$ non-trivial, $\rho(\gamma) \in \mathrm{PSL}(n, \mathbb{F})$ admits a lift $\rho(\gamma)'$ to $\mathrm{SL}(n, \mathbb{F})$ that is conjugate (in $\mathrm{GL}(n, \mathbb{F})$) to a totally positive matrix.

The idea of the proof is to recover $\mathbb{F}$-positive representation from their limit maps which are transverse and equivariant with respect to the representation. This allows us to explicitly describe the image of the representation in terms of the triple and double ratios.

Proof of Proposition 7.3.2. Let $\gamma \in \pi_1(S)$ non-trivial. Fix the topological data λ on S that is needed to define the Bonahon-Dreyer coordinates, see Section 6.1, such that the geodesic representative of γ is not a leaf of λ.

We begin by explaining how to associate to γ and λ a finite polygon P_{γ,λ,x_0} inscribed in $\partial\tilde{S}$. This description follows [Mar19b, Section 5.2]. Choose a point x_0 on the axis of γ in the interior of an ideal triangle t_0 of $\tilde{S} \setminus \tilde{\lambda}$. We remark that all sides of t_0 are lifts of open leaves of λ. In the following we are considering the two triangles t_0 and its γ-translate γt_0, see Figure 7.1. We make two observations.

First, there are only finitely many lifts of closed leaves in λ that separate t_0 and γt_0. Indeed, if a lift of a closed leaf separates t_0 and γt_0 it must intersect the axis of γ, but the geodesic representative of γ intersects only finitely many times the (finitely many) closed leaves in λ. Since the preimage under the covering map of a closed geodesic is a discrete subset of $\tilde{S}$ there are finitely many lifts of a closed geodesic that separate t_0 and γt_0, since x_0 and γx_0 have a finite distance, namely the length of γ in the chosen auxiliary hyperbolic metric on S.

Secondly, if we choose for every lift $\tilde{\eta}$ of a closed leaf η of λ, that separates the two ideal triangles, and a lift $\tilde{k}_\eta$ of a transverse arc k_η, that intersects $\tilde{\eta}$ but none of the sides of t_0 and γt_0, then there exists only finitely many lifts of infinite leaves that separate t_0 and γt_0, and that do not intersect any of the chosen lifts $\tilde{k}_\eta$. In fact, since λ is a closed subset of S, both ends of an open leave spiral towards closed leaves in λ. If infinitely many lifts of an open leaf separate t_0 and γt_0, then so does a lift of the closed leaf towards which the open leaf spirals, call it $\eta \in \lambda$. By the choice of $\tilde{k}_\eta$ and the condition that we only consider those lifts of open leaves that do not intersect $\tilde{k}_\eta$, we single out a finite number of lifts of open leaves.

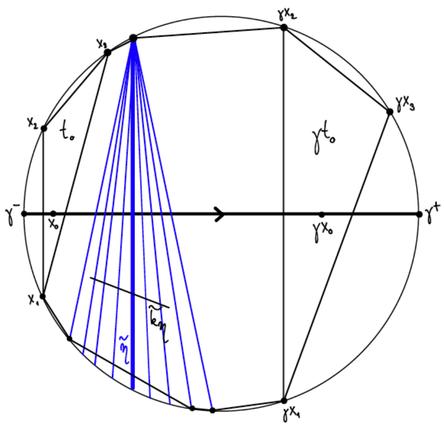

Figure 7.1.: Definition of the finite polygon P_{γ,λ,x_0} in the case that one lift of a closed leaf η in λ separates t_0 and γt_0

We label by $(e_1, \dots, e_p)$, by proximity to t_0, the finite collection of lifts of those open leaves of λ that separate the triangles t_0 and γt_0, and that do not intersect any of the lifts $\tilde{k}_\eta$ of the transverse arcs, where we assume that e_1 is the edge of t_0 closest to γt_0, and e_p is the edge of γt_0 closest to t_0. We set $e_0 := \gamma^{-1} e_p$ and $e_{p+1} := \gamma e_1$, and define

$$\mathcal{E} = \mathcal{E}_{\gamma,\lambda,x_0} = (e_0, \dots, e_{p+1}).$$

The vertices that are endpoints of elements in $\mathcal{E}$ determine a finite cyclically ordered list of points $(x_1, \ldots, x_k)$ on $\partial\tilde{S}$, which defines the desired polygon $P = P_{\gamma,\lambda,x_0}$. It is important to remark that the set of triangles defined by $\mathcal{E}$ is not necessarily a subset of $\tilde{S}\setminus\tilde{\lambda}$ as soon as one of the elements of $\mathcal{E}$ is the lift of a closed leaf in λ. Label the vertices of t_0 by x_1, x_2, x_3 in clockwise order, and the vertices of γt_0 by $y_1 = \gamma x_1$, $y_2 = \gamma x_2$, $y_3 = \gamma x_3$ such that the open leaf $e_1 \in \tilde{\lambda}$ connects x_1 to x_3. We have the two possible cases depicted in Figure 7.2, since the geodesic representative of γ is not a leaf of λ.

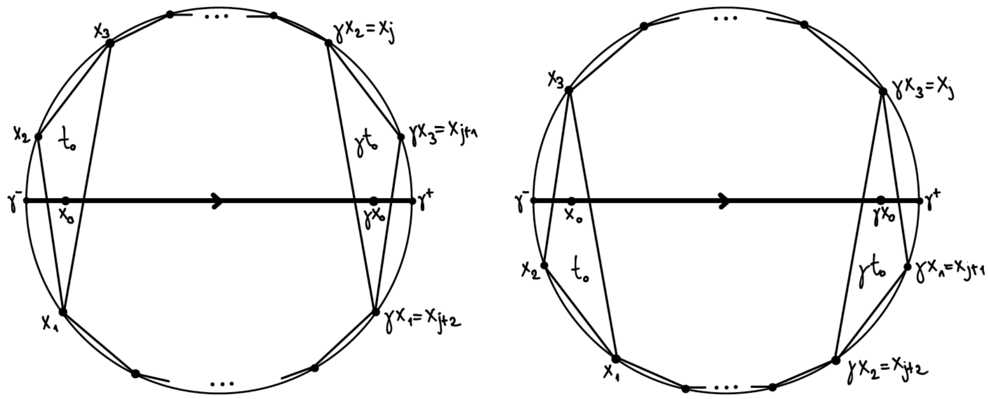

Figure 7.2.: The possible cases of how the axis of γ intersects the triangles t_0 and γt_0.

Let ξ_ρ be the limit map associated to the $\mathbb{F}$-positive representation ρ as in Definition 1.4.3. The important observation is that we can recover ρ from ξ_ρ. Indeed by ρ-equivariance of ξ_ρ, we obtain that there exists $\varphi_\gamma \in \mathrm{PGL}(\mathbb{F}^n)$ such that

$$(\xi_\rho(y_1), \xi_\rho(y_2), \xi_\rho(y_3)) = \varphi_\gamma(\xi_\rho(x_1), \xi_\rho(x_2), \xi_\rho(x_3)).$$

By uniqueness (Proposition 4.1.1) there exists a basis of $\mathbb{F}^n$ in which φ_γ is represented by $\rho(\gamma) \in \mathrm{PSL}(n, \mathbb{F})$.

Since ξ_ρ is positive, it sends the vertices of the polygon P_{γ,λ,x_0} to a positive tuple of flags, see Remark 5.1.5. Since γ is not a leaf of λ and the surface is closed, we are in the setting of Theorem 5.2.3 applied to the finite polygon P_{γ,λ,x_0} and the two triples of flags $(\xi_\rho(x_1), \xi_\rho(x_2), \xi_\rho(x_3))$ and $(\xi_\rho(y_1), \xi_\rho(y_2), \xi_\rho(y_3))$. The theorem tells us that there exists a (potentially different) basis of $\mathbb{F}^n$ in which a lift of φ_γ is represented by a totally positive matrix M_{φ_γ}. Since φ_γ is as well represented by $\rho(\gamma)$, also $\rho(\gamma)$ admits a lift $\rho(\gamma)'$ to $\mathrm{SL}(n, \mathbb{F})$ that is conjugate to M_{φ_γ}, which is what we had to prove. □

7.4. Equivalence of $\mathbb{F}$-Hitchin and $\mathbb{F}$-positive weakly dynamics preserving representations

We recall Theorem A.

Theorem A. *Let $\mathbb{F}$ be a real closed extension of $\mathbb{R}$. A representation $\rho\colon \pi_1(S) \to \mathrm{PSL}(n, \mathbb{F})$ is $\mathrm{PGL}(n, \mathbb{F})$-conjugate to an $\mathbb{F}$-Hitchin representation if and only if it is $\mathbb{F}$-positive and weakly dynamics preserving.*

Proof ($\Longrightarrow$). We first prove the "only if" direction, namely we show that an $\mathbb{F}$-Hitchin representation (Definition 1.4.1) is $\mathbb{F}$-positive weakly dynamics preserving (Definition 1.4.3). Let thus $\rho\colon \pi_1(S) \to \mathrm{PSL}(n, \mathbb{F})$ be an $\mathbb{F}$-Hitchin representation. By Proposition 7.2.1, we know that for every non-trivial $\gamma \in \pi_1(S)$ we can lift $\rho(\gamma)$ to a matrix $\rho(\gamma)' \in \mathrm{SL}(n, \mathbb{F})$ which is diagonalizable over $\mathbb{F}$ with eigenvalues $\lambda_1 > \ldots > \lambda_n > 0$. We can now construct a limit map $\xi_\rho\colon \mathrm{Fix}(S) \to \mathrm{Flag}(\mathbb{F}^n)$ associated to ρ that has the desired properties. For $x \in \mathrm{Fix}(S)$, choose a non-trivial $\gamma \in \pi_1(S)$ with $x = \gamma^+$, where we write γ^+, γ^- for the attracting respectively repelling fixed point of γ acting on $\mathrm{Fix}(S) \subseteq \partial\tilde{S}$. Then we define ξ_ρ by

$$\xi_\rho\colon \mathrm{Fix}(S) \to \mathrm{Flag}(\mathbb{F}^n), \quad x \mapsto F^+_{\rho(\gamma)'},$$

where $F^+_{\rho(\gamma)'}$ denotes the stable flag of $\rho(\gamma)'$ (Definition 5.3.1). We need to check that this map is well-defined since we made several choices (the choice of γ and the choice of a lift of $\rho(\gamma)$). Indeed, first note that the definition of ξ_ρ is independent of the choice of the lift of $\rho(\gamma)$ to $\mathrm{SL}(n, \mathbb{F})$, since two lifts differ by multiplication with ± 1, which preserves the eigenspaces and the moduli of the eigenvalues. Furthermore, for any non-trivial $\gamma_1, \gamma_2 \in \pi_1(S)$ with $x = \gamma_1^+ = \gamma_2^+$, we have $\gamma_1^- = \gamma_2^-$, and thus $\gamma_1^{k_1} = \gamma_2^{k_2}$ for some $k_1, k_2 \in \mathbb{Z}$. But then, since we know that both $\rho(\gamma_1)$ and $\rho(\gamma_2)$ are diagonalizable and they agree up to some power (since γ_1 and γ_2 do), they must have the same eigenspaces and hence $F^+_{\rho(\gamma_1)'} = F^+_{\rho(\gamma_2)'}$, which shows that the map ξ_ρ is well-defined.

The map ξ_ρ is ρ-equivariant: Let $\gamma \in \pi_1(S)$ be a non-trivial element and $\eta^+ \in \mathrm{Fix}(S)$ for some non-trivial $\eta \in \pi_1(S)$. Then

$$\xi_\rho(\gamma \cdot \eta^+) = \xi_\rho((\gamma\eta\gamma^{-1})^+) = \rho(\gamma)\xi_\rho(\eta^+),$$

by observing that $\gamma \cdot \eta^+$ is the attracting fixed point of $\gamma\eta\gamma^{-1}$, and that the stable flag of $\rho(\gamma\eta\gamma^{-1}) = \rho(\gamma)\rho(\eta)\rho(\gamma)^{-1}$ is obtained by applying $\rho(\gamma)$ to the stable flag of $\rho(\eta)$.

To verify that ξ_ρ is $\mathbb{F}$-positive, we use the Tarski-Seidenberg transfer principle, in the same way as we did in Proposition 7.2.1. By Remark 5.1.5 it suffices to check that ξ_ρ sends any triple and quadruple of distinct points to a triple respectively quadruples of positive flags.

(a) We first show that the image of any two distinct points is transverse (Definition 4.0.2). Let $x = \gamma^+$, $y = \eta^+$ be distinct points in $\mathrm{Fix}(S)$ for $\gamma \neq \eta \in \pi_1(S)$. We need to show that the flag tuple $(\xi_\rho(\gamma^+), \xi_\rho(\eta^+))$ is transverse, i.e. for all $j = 0, \ldots, n$ we have

$$\xi_\rho(\gamma^+)^{(j)} + \xi_\rho(\eta^+)^{(n-j)} = \mathbb{F}^n.$$

Since $\xi_\rho(\gamma^+)$ is defined as the stable flag of $\rho(\gamma)$ the above transversality condition can be encoded in the following polynomial inequality

$$\det(v_1|\ldots|v_j|w_1|\ldots|w_{n-j}) \neq 0 \text{ for all } j = 0, \ldots, n,$$

where v_i and w_i are eigenvectors of $\rho(\gamma)$ respectively $\rho(\eta)$ with eigenvalues λ_i respectively μ_i with $|\lambda_1| > \ldots > |\lambda_n| > 0$ and $|\mu_1| > \ldots > |\mu_n| > 0$. We should in fact first lift ρ to a Hitchin representation into $\mathrm{SL}(n, \mathbb{F})$, but this is possible without problems by the considerations in Section 7.1, compare also the proof of

Proposition 7.2.1 for more details. By the Tarski-Seidenberg transfer principle (Theorem 2.1.13) this inequality holds true over $\mathbb{F}$ since it holds true over $\mathbb{R}$; see Theorem 1.3.3.

(b) We verify that any three distinct points positively oriented around the circle are mapped to a positive triple of flags. Let $x_1 = \gamma_1^+$, $x_2 = \gamma_2^+$, $x_3 = \gamma_3^+$ be distinct points in $\mathrm{Fix}(S)$ for some non-trivial $\gamma_1, \gamma_2, \gamma_3 \in \pi_1(S)$, positively oriented around the circle. We need to show that the flag triple $(\xi_\rho(\gamma_1^+), \xi_\rho(\gamma_2^+), \xi_\rho(\gamma_3^+))$ is positive, i.e. it is transverse and all triple ratios are positive. In other words, for all integers $a, b, c \geq 0$ with $a + b + c = n$ we have

$$\xi_\rho(\gamma_1^+)^{(a)} + \xi_\rho(\gamma_2^+)^{(b)} + \xi_\rho(\gamma_3^+)^{(c)} = \mathbb{F}^n,$$

and for all integers $a, b, c \geq 1$ with $a + b + c = n$ we have

$$T_{abc}\left(\xi_\rho(\gamma_1^+), \xi_\rho(\gamma_2^+), \xi_\rho(\gamma_3^+)\right) > 0.$$

We have already seen in (a) how to encode the transversality condition in a semi-algebraic way. The positivity condition on the triple ratios can be encoded by the following boolean combination of polynomial inequalities

$$\begin{aligned}
&\big[\det(v_1|\dots|v_{a+1}|w_1|\dots|w_b|u_1|\dots|u_{c-1})\\
&\qquad\cdot\det(v_1|\dots|v_a|w_1|\dots|w_{b-1}|u_1|\dots|u_{c+1})\\
&\qquad\cdot\det(v_1|\dots|v_{a-1}|w_1|\dots|w_{b+1}|u_1|\dots|u_c) > 0 \text{ and}\\
&\ \det(v_1|\dots|v_{a-1}|w_1|\dots|w_b|u_1|\dots|u_{c+1})\\
&\qquad\cdot\det(v_1|\dots|v_a|w_1|\dots|w_{b+1}|u_1|\dots|u_{c-1})\\
&\qquad\cdot\det(v_1|\dots|v_{a+1}|w_1|\dots|w_{b-1}|u_1|\dots|u_c) > 0\big]\\
&\ \text{or}\\
&\big[\det(v_1|\dots|v_{a+1}|w_1|\dots|w_b|u_1|\dots|u_{c-1})\\
&\qquad\cdot\det(v_1|\dots|v_a|w_1|\dots|w_{b-1}|u_1|\dots|u_{c+1})\\
&\qquad\cdot\det(v_1|\dots|v_{a-1}|w_1|\dots|w_{b+1}|u_1|\dots|u_c) < 0 \text{ and}\\
&\ \det(v_1|\dots|v_{a-1}|w_1|\dots|w_b|u_1|\dots|u_{c+1})\\
&\qquad\cdot\det(v_1|\dots|v_a|w_1|\dots|w_{b+1}|u_1|\dots|u_{c-1})\\
&\qquad\cdot\det(v_1|\dots|v_{a+1}|w_1|\dots|w_{b-1}|u_1|\dots|u_c) < 0\big]
\end{aligned}$$

for all integers $a, b, c \geq 1$ with $a+b+c = n$, where v_i, w_i and u_i are eigenvectors of $\rho(\gamma_1)$, $\rho(\gamma_2)$ respectively $\rho(\gamma_3)$ with eigenvalues λ_i, μ_i respectively ν_i with $|\lambda_1| > \dots > |\lambda_n| > 0$, $|\mu_1| > \dots > |\mu_n| > 0$ and $|\nu_1| > \dots > |\nu_n| > 0$. As in (a), we should consider a lift of ρ to an $\mathbb{F}$-Hitchin representation into $\mathrm{SL}(n, \mathbb{F})$. Again by the Tarski-Seidenberg transfer principle, this property holds true over $\mathbb{F}$ since it holds true over $\mathbb{R}$.

(c) Let x_1, x_2, x_3, x_4 be distinct points in $\mathrm{Fix}(S)$ occurring in this order around the circle. We need to show that the flag quadruple

$$(\xi_\rho(x_1), \xi_\rho(x_2), \xi_\rho(x_3), \xi_\rho(x_4))$$

is positive, i.e. the flag quadruple is transverse, the two flag subtriples $(\xi_\rho(x_1), \xi_\rho(x_2), \xi_\rho(x_3))$ and $(\xi_\rho(x_1), \xi_\rho(x_3), \xi_\rho(x_4))$ are positive, and all double ratios are positive. In particular, for all $a = 1, \ldots, n-1$ we have

$$D_a\left(\xi_\rho(x_1), \xi_\rho(x_3), \xi_\rho(x_2), \xi_\rho(x_4)\right) > 0.$$

Choose $\gamma_j \in \pi_1(S)$ non-trivial distinct so that $\gamma_j^+ = x_j$ for all $j = 1, 2, 3, 4$. We have already seen in (a) and (b) how the transversality condition and the positivity of the triple ratios can be expressed using polynomial equalities and inequalities. The positivity of the double ratios is expressed similarly as the positivity of the triple ratios in (b). Again by the Tarski-Seidenberg transfer principle, this property holds true over $\mathbb{F}$ since it holds true over $\mathbb{R}$.

This proves that ρ is $\mathbb{F}$-positive. Furthermore, by definition of ξ_ρ, we immediately have that ρ is weakly dynamics preserving, which proves one direction. □

For the "if" direction we establish Theorem C, which we recall here.

Theorem C. *The set of* $\mathrm{PGL}(n, \mathbb{F})$*-equivalence classes of* $\mathbb{F}$*-positive weakly dynamics preserving representations is described by the Bonahon-Dreyer coordinates over* $\mathbb{F}$ *and hence homeomorphic to a closed semi-algebraic subset of some* $\mathbb{F}^N$*.*

Proof. For the proof we use Proposition 7.3.1 and the multiplicative variant of the Bonahon-Dreyer coordinates for parametrizing the Hitchin component as in Chapter 6. Fix therefore the necessary topological data λ on the surface S, see Section 6.1. We use the same notation as in Chapter 6. Suppose the geodesic lamination λ has p closed leaves and q infinite leaves, and its complement $S \setminus \lambda$ consists of r ideal triangles. Set

$$N = 3r\frac{(n-1)(n-2)}{2} + (p+q)(n-1).$$

The idea of the proof is as follows. We define a map Ψ from the set of $\mathbb{F}$-positive weakly dynamics preserving representations to $\mathbb{F}^N$ by associating to a representation the $\mathbb{F}$-valued triangle and shear invariants using the $\mathbb{F}$-positive limit map as in Section 6.1. If $\rho\colon \pi_1(S) \to \mathrm{PSL}(n, \mathbb{F})$ is an $\mathbb{F}$-positive weakly dynamics preserving representation, we show that $\Psi(\rho)$ satisfies the relations (i)-(iv) of Section 6.2 . In other words, $\Psi(\rho) \in \mathcal{P}_{\mathbb{F}}$—the $\mathbb{F}$-extension of the polytope $\mathcal{P} \subseteq \mathbb{R}^N$ that is homeomorphic to the Hitchin component $\mathrm{Hit}(S, n)$, compare Section 6.3 and Section 6.4.

The triangle and shear invariants can be defined using a flag decoration, by which we mean an equivariant map from $\partial\tilde{\lambda} \to \mathrm{Flag}(\mathbb{F}^n)$. Thus, in the same way as in (1)-(3) in Section 6.1 we can associate triangle and shear invariants to $\mathbb{F}$-positive representations using only the associated limit map restricted to $\partial\tilde{\lambda} \subseteq \mathrm{Fix}(S)$. This defines the map Ψ.

Let us show that Ψ satisfies (A)-(D) in Section 6.3 over $\mathbb{F}$. The positivity conditions (A) and (B) follow from the $\mathbb{F}$-positivity of the limit map. Note that the rotation condition (C) follows directly since it is a property of triple ratios and how they behave under permutation of the flags, see Lemma 4.2.2 (1). Left to show are therefore the closed leaf equality and closed leaf inequality (D). For Hitchin representations, the proof of the closed leaf equalities relies on [BD14, Proposition 13]. We prove in a complete analogue fashion the multiplicative version of this proposition for the coordinates associated to $\mathbb{F}$-positive representations; see also [BD14, Remark 15] for a multiplicative version of the statement of [BD14, Proposition 13].

Proposition 7.4.1. Let $\rho\colon \pi_1(S) \to \mathrm{PSL}(n,\mathbb{F})$ be an $\mathbb{F}$-positive representation with limit map ξ_ρ. For every closed leaf $\gamma \in \lambda$ and every $a = 1,\dots,n-1$, we have

$$L_a^{\text{right}}(\gamma) = \frac{\lambda_a(\rho(\gamma)')}{\lambda_{a+1}(\rho(\gamma)')} = L_a^{\text{left}}(\gamma),$$

where $\lambda_a(\rho(\gamma)')$ is the eigenvalue of a lift $\rho(\gamma)'$ of $\rho(\gamma)$ to $\mathrm{SL}(n,\mathbb{F})$ corresponding to the one-dimensional eigenspace $\xi_\rho(\gamma^+)^{(a)} \cap \xi_\rho(\gamma^-)^{(n-a+1)}$.

Proof. By Proposition 7.3.1, we can lift $\rho(\gamma) \in \mathrm{PSL}(n,\mathbb{F})$ to $\rho(\gamma)' \in \mathrm{SL}(n,\mathbb{F})$ such that the matrix $\rho(\gamma)'$ is diagonalizable with eigenvalues $\lambda_a(\rho(\gamma)') > 0$ and $\rho(\gamma)'$ acts by multiplication with $\lambda_a(\rho(\gamma)')$ on $\xi_\rho(\gamma^+)^{(a)}/\xi_\rho(\gamma^+)^{(a-1)}$. We will show that $L_a^{\text{right}}(\gamma) = \frac{\lambda_a(\rho(\gamma)')}{\lambda_{a+1}(\rho(\gamma)')}$ and $L_a^{\text{left}}(\gamma) = \frac{\lambda_a(\rho(\gamma)')}{\lambda_{a+1}(\rho(\gamma)')}$. Note that this quotient is independent of the lift of $\rho(\gamma)$ since two lifts differ by multiplication with $\pm\mathrm{Id}_n$.

