

AFFINENESS OF THE ÉTALE LOCUS

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1. INTRODUCTION

The classic purity theorem concerns the non-étale locus of a finite type morphism and dates back to Nagata ([Nag59]) and van der Waerden ([GD67, 21.12.12]). The most general version up to date is due to the first author and can be stated as follows:

Theorem 1 (Gabber). *Let $f : X \rightarrow Y$ be a morphism (essentially) of finite type, from a normal scheme X to a locally noetherian regular scheme Y such that f is (essentially) étale at all points of X of codimension ≤ 1 . Then f is (essentially) étale.*

Put it differently, under the hypothesis of Theorem 1, the non-étale locus is pure of codimension 1. The discussions in [GD67, §21.12] raised the question whether the étale locus is affine. More specifically:

Question 2. Let Y be a noetherian regular scheme (that is, each local ring is a noetherian regular local ring). Let $f : X \rightarrow Y$ be a finite type morphism of schemes with X normal. Let $V \subset X$ be the maximal open subscheme where f is étale. Is it true that the inclusion $V \rightarrow X$ is affine?

This question was answered in the affirmative in the case when Y is *excellent* regular over a field in [Sta23, Tag 0ECB] which also expresses an expectation that the analogue should hold for all excellent regular schemes. The goal of the current article is to remove the excellent hypothesis and to extend the result in [Sta23, Tag 0ECB] to the unramified case in mixed characteristic.

2. PRELIMINARIES ON D -MODULES

In this section we collect some preliminary materials regarding D -modules that will be needed in the sequel.

Let Λ be a commutative ring and let A be any commutative Λ -algebra. Let $D(A; \Lambda)$ denote the ring of Λ -linear differential operators on A , which can be spelled out as follows (cf. [GD67, §16]). Let Δ_A denote the kernel of the multiplication map $A \otimes_{\Lambda} A \rightarrow A$ and set $P_A^n := A \otimes_{\Lambda} A / \Delta_A^{n+1}$. Then

$$D(A; \Lambda) = \bigcup_{n \geq 0} \operatorname{Hom}_A(P_A^n, A),$$

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where $\mathrm{Hom}_A(P_A^n, A)$ consists of R -linear differential operators of order at most n .

Remark 3. Let $A \rightarrow B$ is an Λ -algebra morphism between commutative rings. Assume that $A \rightarrow B$ is étale. Then one has (cf. [Mas91])

$$P_B^n \cong P_A^n \otimes_A B.$$

As a consequence, one has

$$\begin{aligned} D(A; \Lambda) &= \bigcup_{n \geq 0} \mathrm{Hom}_A(P_A^n, A) \\ &\rightarrow \bigcup_{n \geq 0} \mathrm{Hom}_A(P_A^n, A) \otimes_A B \\ &\rightarrow \bigcup_{n \geq 0} \mathrm{Hom}_B(P_A^n \otimes_A B, B) \\ &\cong \bigcup_{n \geq 0} \mathrm{Hom}_B(P_B^n, B) = D(B; \Lambda) \end{aligned}$$

In particular, B is a $D(A; \Lambda)$ -module.

Remark 4. Let $(A, \mathfrak{m})$ be an unramified complete regular local ring of mixed characteristic. By Cohen's Structure Theorem, A admits a coefficient ring (a complete p -ring) Λ such that $A \cong \Lambda[[x_1, \dots, x_d]]$. Assume that a $D(A; \Lambda)$ -module M is supported in $\{\mathfrak{m}\}$. Then

$$\mathrm{inj. dim}_A(M) \leq 1.$$

We prove this as follows. Set $K := \ker(M \xrightarrow{p} M)$ and $C := \mathrm{coker}(M \xrightarrow{p} M)$. Then both K and C are $D(\bar{A}; \bar{\Lambda})$ -modules where $\bar{A} = A/pA$ and $\bar{\Lambda} = \Lambda/p\Lambda$. Since $\bar{A}$ is a ring of formal power series over a field, it follows from the main theorem in [Lyu00] that $\mathrm{inj. dim}_{\bar{A}}(K) = 0 = \mathrm{inj. dim}_{\bar{A}}(C)$. Hence $\mathrm{inj. dim}_A(K) \leq 1$ and $\mathrm{inj. dim}_A(C) \leq 1$. The short exact sequence $0 \rightarrow K \rightarrow M \rightarrow pM \rightarrow 0$ shows the surjectivity of $\mathrm{Ext}_A^i(A/\mathfrak{m}, M) \rightarrow \mathrm{Ext}_A^i(A/\mathfrak{m}, pM)$ for $i \geq 2$. Likewise the short exact sequence $0 \rightarrow pM \rightarrow M \rightarrow C \rightarrow 0$ shows the surjectivity of $\mathrm{Ext}_A^i(A/\mathfrak{m}, pM) \rightarrow \mathrm{Ext}_A^i(A/\mathfrak{m}, M)$ for $i \geq 2$. Therefore the map

