Invariants of Quiver Representations

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Zusammenfassung

In dieser Bachelorarbeit wollen wir endlichdimensionale Darstellungen eines Köchers Q über einem algebraisch abgeschlossenem Körper k betrachten und diese bis auf Isomorphie klassifizieren. Zwei Darstellungen können höchstens dann isomorph sein, wenn sie den gleichen Dimensionsvektor besitzen. Zu einem Dimensionsvektor $\underline{d} = (d_1, \ldots, d_n) \in \mathbb{N}^n$ definieren wir den Darstellungsraum $\mathrm{Rep}(Q, \underline{d})$. Es wird sich zeigen, dass die Frage der Klassifizierung isomorpher Darstellungen von Q zur Beschreibung des Bahnenraumes unter der Wirkung von $\mathrm{GL}(Q, \underline{d}) = \mathrm{GL}_{d_1}(k) \times \cdots \times \mathrm{GL}_{d_n}(k)$ auf dem Darstellungsraum führt. Außerdem induziert die Projektion auf den Quotienten $\mathrm{Rep}(Q, \underline{d})/\mathrm{GL}(Q, \underline{d})$ eine Bijektion zwischen der Menge der abgeschlossenen $\mathrm{GL}(Q, \underline{d})$-Bahnen und der Menge der maximalen Ideale im Invariantenring $k[\mathrm{Rep}(Q, \underline{d})]^{\mathrm{GL}(Q,\underline{d})}$. Es liegt nun nahe, diesen Ring genauer zu untersuchen. Dies wird das Hauptthema dieser Arbeit sein. Ein erster Ansatz ist ein Theorem von Hilbert, welches besagt, dass der Invariantenring für $k = \mathbb{C}$ endlich erzeugt ist. Nun stellt sich die Frage, ob wir diese Erzeuger in dem geschilderten Fall explizit beschreiben können. Eine Beschreibung dieser Elemente liefert das Theorem von Le Bruyn und Procesi. Es besagt, dass der Invariantenring $\mathbb{C}[\mathrm{Rep}(Q, \underline{d})]^{\mathrm{GL}(Q,\underline{d})}$ von Spuren von Zykeln des Köchers erzeugt wird. Ziel dieser Bachelorarbeit ist der Beweis dieses Satzes. Hierfür wird ein Beweis ausführlich vorgestellt und erarbeitet und ein zweiter skizziert.

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Contents

1 Introduction

This thesis deals with the problem of classifying finite-dimensional representations of a quiver Q over an algebraically closed field k. Two such representations can only be isomorphic, if they have the same dimension vector. We fix some dimension vector $\underline{d} = (d_1, \dots, d_n) \in \mathbb{N}^n$ and we define the representation space $\text{Rep}(Q, \underline{d})$ with respect to Q and $\underline{d}$. The question of classifying isomorphic representations of Q amounts to describing the orbit space of $\text{Rep}(Q, \underline{d})$ under the action of $\text{GL}(Q, \underline{d}) = \text{GL}_{d_1}(k) \times \cdots \times \text{GL}_{d_n}(k)$. Lemma 3.5 tells us, that the projection map $\text{pr}\colon \text{Rep}(Q, \underline{d}) \to \text{Rep}(Q, \underline{d})/\text{GL}(Q, \underline{d})$ induces a bijection between the set of closed orbits and the set of maximal ideals in the invariant ring $k[\text{Rep}(Q, \underline{d})]^{\text{GL}(Q,\underline{d})}$. It seems natural to have a closer look at this ring, which will be the main focus of this thesis. Since $\text{Rep}(Q, \underline{d})$ can be considered as an affine variety, we see that the problem can be integrated into the context of Geometric Invariant Theory. In Section 3 we introduce some basics, which we need when talking about geometric aspects in algebra. In particular, we state the following important theorem by Hilbert, which gives some first information about the ring of polynomial invariants, which we are interested in.

Theorem 1.1 (Hilbert)**.** *Let V be a finite-dimensional $\mathbb{C}$-vector space and G an algebraic group acting regularly on V. Then $\mathbb{C}[V]^G$ is finitely generated if G is reductive.*

The proof of this theorem will not be part of this thesis, but can be found in the paper by Schmitt [Sch, Chapter 1.4.2], which is also our main reference for Section 3.

As $\text{GL}(Q, \underline{d}) = \text{GL}_{d_1}(k) \times \cdots \times \text{GL}_{d_n}(k)$ is a product of general linear groups, it is reductive and for $k = \mathbb{C}$ we can apply Hilbert's theorem. Now the question arises whether it is possible to describe the generators of $\mathbb{C}[\text{Rep}(Q, \underline{d})]^{\text{GL}(Q,\underline{d})}$. This will be discussed explicitly in Section 5. We will state and prove the following theorem by Le Bruyn and Procesi, see also [PrBr, Section 3, Theorem 1].

Theorem 1.2 (Le Bruyn-Procesi)**.** *Let Q be a quiver and $\underline{d} \in \mathbb{N}^n$ a dimension vector. Then the invariant ring $\mathbb{C}[\text{Rep}(Q, \underline{d})]^{\text{GL}(Q,\underline{d})}$ is generated by the elements t_σ for σ an oriented cycle in Q.*

The elements $t_\sigma\colon \text{Rep}(Q, \underline{d}) \to \mathbb{C}$ map a fixed representation with dimension vector $\underline{d}$ to the trace of the product of the linear maps assigned to the arrows in the oriented cycle σ in Q by this representation. It is clear, that this is an invariant.

Our first proof of this theorem follows a paper by Aslaksen, Tan and Zhu [ATZ, Section 2, Theorem 1], who elaborated the statement in an even more general context. The main idea is to study multilinear invariants instead of polynomial invariants. In Section 3.1 we explain why this is possible. We reduce the problem to the point where we are able to apply Schur-Weyl-duality. The proof presents an algorithm to find the oriented cycles in Q, which generate the ring of invariants. After a detailed proof in Section 5.1, we will give an example to illustrate the general notion.

The original proof by Le Bruyn and Procesi uses the fact that every polynomial invariant comes from a matrix invariant. We can describe representations of Q by representations of the path algebra $\mathbb{C}Q$. This will be explained in detail in Section 4.3. Unfortunately, the proof given in [PrBr, Section 3, Theorem 1] is rather sketchy, omitting many details. Their idea was presumably to extend $\mathbb{C}Q$ to an algebra TQ, which has the same representations as $\mathbb{C}Q$. The algebra TQ is generated by all paths in Q and by formal elements T_c for c an oriented cycle in Q. Now defining our polynomial invariants, we get that all generators of TQ are mapped to 0 except for those paths which are oriented cycles. More details can be found in the last Subsection 5.2.3.

In the original paper by Le Bruyn and Procesi, the authors stated a bound on the length of the oriented cycles needed as generators. This shows that the ring is in fact finitely generated. The proof of this result will not be part of this thesis, but can be found in [Pro3, Section 3, Theorem 3.4].

To conclude, we may say that it is reassuring to see that the theorem of Le Bruyn and Procesi can be shown using totally different methods in Geometric Invariant Theory.

2 Preliminaries

In this section we introduce some results which will be used later on. As they are not restricted to specific conditions, we will state and prove them under general assumptions.

From now on let k be an algebraically closed field and V and W finite-dimensional k-vector spaces. Let $\dim_k(V) = n$, $\dim_k(W) = m$. Choose a basis $v_1, \dots, v_n$ of V and a basis $w_1, \dots, w_m$ of W. Then the set $\{v_{i_1} \otimes \cdots \otimes v_{i_l} \mid 1 \leq i_1, \dots, i_l \leq n\}$ is a basis of $V^{\otimes l}$ and $v_1^*, \dots, v_n^*$ is the dual basis of the dual vector space V^* and $w_1^*, \dots, w_m^*$ the dual basis of W^*. Let G be a group which acts linearly on V and W. An important example to have in mind is $G = \mathrm{GL}(Z)$ acting on Z in the natural way, for $Z = V$ or $Z = W$. For the set of k-linear maps from V to W, we write $\mathrm{Hom}(V, W)$. We get an induced action of G on $\mathrm{Hom}(V, W)$ by setting $[g.f](v) \coloneqq g.f(g^{-1}.v)$ for $g \in G$, $v \in V$ and $f \in \mathrm{Hom}(V, W)$. Then $\mathrm{Hom}_G(V, W)$ denotes the set of G-equivariant maps from V to W, i.e. for $f \in \mathrm{Hom}_G(V, W)$ it holds that $g.f(v) = f(g.v)$ for all $g \in G$ and $v \in V$.

2.1 On Homomorphism Spaces

Lemma 2.1. *Given V, W and a group G as above. Then there are isomorphisms of k-vector spaces as follows:*

1. $\mathrm{Hom}_G(k, V) \cong V^G$, *where G acts trivially on k.*
2. $V^* \otimes W \cong \mathrm{Hom}(V, W)$.
3. $\mathrm{Hom}_G(V, W) \cong \mathrm{Hom}_G(k, V^* \otimes W)$ *with the trivial action of G on k.*
4. $\mathrm{Hom}_k(V^{\otimes m}, V^{\otimes m'}) \cong ((V^*)^{\otimes m} \otimes V^{\otimes m'})^*$.

PROOF.

1. We claim that the assignment $\varphi \mapsto \varphi(1)$ for $\varphi \in \mathrm{Hom}_G(k, V)$, defines the required isomorphism. First of all, we have $\varphi(1) \in V^G$ since $g.\varphi(1) = \varphi(g^{-1}.1) = \varphi(1)$ for all $g \in G$.

   Conversely, for $v \in V^G$ we can define $\varphi_v \in \mathrm{Hom}_G(k, V)$ by sending $1 \mapsto v$. Then $(g.\varphi_v)(1) = \varphi_v(g^{-1}.1) = \varphi_v(1) = v = g.v = g.(\varphi_v(1))$.

   Obviously, the maps are mutually inverse, so we get the desired isomorphism of k-vector spaces.

2. We define $\Phi\colon V^* \otimes W \to \mathrm{Hom}(V, W)$, $\varphi \otimes w \mapsto \varphi_w$ with $\varphi_w(v) := \varphi(v)w$. This map is well-defined and linear and since our vector spaces are finite-dimensional and of the same dimension, it is enough to check that Φ is injective. Hence assume $0 \neq \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_{i,j}(v_i^* \otimes w_j) \in \ker(\Phi)$. Then there exist l and l', $1 \leq l \leq n$ and $1 \leq l' \leq m$, such that $\alpha_{l,l'} \neq 0$. But then

   $$\sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_{i,j} v_i^*(v_l) w_j = \sum_{j=1}^{m} \alpha_{l,j} w_j \neq 0,$$

   since $\alpha_{l,l'} \neq 0$ and $w_1, \ldots, w_m$ are linearly independent. This contradicts the assumption that the element is in the kernel of Φ. Thus we get that Φ is an isomorphism of k-vector spaces.

3. We have $\mathrm{Hom}_G(V, W) = (\mathrm{Hom}(V, W))^G$ by definition of the action:

   $$\begin{aligned} g.f = f &\Leftrightarrow \forall v \in V \quad [g.f](v) = f(v) \\ &\Leftrightarrow \forall v \in V \quad gf(g^{-1}v) = f(v) \\ &\Leftrightarrow \forall v \in V \quad f(g^{-1}v) = g^{-1}f(v) \\ &\Leftrightarrow f \text{ is } g\text{-equivariant.} \end{aligned}$$

   It is left to show that the isomorphism Φ in 2. is G-equivariant. Then we get by 2. and 1. that

   $$\mathrm{Hom}_G(V, W) = (\mathrm{Hom}(V, W))^G \cong (V^* \otimes W)^G \cong \mathrm{Hom}_G(k, V^* \otimes W),$$

which proves the claim. Indeed, let $g \in G$, $\varphi \in V^*$, $w \in W$. For all $v \in V$. On the one hand it holds that

$$\Phi(g.(\varphi \otimes w))(v) = \Phi(g.\varphi \otimes g.w)(v) = \varphi(g^{-1}.v)(g.w)$$

and on the other hand

$$\begin{aligned}[g.\Phi(\varphi \otimes w)](v) &= [g.\varphi_w](v) \\ &= g.\varphi_w(g^{-1}.v) = g.\varphi(g^{-1}.v)w = \varphi(g^{-1}.v)(g.w).\end{aligned}$$

This shows that Φ is G-equivariant and hence the claim follows.

4. We define

$$\Psi\colon \mathrm{Hom}_k(V^{\otimes m}, V^{\otimes m'}) \to ((V^*)^{\otimes m} \otimes V^{\otimes m'})^*, A \mapsto \Psi(A),$$

with $\Psi(A)(\varphi_1 \otimes \cdots \otimes \varphi_m \otimes v_1 \otimes \cdots \otimes v_{m'}) := (\varphi_1 \otimes \cdots \otimes \varphi_m)(A(v_1 \otimes \cdots \otimes v_{m'})) \in k$ for $\varphi_i \in V^*$ with $1 \leq i \leq m$ and $v_j \in V$ with $1 \leq j \leq m'$. This map is in fact an isomorphism of k-vector spaces. It is clear, that Ψ is linear. By dimension, it is enough to show that Ψ is injective.

Thus let $A \in \mathrm{Hom}_k(V^{\otimes m}, V^{\otimes m'})$, such that $\Psi(A) = 0$. Assume that $A \neq 0$, i.e. there exists $w_1 \otimes \cdots \otimes w_m \in V^{\otimes m}$ such that $0 \neq A(w_1 \otimes \cdots \otimes w_m) = \sum_{\underline{j}=(j_1,\ldots,j_{m'})\in \underline{n}^{m'}} \alpha_{\underline{j}}(v_{j_1} \otimes \cdots \otimes v_{j_{m'}})$. Thus there is $\underline{j}' \in \underline{n}^{m'}$ such that $\alpha_{\underline{j}'} \neq 0$. Now for $\varphi_i := v^*_{j'_i}$, we get that

$$\sum_{\underline{j}=(j_1,\ldots,j_{m'})\in \underline{n}^{m'}} \alpha_{\underline{j}}(v_{j_1} \otimes \cdots \otimes v_{j_{m'}}) = \alpha_{\underline{j}'} \neq 0,$$

which is a contradiction to the fact that $A \in \ker(\Psi)$. Hence Ψ is injective.

□

This lemma leads to the following corollary.

Corollary 2.2. *For $V = W$ we get by Lemma 2.1.2. that*

$$V^* \otimes V \cong \mathrm{Hom}_k(V, V) = \mathrm{End}_k(V)$$

via the map $\varphi \otimes v \mapsto (w \mapsto \varphi(w)v)$ for all $v, w \in V$, $\varphi \in V^*$.

If we identify $\mathrm{Hom}_k(V, W)$ by $m \times n$ matrices and $\mathrm{Hom}_k(U, V)$ by $n \times l$ matrices, where U is a k-vector space of dimension l, then the question arises how matrix multiplication transfers under the isomorphism as in Lemma 2.1.2. Furthermore for $V = W$, one may ask how the linear map $\mathrm{tr}\colon M_n(k) \to k$ translates to a map $\tilde{\mathrm{tr}} := \mathrm{tr} \circ \Phi$ from $V^* \otimes V$ to k. To do so we state the following lemma.

Lemma 2.3. *Let $\varphi \in U^*$, $v \in V$, $\psi \in V^*$ and $w \in W$ and denote by Φ the isomorphism of Lemma 2.1.2. Then we have*

$$U \xrightarrow{\Phi(\varphi \otimes v)} V \xrightarrow{\Phi(\psi \otimes w)} W$$

and the following identity holds

$$\Phi(\psi \otimes w) \cdot \Phi(\varphi \otimes v) = \psi(v) \cdot \Phi(\varphi \otimes w) \qquad \in \mathrm{Hom}_k(U, W).$$

Moreover, for $V = W$ and $\varphi \in V^*$, $v \in V$ the following equation holds:

$$\tilde{\mathrm{tr}}(\varphi \otimes v) = \varphi(v),$$

where $\tilde{\mathrm{tr}}$ is defined as above.

Proof. Let $u_1, \ldots, u_l \in U$ be a basis of U and $u_1^*, \ldots, u_l^*$ the dual basis of U^*. Then the map Φ implies that

$$u_i^* \otimes v_j \mapsto (u_r \mapsto u_i^*(u_r)v_j = \delta_{ir} v_j). \tag{1}$$

for $1 \leq i, r \leq l$ and $1 \leq j \leq n$. We see that the ith basis vector of U gets mapped to the jth basis vector, so the representing matrix of this linear map is the unit matrix $E_{ji} \in M_{n \times l}(k)$, which is the matrix with a 1 in position (j, i) for $1 \leq i \leq l$, $1 \leq j \leq n$ and 0 otherwise, i.e.

$$\Phi(u_i^* \otimes v_j) = E_{ji} \in M_{n \times l}(k). \tag{2}$$

Now let φ, ψ, v, w as given above with $\varphi = \sum_{i=1}^{l} a_i u_i^*$, $\psi = \sum_{i=1}^{n} a_i' v_i^*$ and $v = \sum_{i=1}^{n} b_i v_i$, $w = \sum_{i=1}^{m} b_i' w_i$ for $a_i, a_i', b_i, b_i' \in k$ for all i. Then by (2) we get that

$$\varphi \otimes v = \sum_{i=1}^{l} \sum_{j=1}^{n} a_i b_j (v_i^* \otimes v_j) \mapsto \sum_{i=1}^{l} \sum_{j=1}^{n} a_i b_j E_{ji}$$

under Φ. The matrix representing $\Phi(\varphi \otimes v)$ with respect to the chosen bases equals

$$\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \begin{pmatrix} a_1 & \ldots & a_l \end{pmatrix} \in M_{n \times l}(k).$$

Analogously, we get for $\psi \otimes w = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i' b_j' (v_i^* \otimes w_j)$ the representing matrix

$$\begin{pmatrix} b_1' \\ \vdots \\ b_m' \end{pmatrix} \begin{pmatrix} a_1' & \ldots & a_n' \end{pmatrix} \in M_{m \times n}(k).$$

The matrix of $\Phi(\psi \otimes w) \cdot \Phi(\varphi \otimes v)$ equals

$$\begin{pmatrix} b'_1 \\ \vdots \\ b'_m \end{pmatrix} \begin{pmatrix} a'_1 & \dots & a'_n \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \begin{pmatrix} a_1 & \dots & a_l \end{pmatrix}$$
$$= \psi(v) \begin{pmatrix} b'_1 \\ \vdots \\ b'_m \end{pmatrix} \begin{pmatrix} a_1 & \dots & a_l \end{pmatrix} \in M_{m \times l}(k),$$

since $\psi(v) = \sum_{i=1}^n \sum_{j=1}^n a'_i b_j v_i^*(v_j) = \sum_{i=1}^n a'_i b_i$. This proves implies the first claim.