The proof works in the same way as the proof of [BD14, Proposition 13] by replacing τ_{abc} by T_{abc} and σ_a by D_a (where $\tau_{abc} = \log(T_{abc})$ and $\sigma_a = \log(D_a)$ following the notation from [BD14]) and by writing everything multiplicatively. We also refer to this reference for more details on the computations. The important observation is how $\rho(\gamma)'$ acts on the flags $\xi_\rho(\gamma^\pm)$ associated to the endpoints of γ; see Equation (7.1).

We only discuss the exemplary case when we are on the right-hand side of γ and the spiralling of the triangles occurs in the direction of the orientation of γ. We recall the definition of

$$L_a^{\text{right}}(\gamma) = \prod_{l=1}^{k} \overline{D}_a(h_l) \prod_{l=1}^{k} \prod_{b+c=n-a} T^{\rho}_{abc}(t_l, v_l),$$

which we will show is equal to $\frac{\lambda_a(\rho(\gamma)')}{\lambda_{a+1}(\rho(\gamma)')}$.

Let $h_1,\dots,h_k$ and $t_1,\dots,t_k$ be the infinite leaves in λ and the ideal triangles in $S\setminus\lambda$ that spiral on the right side of γ and assume the spiralling occurs in the direction of the orientation of γ. Let the labelling be such that t_l is bounded on the left by h_{l-1}. Let v_l be the vertex of t_l in the direction of the spiralling. See Figure 7.3 for the corresponding picture in the universal cover. We set $E := \xi_\rho(\gamma^+)$, $F_l := \xi_\rho(x_l)$, where x_l is the endpoint of $\tilde{h}_l$ different from γ^+, and it is convenient to also define $G_1 := F_k$ and $G_l := F_{l-1}$ for $l = 2,\dots,k+1$. By the choice of orientation we have $\gamma\,\tilde{h}_1 = \tilde{h}_{k+1}$ and $\gamma\,\tilde{t}_1 = \tilde{t}_{k+1}$. We first remark that

$$\begin{aligned}
&\overline{D}_a(h_l) = D_a(E, F_l, F_{l-1}, F_{l+1}) = D_a(E, G_{l+1}, G_l, F_{l+1}), \text{ and}\\
&T^{\rho}_{abc}(t_l, v_l) = T_{abc}(E, F_l, F_{l+1}) = T_{abc}(E, F_l, G_l),
\end{aligned}$$

where the first equality follows by checking the two cases (h_l is oriented towards γ or not) and using the properties of the double ratios under permutation of flags. We rewrite $D_a(E, F_l, F_{l-1}, F_{l+1})$ to obtain

$$\begin{aligned}
\overline{D}_a(h_l) = D_a(E, F_l, F_{l-1}, F_{l+1}) = &\\
&\left[\frac{e^{(a)} \wedge f_l^{(n-a-1)} \wedge g_l^{(1)}}{e^{(a-1)} \wedge f_l^{(n-a)} \wedge g_l^{(1)}} \cdot \frac{e^{(a)} \wedge f_l^{(n-a)}}{e^{(a+1)} \wedge f_l^{(n-a-1)}}\right]\\
&\cdot \left[\frac{e^{(a-1)} \wedge f_{l+1}^{(1)} \wedge g_{l+1}^{(n-a)}}{e^{(a)} \wedge f_{l+1}^{(1)} \wedge g_{l+1}^{(n-a-1)}} \cdot \frac{e^{(a+1)} \wedge g_{l+1}^{(n-a-1)}}{e^{(a)} \wedge g_{l+1}^{(n-a)}}\right].
\end{aligned}$$

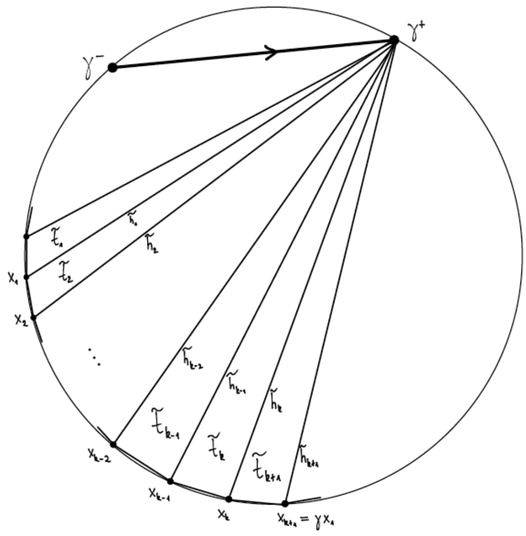

Figure 7.3.: Spiralling in the universal cover.

Note that the minus sign disappears when we swap the order in the wedge product in the nominator and denominator of the third factor, since it involves exterior powers of degrees that differ by one. What is important to note is that within the first bracket only the index l appears, and in the second bracket only the index $l+1$.

The second step is to reorder $\prod_{l=1}^{k} \overline{D}_a(h_l) = \prod_{l=1}^{k} D_a(E, F_l, F_{l-1}, F_{l+1})$. Using the last equality and an index shift, we obtain

$$\begin{aligned}
\prod_{l=1}^{k} \overline{D}_a(h_l) = &\left[\frac{e^{(a)} \wedge f_1^{(n-a-1)} \wedge g_1^{(1)}}{e^{(a-1)} \wedge f_1^{(n-a)} \wedge g_1^{(1)}} \cdot \frac{e^{(a)} \wedge f_1^{(n-a)}}{e^{(a+1)} \wedge f_1^{(n-a-1)}}\right] \\
&\cdot \prod_{l=2}^{k} \Bigg(\left[\frac{e^{(a)} \wedge f_l^{(n-a-1)} \wedge g_l^{(1)}}{e^{(a-1)} \wedge f_l^{(n-a)} \wedge g_l^{(1)}} \cdot \frac{e^{(a)} \wedge f_l^{(n-a)}}{e^{(a+1)} \wedge f_l^{(n-a-1)}}\right] \\
&\qquad \cdot \left[\frac{e^{(a-1)} \wedge f_l^{(1)} \wedge g_l^{(n-a)}}{e^{(a)} \wedge f_l^{(1)} \wedge g_l^{(n-a-1)}} \cdot \frac{e^{(a+1)} \wedge g_l^{(n-a-1)}}{e^{(a)} \wedge g_l^{(n-a)}}\right] \Bigg) \\
&\cdot \left[\frac{e^{(a-1)} \wedge f_{k+1}^{(1)} \wedge g_{k+1}^{(n-a)}}{e^{(a)} \wedge f_{k+1}^{(1)} \wedge g_{k+1}^{(n-a-1)}} \cdot \frac{e^{(a+1)} \wedge g_{k+1}^{(n-a-1)}}{e^{(a)} \wedge g_{k+1}^{(n-a)}}\right].
\end{aligned}$$

Since $x_{k+1} = \gamma\, x_1$ we have $F_{k+1} = \xi_\rho(\gamma\, x_1) = \rho(\gamma)\xi_\rho(x_1) = \rho(\gamma)F_1$ by ρ-equivariance of

ξ_ρ. The last bracket simplifies to

$$\begin{aligned}
&\frac{e^{(a-1)} \wedge f_{k+1}^{(1)} \wedge g_{k+1}^{(n-a)}}{e^{(a)} \wedge f_{k+1}^{(1)} \wedge g_{k+1}^{(n-a-1)}} \cdot \frac{e^{(a+1)} \wedge g_{k+1}^{(n-a-1)}}{e^{(a)} \wedge g_{k+1}^{(n-a)}} \\
&= \frac{e^{(a-1)} \wedge \rho(\gamma)' f_1^{(1)} \wedge \rho(\gamma)' g_1^{(n-a)}}{e^{(a)} \wedge \rho(\gamma)' f_1^{(1)} \wedge \rho(\gamma)' g_1^{(n-a-1)}} \cdot \frac{e^{(a+1)} \wedge \rho(\gamma)' g_1^{(n-a-1)}}{e^{(a)} \wedge \rho(\gamma)' g_1^{(n-a)}} \\
&= \frac{(\rho(\gamma)'^{-1} e^{(a-1)}) \wedge f_1^{(1)} \wedge g_1^{(n-a)}}{(\rho(\gamma)'^{-1} e^{(a)}) \wedge f_1^{(1)} \wedge g_1^{(n-a-1)}} \cdot \frac{(\rho(\gamma)'^{-1} e^{(a+1)}) \wedge g_1^{(n-a-1)}}{(\rho(\gamma)'^{-1} e^{(a)}) \wedge g_1^{(n-a)}} \\
&= \lambda_a(\rho(\gamma)') \frac{e^{(a-1)} \wedge f_1^{(1)} \wedge g_1^{(n-a)}}{e^{(a)} \wedge f_1^{(1)} \wedge g_1^{(n-a-1)}} \cdot \frac{1}{\lambda_{a+1}(\rho(\gamma)')} \frac{e^{(a+1)} \wedge g_1^{(n-a-1)}}{e^{(a)} \wedge g_1^{(n-a)}},
\end{aligned}$$

since $\rho(\gamma)'^{-1}$ acts trivially on $\bigwedge^n(\mathbb{F}^n)$, and for all $a = 1, \ldots, n$, we have

$$\rho(\gamma)' \left(e^{(a)} \right) = \left(\prod_{i=1}^{a} \lambda_i(\rho(\gamma)') \right) e^{(a)}. \tag{7.1}$$

Indeed, since the spiralling occurs in the direction of the orientation of γ and the flag associated with the positive endpoint of γ is the stable flag of $\rho(\gamma)$, see the proof of Proposition 7.3.1, Equation (7.1) holds. Thus

$$\begin{aligned}
\prod_{l=1}^{k} \overline{D}_a(h_l) = &\frac{\lambda_a(\rho(\gamma)')}{\lambda_{a+1}(\rho(\gamma)')} \\
&\cdot \prod_{l=1}^{k} \Bigg(\left[\frac{e^{(a)} \wedge f_l^{(n-a-1)} \wedge g_l^{(1)}}{e^{(a-1)} \wedge f_l^{(n-a)} \wedge g_l^{(1)}} \cdot \frac{e^{(a)} \wedge f_l^{(n-a)}}{e^{(a+1)} \wedge f_l^{(n-a-1)}} \right] \\
&\cdot \left[\frac{e^{(a-1)} \wedge f_l^{(1)} \wedge g_l^{(n-a)}}{e^{(a)} \wedge f_l^{(1)} \wedge g_l^{(n-a-1)}} \cdot \frac{e^{(a+1)} \wedge g_l^{(n-a-1)}}{e^{(a)} \wedge g_l^{(n-a)}} \right] \Bigg) \\
&= \frac{\lambda_a(\rho(\gamma)')}{\lambda_{a+1}(\rho(\gamma)')} \prod_{l=1}^{k} Q_a(E, F_l, G_l)^{-1},
\end{aligned}$$

by definition of Q_a. By Lemma 4.2.2 (2), we have

$$\prod_{l=1}^{k} \overline{D}_a(h_l) = \frac{\lambda_a(\rho(\gamma)')}{\lambda_{a+1}(\rho(\gamma)')} \prod_{l=1}^{k} \prod_{b+c=n-a} T_{abc}(E, F_l, G_l)^{-1},$$

which, by definition of F_l and G_l, yields

$$\begin{aligned}
\frac{\lambda_a(\rho(\gamma)')}{\lambda_{a+1}(\rho(\gamma)')} &= \prod_{l=1}^{k} \overline{D}_a(h_l) \prod_{l=1}^{k} \prod_{b+c=n-a} T_{abc}^{\rho}(E, F_l, G_l), \\
&= \prod_{l=1}^{k} \overline{D}_a(h_l) \prod_{l=1}^{k} \prod_{b+c=n-a} T_{abc}^{\rho}(t_l, v_l) = L_a^{\text{right}}(\gamma),
\end{aligned}$$

which was to prove.

If the spiralling occurs in the opposite direction of the orientation of γ, then we consider the flag $E = \xi_\rho(\gamma^-)$ associated with the negative endpoint of γ, which is therefore the unstable flag of $\rho(\gamma)$. Thus

$$\rho(\gamma)'\left(e^{(a)}\right) = \left(\prod_{i=n-a+1}^{n} \lambda_i(\rho(\gamma)')\right) e^{(a)}$$

for all $a = 1, \ldots, n$. Adapting the above computation to this case concludes the argument.

To consider the left-hand side of γ, we can reverse the orientation of γ and consider the right-hand side to obtain the result by observing that $D_a^\rho(\gamma)$ is replaced by $D_{n-a}^\rho(\gamma)$. $\square$

The above proposition immediately implies the closed leaf equality in (D). Furthermore by Proposition 7.4.1, the closed leaf inequality amounts to showing that

$$\frac{\lambda_a(\rho(\gamma)')}{\lambda_{a+1}(\rho(\gamma)')} > 1$$

for all $a = 1, \ldots, n-1$. Since ρ is assumed to be weakly dynamics preserving (Definition 1.4.3), it follows that $\frac{|\lambda_a(\rho(\gamma))|}{|\lambda_{a+1}(\rho(\gamma))|} \geq 1$ for all $a = 1, \ldots, n-1$. Proposition 7.3.1 implies that $\lambda_a(\rho(\gamma)')$ are all of the same sign and distinct, and thus the closed leaf inequality holds. Thus Ψ satisfies conditions (A)-(D), in other words $\mathrm{Im}(\Psi) \subseteq \mathcal{P}_{\mathbb{F}}$. The forward direction of Theorem A imply that $\mathrm{Im}(\Psi) = \mathcal{P}_{\mathbb{F}}$.

Let $\rho, \rho' \colon \pi_1(S) \to \mathrm{PSL}(n, \mathbb{F})$ be two $\mathbb{F}$-positive representations that are weakly dynamics preserving with limit map ξ_ρ respectively $\xi_{\rho'}$. Left to show is that if $\Psi(\rho) = \Psi(\rho')$, then ρ and ρ' are conjugate in $\mathrm{PGL}(n, \mathbb{F})$. Let $\Psi(\rho) = \Psi(\rho') = (x_{abc,t,v}, y_{m,l}) \in \mathbb{F}^N$, where $a, b, c \geq 1$ are integers with $a + b + c = n$, $m = 1, \ldots, n-1$, v is a vertex of $t \subset S \setminus \lambda$ and l is a leaf of λ. The idea of the proof is to reconstruct a flag decoration $\mathcal{F} \colon \partial\tilde{\lambda} \to \mathrm{Flag}(\mathbb{F}^n)$ from the triple and double ratios, which will agree up to an element of $\mathrm{PGL}(n, \mathbb{F})$ with ξ_ρ and $\xi_{\rho'}$, when the latter are restricted to $\partial\tilde{\lambda}$.

Recall the notations (1)-(3) from Section 6.1 in the definition of the Bonahon-Dreyer coordinates. The statement of the following lemma in the real case is [BD14, Lemma 24]. It can be proven in a complete analogue fashion for general real closed fields by writing everything multiplicatively.

Lemma 7.4.2. Let $(x_{abc,t,v}, y_{m,l}) \in \mathcal{P}_{\mathbb{F}}$. Then there exists a flag decoration $\mathcal{F} \colon \partial\tilde{\lambda} \to \mathrm{Flag}(\mathbb{F}^n)$ such that

(1) $x_{abc,t,v} = T^{\mathcal{F}}_{abc}(t, v)$ for every component t of $S \setminus \lambda$, for every vertex v of t, and integers $a, b, c \geq 1$ with $a + b + c = n$;

(2) $y_{m,l} = D^{\mathcal{F}}_m(l)$ for every leaf l of λ and integer $m = 1, \ldots, n-1$.

Furthermore, $\mathcal{F}$ is unique up to postcomposition by an element of $\mathrm{PGL}(n, \mathbb{F})$.

Since $\Psi(\rho) = \Psi(\rho')$, the uniqueness statement in the above lemma implies that there exists $g \in \mathrm{PGL}(n, \mathbb{F})$ with $g\xi_\rho|_{\partial\tilde{\lambda}} = \xi_{\rho'}|_{\partial\tilde{\lambda}}$. Thus for $x \in \partial\tilde{\lambda}$ and $\gamma \in \pi_1(S)$ we have

$$(g\rho(\gamma)g^{-1})(\xi_{\rho'}(x)) = g\rho(\gamma)\xi_\rho(x) = g\xi_\rho(\gamma x) = \xi_{\rho'}(\gamma x).$$

Similarly, for a triple of distinct points $x, y, z \in \partial\tilde{\lambda}$ we have

$$(g\rho(\gamma)g^{-1})(\xi_{\rho'}(x), \xi_{\rho'}(y), \xi_{\rho'}(z)) = (\xi_{\rho'}(\gamma x), \xi_{\rho'}(\gamma y), \xi_{\rho'}(\gamma z)),$$

which implies that $g\rho(\gamma)g^{-1} = \rho'(\gamma)$ by ρ'-equivariance of $\xi_{\rho'}$ and Proposition 4.1.1. Thus ρ and ρ' are conjugate, which finishes the proof. □

Corollary 7.4.3. Let $\rho, \rho' \colon \pi_1(S) \to \mathrm{PSL}(n, \mathbb{F})$ be two representations, where ρ is $\mathbb{F}$-positive weakly dynamics preserving and ρ' is $\mathbb{F}$-Hitchin. Assume that $\Psi(\rho) = \mathrm{pr}_{\mathbb{F}}^{\mathrm{BD}}(\rho')$. Then ρ and ρ' are conjugate in $\mathrm{PGL}(n, \mathbb{F})$.

Proof. Denote the limit map of ρ' constructed in the proof of the forward direction of Theorem A by $\xi_{\rho'}$, and let ξ_ρ denote the limit map of ρ. Viewing ρ' as an $\mathbb{F}$-positive weakly dynamics preserving representation, we see immediately from the definitions of Ψ and $\mathrm{pr}_{\mathbb{F}}^{\mathrm{BD}}$ that

$$\Psi(\rho') = \mathrm{pr}_{\mathbb{F}}^{\mathrm{BD}}(\rho') = \Psi(\rho).$$

We conclude using Theorem C. □

We can now finish the proof of Theorem A.

Proof of Theorem A ($\Longleftarrow$). Let ρ be an $\mathbb{F}$-positive representation that is weakly dynamics preserving. Then its $\mathrm{PGL}(n, \mathbb{F})$-equivalence class $[\rho]$ corresponds under the above homeomorphism from Theorem C to a point in $\mathcal{P}_{\mathbb{F}}$. But since $\mathcal{P}_{\mathbb{F}}$ is homeomorphic to $\mathrm{Hit}(S, n)_{\mathbb{F}}$ (Corollary 6.4.3) this point gives rise to $[\rho'] \in \mathrm{Hit}(S, n)_{\mathbb{F}}$. Now $\Psi(\rho) = \mathrm{pr}_{\mathbb{F}}^{\mathrm{BD}}(\rho')$, and thus by Corollary 7.4.3 we have that ρ and ρ' are conjugate in $\mathrm{PGL}(n, \mathbb{F})$. Since $\mathrm{Hit}(S, n)_{\mathbb{F}}$ is the space of $\mathrm{PSL}(n, \mathbb{F})$-equivalence classes of $\mathbb{F}$-Hitchin representations, it follows that ρ is $\mathrm{PGL}(n, \mathbb{F})$-conjugate to an $\mathbb{F}$-Hitchin representation. □

7.5. $\mathbb{F}$-positive weakly dynamics preserving representations are irreducible

Let $\mathbb{F}$ be a field and Γ a finitely generated group.

Definition 7.5.1. A representation $\rho \colon \Gamma \to \mathrm{GL}(\mathbb{F}^n)$ is *irreducible* if the only $\rho(\Gamma)$-invariant subspaces of $\mathbb{F}^n$ is $\{0\}$ or $\mathbb{F}^n$. Denote by $\mathrm{Gr}_{\mathbb{F}}$ the space of all non-trivial subspaces of $\mathbb{F}^n$. A representation $\rho \colon \Gamma \to \mathrm{PGL}(\mathbb{F}^n)$ is *irreducible* if its action on $\mathrm{Gr}_{\mathbb{F}}$ has no fixed point.

Remark 7.5.2. Let $\gamma_1, \ldots, \gamma_k$ be a finite generating set for Γ and $W \subseteq \mathbb{F}^n$ a vector subspace. Then $\rho(\Gamma)W \subseteq W$ if and only if $\rho(\gamma_j)W \subseteq W$ for all $j = 1, \ldots, k$. A similar statement holds true for projective representations.

From now on let $\mathbb{F}$ be real closed. We would like to show that $\mathbb{F}$-positive weakly dynamics preserving representations are irreducible. We already know by [Lab06, Lemma 10.1] that Hitchin representations are irreducible. Thus the following proposition is an easy consequence of the Tarski-Seidenberg transfer principle and Theorem A.

Proposition 7.5.3. Let $\rho\colon \pi_1(S) \to \mathrm{PSL}(n,\mathbb{F})$ be $\mathbb{F}$-positive and weakly dynamics preserving. Then the restriction of ρ to every finite index subgroup is irreducible.

Proof. Let $\gamma_1, \ldots, \gamma_{2g}$ be a finite set of generators for $\pi_1(S)$. We saw in Lemma 5.3.5 that $\mathrm{Gr}(k,n)$ is algebraic for all $k = 1, \ldots, n-1$. Since $\mathrm{Gr} = \mathrm{Gr}_{\mathbb{R}}$ is the finite union of $\mathrm{Gr}(1,n)_{\mathbb{R}}, \ldots, \mathrm{Gr}(n-1,n)_{\mathbb{R}}$ it is also algebraic. Consider the semi-algebraic set

$$\{(\rho, V) \in \mathrm{Hom}_{\mathrm{Hit}}(\pi_1(S), \mathrm{PSL}(n,\mathbb{R})) \times \mathrm{Gr} \mid \rho(\gamma_1)V = V, \ldots, \rho(\gamma_{2g})V = V\}.$$

Then [Lab06, Theorem 10.1] implies that this set is empty. By the Tarski-Seidenberg transfer principle (Theorem 2.1.13), so is its $\mathbb{F}$-extension. Since $\mathbb{F}$-positive weakly dynamics preserving representations are $\mathbb{F}$-Hitchin, see Theorem A, we conclude that $\mathbb{F}$-positive weakly dynamics preserving representations are irreducible. The restriction of ρ to every finite index subgroup of $\pi_1(S)$ corresponds to an $\mathbb{F}$-Hitchin representation of the corresponding cover of S, thus the restricted representation is also irreducible. $\square$

8. Intersection geodesic currents for boundary representations

8.1. Positive cross-ratios and intersection currents

More information on geodesic currents can be found in [ES22] or [Mar22b].

As was hinted at in the introduction, the space of geodesic currents $\mathscr{C}(S)$, see Definition 1.4.5, comes equipped with an intersection form generalizing the geometric intersection number. Recall that we endow S with an auxiliary hyperbolic structure. Denote by $\mathcal{G}$ the space of (unoriented and unparametrized) geodesics of $\tilde{S}$.

Definition 8.1.1. Consider $\mathcal{G}^{(2)} \subseteq \mathcal{G} \times \mathcal{G}$ the set of pairs of transversely intersecting geodesics. We define the *intersection form*

$$i : \mathscr{C}(S) \times \mathscr{C}(S) \to \mathbb{R}_{\geq 0}, \quad i(\mu, \eta) := (\mu \times \eta)(\mathcal{G}^{(2)}/\pi_1(S)),$$

where $\mu \times \eta$ is the product measure.

Bonahon proved that i is finite, continuous, symmetric and bilinear, and generalizes the geometric intersection number of homotopy classes of closed curves on S [Bon86, Proposition 4.5].