$$\mathrm{Ext}_A^i(A/\mathfrak{m}, M) \xrightarrow{p} \mathrm{Ext}_A^i(A/\mathfrak{m}, M)$$

is surjective for $i \geq 2$; it follows that $\mathrm{Ext}_A^i(A/\mathfrak{m}, M) = 0$ for $i \geq 2$. This shows $\mathrm{inj. dim}_A(M) \leq 1$.

Remark 5. Let $A, M, \bar{A}, \bar{\Lambda}$ be as in Remark 4. Suppose M is supported in $\{\mathfrak{m}\}$ and $M \xrightarrow{p} M$ is surjective. Then

$$\mathrm{inj. dim}_A(M) = 0.$$

We reason as follows. We have shown that $\mathrm{inj. dim}_A(M) \leq 1$. Hence M admits an injective resolution of length at most 1: $0 \rightarrow M \rightarrow E_0 \rightarrow E_1 \rightarrow 0$, where E_0 is a maximal essential extension of M . The support condition on M implies that both E_0 and E_1 are direct sums of copies of the injective hull of $A/\mathfrak{m}$. Since $M \xrightarrow{p} M$ is surjective, we have an exact sequence

$$0 \rightarrow \mathrm{Hom}_A(\bar{A}, M) \rightarrow \mathrm{Hom}_A(\bar{A}, E_0) \rightarrow \mathrm{Hom}_A(\bar{A}, E_1) \rightarrow 0.$$

Since $\mathrm{Hom}_A(\bar{A}, M) \cong \ker(M \xrightarrow{p} M)$ is a $D(\bar{A}, \bar{A})$ -module that is supported in $\{\mathfrak{m}\}$, it is an injective $\bar{A}$ -module and E_0 is an essential extension of M , it follows that $\mathrm{Hom}_A(\bar{A}, E_1) = 0$. Consequently $E_1 = 0$. That is, $M \cong E_0$ is an injective A -module (and is a direct sum of the injective hull of $A/\mathfrak{m}$).

3. A REDUCTION THEOREM

Theorem 6. *Question 2 can be reduced to the case when $Y = \mathrm{Spec}(R)$ where R is a complete strictly Henselian regular local ring and f is a finite morphism.*

Proof. Let Y, X, V be as in Question 2. Let $x \in X$ with image $y \in Y$. It suffices to show that $V \cap W$ is affine for some affine open neighborhood W of x , which in turn is equivalent to the affineness of $V \times_X \mathrm{Spec}(\mathcal{O}_{X,x})$. This reduces the question to the case $X \times_Y \mathrm{Spec}(\mathcal{O}_{Y,y}) \rightarrow \mathrm{Spec}(\mathcal{O}_{Y,y})$. By an induction on the dimension of $\mathcal{O}_{Y,y}$, one may assume that V is affine away from $f^{-1}(y)$. By replacing X with an affine open neighborhood of x , one may assume that X is affine. Denote $\mathcal{O}_{Y,y}$ by A and X by $\mathrm{Spec}(B)$. Note that $f^{-1}(\{y\}) \cap V$ is a finite set of closed points. Thus, one may assume that f is a finite morphism after replacing X by an affine open neighborhood of x , by the Zariski's Main Theorem. (That is, one may assume that B is a finitely generated A -module.)

Let $A \rightarrow R$ be a faithfully flat extension of local rings such that R is a complete strictly Henselian regular local ring. For instance, one can choose R to be the completion of the strict Henselization of the completion of A .

Set S to be the normalization of $R \otimes_A B$. Since B is a finitely generated A -module, $R \otimes_A B$ is a finitely generated R -module. Hence $R \otimes_A B$ is excellent. Therefore S is a finitely generated $R \otimes_A B$ -module. Consequently $\mathrm{Spec}(S) \rightarrow \mathrm{Spec}(R)$ is a finite morphism. Let $U \subseteq \mathrm{Spec}(S)$ denote the étale locus of $\mathrm{Spec}(S) \rightarrow \mathrm{Spec}(R)$. It follows from the assumption that $U \rightarrow \mathrm{Spec}(S)$ is affine. It remains to show that $V \rightarrow \mathrm{Spec}(B)$ is affine.

