NOTES ON SOME t-STRUCTURES

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In some cases t-structures were associated to perversities. Recall ([BBD], 3.2.20) that for every ringed space $(X, \mathcal{O})$, a locally finite partition of X into locally closed subspaces, and a bounded integer valued function on the set of strata, one has a t-structure on $D(X, \mathcal{O})$. This t-structure depends only on the resulting function $X \to \mathbb{Z}$ and not on the choice of partition. One can think that more generally there are t-structures associated to perverse data on toposes. This is not defined here in general. When X is a noetherian sober space the construction of loc.cit. can be extended to an arbitrary lower-semicontinuous function for the constructible topology. We also treat the case of étale sheaves. See [B] for the equivariant case. Using iterated Godement resolutions we construct perverse truncation functors on the category of complexes of sheaves of modules on the considered ringed site. When X is a noetherian scheme admitting a coherent (resp. étale) dualizing complex, the sheaf of rings being the structure sheaf (resp. the constant étale sheaf $\mathbb{Z}/n\mathbb{Z}$), for suitable perversities the truncations preserve coherence (resp. constructibility) of the cohomology sheaves. For a good $\mathbb{F}_p$-scheme X we use the coherent case and p-linear algebra to get a t-structure on $D^b_c(X_{\acute{e}t}, \mathbb{F}_p)$. This includes the t-structures used in [EK]. The fact that in the classical middle perversity case the perverse sheaves are of finite length ([BBD] 4.3.1, which works also for finite coefficients) is extended by giving criteria for the perverse sheaves to be noetherian or artinian.

1. Let X be a noetherian scheme, $\tau \in \{Zar, \acute{e}t\}$, and R a sheaf of rings on X_τ. By a strong perversity function on X we mean a function $p : X \to \mathbb{Z} \cup \{+\infty\}$ such that

$$y \in \overline{\{x\}} \implies p(y) \geq p(x).$$

A weak perversity function is required to satisfy for every point x and integer m that $p(y) \geq \min(p(x), m)$ on some non-empty open in $\overline{\{x\}}$.

A weak perversity function is bounded below. We construct t-structures defined by p on the derived categories $D(X_\tau, R)$. The construction also works for the Nisnevich topology. A variant can be given for X not necessarily noetherian; the condition for a weak (resp. strong) perversity function should be replaced by $(\forall n \in \mathbb{Z})$ $\{x \in X | p(x) \geq n\}$ is ind-constructible (resp. ind-constructible and closed under specialization).

To make certain constructions strictly functorial we use the Godement resolution $F \to C^0(F) \to C^1(F) \dots$ of a sheaf of R-modules, defined by a conservative family of fibre functors on the site considered. This construction can be extended to bounded below complexes of sheaves by taking the associated total complex of a double complex $\operatorname{Tot} C^\bullet(F^\bullet) = C(F^\bullet)$. One can iterate the construction: We use the canonical maps $C^{(n)}(F^\bullet) \overset{\epsilon_{C^{(n)}(F^\bullet)}}{\to} C^{(n+1)}(F^\bullet)$ where $\epsilon_{F^\bullet} : F^\bullet \to C(F^\bullet)$ is the augmentation. (Here the superscript in $C^{(n)}(F^\bullet)$ refers to iteration, not degree in the complex.) Then one can also iterate over ordinals letting $C^{(0)}(F^\bullet) = F^\bullet$, $C^{(\alpha+1)}(F^\bullet) = C(C^{(\alpha)}(F))$ and $C^{(\lambda)}(F^\bullet) = \varinjlim_{\alpha<\lambda} C^{(\alpha)}(F^\bullet)$ for λ a limit ordinal. This will be used in this version only for ordinals up to ω. For unbounded below complexes set

$$C^{(\lambda)}(F^\bullet) = \varinjlim_n C^{(\lambda)}(\sigma_{\geq -n}F^\bullet) \text{ and } \hat{C}(F^\bullet) = \varprojlim_n C(\tau_{\geq -n}F^\bullet).$$

The terms in all these complexes are flasque. That is, they have 0 higher cohomology on objects of the topos. In some cases it is convenient to use a modified resolution C_ℓ that commutes with filtered direct limits.This is constructed in [SGA 4] XVIII 3.1. $C^*_\ell(F) = \varinjlim C^*(F_\alpha)$ when as an abelian sheaf F is a filtered direct limit of constructible abelian sheaves F_α. It is an exact functor.

The notion of a constructible R-Module is defined using the following equivalent conditions on F:

(1) $\mathrm{Hom}(F, \_)$ commutes with filtered direct limits;
(2) X is a finite union of locally closed subschemes X_i such that $F|_{X_i}$ is of finite presentation (i.e. admits local resolutions $R^a \to R^b \to F \to 0$);
(3) F has a presentation

$$\oplus_{i=1}^m j_! R_{V_i} \to \oplus_{i=1}^n j_! R_{U_i} \to F \to 0$$

where U_i and V_i are open in X in the case of the Zariski topology and quasi-compact objects of the étale site in the case of the étale topology.

In the non-noetherian case this is to be modified as in [SGA 4] IX 2.3. As in [SGA 4] XVIII 3.1.1., for X quasi-compact and quasi-separated, Ind(constr. sheaves) $\to$ (all R-Modules) is an equivalence of categories.