Using the first part of this lemma we get that

$$\tilde{\text{tr}}(v_i^* \otimes v_j) = \text{tr}(\Phi(v_i^* \otimes v_j)) = \text{tr}(E_{ji}) = \delta_{ji} = v_i^*(v_j),$$

for all $1 \leq i, j \leq n$. For general $\varphi \in V^*$ and $v \in V$, given as linear combination of the chosen basis, we get

$$\tilde{\text{tr}}(\varphi \otimes v) = \text{tr}(\sum_{i,j=1}^n a_i b_j E_{ji}) = \sum_{i,j=1}^n a_i b_j \text{tr}(E_{ji})$$
$$= \sum_{i,j=1}^n a_i b_j \delta_{ij} = \sum_{i,j=1}^n a_i b_j v_i^*(v_j) = \varphi(v).$$

This proves the second part of the lemma. □

Now we extend this theory to several tensor factors and obtain a result similar to Lemma 2.1.4.

Corollary 2.4. *Let $m \in \mathbb{N}$. Then we have $(V^*)^{\otimes m} \otimes V^{\otimes m} \cong \text{End}_k(V)^{\otimes m}$ via the map*

$$\Phi_m \colon (V^*)^{\otimes m} \otimes V^{\otimes m} \to \text{End}_k(V)^{\otimes m},$$
$$\varphi_1 \otimes \cdots \otimes \varphi_m \otimes v_1 \otimes \cdots v_m \mapsto \Phi(\varphi_1 \otimes v_1) \otimes \cdots \otimes \Phi(\varphi_m \otimes v_m),$$

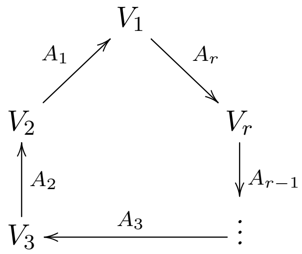
where $\Phi \colon V^* \otimes V \to \text{End}_k(V)$ as in Corollary 2.2.

In the next step we are interested how the trace of a product of matrices translates under the above isomorphism using the equalities from Lemma 2.3.

Lemma 2.5. *Let $V_1, \dots, V_r$ be finite-dimensional k-vector spaces with $\dim(V_i) = d_i$, $1 \leq i \leq r$. Let $A_i \in M_{d_i \times d_{i+1}}(k)$ for $1 \leq i \leq r$, where A_i represents the homomorphism $f_i \colon V_{i+1} \to V_i$ (with respect to chosen bases)* *given by* $\varphi_i \otimes v_i$ *with* $\varphi_i \in V_{i+1}^*$, $v_i \in V_i$ *for* $1 \leq i \leq r$ *under the isomorphism* Φ *at Lemma 2.1.2. Then the following identity holds:*

$$\operatorname{tr}(A_1 \cdots A_r) = \prod_{i=1}^{r} \varphi_i(v_{i+1}),$$

where all indices are taken modulo r*.*



Proof. Because of the fact that $A_i \in M_{d_i \times d_{i+1}}(k)$ (indices taken modulo r) the product $A_1 \cdots A_r$ is defined and an endomorphism of V_1. Hence we get

$$\begin{aligned}
\operatorname{tr}(A_1 \cdots A_r) &= \tilde{\operatorname{tr}}((\varphi_1 \otimes v_1) \cdot (\varphi_2 \otimes v_2) \cdots (\varphi_r \otimes v_r)) \\
&= \operatorname{tr}(\varphi_1(v_2) \cdot \Phi(\varphi_2 \otimes v_1)\Phi(\varphi_3 \otimes v_3) \cdots \Phi(\varphi_r \otimes v_r)) \\
&= \varphi_1(v_2)\varphi_2(v_3) \cdots \varphi_{r-1}(v_r) \cdot \tilde{\operatorname{tr}}(\varphi_r \otimes v_1) \\
&= \prod_{i=1}^{r} \varphi_i(v_{i+1}),
\end{aligned}$$

by using the identities stated in Lemma 2.3 several times and using the fact that multiplication of matrices is associative. □

2.2 Schur-Weyl-Duality

Let $m \in \mathbb{N}$. The group $\operatorname{GL}(V)$ acts on V in the natural way and we can extend this action to an action on $V^{\otimes m}$ via the usual tensor product action:

$$g.(v_1 \otimes \cdots \otimes v_m) \coloneqq g.v_1 \otimes \cdots \otimes g.v_m,$$

for $g \in \operatorname{GL}(V), v_i \in V$ for $1 \leq i \leq m$.

Denote by S_m the symmetric group on m elements. Then S_m acts on $V^{\otimes m}$ by permuting the factors, namely for $\pi \in S_m$, we have

$$\pi.(v_1 \otimes \cdots \otimes v_m) \coloneqq v_{\pi(1)} \otimes \cdots \otimes v_{\pi(m)}.$$

It is easy to see that these actions commute, i.e.

$$\pi.g.(v_1 \otimes \cdots \otimes v_m) = g.\pi.(v_1 \otimes \cdots \otimes v_m),$$

for all $g \in \mathrm{GL}(V), \pi \in S_m$.

For $g \in \mathrm{GL}(V)$, we denote by $\bar{g}$ the endomorphism $V^{\otimes m} \to V^{\otimes m}$, $\bar{g}(v_1 \otimes \cdots \otimes v_m) = g.v_1 \otimes \cdots \otimes g.v_m \in \mathrm{End}_k(V^{\otimes m})$. For $\pi \in S_m$, we set $\bar{\pi} : V^{\otimes m} \to V^{\otimes m}$, $\bar{\pi}(v_1 \otimes \cdots \otimes v_m) = v_{\pi(1)} \otimes \cdots \otimes v_{\pi(m)}$. Then $\bar{\pi} \in \mathrm{End}_k(V^{\otimes m})$.

Now we are ready to state the duality theorem.

Theorem 2.6 (Schur-Weyl-duality)**.**

1. $\mathrm{End}_{S_m}(V^{\otimes m}) = \langle \mathrm{GL}(V) \rangle$, *where* $\langle \mathrm{GL}(V) \rangle$ *denotes the image of the map* $k\mathrm{GL}(V) \to \mathrm{End}_k(V^{\otimes m}), g \mapsto \bar{g}$ *for* $g \in \mathrm{GL}(V)$,
2. *If* $\mathrm{char}(k) = 0$, *then* $\mathrm{End}_{\mathrm{GL}(V)}(V^{\otimes m}) = \langle S_m \rangle$, $\langle S_m \rangle$ *being the image of the map* $kS_m \to \mathrm{End}_k(V^{\otimes m}), \pi \mapsto \bar{\pi}$ *for* $\pi \in S_m$.

PROOF. For an exact proof and more details on this duality we refer to [Gre, Chapter 2, Theorem 2.6c] □

3 (Some) Basics from Geometric Invariant Theory

In this section we give some basic definitions from geometric invariant theory and then explain motivating questions and problems arising in this context. Our main reference is [Sch, Chapter 1.2.1]. Let k be an algebraically closed field and V a finite-dimensional k-vector space. We start with studying polynomial maps from V to k.

3.1 Polynomial Maps

Let

$$T(V^*) = \bigoplus_{l \geq 0} T^l(V^*)$$

with

$$T^l(V^*) := \underbrace{(V^* \otimes V^* \otimes \cdots \otimes V^*)}_{l \text{ factors}}$$

denote the tensor algebra of V^*. The symmetric algebra of V^* is defined as

$$S(V^*) = \bigoplus_{l \geq 0} S^l(V^*),$$

where $S^l(V^*)$ is the quotient of $T^l(V^*)$ by the subspace of symmetric tensors. The image of the map

$$\Psi_l \colon T^l(V^*) \to \mathrm{Maps}(V, k), \quad \phi_1 \otimes \cdots \otimes \phi_l \mapsto \left(v \mapsto \prod_{i=1}^{l} \phi_i(v) \right)$$

is denoted by $\mathcal{P}^l[V]$. The kernel of this map is the set of symmetric tensors of $T^l(V^*)$ because k is infinite, hence $\mathcal{P}^l[V] \cong S^l(V^*)$. An element $\phi \in \mathcal{P}^l[V]$ is called homogeneous of degree l, i.e. $\phi(\alpha v) = \alpha^l \phi(v)$ for all $\alpha \in k$ and $v \in V$. We define the set of polynomial functions $\mathcal{P}[V]$ by $\mathcal{P}[V] \coloneqq \sum_{l \geq 0} \mathcal{P}^l[V] \subset \mathrm{Maps}(V, k)$.

Since k is infinite, we get that this sum is even direct, i.e.

$$\mathcal{P}[V] = \bigoplus_{l \geq 0} \mathcal{P}^l[V]. \tag{3}$$

By construction, this is a graded algebra, since

$$\mathcal{P}^l[V]\mathcal{P}^{l'}[V] \subseteq \mathcal{P}^{l+l'}[V].$$

Because V is finite-dimensional, $T^l(V^*)$ is naturally isomorphic to $(V \otimes \cdots \otimes V)^*$ via the map

$$\Phi_l \colon T^l(V^*) \to (V \otimes \cdots \otimes V)^*, \quad \phi_1 \otimes \cdots \otimes \phi_l \mapsto \left(v_1 \otimes \cdots \otimes v_l \mapsto \prod_{i=1}^{l} \phi_i(v_i) \right),$$

for $v_1, \ldots, v_l \in V$. There is a natural map

$$\Lambda_l \colon \underbrace{(V \otimes V \otimes \cdots \otimes V)}_{l \text{ factors}}{}^* \to \mathrm{Maps}(V, k), \tag{4}$$

$$\phi \mapsto f_\phi, \text{ where } f_\phi(v) \coloneqq \phi(v \otimes v \otimes \cdots \otimes v), v \in V.$$

Lemma 3.1. *Given notation as above, we have that the following diagram commutes.*

$$\begin{array}{ccc} V^* \otimes \cdots \otimes V^* & \xrightarrow{\quad \Phi_l \quad} & (V \otimes \cdots \otimes V)^* \\ & \underset{\Psi_l}{\searrow} \quad \underset{\Lambda_l}{\swarrow} & \\ & \mathrm{Maps}(V, k) & \end{array}$$

Proof. It is enough to check this on elementary tensors. Hence let $\phi_1 \otimes \cdots \otimes \phi_l \in (V^*)^{\otimes l}$. Then

$$\begin{aligned} [\Lambda_l \circ \Phi_l](\phi_1 \otimes \cdots \otimes \phi_l)(v) \\ &= f_{\Phi_l(\phi_1 \otimes \cdots \otimes \phi_l)}(v) \\ &= \Phi_l(\phi_1 \otimes \cdots \otimes \phi_l)(v \otimes \cdots \otimes v) \\ &= \prod_{i=1}^{l} \phi_i(v) \\ &= \Psi_l(\phi_1 \otimes \cdots \otimes \phi_l)(v) \end{aligned}$$

by definiton of Ψ_l. □

Since Φ_l is an isomorphism and the diagram in Lemma 3.1 commutes, it follows that Λ_l maps onto $\mathcal{P}^l[V]$.

Remark 3.2. An element of $\mathcal{P}[V]$ can be expressed as a polynomial in the coordinate functions with respect to a fixed basis. This fact suggests, why we call those maps from V to k polynomial maps. Since V is finite-dimensional, we can then identify $\mathcal{P}[V]$ with the polynomial ring over k in n variables, where $n = \dim_k(V)$. Given the Zariski topology on V, then the regular functions from $V \to k$, i.e. the maps which are continuous in this topology, are exactly the polynomial maps. The set of regular functions on any affine variety X over k is denoted by $k[X]$. Hence by the above we get that

$$k[V] = \mathcal{P}[V] = k[x_1, \ldots, x_n].$$

This result is useful, since we already know about properties of polynomial rings.

In the next section, we will use this fact and identify the coordinate ring of an affine variety with the polynomial ring.

3.2 Polynomial Invariants

Let G be a group acting linearly on V. Then we get an induced action on $\mathcal{P}[V]$ by $g.\phi(v) \coloneqq \phi(g^{-1}.v)$ for $\phi \in \mathcal{P}[V]$, $g \in G$ and $v \in V$. The decomposition (3) even holds as G-module, because G acts linearly: For $\phi \in \mathcal{P}^l[V]$, $g \in G$ and $v \in V$ we have

$$g.\phi(\alpha v) = \phi(g^{-1}.(\alpha v)) = \phi(\alpha(g^{-1}.v)) = \alpha^l \phi(g^{-1}.v) = \alpha^l g.\phi(v),$$

for all $\alpha \in k$. Thus we get $g.\phi \in \mathcal{P}^l[V]$. It follows that

$$\mathcal{P}[V]^G = \bigoplus_{l \geq 0} \mathcal{P}^l[V]^G.$$

In order to describe $\mathcal{P}^l[V]^G$, we notice that the map Λ_l is G-equivariant and hence induces a surjection

$$((\underbrace{V \otimes \cdots \otimes V}_{l \text{ factors}})^*)^G \to \mathcal{P}^l[V]^G. \tag{5}$$

3.3 Motivating Question

Problem 3.3. Since we can identify V with k^n for $n = \dim(V)$, V carries the structure of an affine variety. Let G act on V via a regular action, i.e. the induced map $\rho\colon G \to \mathrm{GL}_n(V)$ is a morphism of linear algebraic groups. Now it seems natural to ask, whether

$$V/G = \{G \cdot v \mid v \in V\},$$

the orbit space regarding this action, carries the structure of an affine variety, such that the projection map

$$\mathrm{pr}\colon V \to V/G$$

is regular, i.e. a morphism of affine varieties.

As first consideration, let us assume that V/G is an affine variety with coordinate ring $k[V/G]$, the regular functions on V/G. Via the following commutative diagram,

$$\begin{array}{ccc} V & \xrightarrow{\mathrm{pr}} & V/G \ , \\ & \overset{f'}{\searrow} & \downarrow f \\ & & k \end{array}$$

with $f' := f \circ \mathrm{pr}$, we obtain a regular function f' on V, which is constant on any G-orbit. Thus, we get

$$\begin{aligned} k[V/G] \subseteq k[V]^G &:= \{f \in k[V] \mid g.f = f\} && (6) \\ &= \{f \in k[V] \mid f \text{ is constant on all } G\text{-orbits}\}. && (7) \end{aligned}$$

On the other hand, a regular function f on V, which is constant on all G-orbits defines a set-theoretic function

$$f'\colon V/G \to k, f'(G \cdot v) = f'(\mathrm{pr}(v)) := f(v).$$

This is well-defined because f is constant on all G-orbits. Now one can ask whether this function is also regular, and thereby an element of $k[V/G]$.

If so, we had $k[V]^G \subseteq k[V/G]$ and by (6) equality.

For a ring R, we denote by

$$\mathrm{Specmax}(R) := \{I \subset R \mid \text{I maximal ideal in R}\}$$

the set of maximal ideals in R. For any affine variety X we have a bijection

$$\begin{aligned} \mathrm{Specmax}(k[X]) &\leftrightarrow \{\text{points in } X\} && (8) \\ I &\mapsto \{x \in X \mid f(x) = 0 \text{ for all } f \in I\} \\ \{f \in k[X] \mid f(x) = 0\} &\mapsfrom x \end{aligned}$$

by Hilbert's Nullstellensatz, see [Kun, §3, Theorem 3.2]. The most natural expectation would be the following:

$$V/G = \mathrm{Specmax}(k[V]^G). \tag{9}$$

Unfortunately, this is not always the case which can be seen by looking at the following easy example [Sch, Chapter 1.2.1].

Example 3.4. Let $k = \mathbb{C}$ and $G = \mathbb{C}^*$ act on the $\mathbb{C}$-vector space $\mathbb{C}^n$ for some $n \in \mathbb{N}$ via scalar multiplication:

$$z.v \coloneqq z \cdot v \text{ for } z \in \mathbb{C}^*,\ v \in \mathbb{C}^n.$$

We have $\mathbb{C}^n/\mathbb{C}^* = \mathbb{P}^{n-1} \cup \{0\}$ as set of points.

Let $f \in \mathbb{C}[\mathbb{C}^n]^{\mathbb{C}^*} = \mathbb{C}[x_1, \ldots, x_n]^{\mathbb{C}^*}$ be a regular function, which is constant on any line. As f is given by a polynomial in n variables, we claim that it is a constant function. Indeed, for $n = 1$ this is true. Now let $n \geq 1$ and view f as a polynomial in one variable x_i for some $1 \leq i \leq n$ by setting the other variables to 0, say $\tilde{f}$. Since f is constant on any line, we get that $f(0, \ldots, 0, 1, 0, \ldots, 0) = f(0, \ldots, 0, \lambda, 0, \ldots, 0)$ for all $\lambda \in \mathbb{C}$. Hence $\tilde{f}(x_i) - f(0, \ldots, 0, 1, 0, \ldots, 0)$ has infinitely many zeroes, hence it is a constant polynomial and the variable x_i does not appear in f. Now f is in fact a polynomial in $n-1$ variables and, by induction, it is constant, which proves the claim. Hence we get $\mathbb{C}[\mathbb{C}^n]^{\mathbb{C}^*} = \mathbb{C}$, the constant polynomials on $\mathbb{C}^n$, and therefore

$$\operatorname{Specmax}(\mathbb{C}[\mathbb{C}^n]^{\mathbb{C}^*}) = \operatorname{Specmax}(\mathbb{C}) = \{point\},$$

as $\mathbb{C}$ is a field. But $\mathbb{C}^n/\mathbb{C}^* = \mathbb{P}^{n-1} \cup \{0\}$ is for $n > 1$ more than just one point.

This example shows that (9) is not true in general. This is easy to understand; if we consider a function $f \in \mathbb{C}[V]^G$, which is continuous in the Zariski topology, then it is not only constant on any G-orbit, but also on its closure. To solve this problem it seems natural to only consider the set of closed orbits instead of the whole orbit space.

Using this idea in the above example, we see that an orbit $\mathbb{C}^* \cdot v \subseteq \mathbb{C}^n$ is closed if and only if $v = 0$, because for $v \neq 0$ the set $\mathbb{C}^* \cdot v$ describes a line without 0. We see that the set of closed $\mathbb{C}^*$-orbits is just one point and the expected equivalence (9) holds. To specify this we state the following lemma.

Lemma 3.5. *The map* $\operatorname{pr}\colon V \to V/G$ *induces a bijection between the set of closed* G*-orbits in* V *and* $\operatorname{Specmax}(k[V]^G)$.

For the statement of this lemma see [Sch, Chapter 1.2, Lemma 1.2.1.5] and for its proof refer to [Sch, Chapter 1.4].

To find closed G-orbits it is hence enough to describe $\operatorname{Specmax}(k[V]^G)$. For convenience we introduce the following notation.

Definition 3.6. We denote by $V/\!/G \coloneqq \operatorname{Specmax}(k[V]^G)$ the set of maximal ideals in $k[V]^G$ or equivalently via the identification in Lemma 3.5 the set of closed orbits.

Equation (8) suggests that it should be possible to view $V//G$ as an affine variety with coordinate ring $k[V]^G$. Note that coordinate rings of affine varieties meet specific conditions; they are in particular finitely generated. In favourable cases this is true by the following theorem.

Theorem 3.7 (Hilbert). *Let V be a finite-dimensional $\mathbb{C}$-vector space and G an algebraic group acting regularly on V. Then $\mathbb{C}[V]^G$ is finitely generated if G is reductive.*

Definition 3.8. The *radical* $R(G)$ of a linear algebraic group G over an algebraically closed field k is its maximal connected solvable normal subgroup. The group G is called *reductive*, if $R(G)$ is a torus, i.e. isomorphic to some finite product $k^* \times \cdots \times k^* = (k^*)^n$ for some $n \in \mathbb{N}$.