Let Z be the complement of V in $\mathrm{Spec}(B)$. Then the codimension of Z in $\mathrm{Spec}(B)$ is 1 by theorem 1. Let V' and Z' denote the base change of V and Z , respectively, in $\mathrm{Spec}(R \otimes_A B)$. Since $R \rightarrow R \otimes_A B$ is flat, the codimension of Z' in $\mathrm{Spec}(R \otimes_A B)$ remains 1. Let Z'' denote the preimage of Z' in $\mathrm{Spec}(S)$. Since S is integral over $R \otimes_A B$, the codimension of Z'' in $\mathrm{Spec}(S)$ remains 1. This shows that V' is open in U with a codimension-1 complement. Since U is the étale locus of $\mathrm{Spec}(S)$ over a regular local ring R , it follows that U is a regular scheme. Therefore, each codimension-1 subscheme of U is locally principal. This shows that $V' \rightarrow U$ is locally principal affine. It follows that $V \rightarrow \mathrm{Spec}(B)$ is affine as V' is a faithfully flat base change of V . $\square$

Corollary 7. *Question 2 has an affirmative answer when Y is a regular scheme over a field.*

Proof. By Theorem 6, we may assume that Y is an excellent regular scheme over a field. In the excellent equal-characteristic case, the answer is affirmative by [Sta23, Tag 0ECB]. $\square$

Corollary 8. *Question 2 has an affirmative answer when $\dim(Y) = 2$.*

Proof. It follows from Theorem 6 and its proof that we can assume $Y = \operatorname{Spec}(A)$ and $X = \operatorname{Spec}(B)$, where A is a complete regular local ring with a separably closed residue field and B is normal and of finite type over A . Hence B is excellent. Let I be the defining ideal of the non-étale locus. By Serre's Criterion, it suffices to show $H_I^j(B) = 0$ for $j \geq 2$.

It follows from the assumptions in Question 2 that $\dim(B) \leq \dim(A)$. Without loss of generality, we may assume that $\dim(B) = \dim(A) = 2$. Hence it remains to show that $H_I^2(B) = 0$. Since B is normal and excellent, the completion of any localization of B is an integral domain. Moreover, it follows from Theorem 1 that I is pure of height 1. Therefore, $H_I^2(B) = 0$ by the Hartshorne-Lichtenbaum vanishing theorem ([Har68, Theorem 3.1]). This completes the proof. $\square$

4. THE UNRAMIFIED CASE

In this section, we will establish the following theorem:

Theorem 9. *Question 2 has an affirmative answer when Y is a regular scheme such that at each point $y \in Y$ the local ring $\mathcal{O}_{Y,y}$ is either an unramified regular local ring or a regular local ring of equal-characteristic.*

Proof. Following the first paragraph in the proof of Theorem 6, we can assume that

- (1) $Y = \operatorname{Spec}(A)$, where $(A, \mathfrak{m})$ is a complete regular local ring with a separably closed residue field.
- (2) $X = \operatorname{Spec}(B)$ is affine,
- (3) f is a finite morphism, and
- (4) V is affine away from $f^{-1}\{\mathfrak{m}\}$.

According to Corollary 8, we can assume that $\dim(A) = \dim(B) \geq 3$. If A contains a field, then the theorem follows from [Sta23, Tag 0ECB]. Therefore, we can assume that A is unramified of mixed characteristic.

Since V is affine away from the closed point, $H^i(V, \mathcal{O}_V)$ ($i > 0$) is $\mathfrak{m}$ -power torsion. Let Λ (a complete p -ring) denote the coefficient ring Λ of A . We denote by $\bar{A}, \bar{B}, \bar{V}$ the base changes of $A, B, V \bmod p$; we will still denote by $\mathfrak{m}$ the maximal ideal of $\bar{A}$. Then $H^i(\bar{V}, \mathcal{O}_{\bar{V}})$ ($i > 0$) is $\mathfrak{m}$ -power torsion. Since $\bar{A}$ is a regular local ring of equal-characteristic p , the assumptions in [Sta23, Tag 0ECC] are satisfied; consequently, $H^i(\bar{V}, \mathcal{O}_{\bar{V}}) = 0$ for all $i > 0$ if and only if $H^i(\bar{V}, \mathcal{O}_{\bar{V}}) = 0$ for $i = \dim(\bar{A}) - 1$. Let I be the defining ideal of the non-étale locus in $\bar{B}$. Note that we may assume that I is a proper ideal; otherwise there is nothing to prove. Since $\dim(\bar{A}) \geq 2$, the sheaf cohomology groups $H^i(\bar{V}, \mathcal{O}_{\bar{V}})$ is isomorphic to the local cohomology groups $H_I^{i+1}(\bar{B})$. It remains to show that $H_I^d(\bar{B}) = 0$ where $d = \dim(\bar{B})$. Since the vanishing of $H_I^d(\bar{B})$ is a local property, we may localize B at a maximal ideal and assume that B is local. Let $\mathfrak{n}$ denote the maximal ideal of B .