2. Given X, the topology, the sheaf of rings R and a weak perversity function p, one defines strictly full subcategories ${}^pD^{\leq n}(X,R)$ (also denoted $D^{\leq p+n}$) and ${}^pD^{\geq n}(X,R)$ (also denoted $D^{\geq p+n}$) of $D(X,R)$ as follows: For $x \in X$ let $\overline{x}$ be a topos point of X_τ localized at x, $X(\overline{x}) = \mathrm{Spec}(\mathcal{O}_{X_\tau,\overline{x}})$, we let $\overline{x}$ also denote the closed point of $X(\overline{x})$, $i_{\overline{x}} : \overline{x} \to X(\overline{x})$ the inclusion, $i^*_{\overline{x}} : D(X(\overline{x}),R) \to D(\overline{x},R)$ and $Ri^!_{\overline{x}} : D^+(X(\overline{x}),R) \to D^+(\overline{x},R)$ the usual functors. For a complex $F^\bullet$, $F^\bullet \in {}^pD^{\leq n}(X,R) \Leftrightarrow \mathcal{H}^i(i^*_{\overline{x}}F^\bullet) = 0\ \forall\, i > p(x)+n\ \forall\, x \in X$, $F^\bullet \in {}^pD^{\geq n}(X,R)$ $\Leftrightarrow F^\bullet$ has bounded below cohomology sheaves and $\mathcal{H}^i(Ri^!_{\overline{x}}F^\bullet) = 0\ \forall\, i < p(x)+n$ $\forall x \in X$.

We will show that $({}^pD^{\leq 0}, {}^pD^{\geq 0})$ is a t-structure on $D(X,R)$ and in the same way one sees that it induces a t-structure on $D^+(X,R)$, and also on $D^b(X,R)$ and $D^-(X,R)$ when p is finite and bounded.

3. Recall that for a locally closed $Z \subset X$ we have $R\underline{\Gamma}_Z : D^+(X,R) \to D^+(Z,R)$ and for $x \in Z$

$$(*) \qquad Ri^!_{\overline{x}} \xrightarrow{\sim} Ri^!_{\overline{x}} R\underline{\Gamma}_Z.$$

Hence

Lemma. $\mathcal{H}^i(R\underline{\Gamma}_Z(F^\bullet)) = 0\ \forall i \leq a$ *iff* $(\forall x \in Z)$ $\mathcal{H}^i(Ri^!_{\overline{x}}(F^\bullet)) = 0\ \forall i \leq a$.

Proof. "Only if" follows from (*). For "if", suppose on the contrary that some sheaf $\mathcal{H}^i_Z(F^\bullet)$ is non-zero for $i \leq a$. For an abelian sheaf F, the support $\text{supp}(F)$ of F is defined as the complement of the maximal open on which $F = 0$, and similarly for a section of a sheaf. We can assume the lower cohomology sheaves of $R\underline{\Gamma}_Z(F^\bullet)$ are zero. Hence there is a U (open in X for $\tau = Zar$, étale over X for $\tau = ét$) such that $\Gamma(U, \mathcal{H}^i_Z(F^\bullet)) \neq 0$. Take $0 \neq s \in \Gamma(U, \mathcal{H}^i_Z(F^\bullet))$. Let $y \in U$ be a maximal point (= a generic point of an irreducible component) of the support of s and x its image in X. Then $i^!_{\overline{x}}\mathcal{H}^i_Z(F^\bullet) \neq 0$. By a spectral sequence derived from (*) $\mathcal{H}^i(Ri^!_{\overline{x}}(F^\bullet)) \neq 0$. Contradiction. □

[The proof of the Lemma extends to the non-noetherian case provided Z is constructible.]

4.

Lemma. *Let Φ be an ind-constructible subset of X, $F \in D(X, R)$, $G \in D^+(X, R)$*

(1) *If $\mathcal{H}^i(i^*_{\overline{x}}F) = 0\ \forall i\ \forall x \notin \Phi$ and $\mathcal{H}^i(Ri^!_{\overline{x}}G) = 0\ \forall i\ \forall x \in \Phi$, then $R\mathcal{H}om(F, G)$ $= 0$.*

(2) *For $n \in \mathbb{N}$, if $\mathcal{H}^i(i^*_{\overline{x}}F) = 0$ if $i > n$ or $x \notin \Phi$ and $\mathcal{H}^i(Ri^!_{\overline{x}}G) = 0$ if $i \leq n$ and $x \in \Phi$, then* $\text{Hom}(F, G) = 0$.

Proof. In case (1), since the assumption is stable under localization it suffices to show that $\text{RHom}(F, G) = 0$. Represent G by a bounded below complex of injectives. Then $\text{Hom}(F, G)$ is the inverse limit of $H_n = \text{Hom}(\tau_{\leq n}F, G)$ which form an inverse system with surjective transition maps. Hence (EGA 0 (13.2.3)) it suffices to prove that each H_n is acyclic. By dévissage one reduces to F having a single non-zero cohomology sheaf. A similar reduction is applicable in (2). Then one reduces to proving (2) for $n = 0$ and F concentrated in degree 0. This is done by induction on the integer m such that $\mathcal{H}^i(G) = 0$ for $i < -m$.