To associate to a representation in a higher rank Teichmüller space a geodesic current, we make use of positive cross-ratios. There are many (non-equivalent definitions) of cross-ratios. The convention of how to arrange the arguments in the cross-ratio follows [MZ19, Definition 2.4]. However the cross-ratio is defined on a smaller set and is not assumed to be continuous as in [BIPP21a, Definition 3.1]. Compare also to [BIPP21a, Remark 3.2] for a comparison between the various definitions in the literature.

Definition 8.1.2. Let $X \subseteq \partial\tilde{S}$ be a $\pi_1(S)$-invariant non-empty subset, e.g. $\mathrm{Fix}(S)$, and let $X^{[4]}$ denote the set of positively oriented, i.e. cyclically ordered (in clockwise direction), quadruples in X. A *cross-ratio* is a $\pi_1(S)$-invariant function

$$B \colon X^{[4]} \to \mathbb{R},$$

that is

(1) symmetric, i.e. for all $x = (x, y, z, w) \in X$ we have $B(x, y, z, w) = B(z, w, x, y)$, and

(2) additive, i.e. $B(x, y, z, w) + B(x, y, w, t) = B(x, y, z, t)$ for all $x, y, z, w, t \in X$ that are positively oriented.

A cross-ratio B is *positive* if $B(x,y,z,w) \geq 0$ for all $(x,y,z,w) \in X^{[4]}$. Denote the set of positive cross-ratios on X by $\mathrm{CR}^+(X)$. The *B-period* of a non-trivial $\gamma \in \pi_1(S)$ with $\gamma^\pm \in X$ is

$$\ell_B(\gamma) \coloneqq B(\gamma^+, \gamma^-, x, \gamma x)$$

for some (any) $x \in X \setminus \{\gamma^\pm\}$ such that $(\gamma^+, \gamma^-, x, \gamma x) \in X^{[4]}$, where γ^+ respectively γ^- is the attracting respectively repelling fixed point of γ. A geodesic current $\mu \in \mathscr{C}(S)$ is an *intersection current for B* if for every non-trivial $\gamma \in \pi_1(S)$ with $\gamma^\pm \in X$ we have $\ell_B(\gamma) = i(\mu, \gamma)$, where we view γ as a geodesic current; see e.g. [Bon88, §1].

Example 8.1.3. The Liouville current is an intersection current for the pull-back of the standard cross-ratio on $\partial\mathbb{H}^2$ via the developing map [Bon88, Proposition 14].

For a general cross-ratio, intersection currents do not need to exist. If we assume $X = \partial\tilde{S}$ and B continuous, then an observation of Hamenstädt in [Ham99], refer to [MZ19, Appendix A] for a detailed proof, implies that if B is positive, then there exists a unique intersection current for B. More generally, we have the following.

Theorem 8.1.4 ([BIPP21a, Theorem 1.6])**.** Let $X \subseteq \partial\tilde{S}$ be a $\pi_1(S)$-invariant non-empty subset and B a positive cross-ratio on X. Then there is a geodesic current μ on S such that for all $e \neq \gamma \in \pi_1(S)$

$$\ell_B(\gamma) = i(\mu, \gamma).$$

The geodesic current μ depends continuously on the cross-ratio B, where $\mathrm{CR}^+(X)$ is endowed with the subspace topology of the topological vector space of cross-ratios on X with the topology of pointwise convergence.

8.2. Valuations and big elements

For an introduction to valuations we refer the reader to [EP05].

Definition 8.2.1. Let $\mathbb{F}$ be a field. A *valuation* on $\mathbb{F}$ is a map

$$v \colon \mathbb{F} \to \mathbb{R} \cup \{\infty\}$$

such that $e^{-v} \colon \mathbb{F} \to \mathbb{R}$ is a norm that satisfies the triangle inequality. The valuation is *trivial* if $v(a) = 0$ for all $a \in \mathbb{F}^\times$, otherwise *non-trivial*.

If $\mathbb{F}$ is ordered, we say that a valuation v is *order-compatible* if for all $0 < x \leq y \in \mathbb{F}$ we have $v(x) \geq v(y)$. We say that v is *non-Archimedean* if $v(x+y) \geq \min\{v(x), v(y)\}$. Two valuations v and v' are *equivalent* if there exists $r \in \mathbb{R}$ positive with $v = rv'$.

Lemma 8.2.2. Let v be a valuation on $\mathbb{F}$. Then $v(1) = 0$, $v(\zeta) = 0$ for ζ a root of unity in $\mathbb{F}$, $v(x^{-1}) = -v(x)$ and $v(-x) = v(x)$ for all $x \in \mathbb{F}$. If v is non-Archimedean and $x, y \in \mathbb{F}$ with $v(x) \neq v(y)$, then $v(x+y) = \min\{v(x), v(y)\}$. If $\mathbb{F}$ is ordered and v is non-Archimedean order-compatible, then for $x, y \in \mathbb{F}_{>0}$ we have $v(x+y) = \min\{v(x), v(y)\}$.

Proof. For the first statements see [EP05, Section 1.3 and (1.3.4)]. For the last statement we can without loss of generality assume that $v(x) < v(y)$. We already know $v(x+y) \geq v(x)$. Since $x, y > 0$ we have $x + y \geq x$, and thus by order-compatibility of v we have $v(x+y) \leq v(x)$. □

Example 8.2.3. The function $-\log$ is an order-compatible valuation on $\mathbb{R}$. The map $\mathbb{R}(X) \to \mathbb{R} \cup \{\infty\}$, $\frac{p}{q} \mapsto \deg(q) - \deg(p)$, where deg denotes the degree of the polynomial, is an order-compatible non-Archimedean valuation on $\mathbb{R}(X)$ with the order $+\infty$, compare Example 2.2.5.

Definition 8.2.4. Let $\mathbb{F}$ be an ordered field. A positive element $b \in \mathbb{F}$ is *big* if for all $x \in \mathbb{F}$ there exists $n \in \mathbb{N}$ such that $x < b^n$. For b a big element, the *logarithm with basis* b is defined by

$$\log_b \colon \mathbb{F}_{>0} \to \mathbb{R}, \quad x \mapsto \inf\{q \in \mathbb{Q} \mid x \leq b^q\}.$$

Note that a big element is always bigger than 1.

Lemma 8.2.5 ([Bru88a, Proposition 5.2.(f)-(g)])**.** Let b be a big element in an ordered field $\mathbb{F}$. To b we can associate the following non-trivial order-compatible valuation

$$v_b \colon \mathbb{F} \to \mathbb{R} \cup \{\infty\}, \quad x \mapsto \begin{cases} -\log_b(|x|) & \text{if } x \neq 0, \\ \infty & \text{if } x = 0. \end{cases}$$

If $\mathbb{F}$ is non-Archimedean, so is v_b.

Lemma 8.2.6 ([Bru88a, Proposition 5.2.(d)])**.** Let b and $b' \in \mathbb{F}$ be two big elements. Then $\log_{b'} = \log_{b'}(b) \cdot \log_b$, i.e. the logarithms differ by a positive scalar multiple.

It follows that the associated valuations v_b and v_b' are equivalent (with scaling factor $-v_{b'}(b) = \log_{b'}(b) > 0$).

Example 8.2.7. In $\mathbb{R}$ every element larger than 1 is a big element in the above sense. If $\mathbb{F}$ is an Archimedean ordered field, i.e. a subfield of $\mathbb{R}$, then $\log_b$ is the usual logarithm, for which the last lemma is known.

In $\mathbb{R}(X)$ with the order $+\infty$, X is a big element. The valuation v_X associated to X is the one from Example 8.2.3. If $\mathbb{F}$ is any non-Archimedean field then no rational number is big.

Lemma 8.2.8. Let $\mathbb{F}$ be an ordered field with a big element $b \in \mathbb{F}$ and v any order-compatible valuation on $\mathbb{F}$. Then

$$v = -v(b) \cdot v_b.$$

In other words, up to scaling, all order-compatible valuations come from the construction in Lemma 8.2.5, hence are equivalent by Lemma 8.2.6.

Proof. Let $x \in \mathbb{F}$ be positive. Then $v_b(x) = -\inf\{q \in \mathbb{Q} \mid x \leq b^q\}$. Assume $x \leq b^q$, then $v(x) \geq qv(b)$ by order-compatibility of v. Since b is big, $v(b) < 0$ and hence $v(x)/v(b) \leq q$. This implies that $v(x)/v(b) \geq -v_b(x)$, as the latter is defined as the infimum over all such q.

Take now $q' \in \mathbb{Q}$ with $q' < v(x)/v(b)$. Assume $x \leq b^{q'}$. By order-compatibility $v(x) \geq q'v(b)$ and hence $v(x)/v(b) \leq q'$, a contradiction. Thus $v(x) = -v(b) \cdot v_b(x)$. By Lemma 8.2.2 the same conclusion holds for $x < 0$, since $v(-x) = v(x)$. □

Lemma 8.2.9 ([Bru88a, §5], [Bou98, Chapter VI, §10])**.** Let $\mathbb{K} \subseteq \mathbb{F}$ be an ordered field of finite transcendence over a subfield $\mathbb{K}$ which contains a big element. Then $\mathbb{F}$ contains a big element.

8.3. Positive cross-ratios for $\mathbb{F}$-positive weakly dynamics preserving representations

Theorem 8.3.1 ([MZ19, Proposition 2.24, Theorem 3.4])**.** Let $\rho\colon \pi_1(S) \to \mathrm{PSL}(n,\mathbb{R})$ be a Hitchin representation. Then for all $k = 1,\dots,n-1$, there exists a unique cross-ratio B_k^ρ such that for all $\gamma \in \pi_1(S)$ non-trivial

$$\ell_{B_k^\rho}(\gamma) = \log \frac{\lambda_1(\rho(\gamma)) \cdot \ldots \cdot \lambda_k(\rho(\gamma))}{\lambda_{n-k+1}(\rho(\gamma)) \cdot \ldots \cdot \lambda_n(\rho(\gamma))},$$

where $\lambda_1(\rho(\gamma)) > \ldots > \lambda_n(\rho(\gamma)) > 0$ are the absolute values of the eigenvalues of $\rho(\gamma)$. Furthermore, the cross-ratio B_k^ρ is positive.

In particular, it follows from this together with Theorem 8.1.4, that there exists an intersection current μ_ρ^k for ρ. Similar results hold for maximal representations in $\mathrm{Sp}(2n,\mathbb{R})$ [Lab08, Section 4.2] and Θ-positive representations in $\mathrm{PO}(p,q)$ [BP21, Theorem 4.9].

The following lemma is a generalization of the above theorem to $\mathbb{F}$-Hitchin representations. Recall that a representation $\rho\colon \pi_1(S) \to \mathrm{PSL}(n,\mathbb{F})$ is called $\mathbb{F}$-Hitchin if it lies in the $\mathbb{F}$-extension of the real semi-algebraic Hitchin component $\mathrm{Hit}(S,n)$, see Definition 1.4.1. Let now $\mathbb{F}$ be a real closed field together with a non-trivial order-compatible valuation v. Recall from 1.4 that for $g \in \mathrm{SL}(n,\mathbb{F})$ totally hyperbolic and for $k = 1,\dots,n-1$ we define the *k-length of g* as

$$L_k(g) := -\sum_{j=1}^{k} v(\lambda_j(g)) + \sum_{j=n-k+1}^{n} v(\lambda_j(g)).$$

Lemma 8.3.2. Let $\mathbb{F} \supseteq \mathbb{R}$ be a real closed field with non-trivial order-compatible valuation v (if $\mathbb{F} = \mathbb{R}$ we assume $v = -\log$). Let $\rho\colon \pi_1(S) \to \mathrm{PSL}(n,\mathbb{F})$ be $\mathbb{F}$-positive weakly dynamics preserving with associated $\mathbb{F}$-positive limit map $\xi_\rho\colon \mathrm{Fix}(S) \to \mathrm{Flag}(\mathbb{F}^n)$. Then for all $k = 1,\dots,n-1$

$$B_k^\rho := \tfrac{1}{2}\big(\tilde{B}_k^\rho + \tilde{B}_{n-k}^\rho\big),$$

is a positive cross-ratio on $\mathrm{Fix}(S)$, where for $x = (x_1,\dots,x_4) \in \mathrm{Fix}(S)^{[4]}$ we define

$$\widetilde{M}_k^\rho(x) := \frac{\xi_\rho(x_1)^{(n-k)} \wedge \xi_\rho(x_3)^{(k)}}{\xi_\rho(x_1)^{(n-k)} \wedge \xi_\rho(x_4)^{(k)}} \cdot \frac{\xi_\rho(x_2)^{(n-k)} \wedge \xi_\rho(x_4)^{(k)}}{\xi_\rho(x_2)^{(n-k)} \wedge \xi_\rho(x_3)^{(k)}}, \text{ and } \tilde{B}_k^\rho(x) = -v(\widetilde{M}_k^\rho(x)).$$

Here we adapt the same notation as in Section 4.1 to define $\widetilde{M}_k^\rho(x)$. Furthermore, for all $e \neq \gamma \in \pi_1(S)$ we have

$$\ell_{B_k^\rho}(\gamma) = L_k(\rho(\gamma)).$$

Proof. Since ξ_ρ is ρ-equivariant, it follows that B_k^ρ is $\pi_1(S)$-invariant. Symmetry and additivity are computations.

For $\mathbb{F} = \mathbb{R}$ and $v = -\log$ in the proof of Theorem 8.3.1, Martone-Zhang prove that already the expression $\tilde{B}_k^\rho(x)$ is positive for all $x \in \mathrm{Fix}(S)^{[4]}$. Equivalently $\widetilde{M}_k^\rho(x) \geq 1$ for all $x \in \mathrm{Fix}(S)^{[4]}$. Thus for fixed $x \in \mathrm{Fix}(S)^{[4]}$ the set

$$S_x := \big\{\rho \in \mathrm{Hom}_{\mathrm{Hit}}(\pi_1(S), \mathrm{PSL}(n,\mathbb{R})) \mid \widetilde{M}_k^\rho(x) \geq 1\big\}$$

is semi-algebraic by Lemma 5.3.6, and equal to $\mathrm{Hom}_{\mathrm{Hit}}(\pi_1(S), \mathrm{PSL}(n,\mathbb{R}))$. Hence by the Tarski-Seidenberg transfer principle (Theorem 2.1.13) the same holds true for its $\mathbb{F}$-extension, i.e. $(S_x)_{\mathbb{F}} = \mathrm{Hom}_{\mathrm{Hit}}(\pi_1(S), \mathrm{PSL}(n,\mathbb{F}))$ for all $x \in \mathrm{Fix}(S)$[4]. Since $\mathbb{F}$-positive weakly dynamics preserving representations are $\mathrm{PGL}(n,\mathbb{F})$-conjugate to $\mathbb{F}$-Hitchin representations (Theorem A), B_k^ρ is $\mathrm{PGL}(n,\mathbb{F})$-invariant, and υ is order-compatible this concludes the positivity of B_k^ρ.

To show that $\ell_{B_k^\rho}(\gamma) = L_k(\rho(\gamma))$ we use similar arguments as in the proof of Proposition 7.4.1. Hence for $y = (\gamma^+, \gamma^-, x, \gamma x)$ we have using Equation (7.1)

$$\begin{aligned}\widetilde{M}_k^\rho(y) &= \frac{\xi_\rho(\gamma^+)^{(n-k)} \wedge \xi_\rho(x)^{(k)}}{\xi_\rho(\gamma^+)^{(n-k)} \wedge \rho(\gamma)\xi_\rho(x)^{(k)}} \cdot \frac{\xi_\rho(\gamma^-)^{(n-k)} \wedge \rho(\gamma)\xi_\rho(x)^{(k)}}{\xi_\rho(\gamma^-)^{(n-k)} \wedge \xi_\rho(x)^{(k)}} \\ &= \frac{\frac{1}{\lambda_{k+1}\cdot\ldots\cdot\lambda_n}}{\frac{1}{\lambda_1\cdot\ldots\cdot\lambda_{n-k}}} = \frac{\lambda_1 \cdot \ldots \cdot \lambda_k}{\lambda_{n-k+1} \cdot \ldots \cdot \lambda_n},\end{aligned}$$

where $\lambda_1 > \ldots > \lambda_n$ are the eigenvalues of $\rho(\gamma)'$, a lift of $\rho(\gamma)$ to $\mathrm{SL}(n,\mathbb{F})$. With the definition of B_k^ρ and the order-compatibility of υ, the claim follows. $\square$

We are ready to prove Theorem E, which we restate here.

Theorem E. *Let $\mathbb{F} \supseteq \mathbb{R}$ be a non-Archimedean real closed field with an order-compatible valuation υ (assumed to be $-\log$ if $\mathbb{F} = \mathbb{R}$) and let $\rho\colon \pi_1(S) \to \mathrm{PSL}(n,\mathbb{F})$ be $\mathbb{F}$-positive and weakly dynamics preserving. Then for every $k = 1, \ldots, n-1$, there exists a geodesic current μ_ρ^k such that for any $e \neq \gamma \in \pi_1(S)$ we have*

$$i(\mu_\rho^k, \gamma) = L_k(\rho(\gamma)).$$

The current μ_ρ^k is non-zero if and only if there exists $\gamma \in \pi_1(S)$ with $\upsilon(|\mathrm{tr}(\rho(\gamma))|) < 0$.

Proof. Lemma 8.3.2 shows that for every $k = 1, \ldots, n-1$ we can associate to ρ a positive cross-ratio B_k^ρ, whose period is equal to the k-length. Theorem 8.1.4 applied to $X = \mathrm{Fix}(S)$ and $B = B_k^\rho$ provides a geodesic current μ_ρ^k with the desired intersection property.

Left to show is the last statement. We only need to consider the non-Archimedean case. If $g \in \mathrm{SL}(n,\mathbb{F})$ is totally hyperbolic with positive eigenvalues $\lambda_1(g) > \ldots > \lambda_n(g) > 0$, then by order-compatibility of υ we have $\upsilon(\lambda_1(g)) \leq \ldots \leq \upsilon(\lambda_n(g))$. Thus by an iterated application of the last statement of Lemma 8.2.2 we have

$$\upsilon(\mathrm{tr}(g)) = \upsilon\left(\sum_{j=1}^n \lambda_j(g)\right) = \min_{j=1,\ldots,n} \upsilon(\lambda_j(g)) = \upsilon(\lambda_1(g)).$$

Furthermore, since $\det(g) = 1$ we have $\lambda_n(g) < 1$ and thus $\upsilon(\lambda_n(g)) \geq 0$.

We first show one direction of the claim for $k = 1$. Recall that $\upsilon(x) = \upsilon(-x)$ for all $x \in \mathbb{F}$ (Lemma 8.2.2), and hence $\upsilon(\mathrm{tr}(\rho(\gamma))$ is independent of a choice of lift of $\rho(\gamma)$ to $\mathrm{SL}(n,\mathbb{F})$. If there exists $\gamma \in \pi_1(S)$ with $\upsilon(\mathrm{tr}(\rho(\gamma))) < 0$, then by the above remark, we have

$$i(\mu_\rho^1, \gamma) = L_1(\rho(\gamma)) = -\upsilon(\lambda_1(\rho(\gamma))) + \upsilon(\lambda_n(\rho(\gamma))) \geq -\upsilon(\mathrm{tr}(\rho(\gamma))) > 0,$$

and hence μ_ρ^1 is non-zero. For $k > 1$ we note that $L_k(\rho(\gamma)) \geq L_1(\rho(\gamma))$ and hence $\mu_\rho^k \neq 0$.

Conversely, assume that μ_ρ^1 is non-zero. Otal [Ota90, Théorème 2] proved that geodesic currents are determined by their intersection function, and hence there exists $e \neq \gamma \in \pi_1(S)$ with

$$0 < i(\mu_\rho^1, \gamma) = L_1(\rho(\gamma)) = -v\left(\frac{\lambda_1(\rho(\gamma))}{\lambda_n(\rho(\gamma))}\right).$$

Now assume by contradiction that $v(\mathrm{tr}(\rho(\gamma^s))) \geq 0$ for all $s \in \mathbb{N}$. Newton's identities, see e.g. [Kal00], imply that the coefficients of the characteristic polynomial of $\rho(\gamma)$ belong to the ring $\mathcal{O} \coloneqq \{x \in \mathbb{F} \mid v(x) \geq 0\}$. Since $\mathcal{O} \subseteq \mathbb{F}$ is a valuation ring, it is integrally closed in $\mathbb{F}$, see [EP05, Theorem 3.1.3.(1)], and hence $\lambda_1(\rho(\gamma)), \ldots, \lambda_n(\rho(\gamma)) \in \mathcal{O}$. Since $\prod_{j=1}^n \lambda_j(\rho(\gamma)) = 1$, it follows that $\frac{\lambda_1(\rho(\gamma))}{\lambda_n(\rho(\gamma))} = \lambda_1(\rho(\gamma))^2 \lambda_2(\rho(\gamma)) \cdots \lambda_{n-1}(\rho(\gamma)) \in \mathcal{O}$, a contradiction. The same argument works for general $k > 1$. □

The above theorem allows us to assign to closed points in the real spectrum compactification of the Hitchin component $\mathrm{RSp}^{\mathrm{cl}}(\mathrm{Hit}(S,n))$ a projective class of a geodesic current. Let us recall the following characterization from Section 3.3:

$$\mathrm{RSp}^{\mathrm{cl}}(\mathrm{Hit}(S,n)) \cong \left\{ (\rho, \mathbb{F}_\rho) \;\middle|\; \begin{array}{l} \rho\colon \pi_1(S) \to \mathrm{PSL}(n, \mathbb{F}_\rho) \text{ is } \mathbb{F}_\rho\text{-Hitchin}, \\ \mathbb{F}_\rho \supseteq \mathbb{R} \text{ real closed, } \rho\text{-minimal}, \\ \mathbb{F}_\rho \text{ Archimedean over } \mathbb{R}[\mathrm{tr}(\mathrm{Ad}(\rho))] \end{array} \right\} \Big/ \sim .$$

where $\mathbb{F}_\rho$ is the ρ-minimal field; see Definition 3.3.1 and Theorem 3.3.4.

Lemma 8.3.3. For all $k = 1, \ldots, n-1$ the map

$$\begin{aligned} \varphi_k \colon \mathrm{RSp}^{\mathrm{cl}}(\mathrm{Hit}(S,n)) &\to \mathbb{P}\mathrm{CR}^+(\mathrm{Fix}(S)), \\ [(\rho, \mathbb{F}_\rho)] &\mapsto [B_k^\rho] \end{aligned}$$

is well-defined and continuous.

Let us show how this lemma implies Corollary F, which we recall here.

Corollary F. *For all $k = 1, \ldots, n-1$ the map*

$$\begin{aligned} \mathrm{RSp}^{\mathrm{cl}}(\mathrm{Hit}(S,n)) &\to \mathbb{P}\mathscr{C}(S), \\ [(\rho, \mathbb{F})] &\mapsto \left[\mu_\rho^k\right] \end{aligned}$$

is well-defined and continuous.

Proof. For every $k = 1, \ldots, n-1$ the map is given as the following composition of maps

$$\mathrm{RSp}^{\mathrm{cl}}(\mathrm{Hit}(S,n)) \to \mathbb{P}\mathrm{CR}^+(\mathrm{Fix}(S)) \to \mathbb{P}\mathscr{C}(S),$$

where the first map is continuous by Lemma 8.3.3, and the second map is given by Theorem 8.1.4 (before projectivizing) and proven to be continuous. □

Before we prove Lemma 8.3.3 we need some preliminary considerations.

Lemma 8.3.4. Let $(\rho, \mathbb{F})$ represent a point in $\mathrm{RSp}^{\mathrm{cl}}(\mathrm{Hit}(S, n))$ with $\mathbb{F} = \mathbb{F}_\rho$ the ρ-minimal field (Definition 3.3.1). Let F be a finite symmetric generating set for $\pi_1(S)$ and $E = F^{2^n-1}$. Set

$$b_\rho := \sum_{\gamma \in E} \mathrm{tr}(\rho(\gamma))^2 \in \mathbb{F}. \tag{8.1}$$

Then b_ρ is positive and a big element in $\mathbb{F}$.