We do not prove Hilbert's Theorem in this thesis. Instead, for a detailed proof we refer to [Sch, Chapter 1.4.2].

The assumption "reductive" is really necessary. In fact, Popov showed that any non-reductive affine algebraic group possesses a representation on a finitely generated $\mathbb{C}$-algebra whose invariant ring is not finitely generated, c.f. [Gro, Chapter 2, §8, Theorem 8.2].

For the general theory of reductive groups, we refer for example to [Hum, Chapter X]. For us it is only relevant that $\mathrm{GL}_n\mathbb{C}$ and finite products of such are reductive.

Example 3.9. Examples for reductive groups are $\mathrm{GL}_n(k)$ or $\mathrm{SL}_n(k)$ for $n \in \mathbb{N}$.

One can easily verify that $R(\mathrm{GL}_n(k)) = k^* \cdot I_n$. Since we have $k^* \cdot I_n \cong k^*$, we get that $\mathrm{GL}_n(k)$ is reductive.

For $\mathrm{SL}_n(k)$ the set $k^* \cdot I_n \cap \mathrm{SL}_n(k) = \mu_n \cdot I_n$, where μ_n denotes the set of all n-th roots of unity, is a maximal solvable normal subgroup. But it is in general not connected, since μ_n is finite. Hence $R(\mathrm{SL}_n(k)) = \{I_n\} \cong (k^*)^0$ and $\mathrm{SL}_n(k)$ is reductive.

The group we will be working with is given as a finite product of general linear groups. The reductivity is ensured by the following proposition.

Proposition 3.10. *Products of reductive linear algebraic groups are reductive.*

Proof. Let $G = G_1 \times \cdots \times G_m$, such that G_i is reductive for all $1 \leq i \leq m$. We need to show that G is reductive. To do so we show that

$$R \coloneqq R(G_1) \times \cdots \times R(G_m) \subseteq G$$

is the radical of G. Since all $R(G_i)$ are isomorphic to some $(k^*)^{n_i}$, we get that R is isomorphic to $(k^*)^{n_1+\cdots+n_m}$, hence a torus, implying our claim.

Notice that R is connected by [Hum, Chapter 7.5, Corollary], and normal in G and solvable, since all $R(G_i)$ are connected, solvable normal subgroups of G_i and group operations in G are component wise. It remains to show that R is maximal with these properties.

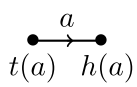
Denote for $1 \leq i \leq m$ by $\mathrm{pr}_i \colon G \to G_i$ the projection map on this factor. Note that pr_i is surjective and a morphism of linear algebraic groups. This implies that $\mathrm{pr}_i(R(G))$ is connected, normal and solvable. Assume now $R \subsetneq R(G)$. Since $R = R(G_1) \times \cdots \times R(G_m)$, there exists i, $1 \leq i \leq m$, such that $R(G_i) \subsetneq \mathrm{pr}_i(R(G))$. But the later is connected, normal and solvable in G_i and $R(G_i)$ maximal with these properties, hence we get a contradiction. □

4 Quivers and Quiver Representations

We now apply the above theory to a specific situation, namely the theory of representations of quivers. We start with the basic definitions.

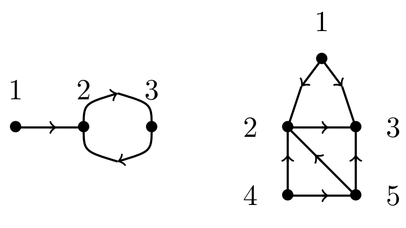
4.1 First Notions

Definition 4.1. A *quiver* is a finite directed graph with possibly loops and multiple edges; that is a quadruple $Q = (Q_0, A, t, h)$ with the finite set Q_0, consisting of the vertices of the graph, and the finite set A, consisting of the arrows, and two maps $t, h \colon A \to Q_0$, which associate to an arrow $a \in A$ its tail $t(a)$ and head $h(a)$, respectively.



For our purpose, we assume the graph to be connected. From now on, we label the vertices of Q by the numbers $\{1, \ldots, n\}$ and identify a vertex with its label.

Example 4.2. The following two pictures show two examples of quivers. The vertices are identified by their labelling. Both quivers are connected.



Examplarily, we explicitly describe the data of the first quiver, i.e. the sets Q_0, A and the two maps t, h. We have $Q_0 = \{1, 2, 3\}$, $A = \{a_1 : 1 \to 2,$

$a_2 : 2 \to 3, a_3 : 3 \to 2\}$ and the two maps $t, h : A \to Q_0$ with $t(a_i) = i$ for all $i = 1, 2, 3$ and $h(a_1) = 2$, $h(a_2) = 3$ and $h(a_3) = 2$.

Next, we define a representation of a quiver and morphisms between representations of quivers.

Definition 4.3. A *representation* V of a quiver Q assigns a finite-dimensional k-vector space V_i to each vertex $1 \leq i \leq n$, and a linear map

$$f_{V,a} \colon V_{t(a)} \to V_{h(a)}$$

to each arrow $a \in A$.

If we talk about a fixed representation V we write f_a for $f_{V,a}$ for all arrows $a \in A$, to abbreviate notation.

Definition 4.4. The tuple $\underline{d} = (\dim V_i)_{1 \leq i \leq n} \in \mathbb{N}^n$ is called the *dimension vector* of the representation V.

Definition 4.5. A *morphism* g between two representations V and W of a quiver Q is a family of linear maps $(g_i \colon V_i \to W_i)_{1 \leq i \leq n}$ such that for all arrows $a \in A$ the following diagram commutes:

$$\begin{array}{ccc} V_{t(a)} & \xrightarrow{f_{V,a}} & V_{h(a)} \\ \downarrow{\scriptstyle g_{t(a)}} & & \downarrow{\scriptstyle g_{h(a)}} \\ W_{t(a)} & \xrightarrow[f_{W,a}]{} & W_{h(a)} \end{array}$$

This means we have $g_{h(a)} \circ f_{V,a} = f_{W,a} \circ g_{t(a)}$.

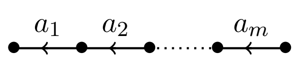
An isomorphism g of two representations is a morphism of representations such that g_i is an isomorphism of vector spaces for all vertices $1 \leq i \leq n$.

4.2 The Path Algebra of a Quiver and Its Representations

In this subsection we define the path algebra of a quiver Q. Furthermore, we show that there is an equivalence of categories between representations of Q and representations of its path algebra. For a more detailed study see [Bri, Section 1.2, Proposition 1.2.2].

To do so we fix a quiver Q, i.e. a set of vertices Q_0, a set of arrows between these vertices A, and two maps $t, h \colon A \to Q_0$, assigning to an arrow its tail and its head, respectively.

Definition 4.6. An *oriented path* in Q is a family $\alpha \coloneqq (a_1, a_2, \ldots, a_m)$ of arrows with $a_i \in A$, $1 \leq i \leq m$, such that $t(a_i) = h(a_{i+1})$ for all $a_i \in \alpha$, $1 \leq i \leq m - 1$. If $t(a_m) = h(a_1)$ then α is called an *oriented cycle* in Q of length m.



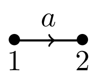
We are now ready to define the path algebra.

Definition 4.7. The *path algebra* kQ of a quiver Q is the vector space with basis the trivial paths e_i of length 0 at every vertex i in Q, $1 \leq i \leq n$, which start and end in i, and all other oriented paths in Q. Addition and scalar multiplication is given by formal linear combination. Multiplication is given by concatenation, if possible. More precisely, for two paths $\alpha = (a_1, \dots, a_m)$, $\beta = (b_1, \dots, b_n)$, we set

$$\alpha \circ \beta := \begin{cases} (a_1, \dots, a_m, b_1, \dots, b_n), & \text{if } h(b_1) = t(a_m), \\ 0, & \text{otherwise.} \end{cases}$$

Remark 4.8. It is easy to verify that kQ becomes an associative k-algebra with the above multiplication.

Example 4.9. For the quiver



with vertices 1 and 2 and arrow $a \colon 1 \to 2$ we get $kQ = ke_1 \oplus k(a) \oplus ke_2$ as vector space. For the paths $\alpha = (a), \beta = (e_2)$ we have

$$\alpha \circ \beta = (a) \circ (e_2) = 0, \text{ but } \beta \circ \alpha = (e_2, a) = (a).$$

In regard to this calculation, we can define a unit element in kQ, namely $1 = e_1 + e_2$. It is easy to check, that this definition satisfies all expected properties of a 1 in an algebra. Hence we get that kQ is a unital k-algebra.

This example already gives us a little intuition about the properties of the path algebra of a quiver. We will recapitulate this in the next proposition.

Proposition 4.10.

1. *The path algebra kQ is an associative k-algebra with 1, namely the element $e := e_1 + \dots + e_n \in kQ$, where e_i is the trivial path at vertex i of length 0 for all $1 \leq i \leq n$, satisfies its properties.*
2. *It is finite-dimensional if and only if Q has no oriented cycles.*

Proof.

1. It is enough to show that e satisfies $e \circ a = a \circ e = a$ for any arrow $a \in A$. Let $a \colon i \to j$ be an arrow from vertex i to j, $1 \leq i, j \leq n$. Then we have

$$e \circ a = (e_1 + \dots + e_n) \circ a = (e_j, a) = a$$
$$a \circ e = a \circ (e_1 + \dots + e_n) = (a, e_i) = a,$$

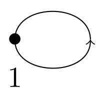
hence e is a neutral element.

2. This is clear since our vector space has basis all paths in Q. Assuming Q has no oriented cycles, then there are only finitely many paths in Q, so Q is finite-dimensional. On the other hand, if kQ is finite-dimensional then there are only finitely many paths by definition, hence no oriented cycles.

□

We can apply this proposition in the following example.

Example 4.11. The given quiver



with one vertex 1 and an arrow $a\colon 1 \to 1$, has an oriented cycle. Hence by the above proposition, we know that its path algebra is infinite-dimensional. In particular, we get that

$$kQ \cong ke_1 \oplus ka \oplus ka^2 \oplus ka^3 \oplus \cdots ,$$

hence $kQ \cong k[x]$, where $k[x]$ denotes the polynomial ring over k in one variable.

Since kQ is an associative k-algebra we have the standard notion of representations.

Definition 4.12. A finite-dimensional representation of kQ is a pair (V, ρ), where V is a finite-dimensional k-vector space and

$$\rho\colon kQ \to \mathrm{End}_k(V)$$

a morphism of k-algebras.

Equivalently we can say that the k-vector space V is a kQ-module via the algebra homomorphism $\rho\colon kQ \to \mathrm{End}_k(V)$ by setting

$$\alpha.v := \rho(\alpha)(v)$$

for all $v \in V$, $\alpha \in kQ$.

4.3 Representations of Q Versus Representations of kQ

The question arises how representations of Q and representations of kQ relate. This leads to the next proposition.

Proposition 4.13. *There is an equivalence between the category of representations of Q, which we denote by* $\mathrm{Rep}(Q)$ *and the category of finite-dimensional representations of its path algebra kQ,* kQ-mod.

PROOF. We first define a functor $F\colon \mathrm{Rep}(Q) \to kQ$-mod. On objects $((V_i)_{1\le i\le n}, (f_a)_{a\in A}) \in \mathrm{Rep}(Q)$ we set

$$V := \bigoplus_{i=1}^{n} V_i.$$

Then V is an N-dimensional vector space for $N := \sum_{i=1}^{n} \dim(V_i)$. For an arrow $a \in A$, we define

$$\hat{f}_a\colon V \to V, \quad \hat{f}_a(v) := (f_a \circ \mathrm{pr}_{t(a)})(v),$$
$$\hat{e}_i\colon V \to V, \quad \hat{e}_i(v) := (\mathrm{pr}_i)$$

where pr_i denotes the projection map from V to V_i for $1 \le i \le n$. Then $\hat{f}_a$ and $\hat{e}_i \in \mathrm{End}_k(V)$ for all $a \in A$ and $1 \le i \le n$. Note that kQ is generated as k-vector space by all paths in Q. Hence it is enough to define $\rho\colon kQ \to \mathrm{End}_k(V)$ for all paths in Q. Let $\alpha = (a_1, \dots, a_m)$ be a path in Q. Then we set

$$\rho(\alpha) = \rho((a_1, \dots, a_m)) := \hat{f}_{a_1} \circ \cdots \circ \hat{f}_{a_m}.$$

Extending ρ linearly to all of kQ, we have that ρ is a homomorphism of k-vector spaces. It is left to check that ρ is in fact a morphism of k-algebras. It is again enough to check this on the given basis. Let $\alpha = (a_1, \dots, a_m)$ and $\beta = (b_1, \dots, b_{m'})$ be two paths in Q. If $h(b_1) = t(a_m)$, we have that

$$\begin{aligned}
\rho(\alpha \circ \beta) &= \rho((a_1, \dots, a_m) \circ (b_1, \dots, b_{m'})) \\
&= \rho((a_1, \dots, a_m, b_1, \dots, b_{m'})), \\
&= \hat{f}_{a_1} \circ \cdots \circ \hat{f}_{a_m} \circ \hat{f}_{b_1} \circ \cdots \circ \hat{f}_{b_{m'}}, \\
&= \rho((a_1, \dots, a_m)) \circ \rho((b_1, \dots, b_{m'})), \\
&= \rho(\alpha) \circ \rho(\beta).
\end{aligned}$$

It is left to check that $\rho((a_1, \dots, a_m)) \circ \rho((b_1, \dots, b_{m'})) = 0$ if $h(b_1) \ne t(a_m)$, because then $\alpha \circ \beta = 0$. Using the definition of $\hat{f}_a$ we get

$$\begin{aligned}
&\rho((a_1, \dots, a_m)) \circ \rho((b_1, \dots, b_{m'})) \\
&\quad = \hat{f}_{a_1} \circ \cdots \circ \hat{f}_{a_m} \circ \hat{f}_{b_1} \circ \cdots \circ \hat{f}_{b_{m'}} \\
&\quad = \hat{f}_{a_1} \circ \cdots \circ \hat{f}_{a_{m-1}} \circ f_{a_m} \circ \mathrm{pr}_{t(a_m)} \circ f_{b_1} \circ \mathrm{pr}_{t(b_1)} \circ \hat{f}_{b_2} \circ \cdots \circ \hat{f}_{b_{m'}} \\
&\quad = 0,
\end{aligned}$$

since $\mathrm{pr}_{t(a_m)} \circ f_{b_1} = 0$, because f_{b_1} has image in $V_{h(b_1)} \ne V_{t(a_m)}$, hence $\mathrm{pr}_{t(a_m)}$ maps the image of f_{b_1} to 0.

To define F on morphisms, let g be a morphism of representations of Q. Recall that a morphism g between two representations $((V_i)_{1\le i\le n}, (f_{V,a})_{a\in A})$

and $((W_i)_{1\leq i\leq n}, (f_{W,a})_{a\in A})$ is a family of linear maps $(g_i\colon V_i \to W_i)_{1\leq i\leq n}$ such that for all arrows $a \in A$ the following diagram commutes:

$$\begin{array}{ccc} V_{t(a)} & \xrightarrow{f_{V,a}} & V_{h(a)} \\ {\scriptstyle g_{t(a)}}\downarrow & & \downarrow{\scriptstyle g_{h(a)}} \\ W_{t(a)} & \xrightarrow[f_{W,a}]{} & W_{h(a)} \end{array}$$

That means we have

$$g_{h(a)} \circ f_{V,a} = f_{W,a} \circ g_{t(a)}. \tag{10}$$

From g we define the morphism of representations of the path algebra $F(g)$ as follows. For $V := \bigoplus_{i=1}^n V_i$ and $W := \bigoplus_{i=1}^n W_i$ we set

$$F(g)\colon V \to W, F(g)(v_1 + \cdots + v_n) := g_1(v_1) + \cdots + g_n(v_n),$$

with componentwise application. This is well-defined since we can write $v = v_1 + \cdots + v_n$ uniquely with $v_i \in V_i$ for all $v \in V$. We have to check that $F(g)$ is in fact a morphism of kQ-modules, i.e. a morphism such that for all $\alpha \in kQ$ and all $v \in V$ we have that

$$F(g)(\alpha.v) = \alpha.(F(g)(v)),$$

where the action of kQ on V and W is given by $\rho_V\colon kQ \to \mathrm{End}_k(V)$ and $\rho_W\colon kQ \to \mathrm{End}_k(W)$, respectively. The maps ρ_V and ρ_W are constructed as in the first part of the proof. Because $F(g)$ is k-linear and kQ is, as an algebra, generated by all arrows in Q, it is enough to check the conditions for these elements. For $a \in A$ and $v = v_1 + \cdots + v_n \in V$, the left hand side simplifies to

$$\begin{aligned} F(g)(a.v) &= F(g)(\rho_V(a)(v)) \\ &= (g_1 \times \cdots \times g_n)(\hat{f}_{V,a}(v)) \\ &= (g_1 \times \cdots \times g_n)((f_{V,a} \circ \mathrm{pr}_{t(a)})(v)) \\ &= g_{h(a)} \circ f_{V,a}(v_{t(a)}) \\ &= f_{W,a} \circ g_{t(a)}(v_{t(a)}) \\ &= f_{W,a} \circ \mathrm{pr}_{t(a)}((g_1 \times \cdots g_n)(v)) \\ &= \hat{f}_{W,a}(F(g)(v)) \\ &= \rho_W(a)(F(g)(v)), \end{aligned}$$

by property (10).

Let $G\colon kQ\text{-mod} \to \mathrm{Rep}(Q)$ be the functor defined as following: For an object (V, ρ) in kQ-mod we set $G((V, \rho)) := ((V_i)_{1\leq i\leq n}, (f_a)_{a\in A})$ with

$$V_i := \rho(e_i)(V)$$

for all vertices $i \in Q_0$, and for all arrows $a \in A$ we define

$$f_a \colon V_{t(a)} \to V_{h(a)}, v \mapsto \rho(a)(v).$$

This is well-defined since $a = e_{h(a)} \circ a$. We find $\rho(a)(v) = \rho(e_{h(a)} \circ a)(v) = \rho(e_{h(a)})(\rho(a)(v)) \in \rho(e_{h(a)})(V) = V_{h(a)}$. The map is linear since ρ is a map of k-algebras. Thus this defines a representation of Q.

Now let $g \colon V \to W$ be a morphism of kQ-modules, which means that

$$g(\rho_V(\alpha)(v)) = \rho_W(\alpha)(g(v)), \tag{11}$$

for all $\alpha \in kQ$ and $v \in V$. For V_i and W_i defined as above for all vertices $i \in Q_0$, we set

$$g_i \colon V_i \to W_i, g_i \coloneqq g_{|V_i}.$$

Since $V_i = \rho_V(e_i)(V)$ we have that for all $v_i \in V_i$ there exists $v \in V$ such that $v_i = \rho_V(e_i)(v)$. Hence we have $g(v_i) = g(\rho_V(e_i)(v)) = \rho_W(e_i)(g(v)) \in W_i$ by definition of W_i and property (11).