The assertion is clear for negative m. Take a family of local sections $s_\alpha \in \Gamma(U_\alpha, F)$ that generates F, where $U_\alpha \xrightarrow{j_\alpha} X$ are quasi-compact objects in the site, and s_α has support Z_α. Then we have an epimorphism

$$\oplus_\alpha j_{\alpha!} R_{Z_\alpha} \xrightarrow{f} F.$$

By §3 and applying $R\Gamma(U_\alpha, \_)$ we have

$$\text{Hom}_{D(U_\alpha, R)}(R_{Z_\alpha}, j^*_\alpha G) = 0.$$

Thus, if we represent G by a bounded below injective complex

$$H^0(\text{Hom}^\bullet(\oplus j_{\alpha!} R_{Z_\alpha}, G)) = 0,$$

so by the Ext sequence one reduces to showing

$$\text{Ext}^{-1}(\ker(f), G) = 0.$$

This holds by induction on m. □

Notice that the assumption that Φ is ind-constructible is not used in the proof, but one can see that if the hypotheses hold, then they hold also for the largest ind-constructible subset of Φ.

[In the non-noetherian case, one should modify the proof by taking U_α to be quasi-compact and quasi-separated (qcqs) and taking Z_α to be a constructible closed subset of U_α s.t. $\mathrm{supp}(s_\alpha) \subset Z_\alpha \subset j_\alpha^{-1}(\Phi)$.]

5. The truncation functors for a weak perversity function are constructed by an iterative procedure using local cohomology with respect to sheaves of families of supports ([Ha] p.222) (of global nature in the case of a strong perversity). It is convenient to use some technical preliminaries. Let X be a scheme, $\tau \in \{Zar, \acute{e}t\}$, Φ an ind-constructible subset of X. We say that an abelian sheaf F on X_τ is Φ-supported iff the stalks of F vanish for τ points localized outside Φ. Let $\underline{\Gamma}_\Phi(F)$ denote the largest Φ-supported subsheaf of F. It is given by

$$U \mapsto \Gamma_\Phi(U,F) = \{s \in F(U) | \,\mathrm{supp}(s) \subset e^{-1}(\Phi)\} = \varinjlim_Z \Gamma_Z(U,F)\,,$$

for every object $U \xrightarrow{e} X$ of the τ topos of X (thought of as an algebraic space étale over X), where the direct limit is over the closed subsets Z of $|U|$ lying above Φ. The condition of ind-constructibility is used to ensure good behaviour relative to limits in X, Φ, and F.

Claim 5.1. *Let $U \xrightarrow{e} X$ be a qcqs algebraic space over X, $Z \subset |U|$ a closed subset s.t. $Z \subset e^{-1}(\Phi)$. Then there is a constructible closed subset $Z' \subset |X|$ s.t. $Z \subset Z' \subset e^{-1}(\Phi)$.*

Proof. Let $(U_\lambda)_{\lambda\in\Lambda}$ be a covering of $|U| - Z$ by quasi-compact opens. The U_λ and $e^{-1}(\Phi)$ form an open covering of $|U|$ for the constructible topology. Since $|U|$ is quasi-compact for this topology (EGA IV 1.9.9, which we leave to the reader to extend to algebraic spaces), there is a finite subset $\Lambda' \subset \Lambda$ s.t. $(U_\lambda)_{\lambda\in\Lambda'}$ and $e^{-1}(\Phi)$ cover U. Take Z' to be the complement of $\cup_{\lambda\in\Lambda'} U_\lambda$. □

One can derive the functors $\Gamma_\Phi(U,-)$ and $\underline{\Gamma}_\Phi$ to $R\Gamma_\Phi(U,-) : D^+(X,\mathcal{O}) \to D^+(\mathcal{O}(U)\text{-modules})$ and

$$R\underline{\Gamma}_\Phi : D^+(X,\mathcal{O}) \to D^+(\Phi\text{-supported } \mathcal{O}\text{-Modules})$$

where $\mathcal{O}$ is any sheaf of rings. These can be computed using flasque resolutions (SGA 4 V 6.6); moreover using claim 5.1 one can see similarly to loc.cit. that $R\Gamma_\Phi(U,-)$ for U coherent and $R\underline{\Gamma}_\Phi$ can be computed using (coh)-acyclic resolutions (SGA 4 V 4) where (coh) is the family of coherent objects of the considered topos.

Proposition 5.2. *Consider the situation of SGA4 VII Th.5.7: $(X_i)_{i\in I}$ is a projective system of qcqs schemes, indexed by a directed set I, with affine transition morphisms $u_{ij} : X_j \to X_i$ $(j \geq i)$, $X = \varprojlim_i X_i$, $u_i : X \to X_i$ the canonical maps. Suppose F is an abelian sheaf on the fibered site of loc.cit.: it is given by abelian sheaves F_i on $(X_i)_\tau$ for every i and maps $u_{ij}^* F_i \to F_j$ compatible with identity and composition. Let $F_\infty = \varinjlim_i u_i^* F_i$. Suppose that for every i, Φ_i is an ind-constructible subset of X_i s.t. $u_{ij}^{-1}(\Phi_i) \subset \Phi_j$ $\forall\, i \leq j$ and let $\Phi = \cup_i u_i^{-1}(\Phi_i)$. Then $\forall\, n$*

$$\varinjlim_i H^n_{\Phi_i}(X_i, F_i) \xrightarrow{\sim} H^n_\Phi(X, F_\infty)\,.$$

Proof. One uses a suitable resolution of F to reduce to the case each F_i is (coh)-acyclic. Then F_∞ is (coh)-acyclic (every coherent object of X_τ comes from a coherent object of some $(X_i)_\tau$). So it is enough to do the case $n = 0$. By loc.cit. (and its analogue for the Zariski topology) the assertion holds without support restriction. So it remains to check that if $s_i \in \Gamma(X_i, F_i)$ induces $s \in \Gamma_\Phi(X, F_\infty)$ then for some $j \geq i$ the induced section $s_j \in \Gamma(X_j, F_j)$ has support $\subset \Phi_j$. But the $\operatorname{supp}(s_j) - \Phi_j$ form an inverse system of compact spaces if endowed with the constructible topology, so if they are all nonempty their inverse limit $\operatorname{supp}(s) - \Phi$ would be nonempty. □

It follows that the stalk of $\mathcal{H}^i_\Phi(X, F)$ at a τ-point ξ is given by $H^i_\Phi(X(\xi), F)$.