Proof. Since b_ρ is a sum of squares, it is clearly positive. If $(\rho, \mathbb{F})$ represents a closed point then $\mathbb{F}$, the minimal field of definition, is non-Archimedean by Theorem 3.3.4 and of finite transcendence degree over $\mathbb{R}$, hence contains a big element (Lemma 8.2.9). Since the trace determines the representation (Theorem 3.1.3), we have that $\mathrm{tr}(\rho(\gamma))^2$ is big for some $\gamma \in \pi_1(S)$. A result of Procesi [Pro76] tells us that there is a finite set of traces that we need to consider (here the traces of the images of $\gamma \in E$ under ρ suffice). Thus b_ρ is big. $\square$

Thus to $(\rho, \mathbb{F})$ we can associate the non-trivial order-compatible valuation $v_\rho := v_{b_\rho}$ from Lemma 8.2.5, which is unique up to scaling by Lemma 8.2.8.

Lemma 8.3.5. If $(\rho, \mathbb{F}_\rho)$ and $(\rho', \mathbb{F}_{\rho'})$ represent the same point in $\mathrm{RSp}^{\mathrm{cl}}(\mathrm{Hit}(S, n))$, then $B_k^\rho = B_k^{\rho'}$.

Proof. Let us abbreviate $\mathbb{F} := \mathbb{F}_\rho$ and $\mathbb{F}' := \mathbb{F}_{\rho'}$. Also write $b := b_\rho$ and $b' := b_{\rho'}$ for the corresponding big elements defined in Equation (8.1). Since $(\rho, \mathbb{F})$ and $(\rho', \mathbb{F}')$ represent the same point, we can without loss of generality assume that there exists an order-preserving isomorphism $\alpha \colon \mathbb{F} \to \mathbb{F}'$ and $g \in \mathrm{PSL}(n, \mathbb{F}')$ such that for all $\gamma \in \pi_1(S)$

$$\alpha(\rho(\gamma)) = g\rho'(\gamma)g^{-1}.$$

We note that if $\xi_{\rho'}$ is the limit map of ρ' then $g\xi_{\rho'}$ is the limit map of $\alpha \circ \rho$. It is easy to see that $\alpha \widetilde{M}_k^\rho = \widetilde{M}_k^{\alpha\circ\rho} = \widetilde{M}_k^{\rho'}$. Since the trace is conjugation invariant we also have $b_{\rho'} = b_{\alpha\circ\rho} = \alpha(b)$. Hence

$$\begin{aligned}
B_k^\rho &= \tfrac{1}{2}\big(\log_b(\widetilde{M}_k^\rho) + \log_b(\widetilde{M}_{n-k}^\rho)\big)\\
&= \tfrac{1}{2}\big(\log_{\alpha(b)}(\widetilde{M}_k^{\rho'}) + \log_{\alpha(b)}(\widetilde{M}_{n-k}^{\rho'})\big)\\
&= \tfrac{1}{2}\big(\log_{b'}(\widetilde{M}_k^{\rho'}) + \log_{b'}(\widetilde{M}_{n-k}^{\rho'})\big)\\
&= B_k^{\rho'}.
\end{aligned}$$
$\square$

Proof of Lemma 8.3.3. For all $k = 1, \dots, n-1$ the map φ_k is well-defined by Lemma 8.3.5. We would like to show that φ_k is continuous, where $\mathrm{RSp}^{\mathrm{cl}}(\mathrm{Hit}(S, n))$ is endowed with the spectral topology, $\mathrm{CR}^+(\mathrm{Fix}(S))$ with the topology of pointwise convergence and $\mathbb{P}\mathrm{CR}^+(\mathrm{Fix}(S))$ with the quotient topology. The following sets form a subbasis of closed sets for the topology on $\mathrm{CR}^+(\mathrm{Fix}(S))$: For $U \subseteq \mathbb{R}_{\geq 0}$ closed and $x = (x_1, x_2, x_3, x_4) \in \mathrm{Fix}(S)^{[4]}$, we define the closed set

$$C(U, x) = \{b \in \mathrm{CR}^+(\mathrm{Fix}(S)) \mid b(x) \in U\}.$$

A basic closed set is of the form

$$C([0,t],x) = \{b \in \mathrm{CR}^+(\mathrm{Fix}(S)) \mid b(x) \leq t\}$$

for $t > 0$.

By Lemma 8.3.5 we can lift φ_k to a map

$$\widetilde{\varphi}_k \colon \mathrm{RSp}^{\mathrm{cl}}(\mathrm{Hit}(S,n)) \to \mathrm{CR}^+(\mathrm{Fix}(S)), \quad [(\rho, \mathbb{F}_\rho)] \mapsto B_k^\rho$$

such that the diagram

$$\begin{tikzcd} \mathrm{RSp}^{\mathrm{cl}}(\mathrm{Hit}(S,n)) \arrow[r, "\widetilde{\varphi}_k"] \arrow[dr, "\varphi_k"'] & \mathrm{CR}^+(\mathrm{Fix}(S)) \arrow[d] \\ & \mathbb{P}\mathrm{CR}^+(\mathrm{Fix}(S)) \end{tikzcd}$$

commutes. It suffices thus to show that $\widetilde{\varphi}_k$ is continuous. We show that $\widetilde{\varphi}_k^{-1}(C([0,t],x))$ is closed in $\mathrm{RSp}^{\mathrm{cl}}(\mathrm{Hit}(S,n))$ for all $t > 0$ and $x \in \mathrm{Fix}(S)^{[4]}$. We have

$$\begin{aligned} \widetilde{\varphi}_k^{-1}(C([0,t],x)) &= \big\{[(\rho,\mathbb{F})] \in \mathrm{RSp}^{\mathrm{cl}}(\mathrm{Hit}(S,n)) \mid \widetilde{\varphi}_k([(\rho,\mathbb{F})]) \in C([0,t],x)\big\} \\ &= \big\{[(\rho,\mathbb{F})] \in \mathrm{RSp}^{\mathrm{cl}}(\mathrm{Hit}(S,n)) \mid B_k^\rho(x) \leq t\big\} \\ &= \big\{[(\rho,\mathbb{F})] \in \mathrm{RSp}^{\mathrm{cl}}(\mathrm{Hit}(S,n)) \mid \log_{b_\rho}\big((\widetilde{M}_k^\rho(x)\widetilde{M}_{n-k}^\rho(x))^{1/2}\big) \leq t\big\} \\ &= \bigcap_{\frac{p}{q}>t} \big\{[(\rho,\mathbb{F})] \in \mathrm{RSp}^{\mathrm{cl}}(\mathrm{Hit}(S,n)) \mid \big(\widetilde{M}_k^\rho(x)\widetilde{M}_{n-k}^\rho(x)\big)^{1/2} \leq {b_\rho}^{p/q}\big\} \\ &= \bigcap_{\frac{p}{q}>t} \big\{[(\rho,\mathbb{F})] \in \mathrm{RSp}^{\mathrm{cl}}(\mathrm{Hit}(S,n)) \mid \big(\widetilde{M}_k^\rho(x)\widetilde{M}_{n-k}^\rho(x)\big)^{q} \leq b_\rho^{2p}\big\} \end{aligned}$$

The left hand side of the expression

$$\big(\widetilde{M}_k^\rho(x)\widetilde{M}_{n-k}^\rho(x)\big)^{q} \leq b_\rho^{2p}$$

depends rationally on ρ, since all involved flags that are needed to define the left hand side are the stable flags of $\rho(\gamma_1), \dots, \rho(\gamma_4)$, where $x_i = \gamma_i^+$. By Lemma 5.3.6, we know that the stable flag depends rationally on the positive hyperbolic element. Thus

$$\bigcap_{\frac{p}{q}>t} \big\{[(\rho,\mathbb{F})] \in \mathrm{RSp}^{\mathrm{cl}}(\mathrm{Hit}(S,n)) \mid \big(\widetilde{M}_k^\rho(x)\widetilde{M}_{n-k}^\rho(x)\big)^{q} \leq b_\rho^{2p}\big\} \subseteq \mathrm{RSp}^{\mathrm{cl}}(\mathrm{Hit}(S,n))$$

is closed, which was to prove. $\square$

Part IV.

Appendices

Appendix A.

Parametrization of configurations of triples of flags by triple ratios

A.1. Triple ratios and low dimensions

The arguments follow closely [Bon23], that describe the real case. Let now $\mathbb{F}$ be any field. At the end of Section 4.1 we already hinted at the fact that low-dimensional cases are more general than it seems at first glance Let us make this precise. For $(E, F, G) \in \mathrm{Flag}(\mathbb{F}^3)^{(3)}$ and integers $a, b, c \geq 1$ with $a + b + c = n$ we consider the $(n-3)$-dimensional subspace

$$W_{abc} \coloneqq E^{(a-1)} + F^{(b-1)} + G^{(c-1)} \subseteq \mathbb{F}^n.$$

We can define three flags $\overline{E}, \overline{F}, \overline{G}$ in the 3-dimensional quotient space $\mathbb{F}^n/W_{abc}$ using the projection $\mathbb{F}^n \to \mathbb{F}^n/W_{abc}$. More precisely we have

$$\begin{aligned}
\overline{E}^{(1)} &\coloneqq E^{(a)}/W_{abc} \cong E^{(a)}/E^{(a-1)} \subseteq \overline{E}^{(2)} \coloneqq E^{(a+1)}/W_{abc} \cong E^{(a+1)}/E^{(a-1)},\\
\overline{F}^{(1)} &\coloneqq F^{(b)}/W_{abc} \cong F^{(b)}/F^{(b-1)} \subseteq \overline{F}^{(2)} \coloneqq F^{(b+1)}/W_{abc} \cong F^{(b+1)}/F^{(b-1)},\\
\overline{G}^{(1)} &\coloneqq G^{(c)}/W_{abc} \cong G^{(c)}/G^{(c-1)} \subseteq \overline{G}^{(2)} \coloneqq G^{(c+1)}/W_{abc} \cong G^{(c+1)}/G^{(c-1)},
\end{aligned}$$

where the isomorphisms follow from the transversality of the flags (E, F, G).

Lemma A.1.1. Let $(E, F, G) \in \mathrm{Flag}(\mathbb{F}^n)^{(3)}$ be a transverse triple of flags, and let $a, b, c \geq 1$ be integers with $a + b + c = n$. Then the triple of flags $(\overline{E}, \overline{F}, \overline{G})$ in $\mathbb{F}^n/W_{abc}$ constructed above is transverse.

Proof. By symmetry we only need to consider two cases.

Assume first that $\overline{E}^{(1)} \subseteq \overline{F}^{(2)}$. Then $E^{(a)}+F^{(b-1)}+G^{(c-1)} \subseteq E^{(a-1)}+F^{(b+1)}+G^{(c-1)}$, which contradicts the transversality of (E, F, G), since it implies that

$$E^{(a)} + F^{(b+1)} + G^{(c-1)} \subseteq E^{(a-1)} + F^{(b+1)} + G^{(c-1)},$$

and the left hand side has dimension $a + (b + 1) + (c - 1) = n$, whereas the right hand side only has dimension $(a - 1) + (b + 1) + (c - 1) = n - 1$.

Assume now that $\overline{E}^{(1)} \subseteq \overline{F}^{(1)} + \overline{G}^{(1)}$. Then $E^{(a)} + F^{(b-1)} + G^{(c-1)} \subseteq E^{(a-1)} + F^{(b)} + G^{(c)}$, which contradicts the transversality of (E, F, G), since it implies that

$$E^{(a)} + F^{(b)} + G^{(c)} \subseteq E^{(a-1)} + F^{(b)} + G^{(c)},$$

and the left hand side has dimension $a + b + c = n$, whereas the right hand side only has dimension $(a - 1) + b + c = n - 1$. $\square$

In the following we relate the triple ratios of (E, F, G) to the triple ratios of $(\overline{E}, \overline{F}, \overline{G})$. More precisely we have the following relation.

Proposition A.1.2. Let $(E, F, G) \in \mathrm{Flag}(\mathbb{F}^n)^{(3)}$ be a transverse triple of flags, and let $a, b, c \geq 1$ be integers with $a + b + c = n$. Then

$$T_{111}(\overline{E}, \overline{F}, \overline{G}) = T_{abc}(E, F, G),$$

where $\overline{E}, \overline{F}, \overline{G}$ are the transverse flags in the 3-dimensional quotient space $\mathbb{F}^n/(E^{(a-1)} + F^{(b-1)} + G^{(c-1)})$.

Proof. Let us choose non-zero generators $e^{(a-1)} \in \bigwedge^{a-1} E^{(a-1)}$, $f^{(b-1)} \in \bigwedge^{b-1} F^{(b-1)}$ and $g^{(c-1)} \in \bigwedge^{c-1} G^{(c-1)}$, and vectors $e_a \in E^{(a)}$, $e_{a+1} \in E^{(a+1)}$, $f_b \in F^{(b)}$, $f_{b+1} \in F^{(b+1)}$, $g_c \in G^{(c)}$ and $g_{c+1} \in G^{(c+1)}$ in such a way that

$$\begin{aligned}
e^{(a-1)} \wedge e_a \neq 0 \text{ in } \bigwedge^{a} E^{(a)}, &\quad e^{(a-1)} \wedge e_a \wedge e_{a+1} \neq 0 \text{ in } \bigwedge^{a+1} E^{(a+1)} \\
f^{(b-1)} \wedge f_b \neq 0 \text{ in } \bigwedge^{b} F^{(b)}, &\quad f^{(b-1)} \wedge f_b \wedge f_{b+1} \neq 0 \text{ in } \bigwedge^{b+1} F^{(b+1)} \\
g^{(c-1)} \wedge g_c \neq 0 \text{ in } \bigwedge^{c} G^{(c)}, &\quad g^{(c-1)} \wedge g_c \wedge g_{c+1} \neq 0 \text{ in } \bigwedge^{c+1} G^{(c+1)}.
\end{aligned}$$

We compute

$$\begin{aligned}
T_{abc}(E, F, G) &= \frac{e^{(a-1)} \wedge e_a \wedge e_{a+1} \wedge f^{(b-1)} \wedge f_b \wedge g^{(c-1)}}{e^{(a-1)} \wedge f^{(b-1)} \wedge f_b \wedge g^{(c-1)} \wedge g_c \wedge g_{c+1}} \\
&\qquad \cdot \frac{e^{(a-1)} \wedge e_a \wedge f^{(b-1)} \wedge g^{(c-1)} \wedge g_c \wedge g_{c+1}}{e^{(a-1)} \wedge e_a \wedge f^{(b-1)} \wedge f_b \wedge f_{b+1} \wedge g^{(c-1)}} \\
&\qquad\qquad \cdot \frac{e^{(a-1)} \wedge f^{(b-1)} \wedge f_b \wedge f_{b+1} \wedge g^{(c-1)} \wedge g_c}{e^{(a-1)} \wedge e_a \wedge e_{a+1} \wedge f^{(b-1)} \wedge g^{(c-1)} \wedge g_c} \\
&= \frac{\bar{e}_a \wedge \bar{e}_{a+1} \wedge \bar{f}_b}{\bar{f}_b \wedge \bar{g}_c \wedge \bar{g}_{c+1}} \cdot \frac{\bar{e}_a \wedge \bar{g}_c \wedge \bar{g}_{c+1}}{\bar{e}_a \wedge \bar{f}_b \wedge \bar{f}_{b+1}} \cdot \frac{\bar{f}_b \wedge \bar{f}_{b+1} \wedge \bar{g}_c}{\bar{e}_a \wedge \bar{e}_{a+1} \wedge \bar{g}_c} \\
&= T_{111}(\overline{E}, \overline{F}, \overline{G}).
\end{aligned}$$

□

A.2. Snakes and their associated bases

Let $A \subseteq \mathbb{F}^n$ be a linear subspace. The *dual subspace* of A is the linear subspace $A^\perp \subseteq (\mathbb{F}^n)^*$, the space of linear functionals on $\mathbb{F}^n$ that vanish on A. If A is a-dimensional, then $A^\perp$ is $(n-a)$-dimensional. Let $(E, F, G) \in \mathrm{Flag}(\mathbb{F}^n)^{(3)}$. We will associate a line decomposition of $(\mathbb{F}^n)^*$ using the flags E, F and G, and a so-called snake. To define snakes, we have to introduce some notation.

Recall that the triple ratios are indexed by integers $a, b, c > 0$ with $a + b + c = n$. It is convenient to think of these arranged in a discrete triangle

$$\Theta_n = \{(a, b, c) \in \mathbb{Z}^3 \mid a + b + c = n,\ a, b, c \geq 0\}.$$

We also define the interior of Θ_n by

$$\mathrm{int}(\Theta_n) = \{(a, b, c) \in \mathbb{Z}^3 \mid a + b + c = n,\ a, b, c > 0\}.$$

To remember the expressions $e^{(a')} \wedge f^{(b')} \wedge g^{(c')} \in \bigwedge^n \mathbb{F}^n$ that appear in the definition of the triples ratios $T_{abc}(E, F, G)$ (Definition 4.1.2), we can visualize them as forming a small hexagon in Θ_n around $(a, b, c) \in \Theta_n$. Whether they appear in the numerator or denominator alternates as we go around the hexagon.

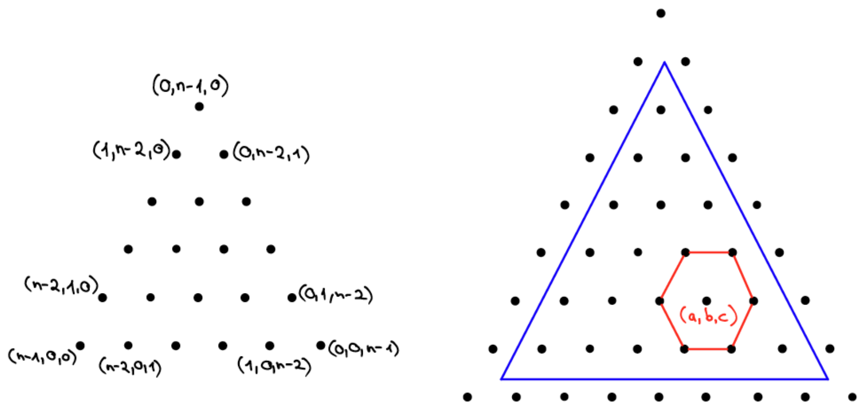

Figure A.1.: The discrete triangle Θ_n (left), its interior $\mathrm{int}(\Theta_n)$ and visualizing the (abc)-triple ratio (right).

Definition A.2.1. A *snake* σ is a sequence of points $\sigma(k) = (\alpha_k, \beta_k, \gamma_k)$ in Θ_{n-1} for $k = 1, \ldots n$ such that

(1) $\sigma(1) = (n-1, 0, 0)$, and

(2) $\sigma(k+1) = \begin{cases} (\alpha_k - 1, \beta_k + 1, \gamma_k), \text{ or} \\ (\alpha_k - 1, \beta_k, \gamma_k + 1). \end{cases}$

The conditions mean that a snake always starts at the bottom left of the discrete triangle Θ_{n-1} and can only go up or right. We always have $\alpha_k = n-k$ for all $k = 1, \ldots, n$. There are two special snakes σ_{top} and σ_{bot}, called the *top* and *bottom* snakes, defined by

$$\sigma_{\mathrm{top}}(k) = (n-k, k-1, 0),$$
$$\sigma_{\mathrm{bot}}(k) = (n-k, 0, k-1) \text{ for all } k = 1, \ldots, n.$$

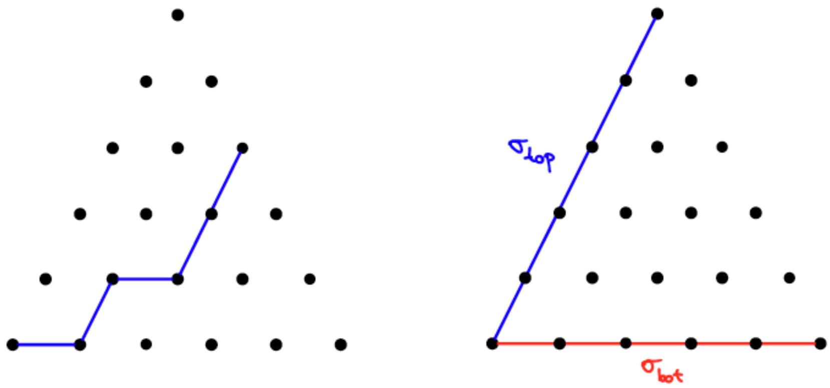

Figure A.2.: A snake (left) and the top and bottom snakes (right) in Θ_{n-1}.

Recall that we would like to define a line decomposition of $(\mathbb{F}^n)^*$ using the transverse flag triple (E, F, G) and a snake σ in Θ_{n-1}. For $\sigma(k) = (\alpha_k, \beta_k, \gamma_k)$ consider

$$L_k := \left(E^{(\alpha_k)} + F^{(\beta_k)} + G^{(\gamma_k)}\right)^{\perp} \subseteq (\mathbb{F}^n)^*$$

for all $k = 1, \ldots, n$. Since (E, F, G) is transverse, $\dim(E^{(\alpha_k)} + F^{(\beta_k)} + G^{(\gamma_k)}) = \alpha_k + \beta_k + \gamma_k = n - 1$ (see the sentence after Definition 4.0.2), and hence $\dim(L_k) = 1$.

Lemma A.2.2. The lines $L_1, \ldots, L_n \subseteq (\mathbb{F}^n)^*$ associated to a snake σ in Θ_{n-1} and $(E, F, G) \in \mathrm{Flag}(\mathbb{F}^n)^{(3)}$ define a line decomposition of $(\mathbb{F}^n)^*$, i.e.

$$(\mathbb{F}^n)^* = \bigoplus_{k=1}^{n} L_k.$$

Proof. We already know that the L_k's are all 1-dimensional. By definition each $L_k \subseteq E^{(\alpha_k)^{\perp}} = E^{(n-k)^{\perp}}$, since $\alpha_k = n - k$. An induction shows that $L_1, \ldots, L_k$ generate $E^{(n-k)^{\perp}}$, and hence $L_1, \ldots, L_n$ generate $E^{(0)^{\perp}} = (\mathbb{F}^n)^*$. For dimension reasons the lines are in direct sum. ☐

For the special snakes σ_{top} and σ_{bot} we obtain the following line decompositions of $(\mathbb{F}^n)^*$ associated to the transverse triple (E, F, G), namely for all $k = 1, \ldots, n$ we have

$$L_k^{\text{top}} = \left(E^{(n-k)} + F^{(k-1)}\right)^{\perp}, \quad L_k^{\text{bot}} = \left(E^{(n-k)} + G^{(k-1)}\right)^{\perp}.$$

We would like to refine this information by choosing non-zero vectors $u_k \in L_k$ in such a way, that $u_2, \ldots, u_n$ are uniquely determined by u_1, (E, F, G) and σ. The way that we will define the vectors u_k will show that if we scale u_1 by $x \in \mathbb{F} \setminus \{0\}$ then $u_2, \ldots, u_n$ will be scaled by x as well. Thus the so constructed basis of $(\mathbb{F}^n)^*$ will be unique up to scaling, and we will call it the basis associated to (E, F, G) by σ.

Let $(E, F, G) \in \mathrm{Flag}(\mathbb{F}^n)^{(3)}$ and σ a snake in Θ_{n-1}. We define inductively the basis of $(\mathbb{F}^n)^*$ associated to (E, F, G) by σ. Choose $u_1 \in L_1 = (E^{(n-1)})^{\perp}$ a non-zero generator.