Left to show is that $g_{h(a)} \circ f_{V,a} = f_{W,a} \circ g_{t(a)}$ for all arrows $a \in A$. Therefore let $a \colon i \to j$ be an arrow in Q. We have to check that $g_j \circ f_{V,a} = f_{W,a} \circ g_i$. Recall that $g_i = g_{|V_i}$. Let $v \in V_i$. Then we have

$$g_j \circ f_{V,a}(v) = g_j(\rho_V(a)(v)) = g(\rho_V(a)(v)) = \rho_W(a)(g(v)) = f_{W,a} \circ g_i(v),$$

hence $G(g)$ is a morphism in $\mathrm{Rep}(Q)$.

It is easy to check that $F \circ G$ and $G \circ F$ are naturally isomorphic to the identity functor. Hence we get the equivalence of categories stated in the claim. □

By the last proposition we see that to describe representations of quivers it is enough to understand representations of the associated path algebra.

Given the quiver as in Example 4.11, we already showed that $kQ \cong k[x]$ the polynomial ring in one variable. A representation of Q assigns a finite-dimensional k-vector space V to the vertex 1 and an endomorphism $f \colon V \to V$ to the only arrow of the quiver. On the other hand, a representation of $k[x]$ is a pair (W, ρ) for W a finite-dimensional k-vector space and $\rho \colon k[x] \to \mathrm{End}_k(W)$. We get

$$\begin{cases} \mathrm{Rep}(Q) \ni (V, f) \mapsto (V, \rho \colon k[x] \to \mathrm{End}_k(V), x \mapsto f) \in k[x]\text{-mod}, \\ k[x]\text{-mod} \ni (W, \rho) \mapsto (W, \rho(x)) \in \mathrm{Rep}(Q). \end{cases}$$

4.4 Isomorphism Classes of Representations of Quivers

In this section we try to understand isomorphism classes of representations of a quiver. We will see that this problem can be put into the context of Geometric Invariant Theory as it is presented in Section 3 focusing on Subsection 3.3.

When trying to classify representations of a given quiver Q up to isomorphism, we notice that two isomorphic representations have the same dimension vector. Thus let $\underline{d} \in \mathbb{N}^n$ be a fixed dimension vector. By choosing bases, we may assume that the assigned vector spaces of a representation are standard vector spaces.

Definition 4.14. The k-vector space

$$\begin{aligned} \operatorname{Rep}(Q, \underline{d}) &:= \bigoplus_{a \in A} \operatorname{Hom}_k(k^{d_{t(a)}}, k^{d_{h(a)}}) \\ &= \bigoplus_{a \in A} M_a(k), \end{aligned}$$

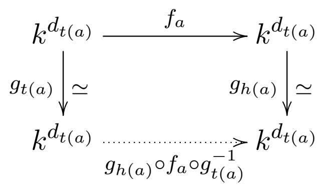
with $M_a(k) = M_{d_{h(a)} \times d_{t(a)}}(k)$, is called the *representation space* of the quiver Q with respect to the dimension vector $\underline{d}$. As $\operatorname{Rep}(Q, \underline{d})$ is a k-vector space, we consider it as an affine variety. We set

$$\operatorname{GL}(Q, \underline{d}) := \prod_{i=1}^{n} \operatorname{GL}_{d_i}(k).$$

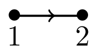
Then the group $\operatorname{GL}(Q, \underline{d})$ acts on the k-vector space $\operatorname{Rep}(Q, \underline{d})$ via the canonical action

$$(g_i)_{1 \leq i \leq n}.(f_a)_{a \in A} = (g_{h(a)} \circ f_a \circ g_{t(a)}^{-1})_{a \in A}. \tag{12}$$

Thus for each arrow $a \in A$ and its assigned linear map f_a, we have



Example 4.15. For the quiver



with $\underline{d} = (d_1, d_2)$ we get $\operatorname{GL}(Q, \underline{d}) = \operatorname{GL}_{d_1}(k) \times \operatorname{GL}_{d_2}(k)$ and $\operatorname{Rep}(Q, \underline{d}) = \operatorname{Hom}_k(k^{d_1}, k^{d_2})$. For $f \in \operatorname{Hom}_k(k^{d_1}, k^{d_2})$, $(g_1, g_2) \in \operatorname{GL}_{d_1}(k) \times \operatorname{GL}_{d_2}(k)$ we have

$$(g_1, g_2).f = g_2 \circ f \circ g_1^{-1}.$$

This is precisely the notion of equivalence of two matrices; Two matrices $A, B \in M_{n \times m}(k)$ are called equivalent, if there exist $G_1 \in \operatorname{GL}_m(k)$ and $G_2 \in \operatorname{GL}_n(k)$ such that $A = G_2 \circ B \circ G_1^{-1}$. The rank of the matrices A,B is the only invariant of the orbits under this action: two matrices are equivalent if and only if they have the same rank.

Example 4.16. For the quiver presented in Example 4.11, we have that two representations (V, φ) and (W, ψ) for V and W finite-dimensional k-vector spaces and $\varphi \in \mathrm{End}_k(V)$, $\psi \in \mathrm{End}_k(W)$, are isomorphic if $\dim_k V = \dim_k W = n$ and it exists $g \in \mathrm{GL}_n(k)$, such that

$$g\varphi g^{-1} = \psi.$$

If so, we call the representing matrices of φ and ψ with respect to a chosen basis similar. We know that two matrices are similiar if and only if they have (up to permutation of the blocks) the same Jordan normal form. Hence we can explicitly determine isomorphic representations of this quiver.

Obviously, the problem of classifying the representations up to isomorphism of a quiver Q with dimension vector $\underline{d}$ amounts to the problem we presented in Section 3. $\mathrm{Rep}(Q, \underline{d})$ is a k-vector space on which the linear reductive group $\mathrm{GL}(Q, \underline{d})$ acts as a product of linear reductive groups, cf. Remark 3.10. Two representations are isomorphic if and only if they lie in the same orbit under this action. We are therefore interested in describing the associated orbit space. By Lemma 3.5, we know that we can describe the set of closed orbits by finding the maximal ideals in the ring of polynomial invariants $k[\mathrm{Rep}(Q, \underline{d})]^{\mathrm{GL}(Q,\underline{d})}$. Hence, we are interested in describing the invariant ring as a first attempt to understand the orbit space.

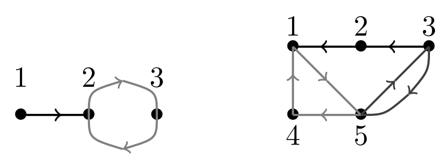
In particular, if $k = \mathbb{C}$ we can apply Hilbert's Theorem 3.7, since $\mathrm{GL}(Q, \underline{d})$ is reductive, and we know that the corresponding invariant ring is finitely generated. In the following section we want to find generators and a bound on their number.

5 Generating the Ring of Invariants

We keep the notation of Section 4. In this section we want to determine generators for $\mathbb{C}[\mathrm{Rep}(Q, \underline{d})]^{\mathrm{GL}(Q,\underline{d})}$. To do so we first need some notions.

Recall that an oriented cycle in Q is an oriented path $(a_1, \ldots, a_m)$ of length $m \in \mathbb{N}$ with $a_i \in A$ for $i \in \{1, \ldots, m\}$, such that $t(a_i) = h(a_{i+1})$ for $i \in \{1, \ldots, m-1\}$ and $t(a_m) = h(a_1)$, i.e. a path that starts and ends in the same vertex.

Example 5.1. The following two graphs represent two quivers, in which we coloured in gray some of their oriented cycles. It is clear that there are a lot more and sometimes they are not easy to find.



Note that we cannot mark all oriented cycles. In the first quiver for example, the two cycles $2 \to 3 \to 2$ and $3 \to 2 \to 3$ are two different paths though they represent the same cycle in our picture. Furthermore, we have to distinguish between oriented cycle of different lengths, i.e. the oriented cycles $2 \to 3 \to 2$ and $2 \to 3 \to 2 \to 3 \to 2$.

Now fix a quiver Q and a dimension vector $\underline{d}$. Our next idea is to describe $\mathrm{GL}(Q, \underline{d})$-invariant polynomial maps from the representation space to k, which arise from oriented cycles in our quiver. Let $\sigma = (a_1, \dots, a_m)$ be an oriented cycle of length m in Q. To σ we can assign a polynomial map

$$\begin{aligned} t_\sigma \colon \mathrm{Rep}(Q, \underline{d}) &\to \mathbb{C} \\ (f_a, a \in A) &\mapsto \mathrm{tr}(f_{a_1} \circ \cdots \circ f_{a_m}), \end{aligned}$$

where tr is the trace map on endomorphisms.

This is well-defined, because $t(a_m) = h(a_1)$ and a polynomial map , since tr is one. Thus we get $t_\sigma \in \mathbb{C}[\mathrm{Rep}(Q, \underline{d})]$ for all oriented cycles σ in Q.

Remark 5.2. *For σ an oriented cycle in Q, the induced map t_σ is invariant under the action of $\mathrm{GL}(Q, \underline{d})$.*

Proof. Let $\sigma = (a_1, \dots, a_m)$ and $g \in \mathrm{GL}(Q, \underline{d})$, so $g = (g_i)_{1 \le i \le n}$, $g_i \in \mathrm{GL}(d_i, \mathbb{C})$. Then for $(f_a, a \in A) \in \mathrm{Rep}(Q, \underline{d})$ we have:

$$\begin{aligned} (g_i).t_\sigma(f_a) &= t_\sigma(((g_i)^{-1}).(f_a)) \\ &= \mathrm{tr}(g_{h(a_1)}^{-1} \circ f_{a_1} \circ g_{t(a_1)} \circ g_{h(a_2)}^{-1} \circ f_{a_2} \circ g_{t(a_2)} \circ \cdots \\ &\quad \cdots \circ g_{t(a_{m-1})} \circ g_{h(a_m)}^{-1} \circ f_{a_m} \circ g_{t(a_m)}) \qquad (13) \\ &= \mathrm{tr}(g_{h(a_1)}^{-1} \circ f_{a_1} \circ f_{a_2} \circ \cdots \circ f_{a_{m-1}} \circ f_{a_m} \circ g_{t(a_m)}) \qquad (14) \\ &= \mathrm{tr}(g_{h(a_1)}^{-1} \circ f_{a_1} \circ f_{a_2} \circ \cdots \circ f_{a_{m-1}} \circ f_{a_m} \circ g_{h(a_1)}) \\ &= \mathrm{tr}(f_{a_1} \circ \cdots \circ f_{a_m}) \qquad (15) \\ &= t_\sigma(f_a), \end{aligned}$$

where (13) holdy by the definition of t_σ and the action described in (12). The equality (14) follows from the fact that σ is an oriented cycle, so we have $t(a_i) = h(a_{i+1})$ for $i \in \{1, \dots, m-1\}$ and $t(a_m) = h(a_1)$. Therefore the elements in the middle cancel out, and we get (15) because tr is invariant under conjugation and hence the claim follows. □

By Remark 5.2 we can state the main theorem of this thesis.

Theorem 5.3 (Le Bruyn-Procesi). *Let Q be a quiver and $\underline{d} \in \mathbb{N}^n$ a dimension vector. Then the invariant ring $\mathbb{C}[\mathrm{Rep}(Q, \underline{d})]^{\mathrm{GL}(Q, \underline{d})}$ is generated by the elements t_σ for σ an oriented cycle in Q.*

Remark 5.4. Procesi showed in [Pro3, Section 3, Theorem 3.4] that we can restrict to those cycles $\sigma = (a_1, \dots, a_m)$ with $m \leq 2^N - 1$, where $N := \sum_{i=1}^n d_i$. In [Sch, Chapter 1.3, Theorem 1.3.3.10] states in his formulation of the Theorem that we can even restrict to those of length $m \leq 1 + N^2$.

We will now give two totally different proofs for this theorem. One is very direct using Schur-Weyl-duality, whereas the other is rather abstract.

The main focus lies on the direct proof based upon the paper of Aslaksen, Tan and Zhu as in [ATZ, Section 2]. We are now going to present this proof, providing considerably more details.

The abstract proof is based on a paper by Le Bruyn and Procesi; see [PrBr, Section 3]. It uses a lot of techniques of which most can be found in [Pro1, Chapter IV, §1].

5.1 Direct Proof

The main idea to directly prove Theorem 5.3 is to find multilinear invariants instead of polynomial invariants.

To find elements in $\mathcal{P}^l[V]^G$, we only need to find G-invariant linear functionals on $V^{\otimes l}, l \geq 0$. More details can be found in Subsection 3.1.

Recall that $\operatorname{Rep}(Q, \underline{d}) = \bigoplus_{a \in A} \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}^{d_{t(a)}}, \mathbb{C}^{d_{h(a)}})$. In particular, this is a finite-dimensional $\mathbb{C}$-vector space. To apply (4) setting $V := \operatorname{Rep}(Q, \underline{d})$, we consider the l-fold tensor product of V for some $l \in \mathbb{N}$. By the distributivity law for direct sums and tensor products, we obtain a direct sum of tensor products with l factors each, such that every factor is of the form $\operatorname{Hom}_{\mathbb{C}}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j})$ for some arrow $a\colon i \to j, \in A$. Let us now consider one of these summands $\mathcal{S}$, where $\operatorname{Hom}_{\mathbb{C}}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j})$ appears exactly m_{ij} times as a factor in $\mathcal{S}$ with $m_{ij} \in \{1, \dots, l\}$ and $\sum_{a\colon i \to j, a \in A} m_{ij} = l$. The tuple $(m_{ij})_{a\colon i \to j}$ is called the degree of the summand. It is enough to consider invariants in one of these summands since $\operatorname{GL}(Q, \underline{d})$ acts linearly as described in Section 3.2. With this notation we get

$$\begin{aligned}
\mathcal{S} &\cong \bigotimes_{a\colon i \to j, a \in A} \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j})^{\otimes m_{ij}} && (16)\\
&\cong \bigotimes_{a\colon i \to j, a \in A} ((\mathbb{C}^{d_i})^* \otimes \mathbb{C}^{d_j})^{\otimes m_{ij}} && (17)\\
&\cong \bigotimes_{i=1}^{n} ((\mathbb{C}^{d_i})^*)^{\otimes O(i)} \otimes (\mathbb{C}^{d_i})^{\otimes I(i)} && (18)\\
&= \bigotimes_{i=1}^{n} \mathcal{S}_i, && (19)
\end{aligned}$$

with

$$O(i) := \sum_{a \in A, t(a) = i} m_{i,h(a)},$$

$$I(i) := \sum_{a \in A, h(a) = i} m_{t(a),i},$$

and

$$\mathcal{S}_i := ((\mathbb{C}^{d_i})^*)^{\otimes O(i)} \otimes (\mathbb{C}^{d_i})^{\otimes I(i)}.$$

The second isomorphism holds because of Lemma 2.1.2. The next step is just reordering; instead of sorting according to arrows in our quiver, we gather together the factors according to the vertices.

Lemma 5.5. *Notice that the isomorphism (17) is a* $\mathrm{GL}(Q, \underline{d})$*-equivariant map, i.e. an isomorphism of* $\mathrm{GL}(Q, \underline{d})$*-modules.*

Proof. It is enough to consider the map

$$\Psi\colon (\mathbb{C}^{d_1})^* \otimes \mathbb{C}^{d_2} \to \mathrm{Hom}(\mathbb{C}^{d_1}, \mathbb{C}^{d_2}),$$
$$\alpha \otimes v \mapsto \varphi_{\alpha,v}$$

for one factor of the tensor product where $\varphi_{\alpha,v}(w) := \alpha(w)v$, for $w \in \mathbb{C}^{d_1}$, and $\mathrm{Hom}(\mathbb{C}^{d_1}, \mathbb{C}^{d_2})$ is considered as $(\mathrm{GL}_{d_1}(\mathbb{C}) \times \mathrm{GL}_{d_2}(\mathbb{C}))$-module via the action

$$(g, h).\varphi := h \circ \varphi \circ g^{-1},$$

for $g \in \mathrm{GL}_{d_1}(\mathbb{C}), h \in \mathrm{GL}_{d_2}(\mathbb{C})$ and $\varphi \in \mathrm{Hom}(\mathbb{C}^{d_1}, \mathbb{C}^{d_2})$. This definition arises from the action of $\mathrm{GL}(Q, \underline{d})$ on the representation space of our quiver Q as in (12). The tensor product $(\mathbb{C}^{d_1})^* \otimes \mathbb{C}^{d_2}$ is $(\mathrm{GL}_{d_1}(\mathbb{C}) \times \mathrm{GL}_{d_2}(\mathbb{C}))$-module via the canonical action on each factor.

Let $G := \mathrm{GL}_{d_1}(\mathbb{C}) \times \mathrm{GL}_{d_2}(\mathbb{C})$. We need to check that Ψ is G-equivariant. Then Ψ^{-1} as map of vector spaces is also G-equivariant and the claim follows.

For $w \in \mathbb{C}^{d_1}$ we have

$$\begin{aligned}
[\Psi((g,h).(\alpha \otimes v))](w) &= \varphi_{g.\alpha, h(v)}(w) \\
&= ([g.\alpha](w))h(v) \\
&= \alpha(g^{-1}(w))h(v),
\end{aligned}$$

$$\begin{aligned}
[(g,h).\Psi(\alpha \otimes v)](w) &= [(g,h).\varphi_{\alpha,v}](w) \\
&= (h \circ \varphi_{\alpha,v} \circ g^{-1})(w) \\
&= (h \circ \varphi_{\alpha,v})(g^{-1}(w)) \\
&= h(\alpha(g^{-1}(w))v) \\
&= \alpha(g^{-1}(w))h(v).
\end{aligned}$$

□

In the next step we want to apply (4). We want to simplify finding invariants under $\mathrm{GL}(Q, \underline{d})$ in $\mathcal{S}$ to finding $\mathrm{GL}_{d_i}(\mathbb{C})$-invariants in $\mathcal{S}_i$ for each vertex i, $1 \leq i \leq n$. To do so, we need the following lemma.

Lemma 5.6. *Let $V_1, \dots, V_r$ be finite-dimensional k-vector spaces. Let $G_i \subseteq \mathrm{GL}(V_i)$, $1 \le i \le r$ be a subgroup of $\mathrm{GL}(V_i)$. Then $G \coloneqq G_1 \times \cdots \times G_r$ acts on $V_1 \otimes \cdots \otimes V_r$ via*

$$(g_1, \dots, g_r).(v_1 \otimes \cdots \otimes v_r) = g_1 v_1 \otimes \cdots \otimes g_r v_r.$$

Then

$$(V_1 \otimes \cdots \otimes V_r)^G = V_1^{G_1} \otimes \cdots \otimes V_r^{G_r}.$$

Proof. We prove this by induction on r. For $r = 1$ this is obviously true. Now let $r \ge 1$. We set $W \coloneqq V_1 \otimes \cdots \otimes V_{r-1}$ and $H \coloneqq G_1 \otimes \cdots \otimes G_{r-1} \subseteq \mathrm{GL}(W)$. Then H is a subgroup of $\mathrm{GL}(W)$ and by induction the claim holds for W. Thus it is enough to proof the claim for two factors.