Assume $H_I^d(\bar{B}) \neq 0$, then it follows from the Hartshorne-Lichtenbaum vanishing theorem ([Har68, Theorem 3.1]), there exists a minimal prime ideal Q in $\bar{B}$ such that $\dim(\bar{B}/(I+Q)) = 0$. Since I defines the non-étale locus, the punctured spectrum of $\bar{B}/Q$ is étale over the punctured spectrum of $\bar{A}$. Since $\bar{A}$ is a regular local ring, there is an equivalence between the category of finite étale over the punctured spectrum

and the one of finite étale over $\bar{A}$. Moreover, since $\bar{A}$ is strictly henselian, it has no non-trivial finite étale covers. This shows that the punctured spectrum of $\bar{B}/Q$ is isomorphic to the punctured spectrum of $\bar{A}$. Set $U_1 = \text{Spec}(\bar{A}) - \{\mathfrak{m}\}$ and $U_2 = \text{Spec}(\bar{B}/Q) - \{\mathfrak{n}\}$; then $U_1 \cong U_2$.

Consider the following diagram

$$\begin{array}{ccccccc}
 & & 0 & & & & 0 \\
 & & \parallel & & & & \parallel \\
 0 & \longrightarrow & H_{\mathfrak{m}}^0(\bar{A}) & \longrightarrow & \bar{A} & \xrightarrow[\cong]{f} & H^0(U_1, \mathcal{O}_{U_1}) \longrightarrow H_{\mathfrak{m}}^1(\bar{A}) \\
 & & \downarrow & & & & \downarrow \cong \\
 0 & \longrightarrow & H_{\mathfrak{n}}^0(\bar{B}/Q) & \longrightarrow & \bar{B}/Q & \xrightarrow[g]{} & H^0(U_2, \mathcal{O}_{U_2}) \longrightarrow H_{\mathfrak{n}}^1(\bar{B}/Q) \\
 & & \parallel & & & & \parallel \\
 & & 0 & & & & 0
 \end{array}$$

Note that both f and g are ring homomorphisms. This commutative diagram produces a ring homomorphism $\bar{B} \rightarrow \bar{B}/Q \rightarrow \bar{A}$ that is a section of $\bar{A} \rightarrow \bar{B}$. But $\bar{B}$ is local; consequently $\bar{A} \cong \bar{B}$. Since p is a non-zero-divisor in both A and B , this shows that $A \cong B$, which implies that the ideal I (that defines the non-étale locus) is not a proper ideal, a contradiction. Therefore, we have shown that $H^i(\bar{V}, \mathcal{O}_{\bar{V}}) = 0$ for $\dim(\bar{A}) - 1$ and hence $H^i(\bar{V}, \mathcal{O}_{\bar{V}}) = 0$ for $i > 0$.

The short exact sequence $0 \rightarrow \mathcal{O}_V \xrightarrow{p} \mathcal{O}_V \rightarrow \mathcal{O}_{\bar{V}} \rightarrow 0$ implies that

- (1) $H^i(V, \mathcal{O}_V) = 0$ for $i \geq 2$, and that
- (2) $H^1(V, \mathcal{O}_V) \xrightarrow{p} H^1(V, \mathcal{O}_V)$ is surjective.

Since $V \rightarrow \text{Spec}(A)$ is étale, $\mathcal{O}_V$ is a $D(A, \Lambda)$ -module (Remark 3); hence so is $H^i(V, \mathcal{O}_V)$ for each i (using the Čech complex). Then it follows from Remark 5 that $H^1(V, \mathcal{O}_V)$ is a direct sum of some copies of the injective hull of $A/\mathfrak{m}$. Now the assumptions in [Sta23, Tag 0ECC] are satisfied; therefore, $H^i(V, \mathcal{O}_V) = 0$ for all $i > 0$ if and only if $H^i(V, \mathcal{O}_V) = 0$ for $\dim(A) - 1$. The latter vanishing holds by the Hartshorne-Lichtenbaum vanishing theorem since B is excellent and normal. $\square$

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