From now on, $u_k \in L_k$ for $k \geq 2$ will be uniquely defined by u_1. By definition of σ we have

$$\sigma(k+1) = \begin{cases} (\alpha_k - 1, \beta_k + 1, \gamma_k), \text{ or} \\ (\alpha_k - 1, \beta_k, \gamma_k + 1). \end{cases}$$

In the first case, the snake slithers up, in the second case, it slithers right. Depending on these two ways the snake can slither, let

$$L_{k+1}^{\mathrm{up}} \coloneqq \left(E^{(\alpha_k-1)} + F^{(\beta_k+1)} + G^{(\gamma_k)}\right)^{\perp},$$
$$L_{k+1}^{\mathrm{rt}} \coloneqq \left(E^{(\alpha_k-1)} + F^{(\beta_k)} + G^{(\gamma_k+1)}\right)^{\perp}$$

be the two possibilities for L_{k+1}. Then L_{k+1}^{up}, L_{k+1}^{rt} and L_k all belong to the 2-dimensional plane $P_k \coloneqq (E^{(\alpha_k-1)} + F^{(\beta_k)} + G^{(\gamma_k)})^{\perp}$, and all three lines are pairwise distinct, i.e. $L_{k+1}^{\mathrm{up}} \neq L_{k+1}^{\mathrm{rt}}$, $L_{k+1}^{\mathrm{up}} \neq L_k$ and $L_{k+1}^{\mathrm{rt}} \neq L_k$. Since they are all contained in P_k which is 2-dimensional, they are linearly dependent, and there exists unique non-zero elements $u_{k+1}^{\mathrm{up}} \in L_{k+1}^{\mathrm{up}}$ and $u_{k+1}^{\mathrm{rt}} \in L_{k+1}^{\mathrm{rt}}$ such that

$$u_k + u_{k+1}^{\mathrm{up}} + u_{k+1}^{\mathrm{rt}} = 0.$$

We define

$$u_{k+1} \coloneqq \begin{cases} u_{k+1}^{\mathrm{up}} & \text{if } L_{k+1} = L_{k+1}^{\mathrm{up}}, \\ -u_{k+1}^{\mathrm{rt}} & \text{if } L_{k+1} = L_{k+1}^{\mathrm{rt}}. \end{cases}$$

Definition A.2.3. The basis $\mathcal{B}^* \coloneqq \{u_1, \ldots, u_n\}$ is the basis of $(\mathbb{F}^n)^*$ *associated to* (E, F, G) *by* σ. For the special snakes σ_{top} and σ_{bot} we obtain two bases $\mathcal{B}^*_{\mathrm{top}} = \{u_1^{\mathrm{top}}, \ldots, u_n^{\mathrm{top}}\}$ and $\mathcal{B}^*_{\mathrm{bot}} = \{u_1^{\mathrm{bot}}, \ldots, u_n^{\mathrm{bot}}\}$ of $(\mathbb{F}^n)^*$.

Example A.2.4. Let (E, F, G) be a triple of transverse flags. We would like to compute $\mathcal{B}^*_{\mathrm{top}} = \{u_1^{\mathrm{top}}, \ldots, u_n^{\mathrm{top}}\}$. To ease notation we denote the vectors by $u_1, \ldots, u_n$. Choose a basis $e_1, \ldots, e_n$ of $\mathbb{F}^n$ such that for all $i = 0, \ldots, n$, we have

$$E^{(i)} = \langle e_1, \ldots, e_i \rangle,\ F^{(i)} = \langle e_n, \ldots, e_{n-i+1} \rangle,\ G^{(1)} = \langle e_1 + \ldots + e_n \rangle.$$

Denote by $e_1^*, \ldots, e_n^*$ its dual basis. By definition of σ_{top}, for all $i = 1, \ldots, n$, we have

$$L_i^{\mathrm{up}} = \left(E^{(n-i)} + F^{(i-1)}\right)^{\perp} = \langle e_1, \ldots, e_{n-i}, e_n, \ldots, e_{n-i+2} \rangle^{\perp} = \langle e_{n-i+1}^* \rangle,$$
$$L_i^{\mathrm{rt}} = \left(E^{(n-i)} + F^{(i-2)} + G^{(1)}\right)^{\perp} = \langle e_1, \ldots, e_{n-i}, e_n, \ldots, e_{n-i+3}, \sum_{j=1}^{n} e_j \rangle^{\perp}$$
$$= \langle e_{n-i+1}^* - e_{n-i+2}^* \rangle.$$

Recall that the vectors $u_1, \ldots, u_n$ are defined recursively. Choose $u_1 = e_n^*$. Then $u_i = u_i^{\mathrm{up}} \in \langle e_{n-i+1}^* \rangle$ with $u_{i-1} + u_i^{\mathrm{up}} + u_i^{\mathrm{rt}} = 0$ and $u_i^{\mathrm{rt}} \in \langle e_{n-i+1}^* - e_{n-i+2}^* \rangle$. Set $u_i = \mu_i e_{n-i+1}^*$, and $u_i^{\mathrm{rt}} = \nu_i (e_{n-i+1}^* - e_{n-i+2}^*)$ for some $\mu_i, \nu_i \in \mathbb{F} \setminus \{0\}$. We need to determine μ_i. Putting it together we obtain

$$0 = \mu_{i-1} e_{n-i+2}^* + \mu_i e_{n-i+1}^* + \nu_i (e_{n-i+1}^* - e_{n-i+2}^*) \implies -\mu_i = \nu_i = \mu_{i-1},$$

and thus, using $\mu_1 = 1$,

$$u_i = \mu_i e_{n-i+1}^* = (-1)^{i-1} e_{n-i+1}^* \text{ for all } i = 1, \ldots, n.$$

A.3. Snake moves

Recall that if $\mathcal{A}$ and $\mathcal{A}'$ are two bases of $\mathbb{F}^n$ the *base change matrix* $T_{\mathcal{A}}^{\mathcal{A}'}$ expressing $\mathcal{A}'$ in $\mathcal{A}$ is defined as follows: we write the i-th basis vector of $\mathcal{A}'$ in the basis $\mathcal{A}$ and the corresponding coefficients form the i-th column of $T_{\mathcal{A}}^{\mathcal{A}'}$. We would like to compute the base change matrix $T_{\mathcal{B}^*_{\mathrm{top}}}^{\mathcal{B}^*_{\mathrm{bot}}}$ expressing the basis $\mathcal{B}^*_{\mathrm{bot}}$ in the basis $\mathcal{B}^*_{\mathrm{top}}$. For this we make use of so-called *snake moves*: the diamond and the boundary move. They allow us to replace a snake by a different snake, which differs from the original one in exactly one point. If we iteratively apply these moves we can transform σ_{top} into σ_{bot}.

Definition A.3.1. A *diamond move at step $k+1$* replaces a snake σ given by

$$\begin{aligned}\sigma(k) &= (\alpha_k, \beta_k, \gamma_k),\\ \sigma(k+1) &= (\alpha_k - 1, \beta_k + 1, \gamma_k),\\ \sigma(k+2) &= (\alpha_k - 2, \beta_k + 1, \gamma_k + 1)\end{aligned}$$

by a snake σ' given by

$$\begin{aligned}\sigma'(j) &= \sigma(j) \text{ for all } j \leq k\\ \sigma'(k+1) &= (\alpha_k - 1, \beta_k, \gamma_k + 1),\\ \sigma'(j) &= \sigma(j) \text{ for all } j \geq k+2.\end{aligned}$$

The snakes σ and σ' differ only at the point $k+1$.

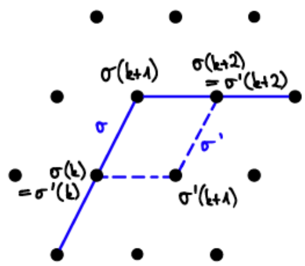

Figure A.3.: A diamond move at step $k+1$.

Let σ and σ' be two snakes in Θ_{n-1} such that σ' is obtained from σ by a diamond move at step $k+1$. Let $\mathcal{B}^*$ and $\mathcal{B}'^*$ be the two bases of $(\mathbb{F}^n)^*$ associated to $(E,F,G) \in \mathrm{Flag}(\mathbb{F}^n)^{(3)}$ by the snakes σ respectively σ'. We now relate the change of basis to the triple ratio $T_{(\alpha_k-1,\beta_k+1,\gamma_k+1)}(E,F,G)$. This can be visualized by drawing Θ_{n-1} and Θ_n in the same picture, and by observing that we obtain the snake σ' from the snake σ in Θ_{n-1} by sweeping over the vertex $(\alpha_k - 1, \beta_k + 1, \gamma_k + 1)$ in Θ_n, see Figure A.4. This is formalized in the following proposition.

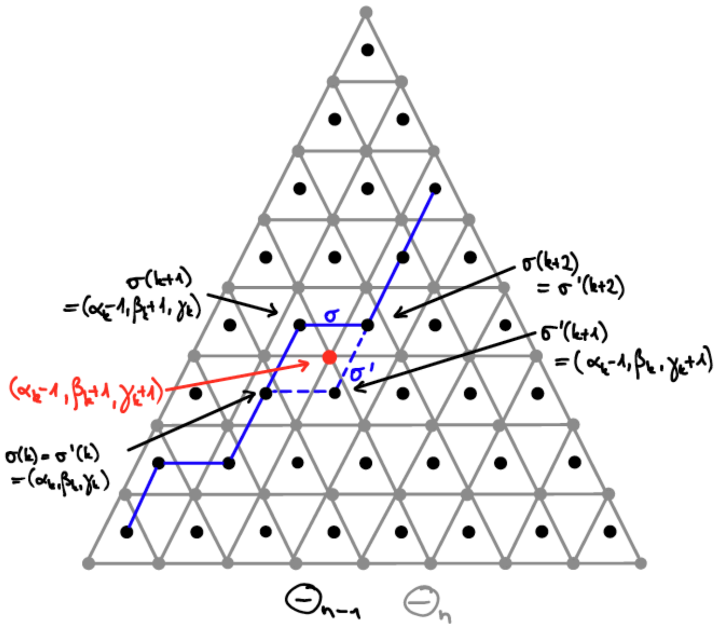

Figure A.4.: The visualization of the base change of a diamond move at step $k+1$: the snake in Θ_{n-1} sweeps through the vertex $(\alpha_k - 1, \beta_k + 1, \gamma_k + 1)$ in Θ_n.

Proposition A.3.2. Let σ and σ' be two snakes in Θ_{n-1} such that σ' is obtained from σ by a diamond move at step $k+1$. Let $\mathcal{B}^* = \{u_1, \dots, u_n\}$ and $\mathcal{B}'^* = \{u'_1, \dots, u'_n\}$ be the two bases of $(\mathbb{F}^n)^*$ associated to $(E, F, G) \in \mathrm{Flag}(\mathbb{F}^n)^{(3)}$ by the snakes σ respectively σ'. Then, after a possible normalization so that $u_1 = u'_1$, we have for all $i = 1, \dots, n$

$$u'_i = \begin{cases} u_i, & \text{if } i \leq k, \\ u_k + u_{k+1}, & \text{if } i = k+1, \\ T_{(\alpha_k - 1, \beta_k + 1, \gamma_k + 1)}(E, F, G) u_i, & \text{if } i \geq k+2, \end{cases}$$

where $\sigma(k) = (\alpha_k, \beta_k, \gamma_k)$ and $x_k := T_{(\alpha_k - 1, \beta_k + 1, \gamma_k + 1)}(E, F, G) \in \mathbb{F} \setminus \{0\}$ denotes the $(\alpha_k - 1, \beta_k + 1, \gamma_k + 1)$-triple ratio associated to (E, F, G). In particular, we have

$$T_{\mathcal{B}^*}^{\mathcal{B}'^*} = \left(\begin{array}{c|cc|c} \mathrm{Id}_k & & & \\ \hline & 1 & 1 & \\ & 0 & 1 & \\ \hline & & & x_k \, \mathrm{Id}_{n-k-2} \end{array} \right),$$

where Id_m denotes the $m \times m$ identity matrix.

Proof. We structure the proof in four steps.

Step 1: $i \leq k$**.** Because σ and σ' coincide at all points $\sigma(i) = \sigma'(i)$ for all $i \leq k$, we have $u'_i = u_i$ for all $i \leq k$ once we have normalized such that $u'_1 = u_1$.

Step 2: $i = k+1$**.** Recall that

$$u_{k+1} := \begin{cases} u_{k+1}^{\mathrm{up}} & \text{if } L_{k+1} = L_{k+1}^{\mathrm{up}}, \\ -u_{k+1}^{\mathrm{rt}} & \text{if } L_{k+1} = L_{k+1}^{\mathrm{rt}}, \end{cases}$$

with u_{k+1}^{up} and u_{k+1}^{rt} unique such that $u_k + u_{k+1}^{\mathrm{up}} + u_{k+1}^{\mathrm{rt}} = 0$. Since $\mathcal{B}^*$ is associated to (E, F, G) by σ we are in the case that

$$L_{k+1} = L_{k+1}^{\mathrm{up}}.$$

Thus $u_{k+1} = u_{k+1}^{\mathrm{up}}$. On the other hand, for σ' we are in the case that

$$L'_{k+1} = (L'_{k+1})^{\mathrm{rt}},$$

and thus $u'_{k+1} = -(u'_{k+1})^{\mathrm{rt}}$. Since $\sigma(k) = \sigma'(k)$ we have by uniqueness that

$$u_{k+1}^{\mathrm{up}} = (u'_{k+1})^{\mathrm{up}} \text{ and } u_{k+1}^{\mathrm{rt}} = (u'_{k+1})^{\mathrm{rt}}.$$

Putting this together we obtain

$$u'_{k+1} = -(u'_{k+1})^{\mathrm{rt}} = -u_{k+1}^{\mathrm{rt}} = u_k + u_{k+1}^{\mathrm{up}} = u_k + u_{k+1},$$

which is what we had to prove.

Step 3: $i > k+2$. Let us assume that we know already that $u'_{k+2} = T_{(\alpha_k-1,\beta_k+1,\gamma_k+1)}(E, F, G)u_{k+2}$ (which will be proven in Step 4). Since $\sigma(i) = \sigma'(i)$ for all $i \geq k+2$, this assumption directly implies that

$$u'_i = T_{(\alpha_k-1,\beta_k+1,\gamma_k+1)}(E, F, G)u_i,$$

for all $i > k+2$, since the definition of u_i only uses u_{i-1} and $\sigma(i)$.

Step 4: $i = k+2$. It is left to show that

$$u'_{k+2} = T_{(\alpha_k-1,\beta_k+1,\gamma_k+1)}(E, F, G)u_{k+2},$$

and we will see how the triple ratios appear. Recall that $\sigma(k+2) = \sigma'(k+2) = (\alpha_k - 2, \beta_k + 1, \gamma_k + 1)$. Thus $u_{k+2}, u'_{k+2} \in L_{k+2} = (E^{(\alpha_k-2)} + F^{(\beta_k+1)} + G^{(\gamma_k+1)})^{\perp}$, and hence there exists $x \in \mathbb{F} \setminus \{0\}$ such that

$$u'_{k+2} = xu_{k+2}.$$

We need to show that $x = x_k = T_{(\alpha_k-1,\beta_k+1,\gamma_k+1)}(E, F, G)$.

By definition of u_{k+2} and u'_{k+2} we have

$$u_{k+1} - u_{k+2} \in \left(E^{(\alpha_k-2)} + F^{(\beta_k+2)} + G^{(\gamma_k)}\right)^{\perp},$$
$$u'_{k+1} + u'_{k+2} \in \left(E^{(\alpha_k-2)} + F^{(\beta_k)} + G^{(\gamma_k+2)}\right)^{\perp}.$$

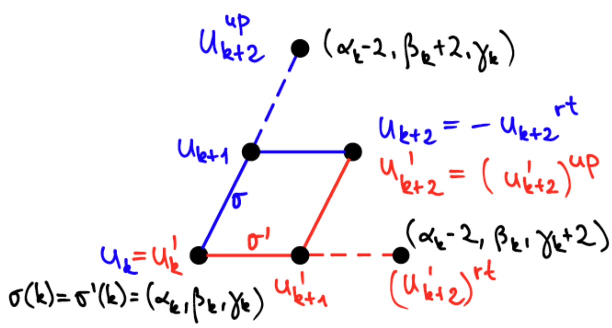

Figure A.5.: The definition of u_{k+2} and u'_{k+2}.

Thus the linear forms $u_k = u'_k$, u_{k+1}, u'_{k+1}, u_{k+2} and u'_{k+2} in $(\mathbb{F}^n)^*$ are elements of $(E^{(\alpha_k-2)} + F^{(\beta_k)} + G^{(\gamma_k)})^\perp$, hence they all vanish on $E^{(\alpha_k-2)} + F^{(\beta_k)} + G^{(\gamma_k)}$ which is $(n-3)$-dimensional. Hence we have induced linear forms

$$\overline{u_k}, \overline{u_{k+1}}, \overline{u_{k+2}}, \overline{u'_k}, \overline{u'_{k+1}}, \overline{u'_{k+2}} \colon V_k \to \mathbb{F}$$

on the 3-dimensional quotient space $V_k := \mathbb{F}^n/(E^{(\alpha_k-2)} + F^{(\beta_k)} + G^{(\gamma_k)})$.

The flags E, F, $G \in \mathrm{Flag}(\mathbb{F}^n)$ induce flags $\overline{E}$, $\overline{F}$, $\overline{G} \in \mathrm{Flag}(V_k)$ by

$$\begin{aligned}
\overline{E}^{(1)} &= E^{(\alpha_k-1)}/E^{(\alpha_k-2)} \subseteq \overline{E}^{(2)} = E^{(\alpha_k)}/E^{(\alpha_k-2)} \\
\overline{F}^{(1)} &= F^{(\beta_k+1)}/F^{(\beta_k)} \subseteq \overline{F}^{(2)} = F^{(\beta_k+2)}/F^{(\beta_k)} \\
\overline{G}^{(1)} &= G^{(\gamma_k+1)}/G^{(\gamma_k)} \subseteq \overline{G}^{(2)} = G^{(\gamma_k+2)}/G^{(\gamma_k)},
\end{aligned}$$

as in Appendix A.1. The transversality of (E, F, G) implies that $(\overline{E}, \overline{F}, \overline{G})$ is transverse in V_k, see Lemma A.1.1. In particular, $(\overline{F}, \overline{G})$ is transverse, and we can choose a basis $\{\bar{f}, \bar{g}, \bar{h}\}$ for V_k such that

$$\bar{f} \in \overline{F}^{(1)}, \ \bar{g} \in \overline{G}^{(1)}, \ \text{and } \bar{h} \in \overline{F}^{(2)} \cap \overline{G}^{(2)}.$$

Choose a generator $\bar{e}_1 \in \overline{E}^{(1)}$. Since $(\overline{E}, \overline{F}, \overline{G})$ is transverse, the coefficients of $\bar{e}_1$ in the basis $\{\bar{f}, \bar{g}, \bar{h}\}$ are non-zero. We can therefore rescale the basis elements in such a way that

$$\bar{e}_1 = \bar{f} + \bar{g} + \bar{h}.$$

Finally, add a vector $\bar{e}_2$ to form a basis $\{\bar{e}_1, \bar{e}_2\}$ of $\overline{E}^{(2)}$. We can without loss of generality assume that $\bar{e}_2 = y\bar{f} + z\bar{g}$ with $y, z \in \mathbb{F} \setminus \{0\}$. By definition of the triple ratios and Proposition A.1.2 we have

$$\begin{aligned}
T_{(\alpha_k-1,\beta_k+1,\gamma_k+1)}(E,F,G) &= T_{111}(\overline{E}, \overline{F}, \overline{G}) \\
&= \frac{\bar{e}^{(2)} \wedge \bar{f}^{(1)}}{\bar{f}^{(1)} \wedge \bar{g}^{(2)}} \cdot \frac{\bar{e}^{(1)} \wedge \bar{g}^{(2)}}{\bar{e}^{(1)} \wedge \bar{f}^{(2)}} \cdot \frac{\bar{f}^{(2)} \wedge \bar{g}^{(1)}}{\bar{e}^{(2)} \wedge \bar{g}^{(1)}} \\
&= \frac{(\bar{f}+\bar{g}+\bar{h}) \wedge (y\bar{f}+z\bar{g}) \wedge \bar{f}}{\bar{f} \wedge \bar{g} \wedge \bar{h}} \cdot \frac{(\bar{f}+\bar{g}+\bar{h}) \wedge \bar{g} \wedge \bar{h}}{(\bar{f}+\bar{g}+\bar{h}) \wedge \bar{f} \wedge \bar{h}} \\
&\qquad \cdot \frac{\bar{f} \wedge \bar{h} \wedge \bar{g}}{(\bar{f}+\bar{g}+\bar{h}) \wedge (y\bar{f}+z\bar{g}) \wedge \bar{g}} \\
&= \frac{\bar{h} \wedge z\bar{g} \wedge \bar{f}}{\bar{f} \wedge \bar{g} \wedge \bar{h}} \cdot \frac{\bar{f} \wedge \bar{g} \wedge \bar{h}}{\bar{g} \wedge \bar{f} \wedge \bar{h}} \cdot \frac{\bar{f} \wedge \bar{h} \wedge \bar{g}}{\bar{h} \wedge y\bar{f} \wedge \bar{g}} = -\frac{z}{y}.
\end{aligned} \tag{A.1}$$

The next step is to express the linear forms $\overline{u_k}, \overline{u_{k+1}}, \overline{u_{k+2}}, \overline{u'_k}, \overline{u'_{k+1}}, \overline{u'_{k+2}}$ in the dual basis $\{\bar{f}^*, \bar{g}^*, \bar{h}^*\}$. Recall $u_{k+1} - u_{k+2} \in (E^{(\alpha_k-2)} + F^{(\beta_k+2)} + G^{(\gamma_k)})^\perp$, thus $\overline{u_{k+1}} - \overline{u_{k+2}}$ vanishes on $\overline{F}^{(2)} = F^{(\beta_k+2)}/F^{(\beta_k)}$. By definition of the dual basis, we have

$$\overline{u_{k+1}} - \overline{u_{k+2}} = a\bar{g}^*$$

for some $0 \neq a \in \mathbb{F}$. Similarly, we have $u'_{k+1} + u'_{k+2} \in (E^{(\alpha_k-2)} + F^{(\beta_k)} + G^{(\gamma_k+2)})^\perp$, thus $\overline{u'_{k+1}} + \overline{u'_{k+2}}$ vanishes on $\overline{G}^{(2)} = G^{(\gamma_k+2)}/G^{(\gamma_k)}$. Hence

$$\overline{u'_{k+1}} + \overline{u'_{k+2}} = b\bar{f}^*$$

for some $0 \neq b \in \mathbb{F}$. Since $u'_{k+2} = xu_{k+2} \in (E^{(\alpha_k-2)} + F^{(\beta_k+1)} + G^{(\gamma_k+1)})^\perp$ the form $\overline{u'_{k+2}} = x\overline{u_{k+2}}$ vanishes on $\overline{F}^{(1)} + \overline{G}^{(1)} = F^{(\beta_k+1)}/F^{(\beta_k)} + G^{(\gamma_k+1)}/G^{(\gamma_k)}$, hence

$$\overline{u'_{k+2}} = x\overline{u_{k+2}} = c\bar{h}^*$$

for some $0 \neq c \in \mathbb{F}$. Putting everything together we obtain

$$\begin{aligned} \overline{u_{k+1}} &= a\bar{g}^* + \overline{u_{k+2}} = a\bar{g}^* + \frac{c}{x}\bar{h}^* \\ \overline{u'_{k+1}} &= b\bar{f}^* - \overline{u'_{k+2}} = b\bar{f}^* - c\bar{h}^* \\ \overline{u_k} &= \overline{u'_{k+1}} - \overline{u_{k+1}} = b\bar{f}^* - a\bar{g}^* - c\left(1 + \frac{1}{x}\right)\bar{h}^* \end{aligned}$$

We recall that

$$\begin{aligned} u_k &\in \left(E^{(\alpha_k)} + F^{(\beta_k)} + G^{(\gamma_k)}\right)^\perp \\ u_{k+1} &\in \left(E^{(\alpha_k-1)} + F^{(\beta_k+1)} + G^{(\gamma_k)}\right)^\perp \\ u'_{k+1} &\in \left(E^{(\alpha_k-1)} + F^{(\beta_k)} + G^{(\gamma_k+1)}\right)^\perp, \end{aligned}$$

and hence $\overline{u_k}$ vanishes on $\overline{E}^{(2)}$, $\overline{u_{k+1}}$ vanishes on $\overline{E}^{(1)} + \overline{F}^{(1)}$ and $\overline{u'_{k+1}}$ vanishes on $\overline{E}^{(1)} + \overline{G}^{(1)}$. Remembering that $\bar{e}_1 = \bar{f} + \bar{g} + \bar{h}$ and $\bar{e}_2 = y\bar{f} + z\bar{g}$ we obtain

$$\begin{aligned} 0 &= \overline{u_k}(\bar{e}_1) = b - a - c(1 + \tfrac{1}{x}) \\ 0 &= \overline{u_{k+1}}(\bar{e}_1) = a + \tfrac{c}{x} \\ 0 &= \overline{u'_{k+1}}(\bar{e}_1) = b - c \\ 0 &= \overline{u_k}(\bar{e}_2) = \overline{u_k}(y\bar{f} + z\bar{g}) = by - az \end{aligned}$$

We solve this system of equations and obtain

$$b = c,\ a = -\tfrac{c}{x},\ cy = -\tfrac{c}{x}z \implies x = -\tfrac{z}{y}.$$

Thus we have shown that $x = T_{(\alpha_k-1,\beta_k+1,\gamma_k+1)}(E, F, G)$, compare to Equation (A.1), which finishes the proof. $\square$

Definition A.3.3. A *boundary move* replaces a snake σ ending with

$$\begin{aligned} \sigma(n-1) &= (1, \beta_{n-1}, \gamma_{n-1}), \\ \sigma(n) &= (0, \beta_{n-1} + 1, \gamma_{n-1}) \end{aligned}$$

by a snake σ' given by

$$\begin{aligned} \sigma'(j) &= \sigma(j) \text{ for all } j \leq n-1 \\ \sigma'(n) &= (0, \beta_{n-1}, \gamma_{n-1} + 1). \end{aligned}$$

The snakes σ and σ' differ only at the point n.