Let V and W be two finite-dimensional k-vector spaces and $G \subseteq \mathrm{GL}(V)$, $H \subseteq \mathrm{GL}(W)$ subgroups. It is clear that

$$V^G \otimes W^H \subseteq (V \otimes W)^{G \times H}.$$

For the other inclusion, take $x \in (V \otimes W)^{G \times H}$. Then we can write $x = \sum_{i=1}^m v_i \otimes w_i$ for $v_i \in V$, $w_i \in W$ and v_i linearly independent. Since x is invariant under the action of $G \times H$, we have $x = (1, h).x = \sum_{i=1}^m v_i \otimes h w_i$ for all $h \in H$. This implies that $h w_i = w_i$ for all $1 \le i \le m$: We have $0 = x - (1, h).x = \sum_{i=1}^m v_i \otimes (w_i - h w_i)$. We can write $w_i' \coloneqq w_i - h w_i$ in a basis $u_1, \dots, u_l$ of W, i.e. $w_i' = \sum_{j=1}^l a_{ij} u_j$. Thus we have $0 = \sum_{i=1}^m \sum_{j=1}^l a_{ij}(v_i \otimes u_j)$ and because v_i and u_j are linearly independent, so is the family $(v_i \otimes u_j)_{i,j}$ and hence $a_{ij} = 0$ for $1 \le i \le m$ and $1 \le j \le l$. Consequently, $w_i \in W^H$ for all $1 \le i \le m$.

Now choose $w_{i_1}, \dots, w_{i_{m'}} \in \{w_1, \dots, w_m\}$ as a basis of $\langle w_1, \dots, w_m \rangle \subseteq W^H$. Then we can write $x = \sum_{j=1}^{m'} v_j' \otimes w_{i_j}$ for suitable $v_i \in V$. We have $x = (g, 1).x = \sum_{j=1}^{m'} g v_j' \otimes w_{i_j}$ and by the same argument as above we get $v_j' \in V^G$ for all $1 \le j \le m'$, i.e. $x \in V^G \otimes W^H$. □

Applying the preceding lemma with $V_i = \mathcal{S}_i^*$ and $G_i = \mathrm{GL}_{d_i}(\mathbb{C})$ we obtain

$$(\mathcal{S}^*)^{\mathrm{GL}(Q,\underline{d})} = \bigotimes_{i=1}^n (\mathcal{S}_i^*)^{\mathrm{GL}_{d_i}(\mathbb{C})}. \tag{20}$$

It is thus enough to study $(\mathcal{S}_i^*)^{\mathrm{GL}_{d_i}(\mathbb{C})}$ for a fixed vertex $i \in Q_0$.

Proposition 5.7. *Fix a vertex $i \in Q_0$.*

1. *If $O(i) \neq I(i)$, then $(\mathcal{S}_i^*)^{\mathrm{GL}_{d_i}(\mathbb{C})} = \{0\}$.*
2. *If $O(i) = I(i)$, then $(\mathcal{S}_i^*)^{\mathrm{GL}_{d_i}(\mathbb{C})} \cong \mathrm{End}_{\mathrm{GL}_{d_i}(\mathbb{C})}((\mathbb{C}^{d_i})^{\otimes O(i)}) \cong \langle S_{O(i)} \rangle$.*

Recall that $\langle S_{O(i)} \rangle$ denotes the image of the map $kS_{O(i)} \to \mathrm{End}_k(\mathbb{C}^{d_i})$, $\pi \mapsto \bar{\pi}$.

Proof. To simplify notation, let $V := \mathbb{C}^{d_i}$ and $G := \mathrm{GL}_{d_i}(\mathbb{C})$ acting naturally on V.

1. Let $m := O(i)$, $m' := I(i)$ such that $m \neq m'$. Then we get

$$(\mathcal{S}_i^*)^G \cong \mathrm{Hom}_G(V^{\otimes m}, V^{\otimes m'}),$$

by Lemma 2.1.4 and the definition of $\mathcal{S}_i^*$. Assume we have a morphism f from $V^{\otimes m}$ to $V^{\otimes m'}$ that commutes with the G-action. In particular, f commutes with $T := \{\text{diagonal matrices}\} \subset G$.

Let $e_j, 1 \leq j \leq d_i$ denote the standard basis vectors of V. Then the $\langle e_j \rangle$ are T-invariant and we define $\varepsilon_j \colon T \to \mathbb{C}$ by $te_j = \varepsilon_j(t)e_j$, $1 \leq j \leq d_i$. Thus $\varepsilon_j(t)$ is the entry on position (j, j) of the matrix t.

Since G acts on $V^{\otimes m}$ and $V^{\otimes m'}$ diagonally, we see that $V^{\otimes m}$ and $V^{\otimes m'}$ can both be decomposed into a direct sum of 1-dimensional T-invariant subspaces with eigenvalues $\varepsilon_{j_1}(t) \cdots \varepsilon_{j_m}(t)$, $\varepsilon_{j_1}(t) \cdots \varepsilon_{j_{m'}}(t)$ for $t \in T$, with $j_l \in \{1, \ldots, d_i\}$, respectively.

Since f commutes with G, we have

$$t.f(v) = f(t.v) = f(\lambda v) = \lambda f(v)$$

for all $t \in T$ and all eigenvectors v of t on $V^{\otimes m}$ with eigenvalue λ. Thus eigenvectors of an element of T are send to eigenvectors of T with the same eigenvalue or to 0.

Consider now a matrix $t = \alpha \cdot I_{d_i}$, such that α is not a root of unity. Then,

$$\varepsilon_{j_1}(t) \cdots \varepsilon_{j_m}(t) = \alpha^m$$

and

$$\varepsilon_{j_1}(t) \cdots \varepsilon_{j_{m'}}(t) = \alpha^{m'}.$$

For $m \neq m'$ we have $\alpha^m \neq \alpha^{m'}$ and by the above $f = 0$.

2. Let $m := O(i) = I(i) \in \mathbb{N}$. We have $\mathcal{S}_i^* = ((V^*)^{\otimes m} \otimes V^{\otimes m})^* \cong \mathcal{S}_i$ just by permuting factors. It is therefore enough to consider invariants in $\mathcal{S}_i$. By the following calculation we get

$$\begin{aligned} \mathcal{S}_i^G &= ((V^*)^{\otimes m} \otimes V^{\otimes m})^G \\ &\cong \mathrm{Hom}_G(\mathbb{C}, (V^*)^{\otimes m} \otimes V^{\otimes m}) \\ &\cong \mathrm{Hom}_G(V^{\otimes m}, W^{\otimes m}) \\ &= \mathrm{End}_G(V^{\otimes m}) \\ &\cong \langle S_m \rangle, \end{aligned}$$

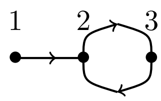
where the second isomorphism holds by Lemma 2.1.1. and the third by Lemma 2.1.3.

The Schur-Weyl-duality 2.6 implies the last step, which proves the proposition.

□

By Proposition 5.7 and Theorem 2.6 we can restrict to the case that $O(i) = I(i)$ for all vertices $1 \leq i \leq n$; otherwise there are no non-trivial invariants in $\mathcal{S}$. This will be illustrated in the following.

Example 5.8. We fix the quiver



with vertices 1, 2 and 3 and three arrows $a_{12}\colon 1 \to 2$, $a_{23}\colon 2 \to 3$ and $a_{32}\colon 3 \to 2$ and a dimension vector $\underline{d} = (d_1, d_2, d_3)$.

The representation space of our quiver is of the form

$$\mathrm{Rep}(Q, \underline{d}) = \mathrm{Hom}(\mathbb{C}^{d_1}, \mathbb{C}^{d_2}) \oplus \mathrm{Hom}(\mathbb{C}^{d_2}, \mathbb{C}^{d_3}) \oplus \mathrm{Hom}(\mathbb{C}^{d_3}, \mathbb{C}^{d_2}).$$

To abbreviate notation, we denote $\mathrm{Rep}(Q, \underline{d})$ by V and $\mathrm{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j})$ by $M_{a_{ij}}$ for an arrow $a\colon i \to j$. Hence $V = M_{a_{12}} \oplus M_{a_{23}} \oplus M_{a_{32}}$.

Now, let $l = 3$. Then we have

$$\begin{aligned} V \otimes V \otimes V &= (M_{a_{12}} \oplus M_{a_{23}} \oplus M_{a_{32}})^{\otimes 3} \\ &= \cdots \oplus (M_{a_{12}} \otimes M_{a_{23}} \otimes M_{a_{32}}) \oplus \cdots . \end{aligned}$$

Let us have a closer look at this summand, i.e.

$$\mathcal{S} := M_{a_{12}} \otimes M_{a_{23}} \otimes M_{a_{32}}.$$

Then we have exactly the degree $(1, 1, 1)$, because $m_{ij} = 1$ for $ij \in \{12, 23, 32\}$.

As explained above we get

$$\begin{aligned} \mathcal{S} &= \Big((\mathbb{C}^{d_1})^* \otimes \mathbb{C}^{d_2}\Big) \otimes \Big((\mathbb{C}^{d_2})^* \otimes \mathbb{C}^{d_3}\Big) \otimes \Big((\mathbb{C}^{d_3})^* \otimes \mathbb{C}^{d_2}\Big) \\ &= (\mathbb{C}^{d_1})^* \otimes \Big((\mathbb{C}^{d_2})^* \otimes (\mathbb{C}^{d_2})^{\otimes 2}\Big) \otimes \Big((\mathbb{C}^{d_3})^* \otimes \mathbb{C}^{d_3}\Big) \end{aligned}$$

This shows that $O(1) = 1 \neq 0 = I(1)$ and also $O(2) = 1 \neq 2 = I(2)$, hence by the last lemma, we cannot find any invariants in this summand.

Note that any summand, in which $M_{a_{12}}$ appears to some positive power m has no invariants, because we always have

$$O(1) = m \neq 0 = I(1).$$

Hence it follows that we have to have a look at those summands where $m_{12} = 0$.

In the case $l = 3$, such a summand can without loss of generality be assumed to be of the form

$$\mathcal{S} = M_{a_{23}} \otimes (M_{a_{32}})^{\otimes 2}.$$

But then,

$$\begin{aligned}\mathcal{S} &= \Big((\mathbb{C}^{d_2})^* \otimes \mathbb{C}^{d_3}\Big) \otimes \Big((\mathbb{C}^{d_3})^* \otimes \mathbb{C}^{d_2}\Big)^{\otimes 2} \\ &= \Big((\mathbb{C}^{d_2})^* \otimes (\mathbb{C}^{d_2})^{\otimes 2}\Big) \otimes \Big(((\mathbb{C}^{d_3})^*)^{\otimes 2} \otimes \mathbb{C}^{d_3}\Big),\end{aligned}$$

and hence $O(2) = 1 \neq 2 = I(2)$ and $O(3) = 2 \neq 1 = I(3)$. This implies that we cannot find any non-trivial invariants for $l = 3$ and every other odd natural number. But for even numbers there exists always at least one summand, such that $O(i) = I(i)$ for all $1 \leq i \leq n$ and the condition for Proposition 5.7 holds.

Schur-Weyl-duality tells us that all elements in $(\mathcal{S}_i^*)^G = (((V^*)^{\otimes m} \otimes V^{\otimes m})^*)^G$ are a linear combination of linear functionals arising from elements in S_m via the map $\mathbb{C}S_m \to \mathrm{End}_{\mathbb{C}}(V^{\otimes m}), \pi \mapsto \bar{\pi}$. We now want to find a concrete representation of these maps.

Proposition 5.9. *The $\mathbb{C}$-vector space $(((V^*)^{\otimes m} \otimes V^{\otimes m})^*)^G$ is generated by the elements μ_π for $\pi \in S_m$ with*

$$\mu_\pi(\varphi_1 \otimes \cdots \otimes \varphi_m \otimes v_1 \cdots \otimes v_m) \coloneqq \prod_{i=1}^{m} \varphi_i(v_{\pi(i)}),$$

for $\varphi_i \in V^*$ and $v_i \in V$, $1 \leq i \leq m$.

Proof. By Schur-Weyl-duality, we get that $(((V^*)^{\otimes m} \otimes V^{\otimes m})^*)^G$ is generated by the elements $\bar{\pi} \in \mathrm{End}_k(V^{\otimes m})$.

We set $\hat{\mu}_\pi \coloneqq \Psi(\bar{\pi})$ with Ψ as in the proof of Lemma 2.1.4. We still have to verify that Ψ is G-equivariant. Hence let $g \in G$. Then we have for $\varphi \in (V^*)^{\otimes m}$, $v \in V^{\otimes m}$, and $A \in \mathrm{End}_k(V^{\otimes m})$,

$$\begin{aligned} g.\Psi(A)(\varphi \otimes v) &= \Psi(A)(g^{-1}.\varphi \otimes g^{-1}.v) \\ &= (g^{-1}.\varphi)(A(g^{-1}.v)) \\ &= \varphi(g.(A(g^{-1}.v))) \\ &= \varphi((g.A)(v)) \\ &= \Psi(g.A)(\varphi \otimes v), \end{aligned}$$

by definition of the action of G on the endomorphisms. By Lemma 2.1.1 and 2.1.3 for $V^{\otimes m}$ we get that

$$(((V^*)^{\otimes m} \otimes V^{\otimes m})^*)^G \cong \mathrm{End}_G(V^{\otimes m}). \tag{21}$$

Then we compute

$$
\begin{aligned}
&\hat{\mu}_\pi(\varphi_1 \otimes \cdots \otimes \varphi_m \otimes v_1 \otimes \cdots \otimes v_m) \\
&\qquad = \Psi(\bar{\pi})(\varphi_1 \otimes \cdots \otimes \varphi_m \otimes v_1 \otimes \cdots \otimes v_m) \\
&\qquad = (\varphi_1 \otimes \cdots \otimes \varphi_m)(\bar{\pi}(v_1 \otimes \cdots \otimes v_m)) \\
&\qquad = (\varphi_1 \otimes \cdots \otimes \varphi_m)(v_{\pi(1)} \otimes \cdots \otimes v_{\pi(m)}) \\
&\qquad = \prod_{i=1}^{m} \varphi_i(v_{\pi(i)}).
\end{aligned}
$$

In particular, we get that $\hat{\mu}_\pi = \mu_\pi$. Since Ψ is an isomorphism and (21) holds, the claim follows. □

The above holds for all vertices in V. By (20), it follows that the space of invariants $\mathcal{S}^{\mathrm{GL}(Q,\underline{d})}$ is generated by elements of the form

$$
t_S := \prod_{i=1}^{n} \prod_{l=1}^{m_i} \varphi_{i,l}(v_{i,\pi_i(l)})
$$

for elements $S \in \mathcal{S}$ of the form

$$
S = \bigotimes_{i=1}^{n} \left(\varphi_{i,1} \otimes \cdots \otimes \varphi_{i,m_i} \otimes v_{i,\pi_i(1)} \otimes \cdots \otimes v_{i,\pi_i(m_i)} \right),
$$

with $m_i = O(i) = I(i)$ and $\pi_i \in S_{m_i}$ for all vertices $1 \le i \le n$.

To proceed we associate to this element a labelled quiver $\tilde{Q}$ as follows. The set of vertices of $\tilde{Q}$ equals the set of vertices of Q. For each $1 \le i, j \le n$, there are m_{ij} arrows from vertex i to vertex j, where m_{ij} is as in (16). Each arrow of $\tilde{Q}$ is labelled by a pair (φ, v) with

$$
\begin{aligned}
&\varphi \in \{\varphi_{1,1}, \ldots, \varphi_{1,m_1}, \ldots, \varphi_{n,1}, \ldots, \varphi_{n,m_n}\} \text{ and} \\
&v \in \{v_{1,1}, \ldots, v_{1,m_1}, \ldots, v_{n,1}, \ldots, v_{n,m_n}\},
\end{aligned}
$$

such that the following conditions are satisfied. If an arrow starts in i for $1 \le i \le n$, then $\varphi \in \{\varphi_{i,1}, \ldots, \varphi_{i,m_i}\}$ and each of these arises exactly once as a first component of a label of such an arrow. If an arrow ends in j for $1 \le j \le n$, then $v \in \{v_{j,1}, \ldots, v_{j,m_j}\}$ and each of these arises exactly once as a second component of a label of such an arrow. Note that this labelling is not unique and the set of oriented cycles generated by the following algorithm depends on the choice of labelling. This does not pose any problems, since we are only interested in finding oriented cycles who generate polynomial invariants.

For any given arrow $a_1 : i \to j$, the element S together with the labelling of $\tilde{Q}$ defines a unique path α ending with a_1. Suppose the first component of its label is $\varphi_{i,l}$. Then $v_{i,\pi_i(l)}$ labelles an arrow b going into vertex i. The first component of the label of b uniquely determines the next arrow as above going into the vertex at the tail of the previous arrow.

As $\tilde{Q}$ has only finitely many arrows, there is an arrow $a_0 \in \alpha$ occuring at least twice. We claim that $a_0 = a_1$, i.e. the arrow we started with. Indeed, take some arrow a_j which ends in some vertex already visited. If this is not the tail of a_1, then ψ_j matches to some w_{j+1} with $j+1 \neq 1$, but also $j+1 \neq i$ for all $i < j$, since w_i was matched to ψ_{i-1} and each ψ matches to only one w and the other way round. Hence the only possibility is that $a_0 = a_1$, which proves the claim. Thus α yields an oriented cycle $\alpha = (a_1, \dots, a_r)$ where the a_i are arrows in $\tilde{Q}$. Suppose the label of a_i equals (ψ_i, w_i). Then $\psi_i \otimes w_i$ defines a linear map f_i from $\mathbb{C}^{d_{t(a_i)}}$ to $\mathbb{C}^{d_{h(a_i)}}$. Let A_i denote the matrix of f_i with respect to the standard basis. Then, the expression $\operatorname{tr}(A_1 \cdots A_r)$ is well-defined since α is an oriented cycle. By Lemma 2.5 it follows that

$$\operatorname{tr}(A_1 \cdots A_r) = \prod_{i=1}^{r} \psi_i(w_{i+1}).$$

Now let $\tilde{Q}'$ denote the labelled quiver obtained by removing α from $\tilde{Q}$. Then we repeat the same procedure as above. This is possible, because we removed an oriented cycle, i.e. $O(i) = I(i)$ still holds for all $i \in \tilde{Q}'_0$. Since $\tilde{Q}$ has only finitely many arrows, we collect in this way a finite number of cycles $\alpha_1, \dots, \alpha_s$, $s \in \mathbb{N}$, one in each step, until all arrows are removed. As explained above, for each cycle α_j, $1 \leq j \leq s$, we have

$$\operatorname{tr}(A_{j,1} \cdots A_{j,r_j}) = \prod_{i=1}^{r_j} \psi_{j,i}(w_{j,i+1})$$

with r_j the length of the cycle α_j, $(\psi_{j,i}, w_{j,i})$ the labels of the arrows in the cycle and $A_{j,i}$ the linear map obtained from this label for all $1 \leq i \leq r_j$. We claim that

$$t_S = \prod_{j=1}^{s} \operatorname{tr}(A_{j,1} \cdots A_{j,r_j}). \tag{22}$$

This implies that we can write the generators of the invariants $\mathcal{S}^{\mathrm{GL}(Q,\underline{d})}$ as a product of traces obtained from oriented cycles in our quiver. It remains to check (22). Recall that

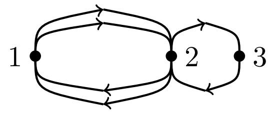
$$t_S = \prod_{i=1}^{n} \prod_{l=1}^{m_i} \varphi_{i,l}(v_{i,\pi_i(l)}), \text{ and } \prod_{j=1}^{s} \operatorname{tr}(A_{j,1} \cdots A_{j,r_j}) = \prod_{j=1}^{l} \prod_{i=1}^{r_j} \psi_{j,i}(w_{j,i+1}).$$

By our labelling of $\tilde{Q}$ there is a bijection between the following two sets

$$\{(\varphi_{i,l}, v_{i,\pi_i(l)}) \mid 1 \leq i \leq n, 1 \leq l \leq m_i\}$$
$$\leftrightarrow \{(\psi_{j,i}, w_{j,i+1}) \mid 1 \leq j \leq s, 1 \leq i \leq r_j\}.$$

Hence the claim holds and the proof of the theorem is complete.