The functor $\underline{\Gamma}_\Phi$: ($\mathcal{O}$-Modules) $\to$ (Φ-supported $\mathcal{O}$-Modules) is right adjoint to the inclusion, hence it defines a comonad ([ML] p.135) in the category of $\mathcal{O}$-Modules consisting of the endofunctor $\underline{\Gamma}_\Phi$ and transformations $\varepsilon : \underline{\Gamma}_\Phi \to \mathrm{id}$, $\delta : \underline{\Gamma}_\Phi \to \underline{\Gamma}_\Phi \circ \underline{\Gamma}_\Phi$ rendering commutative certain diagrams. Also, δ is an isomorphism. This can be derived to a comonad in $D^+(X, \mathcal{O})$. If $K^\bullet$ is a Φ-supported bounded below complex of $\mathcal{O}$-Modules we have a map in the derived category

$$K^\bullet = \underline{\Gamma}_\Phi(K^\bullet) \to R\underline{\Gamma}_\Phi(K^\bullet)\,.$$

Applying this to complexes realizing $R\underline{\Gamma}_\Phi$ we get $\delta : R\underline{\Gamma}_\Phi \to R\underline{\Gamma}_\Phi \circ R\underline{\Gamma}_\Phi$. Together with $\varepsilon : R\underline{\Gamma}_\Phi \to \mathrm{id}$ this constitutes a comonad in $D^+(X, \mathcal{O})$ but δ is not an isomorphism in general; it is an isomorphism when Φ is locally closed or directed increasing union of locally closed subsets.

Proposition 5.3.

(1) *δ is an isomorphism iff for every Φ-supported $\mathcal{O}$-Module F, $F \simeq R\underline{\Gamma}_\Phi(F)$.*
(2) *If F is a Φ-supported abelian sheaf then $R^1\underline{\Gamma}_\Phi(F) = 0$.*
(3) *If the following condition*

$$(*) \qquad (x \in \overline{\{y\}},\ y \in \overline{\{z\}},\ x \in \Phi,\ z \in \Phi) \Rightarrow y \in \Phi$$

holds then in (2) $R^i\underline{\Gamma}_\Phi(F) = 0\ \forall\, i > 0$.

Proof. (1) Necessity: Since $\delta(F)$ is an isomorphism we have a spectral sequence

$$E_2^{p,q} = R^p\underline{\Gamma}_\Phi R^q\underline{\Gamma}_\Phi(F) \Rightarrow R^{p+q}\underline{\Gamma}_\Phi(F)\,.$$

$E_2^{0,q} = R^q\underline{\Gamma}_\Phi(F)$ and the edge homomorphism $E_\infty^q \to E_2^{0,q}$ corresponds to $\varepsilon \circ R\underline{\Gamma}_\Phi$: $R\underline{\Gamma}_\Phi \circ R\underline{\Gamma}_\Phi \to R\underline{\Gamma}_\Phi$ so by the comonad condition it gives the identity on $R^q\underline{\Gamma}_\Phi(F)$. If for some $p > 0$ we have $R^p\underline{\Gamma}_\Phi(F) \neq 0$ and p is the minimal such, one sees that $E_2^{p,0} = E_\infty^{p,0}$ is the kernel of the edge homomorphism so it is 0.

Sufficiency: Easy since $\varepsilon \circ R\underline{\Gamma}_\Phi$ is an isomorphism by the spectral sequence.

(2) To compute the stalk of $R^1\underline{\Gamma}_\Phi(F)$ at a τ point $\overline{x}$ we may assume $X = X(\overline{x})$. Then $H^1_\Phi(X, F) = \varinjlim_Z \operatorname{Coker}[F(X) \to F(X - Z)]$, the direct limit is over the closed subsets of X contained in Φ. The direct limit vanishes since if $s \in F(X - Z)$ then $\operatorname{supp}(s) \subset \Phi$ and we can increase Z to $Z \cup \operatorname{supp}(s)$.

(3) To treat the case $i \geq 2$ we reduce as in (2) to the case $X = X(\overline{x})$ ((∗) is inherited) and to showing that $\lim\limits_{\to} {}_Z H^{i-1}(X - Z, F) = 0$. If $x \notin \Phi$ only $Z = \phi$ is possible and the assertion is clear. If $x \in \Phi$ then (∗) implies that Φ is a union of closed sets so by 5.1 Φ is a directed union of constructible closed subsets Φ_j so $F = \underline{\Gamma}_\Phi(F) = \lim\limits_{\to} {}_j \underline{\Gamma}_{\Phi_j}(F)$. By 5.2, $\mathcal{H}^i_\Phi(F) = \lim\limits_{\to} {}_j \mathcal{H}^i_\Phi(\underline{\Gamma}_{\Phi_j}(F))$ and the latter vanishes as in the proof of (2). □

As an example when δ is not an isomorphism consider $X = \mathbb{A}^2_k$ over a field k, U the open $\{xy \neq 0\}$, $\Phi = \{(0,0)\} \cup U$, τ arbitrary, $F = j_! \mathbb{Z}_U$. Then $\mathcal{H}^2_\Phi(F) = \mathbb{Z}_{\{(0,0)\}}$.