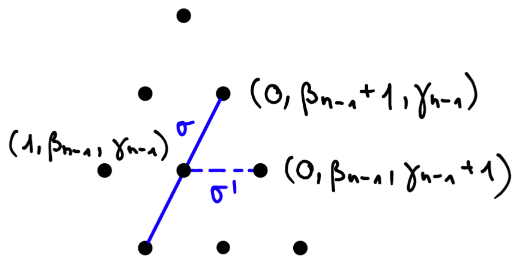

Figure A.6.: A boundary move.

Proposition A.3.4. Let σ and σ' be two snakes in Θ_{n-1} such that σ' is obtained from σ by a boundary move. Let $\mathcal{B}^* = \{u_1, \dots, u_n\}$ and $\mathcal{B}'^* = \{u'_1, \dots, u'_n\}$ be the two bases of $(\mathbb{F}^n)^*$ associated to $(E, F, G) \in \mathrm{Flag}(\mathbb{F}^n)^{(3)}$ by the snakes σ respectively σ'. Then, after a possible normalization so that $u_1 = u'_1$, we have

$$u'_i = \begin{cases} u_i, & \text{if } i \leq n-1, \\ u_{n-1} + u_n, & \text{if } i = n \end{cases}$$

for all $i = 1, \dots, n$. In particular, we have

$$T_{\mathcal{B}^*}^{\mathcal{B}'^*} = \left(\begin{array}{c|cc} \mathrm{Id}_{n-2} & & \\ \hline & 1 & 1 \\ & 0 & 1 \end{array}\right).$$

Proof. For $i \leq n-1$ the snakes σ and σ' agree. Hence $u'_i = u_i$ for all $i \leq n-1$.

Let now $i = n$. By definition of σ, we have $u_n = u_n^{\mathrm{up}}$ with $u_n^{\mathrm{up}} \in (E^{(0)} + F^{(\beta_{n-1}+1)} + G^{(\gamma_{n-1})})^\perp$ such that $u_{n-1} + u_n^{\mathrm{up}} + u_n^{\mathrm{rt}} = 0$. For σ' we have $u'_n = -(u'_n)^{\mathrm{rt}}$ with $(u'_n)^{\mathrm{rt}} \in (E^{(0)} + F^{(\beta_{n-1})} + G^{(\gamma_{n-1}+1)})^\perp$ such that $u'_{n-1} + (u'_n)^{\mathrm{up}} + (u'_n)^{\mathrm{rt}} = 0$. Since $u_{n-1} = u'_{n-1}$, we have by uniqueness that $u_n^{\mathrm{up}} = (u'_n)^{\mathrm{up}}$ and $u_n^{\mathrm{rt}} = (u'_n)^{\mathrm{rt}}$. Thus

$$u'_n = -(u'_n)^{\mathrm{rt}} = -u_n^{\mathrm{rt}} = u_{n-1} + u_n^{\mathrm{up}} = u_{n-1} + u_n,$$

which is what we had to prove. $\square$

A.4. Top and bottom snakes

Recall the two special snakes σ_{top} and σ_{bot} defined for all $k = 1, \dots, n$ by

$$\sigma_{\mathrm{top}}(k) = (n-k, k-1, 0),$$
$$\sigma_{\mathrm{bot}}(k) = (n-k, 0, k-1),$$

and the associated bases $\mathcal{B}^*_{\mathrm{top}}$ and $\mathcal{B}^*_{\mathrm{bot}}$ of $(\mathbb{F}^n)^*$. Denote by $\mathcal{B}_{\mathrm{top}} = \{v_1^{\mathrm{top}}, \dots, v_n^{\mathrm{top}}\}$ and $\mathcal{B}_{\mathrm{bot}} = \{v_1^{\mathrm{bot}}, \dots, v_n^{\mathrm{bot}}\}$ the bases of $(\mathbb{F}^n)^{**} \cong \mathbb{F}^n$ dual to $\mathcal{B}^*_{\mathrm{top}}$ respectively $\mathcal{B}^*_{\mathrm{bot}}$, i.e.

$$u_i^{\mathrm{top}}(v_j^{\mathrm{top}}) = \delta_{ij},\ u_i^{\mathrm{bot}}(v_j^{\mathrm{bot}}) = \delta_{ij}.$$

The following proposition tells us how we can recover the transverse triple (E, F, G) from $\mathcal{B}_{\mathrm{top}}$ and $\mathcal{B}_{\mathrm{bot}}$.

Lemma A.4.1. For all integers $a, b, c \geq 0$ we have

(1) $E^{(a)} = \langle v_{n-a+1}^{\text{top}}, \ldots, v_n^{\text{top}} \rangle = \langle v_{n-a+1}^{\text{bot}}, \ldots, v_n^{\text{bot}} \rangle$,

(2) $F^{(b)} = \langle v_1^{\text{top}}, \ldots, v_b^{\text{top}} \rangle$,

(3) $G^{(c)} = \langle v_1^{\text{bot}}, \ldots, v_c^{\text{bot}} \rangle$.

Proof.

(1) By definition of σ_{top} and $\mathcal{B}_{\text{top}}^*$ we have $u_k^{\text{top}} \in L_k^{\text{top}} = (E^{(n-k)} + F^{(k-1)})^\perp$, which implies that $u_1^{\text{top}}, \ldots, u_k^{\text{top}} \in (E^{(n-k)})^\perp$ (since if $i \leq k$ then $E^{(n-k)} \subseteq E^{(n-i)}$ and hence $(E^{(n-i)})^\perp \subseteq (E^{(n-k)})^\perp$). Now $u_i^{\text{top}}(v_j^{\text{top}}) = 0$ if $i \neq j$ shows that

$$v_{k+1}^{\text{top}}, \ldots, v_n^{\text{top}} \in ((E^{(n-k)})^\perp)^\perp = E^{(n-k)}.$$

Dimensionality reasons now imply that $\langle v_{k+1}^{\text{top}}, \ldots, v_n^{\text{top}} \rangle = E^{(n-k)}$, which shows the first claim. The same argument applied to σ_{bot} and $\mathcal{B}_{\text{bot}}$ using $L_k^{\text{bot}} = (E^{(n-k)} + G^{(k-1)})^\perp$ shows that $\langle v_{k+1}^{\text{bot}}, \ldots, v_n^{\text{bot}} \rangle = E^{(n-k)}$.

(2) As in the proof of (1), we have $u_k^{\text{top}} \in (E^{(n-k)} + F^{(k-1)})^\perp$. Hence $u_k^{\text{top}}, \ldots, u_n^{\text{top}} \in (F^{(k-1)})^\perp$ (since if $i \geq k$ then $F^{(k-1)} \subseteq F^{(i-1)}$ and hence $(F^{(i-1)})^\perp \subseteq (F^{(k-1)})^\perp$). Again, $u_i^{\text{top}}(v_j^{\text{top}}) = 0$ if $i \neq j$ shows that

$$v_1^{\text{top}}, \ldots, v_{k-1}^{\text{top}} \in ((F^{(k-1)})^\perp)^\perp = F^{(k-1)},$$

and by dimension reasons we have $\langle v_1^{\text{top}}, \ldots, v_{k-1}^{\text{top}} \rangle = F^{(k-1)}$.

(3) The same argument as in the proof of (2) applied to σ_{bot} and $\mathcal{B}_{\text{bot}}$ implies (3).

$\square$

We defined in Definition 4.3.2 the matrices $E_k := \mathrm{Id}_n + E_{k,k+1}$ for $k = 1, \ldots, n-1$ and $F_k := E_k^\top$, the transpose of E_k, in $\mathrm{GL}(n, \mathbb{F})$. Furthermore, for $x \in \mathbb{F} \setminus \{0\}$ and $k = 1, \ldots, n$ we set

$$H_k(x) := \mathrm{diag}(\underbrace{1, \ldots, 1}_{k}, \underbrace{x, \ldots, x}_{n-k}).$$

With these notations we see from Proposition A.3.2 and Proposition A.3.4 that the matrix representing the base change of a diamond move at the $(k+1)$-st step is

$$\left(\begin{array}{c|cc|c} \mathrm{Id}_k & & & \\ \hline & 1 & 1 & \\ & 0 & 1 & \\ \hline & & & x_k \,\mathrm{Id}_{n-k-2} \end{array}\right) = E_k H_{k+1}(x_k) = H_{k+1}(x_k) E_k,$$

and the matrix representing the base change of a boundary move is

$$\left(\begin{array}{c|cc} \mathrm{Id}_{n-2} & & \\ \hline & 1 & 1 \\ & 0 & 1 \end{array}\right) = E_{n-1}.$$

Proposition A.4.2. Let σ_{top} and σ_{bot} be the top and bottom snakes in Θ_{n-1}. Le $\mathcal{B}^*_{\text{top}}$ and $\mathcal{B}^*_{\text{bot}}$ be the corresponding bases of $(\mathbb{F}^n)^*$. After normalizing such that $u_1^{\text{top}} = u_1^{\text{bot}}$, the base change matrix expressing $\mathcal{B}^*_{\text{bot}}$ in $\mathcal{B}^*_{\text{top}}$ is given by

$$T_{\mathcal{B}^*_{\text{top}}}^{\mathcal{B}^*_{\text{bot}}} = \prod_{k=1}^{n-1} E_{n-1} \left(\prod_{i=1}^{n-k-1} H_{n-i}(x_{i,n-i-k,k}) E_{n-i-1} \right), \tag{A.2}$$

where $x_{i,n-i-k,k} \coloneqq T_{i,n-i-k,k}(E,F,G)$ denotes the $(i, n-i-k, k)$-triple ratio of (E,F,G).

Proof. The idea of the proof is to modify the top snake σ_{top} by a sequence of diamond and boundary moves until we arrive at the bottom snake σ_{bot}. Let us denote $\sigma_1 \coloneqq \sigma_{\text{top}}$. We start with one boundary move, followed by $n-2$ diamond moves starting at step $n-1$ up to step 2, and we call the snake obtained in this way σ_2. It is parallel to σ_1, slithering only upwards from position 2 on, more precisely,

$$\sigma_2(1) = (n-1, 0, 0),\ \sigma_2(k) = (n-k, k-2, 1) \text{ for all } k \geq 2.$$

We modify σ_2 by one boundary move, followed by $n-3$ diamond moves at step $n-1$ up to step 3 to obtain σ_3. We continue this process and obtain for all $i = 1, \ldots, n$

$$\sigma_i(k) = \begin{cases} (n-k, 0, k-1), & \text{if } k < i, \\ (n-k, k-i, i-1), & \text{if } k \geq i \end{cases}$$

with $\sigma_n = \sigma_{\text{bot}}$.

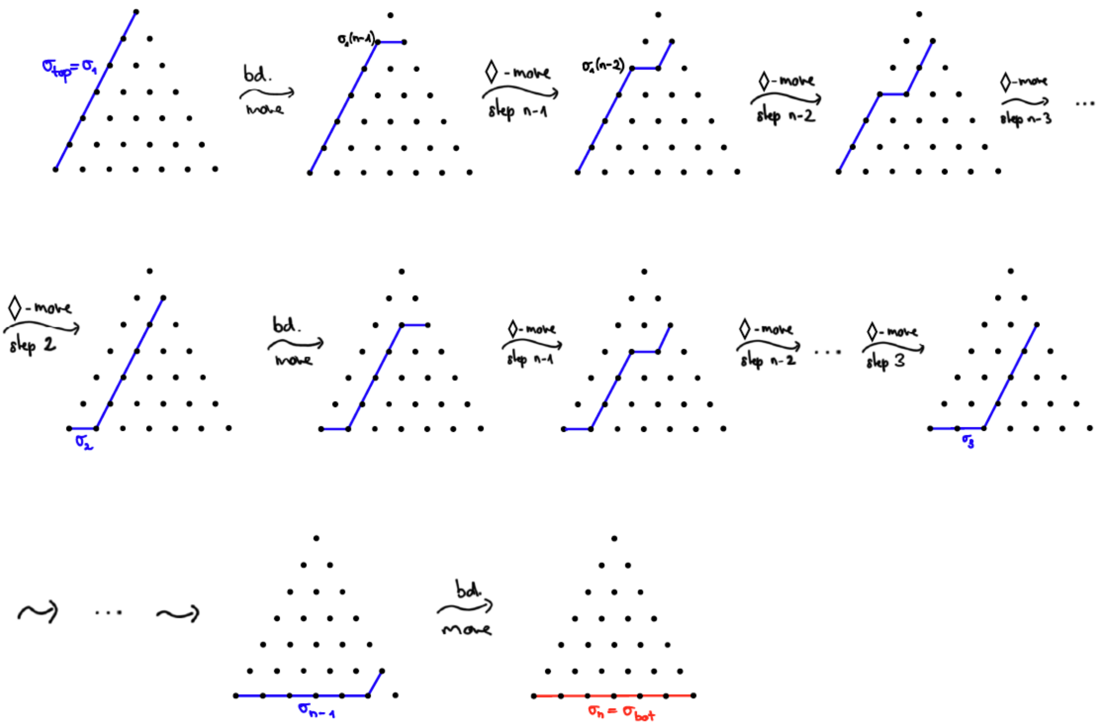

Figure A.7.: The sequence of snake moves from σ_{top} to σ_{bot}.

We make use of the following observation: If $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ are three bases of a vector space V then

$$T_{\mathcal{A}_1}^{\mathcal{A}_3} = T_{\mathcal{A}_1}^{\mathcal{A}_2} T_{\mathcal{A}_2}^{\mathcal{A}_3}.$$

Denote by $\mathcal{B}_k^*$ the bases associated to (E, F, G) of $(\mathbb{F}^n)^*$ by the snake σ_k for all $k = 1, \ldots, n$. Let us describe $T_{\mathcal{B}_1^*}^{\mathcal{B}_2^*}$. By the above remark, Proposition A.3.2 and Proposition A.3.4 we obtain the following, remembering the sequence of snake moves from σ_1 to σ_2:

$$\begin{aligned} T_{\mathcal{B}_1^*}^{\mathcal{B}_2^*} &= E_{n-1} \cdot (H_{n-1}(x_{1,n-2,1})E_{n-2}) \cdot (H_{n-2}(x_{2,n-3,1})E_{n-3}) \\ &\quad \cdot (H_{n-3}(x_{3,n-4,1})E_{n-4}) \cdot \ldots \cdot (H_2(x_{n-2,1,1})E_1) \\ &= E_{n-1} \left(\prod_{i=1}^{n-2} H_{n-i}(x_{i,n-i-1,1})E_{n-i-1} \right), \end{aligned}$$

where $x_{abc} = T_{abc}(E, F, G)$ denotes the (abc)-triple ratio for all integers $a, b, c \geq 1$ with $a + b + c = n$. Similarly, we obtain

$$\begin{aligned} T_{\mathcal{B}_2^*}^{\mathcal{B}_3^*} &= E_{n-1} \cdot (H_{n-1}(x_{1,n-3,2})E_{n-2}) \cdot (H_{n-2}(x_{2,n-4,2})E_{n-3}) \\ &\quad \cdot (H_{n-3}(x_{3,n-5,2})E_{n-4}) \cdot \ldots \cdot (H_3(x_{n-3,1,2})E_2) \\ &= E_{n-1} \left(\prod_{i=1}^{n-3} H_{n-i}(x_{i,n-i-2,2})E_{n-i-1} \right). \end{aligned}$$

More generally, from $\mathcal{B}_k^*$ to $\mathcal{B}_{k+1}^*$ the base change is given by

$$T_{\mathcal{B}_k^*}^{\mathcal{B}_{k+1}^*} = E_{n-1} \left(\prod_{i=1}^{n-k-1} H_{n-i}(x_{i,n-i-k,k})E_{n-i-1} \right).$$

Putting everything together, we obtain

$$T_{\mathcal{B}_{\mathrm{top}}^*}^{\mathcal{B}_{\mathrm{bot}}^*} = \prod_{k=1}^{n-1} T_{\mathcal{B}_k^*}^{\mathcal{B}_{k+1}^*} = \prod_{k=1}^{n-1} E_{n-1} \left(\prod_{i=1}^{n-k-1} H_{n-i}(x_{i,n-i-k,k})E_{n-i-1} \right).$$

□

Corollary A.4.3. Let σ_{top} and σ_{bot} be the top and bottom snakes in Θ_{n-1}. Let $\mathcal{B}_{\mathrm{top}}$ and $\mathcal{B}_{\mathrm{bot}}$ be the to $(E, F, G) \in \mathrm{Flag}(\mathbb{F}^n)^{(3)}$ by σ_{top} and σ_{bot} associated bases of $\mathbb{F}^n$. After normalizing such that $v_1^{\mathrm{top}} = v_1^{\mathrm{bot}}$, the base change matrix expressing $\mathcal{B}_{\mathrm{top}}$ in $\mathcal{B}_{\mathrm{bot}}$ is given by

$$T_{\mathcal{B}_{\mathrm{bot}}}^{\mathcal{B}_{\mathrm{top}}} = \left(T_{\mathcal{B}_{\mathrm{top}}^*}^{\mathcal{B}_{\mathrm{bot}}^*} \right)^{\top} = \prod_{k=1}^{n-1} \left(\left(\prod_{i=1}^{k-1} F_{n-k+i-1} H_{n-k+i}(x_{k-i,i,n-k}) \right) F_{n-1} \right), \tag{A.3}$$

where $x_{abc} = T_{abc}(E, F, G)$ denotes the (abc)-triple ratio for all integers $a, b, c \geq 1$ with $a + b + c = n$, and $F_k = E_k^{\top}$.

Proof. This follows from Proposition A.4.2 and the following general fact from linear algebra: If $\mathcal{A}$ and $\mathcal{A}'$ are two bases of a vector space V and $\mathcal{A}^*$ and $\mathcal{A}'^*$ their dual bases, then

$$T_{\mathcal{A}}^{\mathcal{A}'} = \left((T_{\mathcal{A}^*}^{\mathcal{A}'^*})^{-1} \right)^{\top} = \left(T_{\mathcal{A}'^*}^{\mathcal{A}^*} \right)^{\top}.$$

□

A.5. Proof that triple ratios parametrize

We are now ready to prove Theorem 4.1.3, which we recall here.

Theorem 4.1.3 ([FG06, Section 9], [Bon23, Theorem 4.1])**.** *Let $\mathbb{F}$ be a field, and (E,F,G), $(E',F',G') \in \mathrm{Flag}(\mathbb{F}^n)^{(3)}$ two triples of transverse flags. Then there exists $\varphi \in \mathrm{PGL}(\mathbb{F}^n)$ with $\varphi(E,F,G) = (E',F',G')$ (which is unique by Proposition 4.1.1) if and only if*

$$T_{abc}(E,F,G) = T_{abc}(E',F',G')$$

for all $a+b+c=n$, $a,b,c \in \mathbb{N}_{>0}$.

Furthermore, for all $(a,b,c) \in \mathbb{N}^3_{>0}$ such that $a+b+c=n$, pick $x_{abc} \in \mathbb{F} \setminus \{0\}$. Then there exists a triple of transverse flags (E,F,G) such that $T_{abc}(E,F,G) = x_{abc}$ for all such $(a,b,c) \in \mathbb{N}^3_{>0}$. Thus there is a one-to-one correspondence between

$$\mathrm{Conf}^{(3)}(\mathbb{F}) \longleftrightarrow (\mathbb{F} \setminus \{0\})^{\frac{(n-1)(n-2)}{2}}.$$

Proof. We only prove the first part of the statement. It is clear by the definition of the triple ratios, that if (E,F,G) and (E',F',G') are two triples of transverse flags that lie in the same $\mathrm{PGL}(\mathbb{F}^n)$-orbit, then their triple ratios agree. For the converse direction, recall that we would like to show that if (E,F,G) and (E',F',G') are two triples of transverse flags with the same triple ratios, then there exists a unique element in $\mathrm{PGL}(\mathbb{F}^n)$ that sends (E,F,G) to (E',F',G'). Let $\mathcal{B}_{\mathrm{top}}$, $\mathcal{B}_{\mathrm{bot}}$ and $\mathcal{B}'_{\mathrm{top}}$, $\mathcal{B}'_{\mathrm{bot}}$ be the bases of $\mathbb{F}^n$ associated to (E,F,G) respectively (E',F',G') by the top and bottom snakes σ_{top} and σ_{bot} in Θ_{n-1}, normalized such that $v_1^{\mathrm{top}} = v_1^{\mathrm{bot}} = (v_1')^{\mathrm{top}} = (v_1')^{\mathrm{bot}}$. Let $\varphi \in \mathrm{GL}(\mathbb{F}^n)$ be a linear automorphism that sends $\mathcal{B}_{\mathrm{top}}$ to $\mathcal{B}'_{\mathrm{top}}$. Since (E,F,G) and (E',F',G') have the same triple ratios, the base change matrix from $\mathcal{B}_{\mathrm{bot}}$ to $\mathcal{B}_{\mathrm{top}}$ is equal to the base change matrix from $\mathcal{B}'_{\mathrm{bot}}$ to $\mathcal{B}'_{\mathrm{top}}$, see Corollary A.4.3: the base change is completely determined by the triple ratios. This implies that φ maps $\mathcal{B}_{\mathrm{bot}}$ to $\mathcal{B}'_{\mathrm{bot}}$. By Lemma A.4.1, φ maps (E,F,G) to (E',F',G'). Let $\psi \in \mathrm{GL}(\mathbb{F}^n)$ be another linear automorphism of $\mathbb{F}^n$ with the property that it maps (E,F,G) to (E',F',G'), then $\psi^{-1} \circ \varphi$ stabilizes the transverse triple (E,F,G). By Proposition 4.1.1, φ and ψ define the same element of $\mathrm{PGL}(\mathbb{F}^n)$, which proves the claim. □

Appendix B.