Example 5.10. In this example, we explicitly construct $\tilde{Q}$ given a quiver Q as follows: Let Q be the following quiver.



Let $\underline{d} = (2, 1, 4)$.

For $l = 6$, the degree $(1, 1, 1, 1, 1, 1)$ leads to a summand $\mathcal{S}$ which satisfies $m_i := O(i) = I(i)$ for $i = 1, 2, 3$. Since for all vertices the number of outgoing and ingoing arrows are the same, we can find non-trivial invariants by Proposition 5.7. We have $m_{12} = 2$, $m_{21} = 2$, $m_{23} = 1$, $m_{32} = 1$, $m_{13} = m_{31} = 0$ and $m_1 = 2$, $m_2 = 3$, $m_3 = 1$. Now the space of invariants $\mathcal{S}^{\mathrm{GL}(Q,\underline{d})}$ is generated by all elements $S \in \mathcal{S}$ of the form

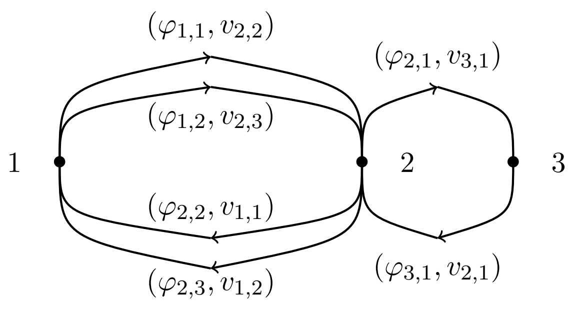
$$\begin{aligned} S = &(\varphi_{1,1} \otimes \varphi_{1,2} \otimes v_{1,\pi_1(1)} \otimes v_{1,\pi_1(2)}) \\ &\otimes(\varphi_{2,1} \otimes \varphi_{2,2} \otimes \varphi_{2,3} \otimes v_{2,\pi_2(1)} \otimes v_{2,\pi_2(2)} \otimes v_{2,\pi_2(3)}) \\ &\otimes(\varphi_{3,1} \otimes v_{3,\pi_3(1)}), \end{aligned}$$

for $\varphi_{1,i} \in (k^{d_1})^* = (k^2)^*$ and $v_{1,i} \in k^{d_1} = k^2$, $i = 1, 2$, $\varphi_{2,i} \in (k^{d_2})^* = k^*$ and $v_{2,i} \in k^{d_2} = k$, $i = 1, 2, 3$ and $\varphi_{3,1} \in (k^{d_3})^* = (k^4)^*$ and $v_{3,1} \in k^{d_3} = k^4$ and permutations π_i for $i = 1, 2, 3$.

Now let $\pi_1 \in S_2$, $\pi_2 \in S_3$ and $\pi_3 \in S_1$ given by $\pi_1 = \begin{pmatrix} 1 & 2 \end{pmatrix}$, $\pi_2 = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ and $\pi_3 = \begin{pmatrix} 1 \end{pmatrix}$. Hence we have

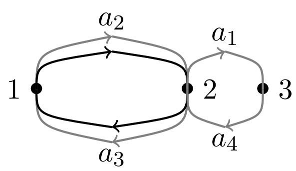
$$\begin{aligned} S = &(\varphi_{1,1} \otimes \varphi_{1,2} \otimes v_{1,2} \otimes v_{1,1}) \\ &\otimes(\varphi_{2,1} \otimes \varphi_{2,2} \otimes \varphi_{2,3} \otimes v_{2,2} \otimes v_{2,3} \otimes v_{2,1}) \\ &\otimes(\varphi_{3,1} \otimes v_{3,1}). \end{aligned}$$

To construct $\tilde{Q}$, we copy Q but label all arrows by pairs (φ, v) following the rules as above. Hence we get:



Note that this labelling is not unique, but the condition $O(i) = I(i)$ for all vertices in Q makes sure that we can indeed find a cycle independent of the chosen labelling.

To construct our cycle α as explained above, we start with one arrow a_1, for example the one which is labelled by $(\varphi_{2,1}, v_{3,1})$. Then the element S tells us that $\varphi_{2,1}$ matches with $v_{2,2}$. The arrow which has $v_{2,2}$ as second component of its label is the upper arrow going from 1 to 2, which we call a_2. The first component of its label is $\varphi_{1,1}$, which is matched by S to $v_{1,2}$. This element labels the lower arrow a_3 going from 2 to 1. The first component of its label $\varphi_{2,3}$ matches to $v_{2,1}$ labelling the arrow a_4 going from 3 to 2. Since $\varphi_{3,1}$ matches to $v_{3,1}$, which is the second component of the label of a_1, the arrow we started with, we are done. This defines the following cycle $\alpha = (a_1, a_2, a_3, a_4)$ of length 4, coloured in gray in the picture below. Note that the arrow we started with defines the last arrow of our cycle.



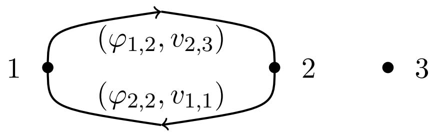
Let $a_i = (\psi_i, w_i)$ with $\psi_i \in \{\varphi_{t(a_i),j} \mid j = 1, \dots, d_{t(a_i)}\}$, $w_i \in \{v_{h(a_i),j} \mid j = 1, \dots, d_{t(a_i)}\}$ for $1 \leq i \leq 4$, and each component appears at most once. Here we have

$$\begin{aligned} a_1 &= (\varphi_{2,1}, v_{3,1}), \\ a_2 &= (\varphi_{1,1}, v_{2,2}), \\ a_3 &= (\varphi_{2,3}, v_{1,2}), \\ a_4 &= (\varphi_{3,1}, v_{2,1}). \end{aligned}$$

We see that ψ_i matches to w_{i+1} as determined by our element S. Let A_i be the matrix representing the linear map f_i given by $\psi_i \otimes w_i$. Then $f_i \colon k^{d_{t(a_i)}} \to k^{d_{h(a_i)}}$. Here $f_1 \colon k \to k^4$, $f_2 \colon k^2 \to k$, $f_3 \colon k \to k^2$ and $f_4 \colon k^4 \to k$, hence $A_1 \in M_{4\times 1}(k)$, $A_2 \in M_{1\times 2}(k)$, $A_3 \in M_{2\times 1}(k)$ and $A_4 \in M_{1\times 4}(k)$. The product $A_1A_2A_3A_4 \in M_4(k)$ is well-defined and since α is an oriented cycle the product of the linear maps is in fact an endomorphism, so that the map tr is defined. By Lemma 2.5 we get

$$\operatorname{tr}(A_1A_2A_3A_4) = \psi_1(w_2)\psi_2(w_3)\psi_3(w_4)\psi_4(w_1).$$

The labelled quiver $\tilde{Q}'$ is obtained from $\tilde{Q}$ by removing $\alpha = (a_1, a_2, a_3, a_4)$:



We now repeat the same procedure as above, i.e. we start with some arrow b_1, for example the arrow labelled with $\varphi_{1,2}$ in the first component. The element

S says that this linear functional is mapped to $v_{1,1}$ which determines the arrow b_2 from 2 to 1. The first component of its label is $\varphi_{2,2}$ which matches to $v_{2,3}$. This label describes the arrow we have started with, hence we have again found an oriented cycle. We label

$$\begin{aligned} b_1 &= (\varphi_{1,2}, v_{2,3}) \\ b_2 &= (\varphi_{2,2}, v_{1,1}), \end{aligned}$$

and we set for $i = 1, 2$, $b_i = (\psi'_i, w'_i)$. Then ψ'_i matches to w'_{i+1}, indices taken modulo 2, and we get matrices A'_1 and A'_2 representing the linear maps given by $\psi'_i \otimes w'_i$. By Lemma 2.5, we get

$$\operatorname{tr}(A'_1 A'_2) = \psi'_1(w'_2)\psi'_2(w'_1).$$

Now we claim, that the linear functional given by the element S can be written as a product of these two traces, i.e. we claim

$$t_S = \operatorname{tr}(A_1 A_2 A_3 A_4)\operatorname{tr}(A'_1 A'_2).$$

We will check this again by computing

$$\begin{aligned} &\operatorname{tr}(A_1 A_2 A_3 A_4)\operatorname{tr}(A'_1 A'_2) \\ &\quad = \psi_1(w_2)\psi_2(w_3)\psi_3(w_4)\psi_4(w_1)\psi'_1(w'_2)\psi'_2(w'_1) \\ &\quad = \varphi_{2,1}(v_{2,2})\varphi_{1,1}(v_{1,2})\varphi_{2,3}(v_{2,1})\varphi_{3,1}(v_{3,1})\varphi_{1,2}(v_{1,1})\varphi_{2,2}(v_{2,3}) \\ &\quad = [\varphi_{1,1}(v_{1,\pi_1(1)})\varphi_{1,2}(v_{1,\pi_1(2)})] \\ &\qquad \cdot [\varphi_{2,1}(v_{2,\pi_2(1)})\varphi_{2,2}(v_{2,\pi_2(2)})\varphi_{2,3}(v_{2,\pi_2(3)})] \\ &\qquad \cdot [\varphi_{3,1}(v_{3,\pi_3(1)})] \\ &\quad = \prod_{i=1}^{3}\prod_{l=1}^{m_i} \varphi_{i,l}(v_{i,\pi_i(l)}) \\ &\quad = t_S, \end{aligned}$$

and thus the claim follows. We have shown that we can write the invariant linear functional t_S as a product of traces coming from oriented cycles in our quiver.

To conclude this chapter, we may say that this proof gives a precise algorithm of how to find the cycles we are interested in. Though we have used Schur-Weyl-duality, this proof is straightforward and its idea is easy to understand, which is a big advantage. Now we introduce a totally different proof in contrast to the ideas mentioned above.

5.2 Abstract Proof

As above, we denote by k an algebraically closed field and by V a finite-dimensional k-vector space of dimension N. In this subsection, we sketch the proof of Le Bruyn and Procesi, [PrBr, Section 3, Theorem 1]. Since it is very abstract, we do not elaborate every detail, but rather explain the main steps of this proof. We work over the complex numbers and we denote by ${}_{\mathbb{C}}\mathcal{A}$ the category of $\mathbb{C}$-algebras and by ${}_{\mathbb{C}}\mathcal{A}^c$ the category of commutative $\mathbb{C}$-algebras. To abbreviate notation we write $\operatorname{Hom}(R, S)$ for the set of morphisms in ${}_{\mathbb{C}}\mathcal{A}$ and in ${}_{\mathbb{C}}\mathcal{A}^c$ for R, S in ${}_{\mathbb{C}}\mathcal{A}$ and ${}_{\mathbb{C}}\mathcal{A}^c$, respectively.

We reduce the problem of finding linear invariants to the problem of finding matrix invariants. This is possible by the following lemma.

Lemma 5.11. *Let V be as above and let $G \subseteq \mathrm{GL}_N(k)$ be a subgroup for some $N \in \mathbb{N}$. Then G acts on V canonically and we have a bijection between the set of G-invariant polynomial functions $k[V]^G$, and the set of G-equivariant polynomial maps $V \to M_N(k)$, where G acts on $M_N(k)$ by conjugation.*

Proof. On the one hand, let $\varphi\colon V \to M_N(k)$ be a G-equivariant polynomial map. Then by the following diagram

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & M_N(k) \\ & \underset{\mathrm{tr}\circ\varphi}{\searrow} & \downarrow \mathrm{tr} \\ & & k \end{array}$$

we get a map $\mathrm{tr} \circ \varphi\colon V \to k$. It is polynomial since φ and tr are and it is G-invariant. Indeed, for $g \in G$, $v \in V$ we calculate

$$[g.(\mathrm{tr} \circ \varphi)](v) = \mathrm{tr} \circ \varphi(g^{-1}.v) = \mathrm{tr}(g^{-1}.\varphi(v)) = \mathrm{tr}(g^{-1}\varphi(v)g) = (\mathrm{tr} \circ \varphi)(v).$$

This implies that $\mathrm{tr} \circ \varphi \in k[V]^G$.

On the other hand, for a G-invariant polynomial map $\psi\colon V \to k$, we define $\tilde{\psi}\colon V \to M_N(k)$, $v \mapsto \frac{1}{N}\mathrm{diag}(\psi(v), \ldots, \psi(v))$ where $\mathrm{diag}(\psi(v), \ldots, \psi(v))$ denotes the $N \times N$ matrix with diagonal entries $\psi(v)$ and 0 otherwise. We have to check that $\tilde{\psi}$ is G-equivariant. Let $g \in G$ and $v \in V$, then

$$\begin{aligned} g.\tilde{\psi}(v) &= g \cdot \frac{1}{N} \cdot \mathrm{diag}(\psi(v), \ldots, \psi(v)) \cdot g^{-1} = \tilde{\psi}(v), \\ \tilde{\psi}(g.v) &= \frac{1}{N} \cdot \mathrm{diag}(\psi(g.v), \ldots, \psi(g.v)) = \tilde{\psi}(v), \end{aligned}$$

because ψ is G-invariant.

We get

$$(\mathrm{tr} \circ \tilde{\psi})(v) = \frac{1}{N} \cdot N \cdot \psi(v) = \psi(v)$$

and hence every G-invariant polynomial map from V to k is given by a G-equivariant polynomial map from V to $M_N(k)$. □

We will use this fact to prove Theorem 5.3.

5.2.1 The Functor $X_{R,N}$

In this subsection, we follow [Pro1, Chapter IV, §1]. We denote by **Set** the category of sets. We introduce some basic properties of functors. For more details see [Ben, Chapter 2.1].

For fixed $R \in {}_{\mathbb{C}}\mathcal{A}$ and $N \in \mathbb{N}$, we consider the functor

$$\begin{aligned} X_{R,N} \colon {}_{\mathbb{C}}\mathcal{A}^c &\to \mathbf{Set} \\ S &\mapsto \mathrm{Hom}(R, M_N(S)) \\ \phi &\mapsto (\psi \mapsto M_N(\phi) \circ \psi). \end{aligned}$$

In the next step we describe some properties of the functor $X_{R,N}$. To do so we need two definitions.

Definition 5.12. A functor $F \colon \mathcal{C} \to \mathbf{Set}$ is called *representable* if it is naturally isomorphic to a functor of the form

$$(X, -) \colon Y \to \mathrm{Hom}(X, Y),$$

for $X, Y \in \mathcal{C}$.

Let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor between two categories $\mathcal{C}$ and $\mathcal{D}$. A functor $G \colon \mathcal{D} \to \mathcal{C}$ which satisfies that there exists a natural isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(-, F(-)) \to \mathrm{Hom}_{\mathcal{C}}(G(-), -)$$

is called the *left adjoint* to F.

Now we can state the following proposition.

Proposition 5.13. *The functor $X_{R,N}$ is representable.*

Proof. For $N \in \mathbb{N}$ consider the functor

$$\begin{aligned} F \colon {}_{\mathbb{C}}\mathcal{A}^c \to {}_{\mathbb{C}}\mathcal{A}, S &\mapsto M_N(S) \\ \phi &\mapsto M_N(\phi), \end{aligned}$$

where $M_N(\phi)$ denotes the $N \times N$ matrix with all entries ϕ. If $\phi \colon S \to S'$, then this defines a map from $M_N(S) \to M_N(S')$ by componentwise application of ϕ. It is clear that F is a covariant functor. To see that $X_{R,N}$ is representable, it is enough to show that F has a left adjoint. Indeed, assume that $F \colon {}_{\mathbb{C}}\mathcal{A}^c \to {}_{\mathbb{C}}\mathcal{A}$, $S \mapsto M_N(S)$ has a left adjoint, i.e. a functor $G \colon {}_{\mathbb{C}}\mathcal{A} \to {}_{\mathbb{C}}\mathcal{A}^c$, such that $\mathrm{Hom}(R, F(S)) \cong \mathrm{Hom}(G(R), S)$ for all $R, S \in {}_{\mathbb{C}}\mathcal{A}$. We then get

$$X_{R,N}(S) = \mathrm{Hom}(R, M_N(S)) \cong \mathrm{Hom}(G(R), S).$$

If we can find $A_{R,N} \coloneqq G(R) \in {}_{\mathbb{C}}\mathcal{A}^c$, then we have that

$$X_{R,N}(S) = \mathrm{Hom}(A_{R,N}, S),$$

hence $X_{R,N}$ representable. The precise construction of $A_{R,N}$ can be found in [Pro1, Chapter IV, §1]. $\square$

It is important to note that $A_{R,N}$ is commutative and finitely generated as $\mathbb{C}$-algebra, if R is finitely generated. Furthermore, there exists a map $j\colon R \to A_{R,N}$ satisfying the following property.

Remark 5.14. *The map $j\colon R \to M_N(A_{R,N})$ is a universal map, i.e. the following universal property holds: For any $\mathbb{C}$-algebra homomorphism $f\colon R \to M_N(B)$, where B is a commutative $\mathbb{C}$-algebra, there exists a unique map $\bar{f}\colon A_{R,N} \to B$, such that the following diagram commutes:*

$$\begin{array}{ccc} R & \xrightarrow{\quad j \quad} & M_N(A_{R,N}) \\ & {\scriptstyle f}\searrow \qquad \swarrow{\scriptstyle M_N(\bar{f})} & \\ & M_N(B) & \end{array}$$

This property follows from the definition of $A_{R,N}$, but we do not prove it in this thesis. It can be found in [Pro2, Section 1].

Now let $G := \mathrm{GL}_N(\mathbb{C})$. Then for any commutative $\mathbb{C}$-algebra B, G acts on $M_N(B)$ by conjugation. In particular, for $B = A_{R,N}$ and $g \in G$ we get a map

$$\pi_g\colon M_N(A_{R,N}) \to M_N(A_{R,N}), M \mapsto gMg^{-1}.$$

From the universal property stated in Remark 5.14, applied to $\pi_g \circ j$, we get a unique map $\varphi_g\colon A_{R,N} \to A_{R,N}$, such that

$$\pi_g \circ j = M_N(\varphi_g) \circ j, \tag{23}$$

i.e. the following diagram commutes:

$$\begin{array}{ccc} R & \xrightarrow{\ j\ } & M_N(A_{R,N}) \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle M_N(\varphi_g)} \\ M_N(A_{R,N}) & \xrightarrow[\pi_g]{} & M_N(A_{R,N}) \end{array}$$

This leads to the following proposition.

Proposition 5.15. *The map*

$$\rho\colon G \to M_N(A_{R,N}), g \mapsto \rho_g,$$

where $\rho_g := \pi_g \circ M_N(\varphi_g^{-1})$ and π_g, φ_g are defined as above for all $g \in G$ defines a representation of G.