Proposition 5.4. *Given $X, \tau, \Phi, \mathcal{O}$, an integer n and $F \in D^+(X, \mathcal{O})$, the following conditions are equivalent:*

(a) $H^i_{\{\overline{x}\}}(X(\overline{x}), F) = 0 \ \forall\, i < n \ \forall\, x \in \Phi$

(b) $\mathcal{H}^i_\Phi(F) = 0 \ \forall\, i < n$

(c) *For every object U of the τ topos and a closed $Z \subset |U|$ lying above Φ, $\mathcal{H}^i_Z(U, F) = 0 \ \forall i < n$.*

(c') *Condition (c) for U open and Z constructible.*

Proof. (a) ⇒ (c). Since the assertion is local we may assume that U is an affine scheme in the τ site. Then by 5.1 enlarge Z to a constructible Z' and use $R\underline{\Gamma}_Z = R\underline{\Gamma}_Z \circ R\underline{\Gamma}_{Z'}$ and that the assertion for $R\underline{\Gamma}_{Z'}$ holds by section 3.

(c) ⇒ (b). If (c) holds we get in the situation of (c), that $\forall\, i < n$, $H^i_Z(U, F) = 0$, hence the same for H^i_Φ which yields (b).

(b) ⇒ (a). We prove (a) by induction on i. It is clear for i small enough. Using 5.2 the assumption in (b) gives $H^i_\Phi(X(\overline{x}), F) = 0$. For a nonempty closed subset Z of $X(\overline{x})$ lying over Φ we have an exact sequence

$$H^{i-1}_{Z-\{\overline{x}\}}(X(\overline{x}) - \{\overline{x}\}, F) \to H^i_{\{\overline{x}\}}(X(\overline{x}), F) \to H^i_Z(X(\overline{x}), F)\,.$$

The first term vanishes by the induction hypothesis and the implication (a) ⇒ (c). Taking the direct limit over Z we get the vanishing of the second term.

Finally (c) ⇒ (c') is clear and (c') ⇒ (a) holds by section 3. □

When the equivalent conditions of Proposition 5.4 hold we say that $\mathrm{depth}_\Phi(F) \geq n$.

Proposition 5.5. *If* $\mathrm{depth}_\Phi(F) \geq n$ *then*

$$\mathrm{depth}_\Phi(\mathrm{Cone}\ (R\underline{\Gamma}_\Phi(F) \to F)) \geq n + 1\,.$$

Proof. Using 5.4(b) as definition the assertion is equivalent to

(1) $\mathcal{H}^i_\Phi(\epsilon(F))$ is an isomorphism for all $i \leq n$ and injective for $i = n + 1$.

Since $\delta(F)$ is a right inverse of $R\underline{\Gamma}_\Phi \circ \epsilon(F)$, (1) is equivalent to

(2) $\mathcal{H}^i(\delta(F))$ is an isomorphism for all $i \leq n + 1$
which is in turn equivalent to

(3) $\epsilon(R\underline{\Gamma}_\Phi(F)) : R\underline{\Gamma}_\Phi R\underline{\Gamma}_\Phi(F) \to R\underline{\Gamma}_\Phi(F)$ induces an isomorphism on $\mathcal{H}^i$ for all $i \leq n$ and a monomorphism for $i = n + 1$.

The map studied in (3) is the edge homomorphism for the usual spectral sequence for the cohomology sheaves of $R\underline{\Gamma}_\Phi R\underline{\Gamma}_\Phi(F)$, and (3) follows from the fact that the $p = 1$ column of the spectral sequence vanishes by 5.3(2).

Proposition 5.6. *Suppose that $\Psi \subset \Phi$ are ind-constructible subsets and* $\mathrm{depth}_\Phi(F) \geq n$*. Then*

(1) $\mathrm{depth}_\Psi(F) \geq n$
(2) $\mathcal{H}^n_\Psi(F) \to \mathcal{H}^n_\Phi(F)$ *is injective and identifies $\mathcal{H}^n_\Psi(F)$ with the largest Ψ-supported subsheaf of $\mathcal{H}^n_\Phi(F)$*
(3) $\mathrm{depth}_\Phi(\mathrm{Cone}\ (R\underline{\Gamma}_\Psi(F) \to F)) \geq n$.

Proof. (1) follows from 5.4. Using 5.5 and (1), the map $R\underline{\Gamma}_\Phi(F) \to F$ induces isomorphisms on the sheaves studied in (2), so (2) is reduced to the case where F is concentrated in degrees $\geq n$, when the studied map is identified by a spectral sequence with

$$\underline{\Gamma}_\Psi(\mathcal{H}^n(F)) \to \underline{\Gamma}_\Phi(\mathcal{H}^n(F)) .$$

(3) By the octahedral axiom it suffices to know the statement for Cone $(R\underline{\Gamma}_\Psi(F) \to R\underline{\Gamma}_\Phi(F))$, which holds because by (1) and (2) this complex is in $D^{\geq n}$, and the statement for Cone $(R\underline{\Gamma}_\Phi(F) \to F)$, which is weaker than 5.5. □