Description of the base change by triple and double ratios

B.1. Associated bases and their base changes

Let (E, F, G) be a transverse triple of flags in $\mathbb{F}^n$. We have seen how we can associate to (E, F, G) two bases of $(\mathbb{F}^n)^*$ (and hence, by taking their duals, of $\mathbb{F}^n$) by the top and bottom snakes. It turns out, that if we consider this triple only up to cyclic permutation, we can in fact, associate to (E, F, G) six bases of $\mathbb{F}^n$ by the top and bottom snakes: Namely, consider the two bases associated to the triple (F, G, E) and to the triple (G, E, F) by the top and bottom snakes.

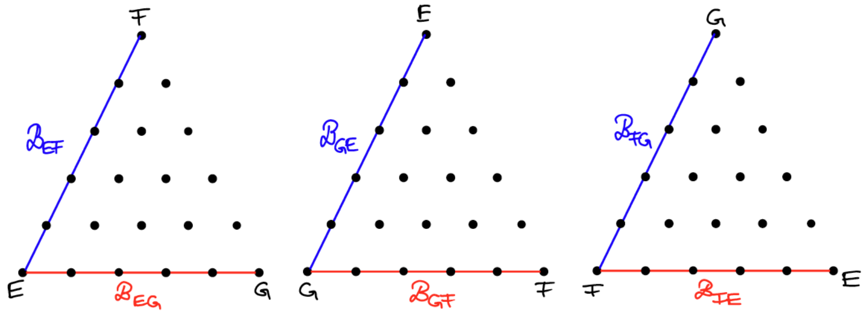

Figure B.1.: The six bases associated to a cyclically ordered triple of transverse flags (E, F, G).

As in the above figure, we denote them by

$$\mathcal{B}_{EF},\ \mathcal{B}_{EG},\ \mathcal{B}_{GE},\ \mathcal{B}_{GF},\ \mathcal{B}_{FG} \text{ and } \mathcal{B}_{FE}.$$

If it is not clear from the context what is the third flag G that determines the basis corresponding to the line decomposition given by E and F, we sometimes write $\mathcal{B}_{EF,G}$ instead of $\mathcal{B}_{EF}$. Their dual bases will be decorated with an $*$. With this new notation, we can give an interpretation of the matrices defined in Definition 4.3.2 and Definition 4.3.3 as base change matrices between the above bases.

Proposition B.1.1 ([FG06, Proposition 9.2])**.** Let (E, F, G) be a transverse triple of flags in $\mathbb{F}^n$, and $\mathcal{B}_{EF}$, $\mathcal{B}_{EG}$, $\mathcal{B}_{GE}$, $\mathcal{B}_{GF}$, $\mathcal{B}_{FG}$ and $\mathcal{B}_{FE}$ the six projective bases of $\mathbb{F}^n$ associated to it by the top and bottom snakes. Then we have the following expressions for the base change matrices:

(1) $T_{\mathcal{B}_{EG}}^{\mathcal{B}_{EF}} = \prod_{k=1}^{n-1}\left(\left(\prod_{i=1}^{k-1} F_{n-k+i-1}H_{n-k+i}(x_{k-i,i,n-k})\right)F_{n-1}\right) = M_{(E,F,G)}$,

(2) $T_{\mathcal{B}_{FE}}^{\mathcal{B}_{EF}} = \begin{pmatrix} & & & & (-1)^{n-1} \\ & & & \iddots & \\ & & -1 & & \\ & 1 & & & \\ -1 & & & & \\ 1 & & & & \end{pmatrix} = S$,

(3) $T_{\mathcal{B}_{FE}}^{\mathcal{B}_{GE}} = SM_{(E,F,G)}^{-1}S^{-1}$.

Proof.

(1) This is Corollary A.4.3.

(2) Choose a basis $e_1, \ldots, e_n$ of $\mathbb{F}^n$ such that for all $i = 0, \ldots, n$

$$E^{(i)} = \langle e_1, \ldots, e_i\rangle,\ F^{(i)} = \langle e_n, \ldots, e_{n-i+1}\rangle,\ G^{(1)} = \langle e_1 + \ldots + e_n\rangle.$$

Denote by $e_1^*, \ldots, e_n^*$ its dual basis. After normalizing so that $u_1 = e_n^*$, we know by Example A.2.4 that

$$u_i = (-1)^{i-1}e_{n-i+1}^* \text{ for all } i = 1, \ldots, n.$$

On the other hand, the vectors $U_1, \ldots, U_n \in \mathcal{B}_{FE}^*$ are defined by the condition $U_i = -U_i^{\mathrm{rt}} \in \langle e_{n-i+1}^*\rangle$ with $U_{i-1} + U_i^{\mathrm{up}} + U_i^{\mathrm{rt}} = 0$ and $U_i^{\mathrm{up}} \in \langle e_{n-i+1}^* - e_{n-i+2}^*\rangle$. Set $U_i = M_i e_{n-i+1}^*$, and $U_i^{\mathrm{up}} = N_i(e_{n-i+1}^* - e_{n-i+2}^*)$. By the same argument as in Example A.2.4 we obtain, using $M_1 = 1$ after normalizing such that $u_1 = U_1$,

$$U_i = M_i e_{n-i+1}^* = e_{n-i+1}^* \text{ for all } i = 1, \ldots, n.$$

Thus we have $T_{\mathcal{B}_{FE}^*}^{\mathcal{B}_{EF}^*} = \begin{pmatrix} & & & & (-1)^{n-1} \\ & & & \iddots & \\ & & -1 & & \\ & 1 & & & \\ -1 & & & & \\ 1 & & & & \end{pmatrix} = S$, and hence

$$T_{\mathcal{B}_{FE}}^{\mathcal{B}_{EF}} = \left(T_{\mathcal{B}_{FE}^*}^{\mathcal{B}_{EF}^*}\right)^{-1} = (S^{-1})^\top = S.$$

(3) This follows from linear algebra and (1) and (2), since

$$T_{\mathcal{B}_{FE}}^{\mathcal{B}_{GE}} = T_{\mathcal{B}_{FE}}^{\mathcal{B}_{EF}}\, T_{\mathcal{B}_{EF}}^{\mathcal{B}_{EG}}\, T_{\mathcal{B}_{EG}}^{\mathcal{B}_{GE}} = SM_{(E,F,G)}^{-1}S^{-1}.$$

□

B.2. Double ratios and base changes

We now turn to the question how double ratios and the associated bases defined in Definition A.2.3 relate. The proof is adapted from [Mar19a, Proposition 2.17]. We recall and reprove it here to keep the notation consistent.

Let (E, F, G, H) be a transverse quadruple of flags in $\mathbb{F}^n$. The subtriples (E, F, G) and (E, G, H) determine two bases $\mathcal{B}^*_{EG,F} = \{u_1, \ldots, u_n\}$ (associated to (E, F, G) by σ_{bot}) and $\mathcal{B}^*_{EG,H} = \{U_1, \ldots, U_n\}$ (associated to (E, G, H) by σ_{top}). Both correspond to the line decomposition $L_k = (E^{(n-k)} + G^{(k-1)})^\perp$ of $(\mathbb{F}^n)^*$.

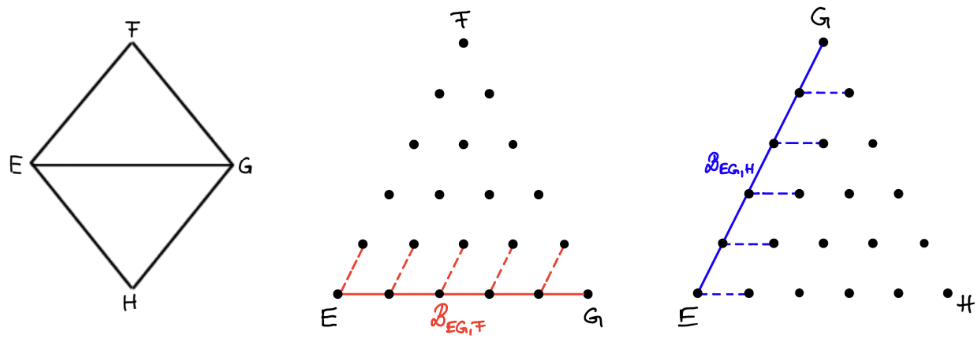

Figure B.2.: The subtriples (E, F, G) and (E, G, H) of the transverse quadruple (E, F, G, H).

Proposition B.2.1 ([FG06, Lemma 9.3])**.** Let (E, F, G, H) be a transverse quadruple of flags in $\mathbb{F}^n$ and $\mathcal{B}^*_{EG,F} = \{u_1, \ldots, u_n\}$ and $\mathcal{B}^*_{EG,H} = \{U_1, \ldots, U_n\}$ as above. After normalizing such that $u_1 = U_1$, we have for all $1 < i \leq n$

$$U_i = \frac{1}{d_{n-i+1} \cdot \ldots \cdot d_{n-1}} u_i,$$

where $d_k := D_k(E, G, F, H)$ is the k-th double ratio of (E, G, F, H) for all $k = 1, \ldots, n-1$. In particular, the base change matrix expressing $\mathcal{B}^*_{EG,H}$ in the basis $\mathcal{B}^*_{EG,F}$ is diagonal, namely

$$\begin{aligned} T^{\mathcal{B}^*_{EG,H}}_{\mathcal{B}^*_{EG,F}} &= \operatorname{diag}\left(1, \tfrac{1}{d_{n-1}}, \tfrac{1}{d_{n-2}d_{n-1}}, \ldots, \tfrac{1}{d_{n-i+1} \cdot \ldots \cdot d_{n-1}}, \ldots, \tfrac{1}{d_1 \cdot \ldots \cdot d_{n-1}}\right) \\ &= \begin{pmatrix} 1 & & & & \\ & \frac{1}{d_{n-1}} & & & \\ & & \frac{1}{d_{n-2}d_{n-1}} & & \\ & & & \ddots & \\ & & & & \frac{1}{d_1 \cdot \ldots \cdot d_{n-1}} \end{pmatrix}. \end{aligned}$$

Proof. Choose a basis $e_1, \ldots, e_n$ of $\mathbb{F}^n$ such that for all $i = 0, \ldots, n$

$$E^{(i)} = \langle e_1, \ldots, e_i \rangle, \; G^{(i)} = \langle e_n, \ldots, e_{n-i+1} \rangle, \; F^{(1)} = \langle e_1 + \ldots + e_n \rangle.$$

Denote by $e_1^*, \ldots, e_n^*$ its dual basis. Pick $0 \neq h \in H^{(1)}$ and write $h = \sum_{j=1}^n h_j e_j$ with $0 \neq h_j \in \mathbb{F}$ for all $j = 1, \ldots, n$. Recall that $D_i(E, G, F, H)$ depends only on the flags E and G, and the one-dimensional subspaces $F^{(1)}$ and $H^{(1)}$ of the flags F respectively H.

We compute

$$
\begin{aligned}
d_i &= D_i(E, G, F, H) \\
&= -\frac{e_1 \wedge \ldots \wedge e_i \wedge e_n \wedge \ldots \wedge e_{i+2} \wedge (\sum_{j=1}^n e_j)}{e_1 \wedge \ldots \wedge e_i \wedge e_n \wedge \ldots \wedge e_{i+2} \wedge (\sum_{j=1}^n h_j e_j)} \\
&\quad \cdot \frac{e_1 \wedge \ldots \wedge e_{i-1} \wedge e_n \wedge \ldots \wedge e_{i+1} \wedge (\sum_{j=1}^n h_j e_j)}{e_1 \wedge \ldots \wedge e_{i-1} \wedge e_n \wedge \ldots \wedge e_{i+1} \wedge (\sum_{j=1}^n e_j)} = -\frac{h_i}{h_{i+1}}.
\end{aligned}
$$

Similar to Example A.2.4 (by swapping the roles of F and G), we have that

$$
\begin{aligned}
&\left(E^{(n-i)} + G^{(i-1)}\right)^\perp = \langle e_1, \ldots, e_{n-i}, e_n, \ldots, e_{n-i+2}\rangle^\perp = \langle e^*_{n-i+1}\rangle, \\
&\left(E^{(n-i)} + F^{(1)} + G^{(i-2)}\right)^\perp = \langle e_1, \ldots, e_{n-i}, \sum_{j=1}^n e_j, e_n, \ldots, e_{n-i+3}\rangle^\perp \\
&\quad = \langle e^*_{n-i+1} - e^*_{n-i+2}\rangle, \text{ and} \\
&\left(E^{(n-i)} + G^{(i-2)} + H^{(1)}\right)^\perp = \langle e_1, \ldots, e_{n-i}, e_n, \ldots, e_{n-i+3}, \sum_{j=1}^n h_j e_j, \rangle^\perp = \langle u\rangle,
\end{aligned}
$$

with $u \in \langle e^*_{n-i+1}, e^*_{n-i+2}\rangle$ and $h \in \ker(u)$. Let $u = e^*_{n-i+1} + \mu e^*_{n-i+2}$ for some $\mu \in \mathbb{F}$. We then have $0 = u(h) = (e^*_{n-i+1} + \mu e^*_{n-i+2})(\sum_{j=1}^n h_j e_j) = h_{n-i+1} + \mu h_{n-i+2}$, which implies that $\mu = -\frac{h_{n-i+1}}{h_{n-i+2}} = d_{n-i+1}$, and thus

$$
u \in \langle e^*_{n-i+1} + d_{n-i+1} e^*_{n-i+2}\rangle.
$$

Recall that the vectors $u_1, \ldots, u_n$ and $U_1, \ldots, U_n$ are defined recursively. We normalize such that $u_1 = e^*_n$. Since $\mathcal{B}^*_{EG,F}$ is associated to (E, F, G) by σ_{bot} (see Figure B.2) we have $u_i = -u_i^{\text{rt}} \in \langle e^*_{n-i+1}\rangle$ with $u_{i-1} + u_i^{\text{up}} + u_i^{\text{rt}} = 0$ and $u_i^{\text{up}} \in \langle e^*_{n-i+1} - e^*_{n-i+2}\rangle$. Set $u_i = \mu_i e^*_{n-i+1}$, and $u_i^{\text{up}} = \nu_i(e^*_{n-i+1} - e^*_{n-i+2})$. Putting it together we obtain

$$
0 = \mu_{i-1} e^*_{n-i+2} + \nu_i(e^*_{n-i+1} - e^*_{n-i+2}) - \mu_i e^*_{n-i+1} \implies \mu_i = \nu_i = \mu_{i-1},
$$

and thus, using $\mu_1 = 1$,

$$
u_i = \mu_i e^*_{n-i+1} = e^*_{n-i+1} \text{ for all } i = 1, \ldots, n.
$$

Similarly $\mathcal{B}^*_{EG,H}$ is associated to (E, G, H) by σ_{top} (see Figure B.2), hence $U_i = U_i^{\text{up}} \in \langle e^*_{n-i+1}\rangle$ with $U_{i-1} + U_i^{\text{up}} + U_i^{\text{rt}} = 0$ and $U_i^{\text{rt}} \in \langle u\rangle = \langle e^*_{n-i+1} + d_{n-i+1} e^*_{n-i+2}\rangle$. Set $U_i = M_i e^*_{n-i+1}$, and $U_i^{\text{rt}} = N_i(e^*_{n-i+1} + d_{n-i+1} e^*_{n-i+2})$. Again we obtain

$$
\begin{aligned}
0 &= M_{i-1} e^*_{n-i+2} + M_i e^*_{n-i+1} + N_i(e^*_{n-i+1} + d_{n-i+1} e^*_{n-i+2}) \\
&\implies M_i = -N_i = \frac{M_{i-1}}{d_{n-i+1}},
\end{aligned}
$$

and thus, using $M_1 = 1$,

$$
\begin{aligned}
U_i &= M_i e^*_{n-i+1} = \frac{M_{i-1}}{d_{n-i+1}} e^*_{n-i+1} \\
&= \frac{1}{d_{n-i+1} \cdot \ldots \cdot d_{n-1}} e^*_{n-i+1} = \frac{1}{d_{n-i+1} \cdot \ldots \cdot d_{n-1}} u_i
\end{aligned}
$$

for all $i = 1, \ldots, n$, which is what we had to prove. $\square$

Note that this result together with the remark in the proof of Corollary A.4.3 implies that the base change matrix is

$$T_{\mathcal{B}_{EG,F}}^{\mathcal{B}_{EG,H}} = \operatorname{diag}(1, d_{n-1}, d_{n-2}d_{n-1}, \ldots, d_1 \cdot \ldots \cdot d_{n-1}) = D_{(E,F,G,H)},$$

where the latter was defined in Definition 4.3.3.

B.3. Proof of the description of the base change

In this appendix we prove Theorem 4.3.4, which we recall here.

Theorem 4.3.4 ([FG06, Proposition 9.2])**.** *Let $(F_1, \ldots, F_k) \in \operatorname{Flag}(\mathbb{F}^n)^{(k)}$ be a transverse k-tuple of flags associated to a polygon P with k vertices $x_1, \ldots, x_k$ in clockwise order around the polygon. Assume either that there exists $4 \leq j \leq k-2$ so that (F_1, F_2, F_3) and*

(1) (F_{j+2}, F_j, F_{j+1}), or

(2) (F_{j+1}, F_{j+2}, F_j)

have the same triple ratios, and let $\varphi \in \operatorname{PGL}(\mathbb{F}^n)$ be the element that maps (F_1, F_2, F_3) to the corresponding triple. Then, in the respective cases, there exists a basis of $\mathbb{F}^n$ in which φ is represented by

(1) $M_\varphi := \left(\prod_{i=1}^{k-j-1} D_i M_i\right)\left(\prod_{i=k-j+1}^{k-3} D_i S M_i^{-1} S^{-1}\right)$, or

(2) $M_\varphi := M_0 \left(\prod_{i=1}^{k-j-1} D_i M_i\right)\left(\prod_{i=k-j+1}^{k-4} D_i S M_i^{-1} S^{-1}\right) D_{k-3}$,

where the D_i and M_i are defined as above associated to the ideal triangulation $\mathcal{E}_j$ as described before.

For the definition of the matrices D_i, M_i and S we refer to Section 4.3. The idea of the proof comes from the following observation in the proof of Theorem 4.1.3 in Appendix A.5, which showed us how to find an explicit matrix in $\operatorname{GL}(n, \mathbb{F})$ representing the linear automorphism φ that maps (E, F, G) to (E', F', G'). Indeed, if $\mathcal{A}$ and $\mathcal{A}$ are two bases of an n-dimensional $\mathbb{F}$-vector space V, and $\psi \in \operatorname{GL}(V)$ is such that $\psi(\mathcal{A}) = \mathcal{A}'$, then

$$M_{\mathcal{A}}^{\mathcal{A}}(\psi) = T_{\mathcal{A}}^{\mathcal{A}'},$$

where $M_{\mathcal{A}}^{\mathcal{A}}(\psi) \in \operatorname{GL}(n, \mathbb{F})$ denotes the matrix representing the linear isomorphism ψ in the basis $\mathcal{A}$. Recall that if (E, F, G) and (E', F', G') have the same triple ratios, $\varphi\colon (E, F, G) \mapsto (E', F', G')$ is the automorphism of $\mathbb{F}^n$ with the property

$$\varphi(\mathcal{B}_{\text{top}}) = \mathcal{B}'_{\text{top}},$$

where $\mathcal{B}_{\text{top}}$ and $\mathcal{B}'_{\text{top}}$ denote the bases of $\mathbb{F}^n$ associated to (E, F, G) respectively (E', F', G') by the top snake σ_{top} in Θ_{n-1}. By the above remark, we obtain

$$M_{\mathcal{B}_{\text{top}}}^{\mathcal{B}_{\text{top}}}(\varphi) = T_{\mathcal{B}_{\text{top}}}^{\mathcal{B}'_{\text{top}}}.$$

We would like to give an explicit description of $T_{\mathcal{B}_{\text{top}}}^{\mathcal{B}'_{\text{top}}}$, which involves the triple and double ratios of all the subtriples and -quadruples of $(F_1, \ldots, F_k)$ singled out by the triangulation $\mathcal{E}$ of the polygon P. Similarly, we could also consider $T_{\mathcal{B}_{\text{bot}}}^{\mathcal{B}'_{\text{bot}}}$.

Proof of Theorem 4.3.4. We use the triangulation $\mathcal{E} = \mathcal{E}_j$ from Section 4.3 to describe the base change matrix from the basis $\mathcal{B} := \mathcal{B}_{F_3F_1,F_2}$ to the basis $\mathcal{B}' := \mathcal{B}_{F_{j+1}F_{j+2},F_j}$ in case (1). By the above remarks this matrix sends (F_1, F_2, F_3) to (F_{j+2}, F_j, F_{j+1}), and we will show that it is equal to the desired product.

Every directed diagonal e_i, $i = 1, \ldots, k-3$, in the triangulation $\mathcal{E}$ defines two bases $\mathcal{B}_i^r$ and $\mathcal{B}_i^l$ of $\mathbb{F}^n$ (corresponding to the line decomposition given by the flags at the endpoints of e_i), which are associated to the triple of flags of the triangle lying to the right and to the left of the diagonal. Observe that $\mathcal{B} = \mathcal{B}_1^r$ and set $\mathcal{B}_{k-2}^r := \mathcal{B}'$. We obtain

$$T_{\mathcal{B}}^{\mathcal{B}'} = \prod_{i=1}^{k-3} T_{\mathcal{B}_i^r}^{\mathcal{B}_i^l} T_{\mathcal{B}_i^l}^{\mathcal{B}_{i+1}^r}.$$

From Proposition B.1.1(1) and (3) it follows that

$$T_{\mathcal{B}_i^l}^{\mathcal{B}_{i+1}^r} = \begin{cases} M_i, & \text{if } 1 \leq i \leq k-j-1, \\ SM_iS^{-1}, & \text{if } k-j \leq i \leq k-3, \end{cases}$$

for all $i = 1, \ldots, k-3$. From Proposition B.2.1 and a close observation of the definition of quadruple of flags associated to an oriented edge, it follows that

$$T_{\mathcal{B}_i^l}^{\mathcal{B}_i^r} = D_i$$

for all $i = 1, \ldots, k-3$, which proves the claim.

For (2), we describe the base change matrix from the basis $\mathcal{B} := \mathcal{B}_{F_3F_2,F_1}$ to the basis $\mathcal{B}' := \mathcal{B}_{F_jF_{j+2},F_{j+1}}$. We observe that

$$T_{\mathcal{B}}^{\mathcal{B}'} = T_{\mathcal{B}}^{\mathcal{B}_1^r} \left(\prod_{i=1}^{k-4} T_{\mathcal{B}_i^r}^{\mathcal{B}_i^l} T_{\mathcal{B}_i^l}^{\mathcal{B}_{i+1}^r} \right) T_{\mathcal{B}_{k-3}^r}^{\mathcal{B}_{k-3}^l} = M_\varphi,$$

by the same arguments as before. □

B.4. Total positivity of the base change

We prove that the matrices M_φ defined in the last theorem are totally positive. This will immediately imply Theorem 5.2.3, which we recall here.