Furthermore, every element of $j(R)$ is invariant under the action defined by ρ.

PROOF. For $g, h \in G$ we have $\pi_{gh} \circ j = M_N(\varphi_{gh}) \circ j$ and on the other hand

$$\begin{aligned}\pi_{gh} \circ j &= \pi_g \circ \pi_h \circ j \\ &= \pi_g \circ M_N(\varphi_h) \circ j \\ &= M_N(\varphi_h) \circ \pi_g \circ j \\ &= M_N(\varphi_h) \circ M_N(\varphi_g) \circ j \\ &= M_N(\varphi_h \circ \varphi_g) \circ j,\end{aligned}$$

so that $\varphi_{gh} = \varphi_h \circ \varphi_g$ by uniqueness. The third equality in the second equation holds, because φ_h is a $\mathbb{C}$-algebra homomorphism for all $h \in G$, hence conjugation by $g \in G$ and component wise application of φ_h commute.

It is left to show that $\rho_{gh} = \rho_g \circ \rho_h$ and that $\rho_g(j(r)) = j(r)$ for all $r \in R$. We have

$$\begin{aligned}\rho_{gh} &= \pi_{gh} \circ M_N(\varphi_{gh}^{-1}) \\ &= \pi_g \circ \pi_h \circ M_N((\varphi_h \circ \varphi_g)^{-1}) \\ &= \pi_g \circ \pi_h \circ M_N(\varphi_g^{-1} \circ \varphi_h^{-1}) \\ &= \pi_g \circ \pi_h \circ M_N(\varphi_g^{-1}) \circ M_N(\varphi_h^{-1}) \\ &= \pi_g \circ M_N(\varphi_g^{-1}) \circ \pi_h \circ M_N(\varphi_h^{-1}) \\ &= \rho_g \circ \rho_h,\end{aligned}$$

hence ρ is a representation of G.

Now for $r \in R$ we get

$$\begin{aligned}\rho_g(j(r)) &= (\pi_g \circ M_N(\varphi_g^{-1}))(j(r)) \\ &= (\pi_g \circ M_N(\varphi_g^{-1}) \circ j)(r) \\ &= (\pi_g \circ \pi_g^{-1} \circ j)(r) \\ &= j(r).\end{aligned}$$

by (23). □

By Proposition 5.15 we get that j maps R into $M_N(A_{R,N})^{\mathrm{GL}_N(\mathbb{C})}$.

In the next step, we will show that $A_{R,N}$ is isomorphic to the coordinate ring of $X_{R,N}(\mathbb{C})$ and precisely describe this isomorphism. To see this, notice that

$$X_{R,N}(\mathbb{C}) = \mathrm{Hom}(R, M_N(\mathbb{C})) \cong \mathrm{Hom}(A_{R,N}, \mathbb{C}). \tag{24}$$

By (8), it is enough to show that there is a bijection

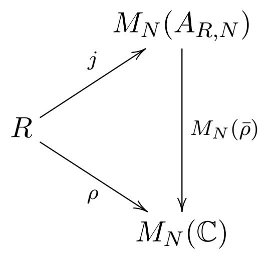
$$\mathrm{Hom}(A_{R,N}, \mathbb{C}) \longleftrightarrow \{\text{maximal ideals in } A_{R,N}\}.$$

On the one hand, if $f \in \mathrm{Hom}(A_{R,N}, \mathbb{C})$, then $\ker(f) \in \mathrm{Specmax}(A_{R,N})$ because $A_{R,N}/\ker(f) \cong \mathbb{C}$. On the other hand, for $\mathfrak{m} \in \mathrm{Specmax}(A_{R,N})$, we set $f_\mathfrak{m} \colon A_{R,N} \twoheadrightarrow A_{R,N}/\mathfrak{m}$ as the projection map onto $A_{R,N}$ modulo $\mathfrak{m}$. Then $\mathbb{C} \hookrightarrow A_{R,N} \twoheadrightarrow A_{R,N}/\mathfrak{m} \cong k$ for some field k, is a non-trivial homomorphism of fields, hence injective. Thus we have $\mathbb{C} \subseteq A_{R,N}/\mathfrak{m}$. Since $A_{R,N}$ is finitely generated as $\mathbb{C}$-algebra by construction, we get that $A_{R,N}/\mathfrak{m}$ is finitely generated. It follows that $(A_{R,N}/\mathfrak{m})/\mathbb{C}$ is an algebraic field extension. But since $\mathbb{C}$ is algebraically closed, we get that $A_{R,N}/\mathfrak{m} = \mathbb{C}$. For more details and a proof of Hilbert's Nullstellensatz, see [Kun, §3, Satz 3.2].

It is easy to verify that these maps are mutually inverse, hence we get that

$$A_{R,N} = \mathbb{C}[X_{R,N}(\mathbb{C})],$$

i.e. we view elements in $A_{R,N}$ as polynomial maps from $X_{R,N}(\mathbb{C})$ to $\mathbb{C}$. More precisely, for $a \in A_{R,N}$, let $\hat{a}$ be the corresponding polynomial map $X_{R,N}(\mathbb{C}) \to \mathbb{C}$. By the following diagram as in Remark 5.14



(25)

and by (24), we have that

$$\hat{a}(\rho) = \bar{\rho}(a) \tag{26}$$

for $\rho \in X_{R,N}$.

In turn, we may interpret $M_N(A_{R,N})$ as the set of polynomial maps from $X_{R,N}(\mathbb{C})$ to $M_N(\mathbb{C})$. Note that this is possible, since a polynomial map to $M_N(\mathbb{C})$ is a family of N^2 polynomial maps to $\mathbb{C}$. Each of these polynomial maps is given as a polynomial in the coordinates.

Using (26), we can describe the polynomial map from $X_{R,N}(\mathbb{C})$ to $M_N(\mathbb{C})$ arising from $j(r)$ for all $r \in R$. Let $r \in R$ and $j(r) = (a_{ij})_{1 \le i,j \le N} \in M_N(A_{R,N})$. Write $\hat{a}_{ij}$ as above for the assigned polynomial map $X_{R,N}(\mathbb{C}) \to \mathbb{C}$ for all $1 \le i, j \le N$. Then $\widehat{j(r)} = (\hat{a}_{ij})_{ij}$ is a polynomial map $X_{R,N}(\mathbb{C}) \to M_N(\mathbb{C})$ and we have for all $\rho \in X_{R,N}(\mathbb{C})$

$$\begin{aligned}\widehat{j(r)}(\rho) &= (\hat{a}_{ij}(\rho))_{ij} = (\bar{\rho}(a_{ij}))_{ij} = M_N(\bar{\rho})((a_{ij})_{ij}) \\ &= M_N(\bar{\rho})(j(r)) = (M_N(\bar{\rho}) \circ j)(r) = \rho(r)\end{aligned} \tag{27}$$

by (25).

Assume now that our ring R has a decomposition of 1 as a sum of orthogonal idempotents $e_1, \ldots, e_n$. We define $B = \mathbb{C}[x_1, \ldots, x_n]/I$, where

I is the ideal generated by the relations $x_i^2 = x_i$, $x_i x_j = x_j x_i = 0$ for $i, j = 1, \dots, n$, $i \neq j$ and $\sum_{i=1}^n x_i = 1$. We denote by $\bar{x}_i$ the image of x_i under the projection map for $1 \leq i \leq n$, so that $B = \mathbb{C}[\bar{x}_1, \dots, \bar{x}_n]$. Then B maps injectively into R via the map $\iota\colon B \hookrightarrow R, \bar{x}_i \mapsto e_i$.

We define $C_n(N) \coloneqq \{(d_1, \dots, d_n) \in \mathbb{N}^n \mid \sum_{i=1}^n d_i = N\}$, the set of n-compositions of N. For $\underline{d} \in C_n(N)$, we set

$$X_{\underline{d}}(\mathbb{C}) \coloneqq \{\varphi \in X_{B,N}(\mathbb{C}) \mid \operatorname{tr}(\varphi(\bar{x}_i)) = d_i, 1 \leq i \leq n\}.$$

To proceed, we introduce further notation.

Definition 5.16. *Fix $\underline{d} \in C_n(N)$.*

1. *For $1 \leq i \leq n$, let $u_i \in M_N(\mathbb{C})$ be the diagonal matrix with 1 in the positions from $d_1 + \cdots + d_{i-1} + 1$ to $d_1 + \cdots + d_i$, and zeros elsewhere.*
2. *Let $\mathrm{GL}(\underline{d})$ denote the stabilizer of $u_1, \cdots, u_n$ in $\mathrm{GL}_N(\mathbb{C})$.*
3. *Let $\varphi_0\colon B \to M_N(\mathbb{C})$ be the morphism defined by $\varphi_0(\bar{x}_i) \coloneqq u_i$ for $1 \leq i \leq n$.*

Note that the u_i are orthogonal idempotents in $M_N(\mathbb{C})$ with $\sum_{i=1}^n u_i = I_N$ and $\operatorname{tr}(u_i) = d_i$ and that $\varphi_0 \in X_{\underline{d}}(\mathbb{C})$. Also, $\mathrm{GL}(\underline{d})$ is isomorphic to the direct product $\mathrm{GL}_{d_1}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{d_n}(\mathbb{C})$. This leads to the following proposition.

Proposition 5.17. *With the notation introduced above, we have*

$$X_{B,N}(\mathbb{C}) = \dot{\bigcup}_{\underline{d} \in C_n(N)} X_{\underline{d}}(\mathbb{C}).$$

Furthermore, for fixed $\underline{d} \in C_n(N)$ there is a bijection

$$\mathrm{GL}_N(\mathbb{C})/\mathrm{GL}(\underline{d}) \to X_{\underline{d}}(\mathbb{C})$$

given by $g\mathrm{GL}(\underline{d}) \mapsto g\varphi_0 g^{-1}$.

Proof. Since $X_{\underline{d}}(\mathbb{C}) \subseteq X_{B,N}(\mathbb{C})$ for all $\underline{d} \in C_n(N)$, it follows that $X_{B,N}(\mathbb{C}) \supseteq \bigcup_{\underline{d} \in C_n(N)} X_{\underline{d}}(\mathbb{C})$. By definition of $X_{\underline{d}}(\mathbb{C})$, this union is disjoint.

Conversely, recall that $B = \mathbb{C}[\bar{x}_1, \dots, \bar{x}_n]$. Thus any $\varphi \in X_{B,N}(\mathbb{C}) = \operatorname{Hom}(B, M_N(\mathbb{C}))$ is determined by the images of $\bar{x}_1, \dots, \bar{x}_n$. Since $\bar{x}_1, \dots, \bar{x}_n$ are idempotents, we get that $\varphi(\bar{x}_i) \in M_N(\mathbb{C})$ for all $1 \leq i \leq n$ is idempotent, hence has only 0 and 1 as eigenvalue. Then $\varphi \in X_{\underline{d}}(\mathbb{C})$ for some $\underline{d} \in C_n(N)$. Indeed, for all $1 \leq i \leq n$ we define $d_i \coloneqq \operatorname{tr}(\varphi(\bar{x}_i)) = \sum_{j=1}^N \lambda_{i,j} \in \mathbb{N}$ for $\lambda_{i,j} \in \{0, 1\}$ eigenvalues of $\varphi(\bar{x}_i)$. Then $N = \operatorname{tr}(I_N) = \operatorname{tr}(\varphi(1)) = \operatorname{tr}(\varphi(\bar{x}_1 + \cdots + \bar{x}_n)) = \sum_{i=1}^n \operatorname{tr}(\varphi(\bar{x}_i) = \sum_{i=1}^n d_i$ and hence $\varphi \in X_{\underline{d}}(\mathbb{C})$ for $\underline{d} = (d_1, \dots, d_n)$.

For the second part of the proposition, we fix $\underline{d} \in C_n(N)$. Now let $\varphi \in X_{\underline{d}}(\mathbb{C})$. We first show that for all $1 \leq i \leq n$, $\varphi(\bar{x}_i)$ is conjugate to

u_i for some $g \in \mathrm{GL}_N(\mathbb{C})$. Note that for $1 \leq i \leq n$, the matrix $\varphi(\bar{x}_i)$ is idempotent, since $\varphi(\bar{x}_i)\varphi(\bar{x}_i) = \varphi(\bar{x}_i\bar{x}_i) = \varphi(\bar{x}_i)$ and $\bar{x}_i$ is idempotent. Thus the polynomial $t^2 - t \in \mathbb{C}[t]$ has the matrix $\varphi(\bar{x}_i)$ as zero. Hence the minimal polynomial of $\varphi(\bar{x}_i)$ divides $t^2 - t = t(t-1)$, which has two different roots, hence $\varphi(\bar{x}_i)$ is diagonalisable. For $i \neq j$, $1 \leq i, j \leq n$, we have $\varphi(\bar{x}_i)\varphi(\bar{x}_j) = \varphi(\bar{x}_i\bar{x}_j) = \varphi(0) = \varphi(\bar{x}_j\bar{x}_i) = \varphi(\bar{x}_j)\varphi(\bar{x}_i)$, hence all $\varphi(\bar{x}_i)$ commute. Consequently, we get that they are simultaneously diagonalisable, i.e. there exists $g \in \mathrm{GL}_N(\mathbb{C})$ such that $g\varphi(\bar{x}_i)g^{-1} = u_i = \varphi_0(\bar{x}_i)$ for all $1 \leq i \leq n$.

Now the map $\mathrm{GL}_N(\mathbb{C}) \to X_{\underline{d}}(\mathbb{C})$, $g \mapsto g\varphi_0 g^{-1}$ is surjective and the fibres are exactly the cosets of $\mathrm{GL}(\underline{d})$ in $\mathrm{GL}_N(\mathbb{C})$: let $g, h \in GL_N(\mathbb{C})$ such that $g\varphi_0 g^{-1} = h\varphi_0 h^{-1}$. Then $(h^{-1}g)\varphi_0(h^{-1}g)^{-1} = \varphi_0$, which implies that $(h^{-1}g)u_i(h^{-1}g)^{-1} = u_i$ for all $1 \leq i \leq n$, hence we get $h^{-1}g \in \mathrm{Stab}(u_1, \ldots, u_n) = \mathrm{GL}_{d_1}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{d_n}(\mathbb{C}) = \mathrm{GL}(\underline{d})$. This proves the claim and in particular that the map given above is well-defined and bijective. □

Note that the map $\iota \colon B \hookrightarrow R$, induces a map $\omega \colon X_{R,N}(\mathbb{C}) \to X_{B,N}(\mathbb{C})$, $\varphi \mapsto \varphi \circ \iota$. Thus we can decompose

$$X_{R,N}(\mathbb{C}) = \dot{\bigcup}_{\underline{d} \in C_n(N)} \omega^{-1}(X_{\underline{d}}(\mathbb{C})), \tag{28}$$

with $\omega^{-1}(X_{\underline{d}}(\mathbb{C})) = \{\varphi \in X_{R,N}(\mathbb{C}) \mid \mathrm{tr}(\varphi(e_i)) = d_i, 1 \leq i \leq n\}$. We define

$$X_{R_{\underline{d}},N}(\mathbb{C}) := \omega^{-1}(X_{\underline{d}}(\mathbb{C})).$$

Denote by $A_{R,\underline{d}}$ the coordinate ring of $X_{R_{\underline{d}},N}(\mathbb{C})$. Because of (28) we have that

$$A_{R,N} \cong \bigoplus_{\underline{d} \in C_n(N)} A_{R,\underline{d}}.$$

This implies that

$$M_N(A_{R,N}) \cong \bigoplus_{\underline{d} \in C_n(N)} M_N(A_{R,\underline{d}})$$

and because $\mathrm{GL}_N(\mathbb{C})$ acts linearly we even get

$$M_N(A_{R,N})^{\mathrm{GL}_N(\mathbb{C})} \cong \bigoplus_{\underline{d} \in C_n(N)} M_N(A_{R,\underline{d}})^{\mathrm{GL}_N(\mathbb{C})}. \tag{29}$$

Thus, let

$$R_{\underline{d}} := M_N(A_{R,\underline{d}})^{\mathrm{GL}_N(\mathbb{C})}. \tag{30}$$

We use the next part to define an algebra with trace and to see that we have in fact $R_{\underline{d}} \subseteq R$, if we assume that R has the following properties.

5.2.2 Algebras with Trace and Cayley-Hamilton Polynomials

Definition 5.18. Let k be any field. A *k-algebra A with trace* is an algebra with a k-linear map $\mathrm{Tr}\colon A \to A$ satisfying

1. $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$,
2. $\mathrm{Tr}(a)b = b\mathrm{Tr}(a)$,
3. $\mathrm{Tr}(\mathrm{Tr}(a)b)) = \mathrm{Tr}(a)\mathrm{Tr}(b)$,

for all $a, b \in A$.

Example 5.19. Let k be a field and $A = M_N(k)$ the k-algebra of N by N matrices. Then

$$Tr\colon M_N(k) \to M_N(k), M \mapsto \mathrm{tr}(M) \cdot 1_N,$$

defines a k-linear map that meets the demands above. Hence $M_N(k)$ is an algebra with trace.

Theorem 5.20. *If R is a $\mathbb{C}$-algebra with trace and satisfies the Cayley-Hamilton polynomial identities for N by N matrices, then the map $j\colon R \to M_N(A_{R,N})^{\mathrm{GL}_N(\mathbb{C})}$ is an isomorphism.*

A proof of this theorem can be found in [Pro2, Section 2, Theorem 2.6].

Definition 5.21. For a k-algebra A with a trace map $\mathrm{Tr}\colon A \to A$, we define for any $a \in A$ the *formal N-characteristic polynomial* $\chi_a^{(N)}(t)$ by

$$\chi_a^{(N)}(t) := \prod_{i=1}^{N}(t - t_i), \quad \text{setting } \sum_{i=1}^{N} t_i^l = \mathrm{Tr}(a^l).$$

A satisfies the *N-th Cayley-Hamilton identity*, if

1. $\mathrm{Tr}(1) = N$
2. $\forall a \in A : \chi_a^{(N)}(a) = 0 \in A$.

Then A is called a *Cayley-Hamilton algebra of degree N*.

If we assume that R is a such an algebra we have

$$R = \bigoplus_{\underline{d} \in C_n(N)} R_{\underline{d}} \tag{31}$$

by (29). This leads to the following lemma.

Lemma 5.22. *We have a bijection*

$$j(R) = M_N(A_{R,N})^{\mathrm{GL}_N(\mathbb{C})}$$
$$\updownarrow$$
$$\{\mathrm{GL}_N(\mathbb{C})\text{-}\textit{equivariant and polynomial maps } X_{R,N}(\mathbb{C}) \to M_N(\mathbb{C})\},$$

$$j(r) \mapsto \left(\widehat{j(r)}\colon X_{R,N}(\mathbb{C}) \to M_N(\mathbb{C}), \quad \rho \mapsto \rho(r)\right).$$

Proof. By Theorem 5.20 we know that $j\colon R \to M_N(A_{R,N})^{\mathrm{GL}_N(\mathbb{C})}$ is an isomorphism, hence the map is well-defined. By (27), we get that this is indeed the right interpretation. It is clear, that $\widehat{j(r)}$ is $\mathrm{GL}_N(\mathbb{C})$-equivariant for all $r \in R$. □

Now, we define a subfunctor $X'_{R_{\underline{d}}}(-)\colon {}_{\mathbb{C}}\mathcal{A} \to \mathbf{Set}$ of $X_{R,N}(-)$,

$$X'_{R_{\underline{d}}}(S) := \{\varphi\colon R_{\underline{d}} \to M_N(S) \mid \varphi(\mathrm{pr}_{\underline{d}}(e_i)) = u_i, 1 \le i \le n\}, \qquad (32)$$

where $\mathrm{pr}_{\underline{d}}\colon R \to R_{\underline{d}}$ denotes the projection map to this summand. Clearly, $X'_{R_{\underline{d}}}(\mathbb{C})$ is invariant under the action of $\mathrm{GL}(\underline{d})$.