6. We verify that the definition in §2 indeed gives a t-structure. The following argument works for a bounded below weak perversity function on an arbitrary scheme. We have to check that $\mathrm{Hom}(F, G) = 0$ if $F \in {}^pD^{\leq n}$ and $G \in {}^pD^{\geq n+1}$. As in §4, we reduce to the case where F has a single non-zero cohomology sheaf. This case holds by §4. We have to construct a truncation ${}^p\tau_{\leq 0}F = \tau_{\leq p}F$ which is in ${}^pD^{\leq 0}$ and such that

$$\mathrm{Cone}({}^p\tau_{\leq 0}F \to F) \in {}^pD^{\geq 1}.$$

Let c be a lower bound for p. For $d \geq c$, consider $p_d = \min(d, p)$ which is a (weak or strong) perversity function if p is. We construct a model for $\tau_{\leq p_d}F$ as a subcomplex of $C^{(d-c)}(F)$ and construct $\tau_{\leq p}F$ as the direct limit. Set $p_c = c$ and $\tau_{\leq p_c} = \tau_{\leq c}$. To go up it suffices to see that if we have $\tau_{\leq p_d}F$ as a subcomplex of F then we can define $\tau_{\leq p_{d+1}}F$ as a subcomplex of CF containing (the image of) $C(\tau_{\leq p_d}F)$. It suffices to do this for $F/\tau_{\leq p_d}F$. So assume $F \in D^{\geq 1}$ (relative to p_d) $= D^{\geq p_d+1}$. Let $\Phi = \{x \in X | p_{d+1}(x) = d + 1\}$. It is ind-constructible.

We take $\tau_{\leq p_{d+1}}F = \tau_{\leq d+1}\underline{\Gamma}_\Phi CF$. It is clear that this is in $D^{\leq p_{d+1}}$. Note that $\mathrm{depth}_\Phi(F) \geq d+1$ so $\tau_{\leq p_{d+1}}F = \mathcal{H}^{d+1}_\Phi(F)[-d-1]$ in the derived category. We have to show that $M = CF/\tau_{\leq p_{d+1}}F \in D^{>p_{d+1}}$. (We use the notation $D^{>p}$ or $D^{\geq p+1}$.) That is

(1) $\mathrm{depth}_\Phi M \geq d + 2$;
(2) for every integer $m \leq d$, if $\Psi = \{x \in X \mid p_d(x) \geq m\}$ then $\mathrm{depth}_\Psi M \geq m+1$.

For this we use the exact sequence

$$(*) \qquad 0 \to \tau'_{\geq d+2}\underline{\Gamma}_\Phi CF \to M \to CF/\underline{\Gamma}_\Phi CF \to 0$$

($\tau'_{\geq n}K^\bullet = K^\bullet/\tau_{\leq n-1}K^\bullet$ is quasi-isomorphic to $\tau_{\geq n}K^\bullet$). The first term in $(*)$ has the required depth properties for degree reasons. Since $\mathrm{depth}_\Phi(F) \geq d+1$ we have by Proposition 5.5 that the last term in $(*)$ has $\mathrm{depth}_\Phi \geq d + 2$ which yields (1). To prove (2), note that $\mathrm{depth}_\Psi F \geq m + 1$ so by Proposition 5.6(3) the last term in $(*)$ has $\mathrm{depth}_\Psi \geq m$.

Remarks 6.1.

(1) The construction shows that $\tau_{\leq p}$ and $\tau_{\geq p}$ preserve $D^{\geq n}$ for an integer n. If $F \in D^{\leq n}$ then $\mathcal{H}^n(\tau_{\leq p}F) = \underline{\Gamma}_\Phi \mathcal{H}^n(F)$ where $\Phi = \{x \in X \mid p(x) \geq n\}$.

(2) If $Z \subset X$ is a closed subscheme and $U = X - Z$ and we restrict the perversity and the sheaf of rings to Z and U one can say that the t-structure on $D^+(X, \mathcal{O})$ is obtained by "recollement" ([BBD] 1.4) from the t-structures on $D^+(Z, \mathcal{O})$ and $D^+(U, \mathcal{O})$.

(3) If p, q are perversities s.t. $\tau_{\leq p}$ preserves $D^{\leq q}$ then $\tau_{\leq p} : D^{\leq q} \to D^{\leq \min(p,q)}$ is right adjoint to the inclusion (cf. [BBD] 1.3.3(i)), and it follows formally that $\tau_{\leq p} \circ \tau_{\leq q}$ is $\tau_{\leq \min(p,q)}$. This holds in particular if $p \geq q$ or $p \leq q$ or p is constant.

(4) Similarly if $\tau_{\geq p}$ preserves $D^{\geq q}$ then $\tau_{\geq p} \circ \tau_{\geq q}$ is $\tau_{\geq \max(p,q)}$. This holds in particular if $p \geq q$ or $p \leq q$ or q is constant.

(5) If $p \leq q$ are weak perversity functions and F any complex we have $\tau_{\leq p}F \subset \tau_{\leq q}F$ if these are constructed as above as subcomplexes of $C^{(\omega)}F$ and $\tau_{\leq q}F/\tau_{\leq p}F$ serves as $\tau_{\geq p+1}\tau_{\leq q}F$ and as $\tau_{\leq q}\tau_{\geq p+1}F$.

(6) When Φ is a union of closed subsets and R is the structure sheaf, $R\underline{\Gamma}_\Phi$ preserves $D^+_{\text{quasicoh}}(X, \mathcal{O}_X)$. The truncation functors defined by strong perversity functions preserve D_{quasicoh}.