Theorem 5.2.3 ([FG06, Theorem 9.3])**.** *Under the hypotheses and conclusions of Theorem 4.3.4, if we additionally assume that $(F_1, \ldots, F_k) \in \mathrm{Flag}(\mathbb{F}^n)^{(k,+)}$ is a positive k-tuple of flags, then M_φ is a totally positive matrix.*

We need some preliminary results about totally positive matrices. We begin by recalling *Cauchy-Binet's formula*, see for example [Pin10, Chapter 1]. Let B be an

$n \times n$-matrix over any field. For $1 \le i_1 < \ldots < i_p \le n$ and $1 \le j_1 < \ldots < j_p \le n$ denote by

$$B\begin{pmatrix} i_1, \ldots, i_p \\ j_1, \ldots, j_p \end{pmatrix}$$

the $p \times p$-minor of B obtained by taking the determinant of the submatrix of B with rows $i_1, \ldots, i_p$ and columns $j_1, \ldots, j_p$. If B, C, D are $n \times n$-matrices over any field and $B = CD$, then Cauchy-Binet's formula states that

$$B\begin{pmatrix} i_1, \ldots, i_p \\ j_1, \ldots, j_p \end{pmatrix} = \sum_{1 \le k_1 < \ldots < k_p \le n} C\begin{pmatrix} i_1, \ldots, i_p \\ k_1, \ldots, k_p \end{pmatrix} D\begin{pmatrix} k_1, \ldots, k_p \\ j_1, \ldots, j_p \end{pmatrix}.$$

If B is invertible, we have

$$B^{-1}\begin{pmatrix} i_1, \ldots, i_p \\ j_1, \ldots, j_p \end{pmatrix} = \frac{(-1)^{\sum_{k=1}^p i_k + j_k}}{\det B} B\begin{pmatrix} j'_1, \ldots, j'_{n-p} \\ i'_1, \ldots, i'_{n-p} \end{pmatrix},$$

where $i'_1 < \ldots < i'_{n-p}$ and $j'_1 < \ldots < j'_{n-p}$ are so that $\{i_1, \ldots, i_p\} \cup \{i'_1, \ldots, i'_{n-p}\} = \{1, \ldots, n\}$ and $\{j_1, \ldots, j_p\} \cup \{j'_1, \ldots, j'_{n-p}\} = \{1, \ldots, n\}$.

The following lemma about products of totally positive/non-negative (upper/lower triangular) matrices (see Definition 5.2.1) is a direct application of this formula and is classical.

Corollary B.4.1. Let $\mathbb{F}$ be an ordered field.

(1) The product of a totally positive matrix with a totally non-negative matrix is totally positive. The same holds true if we restrict to the set of upper or lower triangular matrices.

(2) The product of a totally positive lower triangular matrix with a totally positive upper triangular matrix is totally positive.

Proposition B.4.2. If $(E, F, G) \in \mathrm{Flag}(\mathbb{F}^n)^{(3,+)}$ then $M_{(E,F,G)}$ is totally positive lower triangular, and $SM^{-1}_{(E,F,G)}S^{-1}$ is totally positive upper triangular.

The second part of the proposition follows from the first part together with the following more general statement.

Lemma B.4.3. Let $\mathbb{F}$ be an ordered field. Let $M \in \mathrm{GL}(n, \mathbb{F})$ be a totally positive lower triangular matrix. Then $SM^{-1}S^{-1}$ is totally positive upper triangular.

Proof. It is clear that $SM^{-1}S^{-1}$ is upper triangular. Note that $S^{-1} = (-1)^{n-1}S$. We make use of Cauchy-Binet's formula, the shape of S and M, and expressing the minors of M^{-1} in terms of the minors of M. By [Pin10, Theorem 2.8] we only need to prove that

$$(SM^{-1}S^{-1})\begin{pmatrix} 1, \ldots, k \\ j+1, \ldots, j+k \end{pmatrix} > 0$$

for all $j = 0, \dots, n-k$ and $k = 1, \dots, n$. We compute

$$\begin{aligned}
&(SM^{-1}S^{-1}) \begin{pmatrix} 1, \dots, k \\ j+1, \dots, j+k \end{pmatrix} \\
&= (-1)^{(n-1)k} \cdot S \begin{pmatrix} 1, \dots, k \\ n-k+1, \dots, n \end{pmatrix} \cdot M^{-1} \begin{pmatrix} n-k+1, \dots, n \\ n-(j+k)+1, \dots, n-j \end{pmatrix} \\
&\quad \cdot S \begin{pmatrix} n-(j+k)+1, \dots, n-j \\ j+1, \dots, j+k \end{pmatrix} \\
&= \frac{1}{\det M} \cdot M \begin{pmatrix} 1, \dots, n-(j+k), n-j+1, \dots, n \\ 1, \dots, n-k \end{pmatrix} > 0,
\end{aligned}$$

since M was assumed to be totally positive lower triangular, and $j \leq n-k$. □

Proof of Proposition B.4.2. We recall the definition of

$$M := M_{(E,F,G)} = \prod_{k=1}^{n-1} \left(\left(\prod_{i=1}^{k-1} F_{n-k+i-1} H_{n-k+i}(x_{k-i,i,n-k}) \right) F_{n-1} \right).$$

To simplify notation we set

$$B_k := \left(\prod_{i=1}^{k-1} F_{n-k+i-1} H_{n-k+i}(x_{k-i,i,n-k}) \right) F_{n-1}.$$

Then $M = B_1 \cdot \ldots \cdot B_{n-1}$. We investigate the structure of B_k more closely, and we see that for all $k = 1, \dots, n-1$

$$B_k = \left(\begin{array}{c|cccccc} \mathrm{Id}_{n-(k+1)} & & & & & & \\ \hline & 1 & & & & & \\ & 1 & 1 & & & & \\ & & x_1 & x_1 & & & \\ & & & x_1 x_2 & x_1 x_2 & & \\ & & & & \ddots & \ddots & \\ & & & & & \prod_{j=1}^{k-1} x_j & \prod_{j=1}^{k-1} x_j \end{array} \right)$$

for some positive elements $x_1, \dots, x_{k-1} \in \mathbb{F}$ (the x_i's that appear in the B_k's are exactly the triple ratios of (E, F, G) but we only care about their positivity and not their exact value, which is why for simplicity we omit the indices). It is clear that M is lower triangular totally non-negative, since all B_k's are. To check total positivity of M it suffices to verify that

$$M \begin{pmatrix} n-j+1, \dots, n \\ 1, \dots, j \end{pmatrix} > 0 \text{ for all } j = 1, \dots, n$$

by [Pin10, Proposition 2.9] (beware of different notions in the reference: what we call total non-negativity in this thesis is called total positivity, and what we call total positivity is called strict total positivity). We again use Cauchy-Binet's formula and the shape of the B_k's. Let us begin with the case $j = 1$ for simplicity:

$$\begin{aligned} M\binom{n}{1} &= \sum_{1\le j\le n} (B_1 \cdot \ldots \cdot B_{n-2}) \binom{n}{j} B_{n-1} \binom{j}{1} \\ &= (B_1 \cdot \ldots \cdot B_{n-2}) \binom{n}{2} B_{n-1} \binom{2}{1} \\ &= B_1 \binom{n}{n-1} B_2 \binom{n-1}{n-2} \cdot \ldots \cdot B_{n-2} \binom{3}{2} B_{n-1} \binom{2}{1} > 0 \end{aligned}$$

by the definition of the B_k's and the fact that all x_i are positive. For $1 \le j \le n$ the general formula is given by

$$\begin{aligned} M\begin{pmatrix} n-j+1,\ldots,n \\ 1,\ldots,j \end{pmatrix} &= (B_1 \cdot \ldots \cdot B_{j-1}) \begin{pmatrix} n-j+1,\ldots,n \\ n-j+1,\ldots,n \end{pmatrix} \\ &\cdot B_j \begin{pmatrix} n-j+1,\ldots,n \\ n-j,\ldots,n-1 \end{pmatrix} B_{j+1} \begin{pmatrix} n-j,\ldots,n-1 \\ n-j-1,\ldots,n-2 \end{pmatrix} \\ &\cdot \ldots \cdot B_{n-2} \begin{pmatrix} 3,\ldots,j+2 \\ 2,\ldots,j+1 \end{pmatrix} B_{n-1} \begin{pmatrix} 2,\ldots,j+1 \\ 1,\ldots,j \end{pmatrix}. \end{aligned}$$

Since all of the above factors are determinants of triangular matrices with positive entries on the diagonal, we obtain for all $j = 1, \ldots, n$ that

$$M\begin{pmatrix} n-j+1,\ldots,n \\ 1,\ldots,j \end{pmatrix} > 0,$$

which is what we had to prove. □

Proof of Theorem 5.2.3. In the first case we are in the situation of Theorem 4.3.4 (1). Using the definitions of D_i, M_i and S as before, we obtain that

$$M_\varphi := \left(\prod_{i=1}^{k-j-1} D_i M_i \right) \left(\prod_{i=k-j}^{k-3} D_i (S M_i^{-1} S^{-1}) \right)$$

is a matrix representing $\varphi \in \mathrm{GL}(\mathbb{F}^n)$ that satisfies $\varphi(F_1, F_2, F_3) = (F_{j+2}, F_j, F_{j+1})$. Since $(F_1, \ldots, F_k)$ is by assumption a positive k-tuple of flags, the triple and double ratios of all subtriples respectively -quadruples of flags are positive (see the discussion after Definition 5.1.3)—in particular those, that we obtain from the triangulation $\mathcal{E}_j$ defined before Theorem 4.3.4.

Proposition B.4.2 then implies that M_i is totally positive lower triangular and $S M_i^{-1} S^{-1}$ is totally positive upper triangular for all $i = 1, \ldots, k-3$. Furthermore, the matrices D_i are diagonal with only positive entries on the diagonal, since all double ratios are assumed to be positive, and conjugation by S^{-1} preserves positivity. Thus, by Corollary B.4.1 (1), $D_i M_i$ and $D_i(S M_i^{-1} S^{-1})$ are totally positive lower respectively upper triangular for all $i = 1, \ldots, k-3$. The assumption $4 \le j \le k-2$ assures that both of the above products in M_φ are non-empty. Corollary B.4.1 (2) implies that M_φ is totally positive.

In the second case, we apply Theorem 4.3.4 (2), and obtain the result by the same arguments as before. □

Bibliography

[BCGP$^+$21] Steve Bradlow, Brian Collier, Oscar Garcia-Prada, Peter Gothen, and André Oliveira. A general Cayley correspondence and higher Teichmüller spaces, 2021. arXiv:2101.09377.

[BCR98] Jacek Bochnak, Michel Coste, and Marie-Françoise Roy. *Real algebraic geometry*, volume 36 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1998. Translated from the 1987 French original, Revised by the authors.

[BD14] Francis Bonahon and Guillaume Dreyer. Parameterizing Hitchin components. *Duke Math. J.*, 163(15):2935–2975, 2014.

[BD17] Francis Bonahon and Guillaume Dreyer. Hitchin characters and geodesic laminations. *Acta Math.*, 218(2):201–295, 2017.

[Bes88] Mladen Bestvina. Degenerations of the hyperbolic space. *Duke Math. J.*, 56(1):143–161, 1988.

[BIPP21a] Marc Burger, Alessandra Iozzi, Anne Parreau, and Maria Beatrice Pozzetti. Positive crossratios, barycenters, trees and applications to maximal representations, 2021. arXiv:2103.17161.

[BIPP21b] Marc Burger, Alessandra Iozzi, Anne Parreau, and Maria Beatrice Pozzetti. The real spectrum compactification of character varieties: characterizations and applications. *C. R. Math. Acad. Sci. Paris*, 359:439–463, 2021.

[BIPP23] Marc Burger, Alessandra Iozzi, Anne Parreau, and Maria Beatrice Pozzetti. The real spectrum compactification of character varieties, 2023. In preparation.

[BIW03] Marc Burger, Alessandra Iozzi, and Anna Wienhard. Surface group representations with maximal Toledo invariant. *C. R. Math. Acad. Sci. Paris*, 336(5):387–390, 2003.

[Bon86] Francis Bonahon. Bouts des variétés hyperboliques de dimension 3. *Ann. of Math. (2)*, 124(1):71–158, 1986.

[Bon88] Francis Bonahon. The geometry of Teichmüller space via geodesic currents. *Invent. Math.*, 92(1):139–162, 1988.

[Bon96] Francis Bonahon. Shearing hyperbolic surfaces, bending pleated surfaces and Thurston's symplectic form. *Ann. Fac. Sci. Toulouse Math. (6)*, 5(2):233–297, 1996.

[Bon01] Francis Bonahon. Geodesic laminations on surfaces. In *Laminations and foliations in dynamics, geometry and topology (Stony Brook, NY, 1998)*, volume 269 of *Contemp. Math.*, pages 1–37. Amer. Math. Soc., Providence, RI, 2001.

[Bon23] Francis Bonahon. Notes on Fock-Goncharov coordinates, 2023. In preparation.

[Bou98] Nicolas Bourbaki. *Commutative algebra. Chapters 1–7*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1998. Translated from the French, Reprint of the 1989 English translation.

[Bou12] Nicolas Bourbaki. *Éléments de mathématique. Algèbre. Chapitre 8. Modules et anneaux semi-simples*. Springer, Berlin, 2012. Second revised edition of the 1958 edition.

[BP21] Jonas Beyrer and Maria Beatrice Pozzetti. Positive surface group representations in $\mathrm{PO}(p, q)$, 2021. arXiv:2106.14725.

[BPR06] Saugata Basu, Richard Pollack, and Marie-Françoise Roy. *Algorithms in real algebraic geometry*, volume 10 of *Algorithms and Computation in Mathematics*. Springer-Verlag, Berlin, second edition, 2006.

[Bru88a] Gregory W. Brumfiel. The real spectrum compactification of Teichmüller space. In *Geometry of group representations (Boulder, CO, 1987)*, volume 74 of *Contemp. Math.*, pages 51–75. Amer. Math. Soc., Providence, RI, 1988.

[Bru88b] Gregory W. Brumfiel. A semi-algebraic Brouwer fixed point theorem for real affine space. In *Geometry of group representations (Boulder, CO, 1987)*, volume 74 of *Contemp. Math.*, pages 77–82. Amer. Math. Soc., Providence, RI, 1988.

[Bru88c] Gregory W. Brumfiel. The tree of a non-Archimedean hyperbolic plane. In *Geometry of group representations (Boulder, CO, 1987)*, volume 74 of *Contemp. Math.*, pages 83–106. Amer. Math. Soc., Providence, RI, 1988.

[Bru92] Gregory W. Brumfiel. A Hopf fixed point theorem for semi-algebraic maps. In *Real algebraic geometry (Rennes, 1991)*, volume 1524 of *Lecture Notes in Math.*, pages 163–169. Springer, Berlin, 1992.

[Can21] Richard D. Canary. Anosov representations: Informal lecture notes. URL: `https://dept.math.lsa.umich.edu/~canary/Anosovlecnotes.pdf`, 2021. Accessed: 2023-07-13.

[CG93] Suhyoung Choi and William M. Goldman. Convex real projective structures on closed surfaces are closed. *Proc. Amer. Math. Soc.*, 118(2):657–661, 1993.

[CTT20] Alex Casella, Dominic Tate, and Stephan Tillmann. *Moduli spaces of real projective structures on surfaces*, volume 38 of *MSJ Memoirs*. Mathematical Society of Japan, Tokyo, [2020] ©2020.

[EP05] Antonio J. Engler and Alexander Prestel. *Valued fields*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2005.

[ES22] Viveka Erlandsson and Juan Souto. *Mirzakhani's curve counting and geodesic currents*, volume 345 of *Progress in Mathematics*. Birkhäuser/Springer, Cham, [2022] ©2022.

[FG06] Vladimir V. Fock and Alexander B. Goncharov. Moduli spaces of local systems and higher Teichmüller theory. *Publ. Math. Inst. Hautes Études Sci.*, (103):1–211, 2006.

[FG07] Vladimir V. Fock and Alexander B. Goncharov. Moduli spaces of convex projective structures on surfaces. *Adv. Math.*, 208(1):249–273, 2007.

[Fla22] Xenia Flamm. Characterizing Hitchin representations over real closed fields, 2022. arXiv:2208.13611.

[FLP12] Albert Fathi, François Laudenbach, and Valentin Poénaru. *Thurston's work on surfaces*, volume 48 of *Mathematical Notes*. Princeton University Press, Princeton, NJ, 2012. Translated from the 1979 French original by Djun M. Kim and Dan Margalit.

[GK02] Felix Gantmacher and Mark Krein. *Oscillation matrices and kernels and small vibrations of mechanical systems*. AMS Chelsea Publishing, Providence, RI, revised edition, 2002. Translation based on the 1941 Russian original, Edited and with a preface by Alex Eremenko.

[GLW21] Olivier Guichard, François Labourie, and Anna Wienhard. Positivity and representations of surface groups, 2021. arXiv:2106.14584.

[Gol80] William M. Goldman. *Discontinuous Groups and the Euler class*. ProQuest LLC, Ann Arbor, MI, 1980. Thesis (Ph.D.)–University of California, Berkeley.

[GS06] Philippe Gille and Tamás Szamuely. *Central simple algebras and Galois cohomology*, volume 101 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2006.

[GW18] Olivier Guichard and Anna Wienhard. Positivity and higher Teichmüller theory. In *European Congress of Mathematics*, pages 289–310. Eur. Math. Soc., Zürich, 2018.

[Ham99] Ursula Hamenstädt. Cocycles, symplectic structures and intersection. *Geom. Funct. Anal.*, 9(1):90–140, 1999.

[Hat91] Allen Hatcher. On triangulations of surfaces. *Topology and its Applications*, 40(2):189–194, 1991.

[Hit92] Nigel J. Hitchin. Lie groups and Teichmüller space. *Topology*, 31(3):449 – 473, 1992.

[HL22] Xuhua He and George Lusztig. Total positivity and conjugacy classes, 2022. arXiv:2201.12479.

[Iva84] Nikolai V. Ivanov. Algebraic properties of the Teichmüller modular group. *Dokl. Akad. Nauk SSSR*, 275(4):786–789, 1984.

[Iva92] Nikolai V. Ivanov. *Subgroups of Teichmüller modular groups*, volume 115 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1992. Translated from the Russian by E. J. F. Primrose and revised by the author.

[JM87] Dennis Johnson and John J. Millson. Deformation spaces associated to compact hyperbolic manifolds. In *Discrete groups in geometry and analysis (New Haven, Conn., 1984)*, volume 67 of *Progr. Math.*, pages 48–106. Birkhäuser Boston, Boston, MA, 1987.

[Kal00] Dan Kalman. A Matrix Proof of Newton's Identities. *Math. Mag.*, 73(4):313–315, 2000.

[Lab06] François Labourie. Anosov flows, surface groups and curves in projective space. *Invent. Math.*, 165(1):51–114, 2006.

[Lab08] François Labourie. Cross ratios, Anosov representations and the energy functional on Teichmüller space. *Ann. Sci. Éc. Norm. Supér. (4)*, 41(3):437–469, 2008.

[Lam84] Tsit Y. Lam. An introduction to real algebra. volume 14, pages 767–814. 1984. Ordered fields and real algebraic geometry (Boulder, Colo., 1983).

[Lan02] Serge Lang. *Algebra*, volume 211 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 2002.

[LRT11] Darren D. Long, Alan W. Reid, and Morwen Thistlethwaite. Zariski dense surface subgroups in $\mathrm{SL}(3, \mathbf{Z})$. *Geom. Topol.*, 15(1):1–9, 2011.

[Lus94] George Lusztig. Total positivity in reductive groups. In *Lie theory and geometry*, volume 123 of *Progr. Math.*, pages 531–568. Birkhäuser Boston, Boston, MA, 1994.

[Mar19a] Giuseppe Martone. Positive configurations of flags in a building and limits of positive representations. *Math. Z.*, 293(3-4):1337–1368, 2019.

[Mar19b] Giuseppe Martone. Sequences of Hitchin representations of Tree-type, 2019. arXiv:1910.13585.

[Mar22a] Arnaud Maret. A note on character varieties. URL: `https://arnaudmaret.com/files/character-varieties.pdf`, 2022. Accessed: 2023-07-21.

[Mar22b] Bruno Martelli. An introduction to geometric topology, 2022. arXiv:1610.02592.

[McC85] John McCarthy. A "Tits-alternative" for subgroups of surface mapping class groups. *Trans. Amer. Math. Soc.*, 291(2):583–612, 1985.

[Mor86] John W. Morgan. Group actions on trees and the compactification of the space of classes of $\mathrm{SO}(n,1)$-representations. *Topology*, 25(1):1–33, 1986.

[MOT21] Giuseppe Martone, Charles Ouyang, and Andrea Tamburelli. A closed ball compactification of a maximal component via cores of trees, 2021. arXiv:2110.06106.

[MS84] John W. Morgan and Peter B. Shalen. Valuations, trees, and degenerations of hyperbolic structures. I. *Ann. of Math. (2)*, 120(3):401–476, 1984.

[MS85] John W. Morgan and Peter B. Shalen. An introduction to compactifying spaces of hyperbolic structures by actions on trees. In *Geometry and topology (College Park, Md., 1983/84)*, volume 1167 of *Lecture Notes in Math.*, pages 228–240. Springer, Berlin, 1985.

[MZ19] Giuseppe Martone and Tengren Zhang. Positively ratioed representations. *Comment. Math. Helv.*, 94(2):273–345, 2019.

[NGSdS03] Juan A. Navarro González and Juan B. Sancho de Salas. *C^∞-differentiable spaces*, volume 1824 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2003.

[OT21] Charles Ouyang and Andrea Tamburelli. Limits of Blaschke metrics. *Duke Math. J.*, 170(8):1683–1722, 2021.

[OT23a] Charles Ouyang and Andrea Tamburelli. Boundary of the Gothen components. *Topology Appl.*, 326:Paper No. 108420, 11, 2023.

[OT23b] Charles Ouyang and Andrea Tamburelli. Length spectrum compactification of the $\mathrm{SO}_0(2,3)$-Hitchin component. *Adv. Math.*, 420:Paper No. 108997, 37, 2023.

[Ota90] Jean-Pierre Otal. Le spectre marqué des longueurs des surfaces à courbure négative. *Ann. of Math. (2)*, 131(1):151–162, 1990.

[Ouy19] Charles Ouyang. High energy harmonic maps and degeneration of minimal surfaces, 2019. arXiv:1910.06999.

[Pal13] Frederic Palesi. Introduction to positive representations and Fock-Goncharov coordinates. *hal-01218570*, 2013.

[Par12] Anne Parreau. Compactification d'espaces de représentations de groupes de type fini. *Math. Z.*, 272(1-2):51–86, 2012.

[Pau88] Frédéric Paulin. Topologie de Gromov équivariante, structures hyperboliques et arbres réels. *Invent. Math.*, 94(1):53–80, 1988.

[Pin10] Allan Pinkus. *Totally positive matrices*, volume 181 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2010.

[Pla13] rspuzio PlanetMath. Archimedean ordered fields are real. URL: https://planetmath.org/archimedeanorderedfieldsarereal#:~:text=using%20Dedekind%20cuts.&text=of%20fields.,n%20such%20that%20x%3Cn%20., 2013. Accessed: 2023-07-08.

[Pro76] Claudio Procesi. The invariant theory of $n \times n$ matrices. *Advances in Math.*, 19(3):306–381, 1976.

[RS90] Roger Richardson and Peter Slodowy. Minimum vectors for real reductive algebraic groups. *J. London Math. Soc. (2)*, 42(3):409–429, 1990.

[Sch75] Gerald W. Schwarz. Smooth functions invariant under the action of a compact Lie group. *Topology*, 14:63–68, 1975.

[Sik12] Adam S. Sikora. Character varieties. *Trans. Amer. Math. Soc.*, 364(10):5173–5208, 2012.

[Sko90] Richard K. Skora. Splittings of surfaces. *Bull. Amer. Math. Soc. (N.S.)*, 23(1):85–90, 1990.

[Thu88] William P. Thurston. On the geometry and dynamics of diffeomorphisms of surfaces. *Bull. Amer. Math. Soc. (N.S.)*, 19(2):417–431, 1988.

[Thu02] William P. Thurston. The geometry and topology of 3-manifolds, 2002. Princeton lecture notes (1978–1981), preprint, http://library.msri.org/books/gt3m.

[Thu22] William P. Thurston. Minimal stretch maps between hyperbolic surfaces. In *Collected works of William P. Thurston with commentary. Vol. I. Foliations, surfaces and differential geometry*, pages 533–585. Amer. Math. Soc., Providence, RI, [2022] ©2022. 1986 preprint, 1998 eprint.

[Wie18] Anna Wienhard. An invitation to higher Teichmüller theory. In *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. II. Invited lectures*, pages 1013–1039. World Sci. Publ., Hackensack, NJ, 2018.

[Wol11] Maxime Wolff. Connected components of the compactification of representation spaces of surface groups. *Geom. Topol.*, 15(3):1225–1295, 2011.