Recall that $X_{R_{\underline{d}},N}(\mathbb{C}) = \omega^{-1}(X_{\underline{d}}(\mathbb{C})) = \{\varphi\colon R \to M_N(\mathbb{C}) \mid \mathrm{tr}(\varphi(e_i)) = d_i, 1 \le i \le n\}$. It follows from (31) that $X'_{R_{\underline{d}}}(\mathbb{C}) \subseteq X_{R_{\underline{d}},N}(\mathbb{C})$.

Proposition 5.23. *There exists an isomorphism*

$$\rho\colon R_{\underline{d}} \to \{f\colon X'_{R_{\underline{d}}}(\mathbb{C}) \to M_N(\mathbb{C}) \mid f \textit{ is } \mathrm{GL}(\underline{d})\textit{-equivariant and polynomial}\}.$$

Proof. Recall that $R_{\underline{d}} = M_N(A_{R,\underline{d}})^{\mathrm{GL}_N(\mathbb{C})}$. Since $A_{R,\underline{d}}$ corresponds to the coordinate ring of $X_{R_{\underline{d}},N}(\mathbb{C})$, we can and will identify $R_{\underline{d}}$ with the set of $\mathrm{GL}_N(\mathbb{C})$-equivariant maps from $X_{R_{\underline{d}},N}(\mathbb{C})$ to $M_N(\mathbb{C})$ as in Lemma 5.22. Let $f \in R_{\underline{d}}$. Then we define $\rho(f)$ as the restriction of f to $X'_{R_{\underline{d}}}(\mathbb{C})$. Since $X'_{R_{\underline{d}}}(\mathbb{C})$ is $\mathrm{GL}(\underline{d})$-invariant and $\mathrm{GL}(\underline{d}) \subseteq GL_N(\mathbb{C})$, we get that $\rho(f)$ is $\mathrm{GL}(\underline{d})$-equivariant.

It remains to show that ρ is injective and surjective. By definition we have $X_{R_{\underline{d}},N}(\mathbb{C}) = \omega^{-1}(X_{\underline{d}}(\mathbb{C}))$. Now let T be a set of representatives for the left cosets of $\mathrm{GL}(\underline{d})$ in $\mathrm{GL}_N(\mathbb{C})$. By the second assertion of Proposition 5.17, we have $X_{\underline{d}}(\mathbb{C}) = \dot{\bigcup}_{g\in T} g\varphi_0 g^{-1}$. This implies

$$\begin{aligned} X_{R_{\underline{d}},N}(\mathbb{C}) &= \omega^{-1}(X_{\underline{d}}(\mathbb{C})) \\ &= \omega^{-1}(\dot{\bigcup}_{g\in T} g.\varphi_0) \\ &= \dot{\bigcup}_{g\in T}(\omega^{-1}(g.\varphi_0)) \\ &= \dot{\bigcup}_{g\in T}(g.(\omega^{-1}(\varphi_0))), \end{aligned}$$

where $(g.\varphi_0)(r) := g\varphi_0(r)g^{-1}$ for $g \in \mathrm{GL}_N(\mathbb{C})$. Notice that $\omega^{-1}(\varphi_0) = X'_{R_{\underline{d}}}(\mathbb{C})$.

For $h \in \{f\colon X'_{R_{\underline{d}}}(\mathbb{C}) \to M_N(\mathbb{C}) \mid f \text{ is } \mathrm{GL}(\underline{d})\text{-equivariant and polynomial}\}$ we define $f \in R_{\underline{d}}$ by

$$f(g.y) \coloneqq g.h(y),$$

for all $y \in X'_{R_{\underline{d}}}(\mathbb{C})$ and $g \in T$. It remains to show that f is indeed $\mathrm{GL}_N(\mathbb{C})$-equivariant. Therefore let $g' \in \mathrm{GL}_N(\mathbb{C})$. Then we have $g'.f(g.y) = (g'g).h(y) = (g_1g_2).h(y)$ with $g_1g_2 = g'g$ with unique elements $g_1 \in T$ and $g_2 \in \mathrm{GL}(\underline{d})$. Because h is $\mathrm{GL}(\underline{d})$-equivariant, it follows that $(g_1g_2).h(y) = g_1.(h(g_2.y)) = f((g_1g_2).y) = f(g'.(g.y))$, which proves what was left to show.

Now, if an element in $R_{\underline{d}}$ is defined by $f(g.y) \coloneqq g.h(y)$ for some $h \in \{f\colon X'_{R_{\underline{d}}}(\mathbb{C}) \to M_N(\mathbb{C}) \mid f \text{ is } \mathrm{GL}(\underline{d})\text{-equivariant}\}$, then $\rho(f)$ is the restriction of f to $X'_{R_{\underline{d}}}(\mathbb{C})$; hence $g = 1$, and it follows that $f(y) = f(1.y) = 1.h(y) = h(y)$, i.e. $\rho(f) = h$. □

5.2.3 The Algebra TQ

We wish to apply this general result in our situation. As Le Bruyn and Procesi do hardly explain the next steps in their paper, this is a first attempt in completing the proof sketched by them.

First, we have to make sure that our chosen $\mathbb{C}$-algebra satisfies the required properties. Obviously, the path algebra $\mathbb{C}Q$ given a quiver Q is finitely generated and has a decomposition of 1 into orthogonal idempotents, just by the definition of 1 in the path algebra; see Proposition 4.10. To apply Theorem 5.20, we need to define a trace map on $\mathbb{C}Q$ satisfying the conditions. Note that there seems to be no obvious or canonical way to achieve this. Of course, one could try to take a faithful representation of $\mathbb{C}Q$, say

$$\iota\colon \mathbb{C}Q \hookrightarrow \mathrm{End}_{\mathbb{C}}(V)$$

for some $\mathbb{C}$-vector space V and define a trace map on $\mathbb{C}Q$ via

$$\mathrm{incl} \circ \mathrm{tr} \circ \iota\colon \mathbb{C}Q \hookrightarrow \mathrm{End}_{\mathbb{C}}(V) \to \mathbb{C} = \mathbb{C}1 \hookrightarrow \mathbb{C}Q,$$

using the inclusion map $\mathrm{incl}\colon \mathbb{C} \hookrightarrow \mathbb{C}Q$. This construction depends on the choice of V and ι. Moreover, it might not even be possible to find a finite-dimensional representation of $\mathbb{C}Q$, since we allow oriented cycles in Q and by Proposition 4.10 we know that then $\mathbb{C}Q$ is not finite-dimensional.

Instead, we want to enlarge our path algebra to turn it into an algebra with trace. This idea makes precise what Le Bruyn and Procesi mean by the sentence "the algebra TQ_0 obtained from $\mathbb{C}Q_0$ by adding traces" [PrBr, Section 3, Theorem 1, page 590, line 1], where for them $\mathbb{C}Q_0$ denotes the path algebra of Q over $\mathbb{C}$. To define TQ we adjoin formal elements T_p where p is a word in the arrows in Q, imposing the relations that $T_p = 0$, if p is not an oriented cycle and $T_p = T_q$, if p and q are cycles and differ by a cyclic permutation. We therefore put

$$TQ \coloneqq \mathbb{C}Q[\,T_c \mid c \text{ oriented cycle in } Q],$$

where the oriented cycle can be written in any way. Note that the trivial cycles e_i for $1 \leq i \leq n$ are oriented cycles and included by this notion. Now we can define a trace map

$$\mathrm{Tr}\colon TQ \to TQ,\ \mathrm{Tr}(p) = \begin{cases} T_p, & \text{if } p \text{ is an oriented cycle} \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathrm{Tr}(a \cdot \bar{T}) = \mathrm{Tr}(a) \cdot \bar{T}, \tag{33}$$

for $a \in kQ$ and $\bar{T}$ a monomial in the T_c's. Now fix $\underline{d} \in C_n(N)$. Then we define

$$T_{\underline{d}}Q \coloneqq TQ/\left(\chi_r^{(N)}(r), \mathrm{Tr}(e_i) - d_i \cdot 1\right),$$

where $\chi_r^{(N)}(t)$ denotes the formal Cayley-Hamilton polynomial of degree N as in 5.21 for all $r \in TQ$. From now on we are only interested in the algebra $T_{\underline{d}}Q$ and we use the same letter for an element in TQ and its image in $T_{\underline{d}}Q$.

Lemma 5.24. *Given the notion as above, the following two properties hold:*

1. *TQ is an algebra with trace,*
2. *$T_{\underline{d}}Q$ is a Cayley-Hamilton algebra of degree N with a decomposition of $1 = e_1 + \cdots + e_n$ into orthogonal idempotents such that $\mathrm{Tr}(e_i) = d_i \cdot 1$.*

PROOF. To see that TQ is an algebra with trace, we have to verify the conditions as in Definition 5.18. The first one is true, since the elements T_c, for c an oriented cycle in Q, lie in the centre of TQ. For the second condition it suffices to show that $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$ for $a, b \in \mathbb{C}Q$. This is a routine computation. We verify the last condition for the elements $a' = p_a \cdot T_{c_a}$ and $b' = p_b \cdot T_{c_b}$ with p_a, p_b paths and c_a, c_b oriented cycles in Q. The general case follows from this because Tr is $\mathbb{C}$-linear.

$$\begin{aligned} \mathrm{Tr}(a'\mathrm{Tr}(b')) &= \mathrm{Tr}(p_a T_{c_a} \mathrm{Tr}(p_b T_{c_b})) \\ &= \mathrm{Tr}(p_a T_{c_a} \mathrm{Tr}(p_b) T_{c_b}) \\ &= \mathrm{Tr}(p_a T_{c_a} T_{p_b} T_{c_b}) \\ &= \mathrm{Tr}(p_a) T_{c_a} T_{p_b} T_{c_b} \\ &= \mathrm{Tr}(p_a) T_{c_a} \mathrm{Tr}(p_b) T_{c_b} \\ &= \mathrm{Tr}(a')\mathrm{Tr}(b'). \end{aligned}$$

The second claim follows from the definition of $T_{\underline{d}}Q$ as a quotient of TQ satisfying the required properties. □

We set

$$T \coloneqq \bigoplus_{\underline{d} \in C_n(N)} T_{\underline{d}}Q$$

and we apply the above theory for T. From now on we work over the category of $\mathbb{C}$-algebras with trace. In particular, we have $X_{T,N}(\mathbb{C}) = \mathrm{Hom}(T, M_N(\mathbb{C}))$. We say that an element $f \in X_{T,N}(\mathbb{C})$ is compatible with the trace map, if the following diagram commutes:

$$\begin{array}{ccc} T & \xrightarrow{f} & M_N(\mathbb{C}) \\ {\scriptstyle \mathrm{Tr}}\downarrow & & \downarrow{\scriptstyle \mathrm{tr}} \\ T & \xrightarrow[f]{} & M_N(\mathbb{C}) \end{array} \tag{34}$$

Here Tr denotes the trace map on T and $\mathrm{tr}\colon M_N(\mathbb{C}) \to \mathbb{C} \hookrightarrow M_N(\mathbb{C})$ the trace map on matrices as in Example 5.19. We set

$$X_{T,N}^{\mathrm{Tr}}(\mathbb{C}) = \{f \in X_{T,N}(\mathbb{C}) \mid f \text{ is compatible with the trace map}\}.$$

Suppose that we can define a universal algebra $A_{T,N}^{\mathrm{Tr}}$ representing the functor $X_{T,N}^{\mathrm{Tr}}(-)$, which has the property stated in Remark 5.14.

As in (30) we set

$$T_{\underline{d}} := M_N(A_{T,\underline{d}}^{\mathrm{Tr}})^{\mathrm{GL_N(\mathbb{C})}}.$$

Proposition 5.25. *There is an isomorphism $T_{\underline{d}} \cong T_{\underline{d}}Q$ as $\mathbb{C}$-algebras with trace.*

Proof. By Theorem 5.20 we know that

$$T \cong M_N(A_{T,N}^{\mathrm{Tr}})^{\mathrm{GL}_N(\mathbb{C})}.$$

On the other hand, we have decompositions $T = \bigoplus_{\underline{d}\in C_n(N)} T_{\underline{d}}Q$ and $M_N(A_{T,N}^{\mathrm{Tr}})^{\mathrm{GL}_N(\mathbb{C})} = \bigoplus_{\underline{d}\in C_n(N)} M_N(A_{T,\underline{d}}^{\mathrm{Tr}})^{\mathrm{GL}_N(\mathbb{C})}$. Because the isomorphism in Theorem 5.20 is compatible with the direct sum decomposition, we get that

$$T_{\underline{d}}Q \cong M_N(A_{T,\underline{d}}^{\mathrm{Tr}})^{\mathrm{GL}_N(\mathbb{C})},$$

hence $T_{\underline{d}}Q \cong T_{\underline{d}}$. □

We use without proof that this isomorphism even holds in the category of $\mathbb{C}$-algebras with trace.

As in (32) we get the subfunctor

$$X_{T_{\underline{d}}}^{\mathrm{Tr}\,\prime}(S) := \{\varphi\colon T_{\underline{d}} \to M_N(S) \mid \varphi \text{ is compatible with trace and } \varphi(\mathrm{pr}_{\underline{d}}(e_i)) = u_i, 1 \le i \le n\}.$$

By Proposition 5.23 we get that

$$\begin{aligned} T_{\underline{d}} &\cong \{f\colon X'_{T_{\underline{d}}}(\mathbb{C}) \to M_N(\mathbb{C}) \mid f \text{ is} \\ &\qquad \mathrm{GL}(\underline{d})\text{-equivariant, polynomial and compatible with trace}\} \\ &= M_N(A_{T,\underline{d}}^{\mathrm{Tr}\,\prime})^{\mathrm{GL}(\underline{d})}. \end{aligned}$$

Proposition 5.26. *For every finite-dimensional $\mathbb{C}Q$-module (V, ρ) satisfying $\rho(e_i) = u_i$ for all $1 \leq i \leq n$ we get a finite-dimensional $T_{\underline{d}}Q$-module (V, ρ') which is compatible with trace with the property $\rho'(e_i) = u_i$ for all $1 \leq i \leq n$, and vice versa.*

Proof. The underlying $\mathbb{C}$-vector space V stays the same and is of dimension N.

Let $\rho\colon \mathbb{C}Q \to \mathrm{End}_{\mathbb{C}}(V)$ be a representation of $\mathbb{C}Q$ of dimension N. For all $v \in V$ we set $\rho'(p)(v) := \rho(p)(v)$ for p a word in the arrows in Q, and $\rho'(T_c)(v) := \mathrm{tr}(\rho(c))v$ for all oriented cycles $c \in Q$. Then this action is compatible with trace because the diagram

$$\begin{array}{ccc} T_c & \xmapsto{\ \rho'\ } & \mathrm{tr}(\rho(c)) \cdot I_N \\ {\scriptstyle \mathrm{Tr}}\downarrow & & \downarrow{\scriptstyle \mathrm{tr}} \\ N \cdot T_c & \xmapsto[\ \rho'\ ]{} & N \cdot \mathrm{tr}(\rho(c)) \cdot I_N \end{array}$$

commutes as required in (34). Since $T_{\underline{d}}Q$ is generated by the images of these elements the claim follows. Conversely, given a $T_{\underline{d}}Q$-action on V, it restricts to an action of $\mathbb{C}Q$ on V since $\mathbb{C}Q \subseteq T_{\underline{d}}Q$ as a subalgebra. The property $\rho(e_i) = u_i$ for all $1 \leq i \leq n$ is obviously preserved. □

Note, that T_c acts as a scalar for every finite-dimensional representation of $T_{\underline{d}}Q$. This follows from the compatibility with trace and the fact that $\mathrm{Tr}(T_c) = N \cdot T_c$. At the moment, we do not see whether the scalar by which T_c acts, is uniquely determined by the restriction of the module to $\mathbb{C}Q$.

The arguments to follow rely on the conjecture that the extension of a $\mathbb{C}Q$-module to a $T_{\underline{d}}Q$-module is unique. Then we may replace the path algebra by the algebra $T_{\underline{d}}Q$. Consider the following commutative diagram for $\rho \in X^{\mathrm{Tr}}_{T_{\underline{d}}}(\mathbb{C})$.

$$\begin{array}{ccc} & & M_N(A^{\mathrm{Tr}}_{T,\underline{d}})^{\mathrm{GL}_N(\mathbb{C})} \cong M_N({A^{\mathrm{Tr}}_{T,\underline{d}}}')^{\mathrm{GL}(\underline{d})} \\ & \overset{j}{\underset{\cong}{\nearrow}} & \downarrow{\scriptstyle M_N(\bar{\rho})} \\ T_{\underline{d}}Q & \xrightarrow[\ \rho\ ]{} & M_N(\mathbb{C}) \\ & \underset{\mathrm{tr}\circ\rho}{\searrow} & \downarrow{\scriptstyle \mathrm{tr}} \\ & & \mathbb{C} \end{array}$$

Following Lemma 5.22, we interpret $M_N(A^{\mathrm{Tr}}_{T,\underline{d}})^{\mathrm{GL}_N(\mathbb{C})}$ as the set of $\mathrm{GL}_N(\mathbb{C})$-equivariant polynomial functions

$$X^{\mathrm{Tr}}_{T_{\underline{d}}}(\mathbb{C}) \to M_N(\mathbb{C}),$$

which are compatible with trace. The isomorphism j defines for all $x \in T_{\underline{d}}Q$ such a map by $\rho \mapsto \rho(x)$ as in (27). By Lemma 5.11, all polynomial invariants are obtained from $\mathrm{GL}(\underline{d})$-equivariant polynomial maps $T_{\underline{d}}Q \to M_N(\mathbb{C})$.

It is now left to show that it is enough to consider oriented cycles in Q to generate the invariant ring. We know that $T_{\underline{d}}Q$ is generated by all paths p in Q and all T_c, for c an oriented cycle in Q. If p is a path which does not contain an oriented cycle, then we get that $\mathrm{tr}(\rho(p)) = 0$, because the entries on the diagonal of a corresponding matrix are 0. If p is a path which contains an oriented cycle c, then $\mathrm{tr}(\rho(p)) = \mathrm{tr}(\rho(c))$, because the arrows not contained in c do not yield any non-zero entries on the diagonal. Hence to get the invariant described by the element p, it is enough to take c into account. If we have T_c for some oriented cycle c, then the corresponding invariant is $\mathrm{tr}(\rho(T_c)) = N \cdot \mathrm{tr}(\rho(c))$, hence already obtained by considering c.

To conclude, the ring of polynomial invariants is generated by traces of cycles as claimed in Theorem 5.3.

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