(7) If $\varepsilon : (X_{\acute{e}t}, \mathcal{O}) \to (X_{Zar}, \mathcal{O})$ is the morphism of ringed toposes we have an equivalence of categories $\epsilon^* : D_{\text{quasicoh}}(X_{Zar}) \to D_{\text{quasicoh}}(X_{\acute{e}t})$; a quasi-inverse is given by $K \mapsto \varepsilon_* \hat{C}(K)$.

This equivalence is compatible with the t-structures defined by strong perversity functions.

(8) If p is a weak perversity function on X and $\tau_{\leq p}$ preserves $D^+_{\text{quasicoh}}(X)$ then p is strong.

Indeed by remark (1) it suffices to see that if $\underline{\Gamma}_\Phi$ preserves quasicoherence then Φ is closed under specialization.

In the case of the Zariski topology, if $x \in \Phi$ and Z is the closed reduced irreducible subscheme with generic point x, $\underline{\Gamma}_\Phi(\mathcal{O}_Z)$ is the extension by 0 of $\mathcal{O}_Z$ from the largest open $U \subset Z$ contained in Φ, which is not quasicoherent if $U \neq Z$.

In the case of the étale topology use $\varepsilon_*\underline{\Gamma}_\Phi = \underline{\Gamma}_\Phi\varepsilon_*$.

Remark 6.2. Suppose X is a scheme with an unbounded below weak perversity function p. Then X is an increasing union of opens U_n ($n \in \mathbb{N}$) s.t. $p \geq -n$ on U_n. We still have a t-structure $({}^pD^{\leq 0}, {}^pD^{\geq 0})$ on $D(X_\tau, R)$. In the definition of "$K \in {}^pD^{\geq 0}$" one requires that K is locally in D^+.

Suppose $K \in {}^pD^{\leq 0}$, $L \in {}^pD^{\geq 1}$ and we want to show that $\operatorname{Hom}(K, L) = 0$. We represent K, L by complexes $K^\bullet, L^\bullet$ with $L^\bullet$ fibrant, i.e. K-injective [Sp] with injective terms. Then

$$\operatorname{Hom}(K, L) = H^0 \operatorname{Hom}^\bullet(K^\bullet, L^\bullet)$$

and

$$\operatorname{Hom}^\bullet(K^\bullet, L^\bullet) = \varprojlim_n \operatorname{Hom}^\bullet(K^\bullet|_{U_n}, L^\bullet|U_n)\,,$$

an inverse limit with surjective transition maps. We know already that $\operatorname{Hom}^\bullet$ $(K^\bullet|_{U_n}, L^\bullet|_{U_n})$ is acyclic in non positive degrees, so the same holds for $\operatorname{Hom}^\bullet(K^\bullet,$ $L^\bullet)$.

Suppose K is a complex of R-Modules and we want to construct $\tau_{\leq p}K$. On U_n one can start with the lower bound $-n$ for p and define as above a subcomplex T_n of $C^{(\omega)}(K|_{U_n})$ realizing $\tau_{\leq p}(K|_{U_n})$. We have $t : C^{(\omega)}K \to C^{(\omega)}K$ obtained by applying $C^{(\omega)}$ to $\varepsilon : K \to CK$, and on U_n one has $t(T_n) \subset T_{n+1}$. Then the subcomplex $\varinjlim j_! T_n$ of $\varinjlim (C^{(\omega)}K \xrightarrow{t} C^{(\omega)}K \xrightarrow{t} \dots)$ has the properties required of $\tau_{\leq p}K$.

7. Fix a *qcqs* scheme X, the topology, the weak perversity function, and the sheaf R. The heart of the t-structure defined above will be denoted $\mathrm{Perv}_p(X_\tau, R)$. Let Q denote the canonical functor from the category of complexes to the derived category. If $(A_i)_{i\in I}$ is a small family of perverse complexes then $\oplus_i A_i$ is perverse and $Q(\oplus A_i)$ serves as the coproduct of the QA_i in the category Perv since it is already a coproduct in the full derived category. Hence Perv has small colimits.

Proposition 7.1.

(1) *If $f : I \to \mathrm{Perv}$ is a functor (I a small category) then f lifts to a functor $I \to C(X, R)$ (the category of complexes).*
(2) *If I is a filtered category and $i \mapsto K_i$ is an inductive system of perverse complexes then $Q(\varinjlim K_i)$ is perverse and is the colimit of the QK_i in* Perv *.*
(3) Perv *has exact filtered colimits and a small set of generators.*

Proof. (1) This can be done by a version of the method of [BBD, 3.2]. We give another argument. The functor ${}^p\mathcal{H}^0 : D(X, R) \to \mathrm{Perv}$ can be realized by a functor θ on complexes, e.g. both $\tau_{\leq p-1}$ and $\tau_{\leq p}$ can be realised as subcomplexes of the same iterated Godement resolution and $\theta = \tau_{\leq p}/\tau_{\leq p-1}$.

Say $f(i) = QK_i$, with K_i injective and bounded below. For $\varphi : i \to j$ one can realize $f(\varphi)$ by a map $K_\varphi : K_i \to K_j$. Consider

$$\oplus_{\varphi:i\to j} K_i \xrightarrow{\alpha} \oplus_{i\in I} K_i.$$

where α is given on the φ component by $i_i \circ \mathrm{id}_{K_i} - i_j \circ K_\varphi$, i_j being the inclusion of the jth component. Then the colimit of the QK_i is the cokernel of $Q\alpha$ in Perv, and thus realized by $\theta(\mathrm{Cone}(\alpha))$.

Form the categories I/i of objects mapping to i. We have

$$QK_i \xrightarrow{\sim} \mathrm{colim}_{j\in I/i}\, QK_j \xrightarrow{\sim} Q(\theta(\mathrm{Cone}(\alpha|_{I/i})))$$

and a lifting of f may be defined by sending i to the term on the right.

(2) It is easily checked that filtered direct limits preserve $D^{\geq n}$ and $D^{\leq n}$ relative to any p. We have to check that in the above $\mathrm{coker}(\alpha)$ serves as ${}^p\mathcal{H}^0$ of $\mathrm{Cone}(\alpha)$ and for that it suffices to see that $\ker(\alpha)$ is perverse. But $\ker(\alpha)$ is a colimit of the corresponding things for I/i so we may assume that I has a final object i. But then a standard homotopy operator shows that $\ker(\alpha)$ is a direct summand of $\oplus_\varphi K_i$.

(3) One can choose θ commuting with filtered colimits. Any complex K is a filtered colimit of bounded complexes with constructible terms K_α. Then $\theta K = \mathrm{colim}\, \theta K_\alpha$, so by (2) the ${}^p\mathcal{H}^0$ of bounded constructible complexes form a set of generators.

Let

$$0 \to A_i \to B_i \to C_i \to 0 \quad i \in I$$

be a filtered inductive system of exact sequences of perverse sheaves. For every i we can realize that as an exact sequence of injective complexes bounded below. Also for $i \to j$ one can find a map of short exact sequences of complexes realizing the derived category maps, e.g. given maps $A_i \to A_j$ and $B_i \to B_j$ realizing the derived category maps,

$$\begin{array}{ccc} A_i & \longrightarrow & B_i \\ \downarrow & & \downarrow \\ A_j & \longrightarrow & B_j \end{array}$$

commutes up to homotopy, but $A_i \subset B_i$ is termwise split so one can replace $B_i \to B_j$ by a homotopic map to make the square strictly commutative.

Then the construction in the proof of (1) gives an inductive system of short exact sequences in $C(X, R)$ realizing

$$0 \to A_i \to B_i \to C_i \to 0.$$

Now take the colimit, use (2) and the fact that a short exact sequence of perverse complexes is exact in Perv. $\square$

Note that Proposition 7.1(2) implies that the functor $K \mapsto \mathcal{H}^n K$ from Perv to R-Mod commutes with filtered colimits.

Remark 7.2.

(1) If R is coherent then the kernel of a homomorphism of constructible R-Modules is constructible, every constructible R-Module F admits an infinite resolution

$$\cdots \to j_{1!}R_{U_1} \to j_{0!}R_{U_0} \to F \to 0$$

with U_i coherent objects of the topos, and the functors $\mathrm{Ext}^p(F, -)$ commute with filtered colimits (cf. SGA 4 IX 2.7.3).

(2) If furthermore the perverse truncation functors preserve the subcategory $D^b_c(X, R) \subset D(X, R)$ of objects with bounded constructible cohomology then the $\varinjlim$ functor (SGA4 I 8.7)

$$\mathrm{Ind}(\mathrm{Perv} \cap D^b_c) \to \mathrm{Perv}$$

is an equivalence of categories.

In (2) the functor is fully faithful using (1) and is essentially surjective using the proof of Proposition 7.1(3).

8. Suppose that $n > 1$ is an integer, $A = \mathbb{Z}/n\mathbb{Z}$, X a noetherian $\mathbb{Z}\left[\frac{1}{n}\right]$ scheme of finite dimension that admits a dualizing complex $K \in D^b_c(X_{\acute{e}t}, A)$ in the sense of SGA5 I. For every $F \in D^b_c(X, A)$ we have $D_K(F) = R\mathcal{H}om(F, K) \in D^b_c$ and the canonical map $F \to D_K D_K(F)$ is an isomorphism. By SGA5 I 1.15 if $f : Y \to X$ is quasi-finite $Rf^! K$ is dualizing on Y. In particular the restriction of K to strict localizations can be shown to be dualizing. For every geometric point $\overline{x}$ of X, $Ri^!_{\overline{x}} K$ is isomorphic to $A[-n(x)]_{\overline{x}}$ for some integer $n(x)$.

Lemma 8.1. *If y is an immediate specialization of x we have $n(y) = n(x) + 2$.*

Proof. One reduces this to the strictly local case, with y the closed point. If $Z = \overline{\{x\}}$ (as a reduced scheme) and $i : Z \hookrightarrow X$ the inclusion then $Ri^!K$ is dualizing on Z so one reduces the assertion to Z. If Z' is the normalization of Z then Z' is the spectrum of a strictly henselian discrete valuation ring and $Z' \to Z$ is a universal homeomorphism and induces an equivalence of étale toposes. One reduces the assertion to Z', in which case K is isomorphic to $A[n]$ for some n (SGA5 I 2.1, 5.1) and the assertion is purity (loc.cit.). $\square$

For points x, y with $y \in \overline{\{x\}}$ let $c(y,x) = \operatorname{codim}(\overline{\{y\}}, \overline{\{x\}})$. Lemma 8.1 implies that X is catenarian. (In fact the strict localizations of X are also catenarian, and one deduces that X is universally catenarian.)

Theorem 8.2. *Let p be a weak perversity function on X. Then the truncation functors associated to p preserve $D^b_c(X,A)$ iff the following condition holds*

$(*)$ $\forall\, x \in X$ *there is an open dense set* $U \subset \overline{\{x\}}$ *s.t.* $\forall\, y \in U$, $p(y) \leq p(x) + 2c(y,x)$.

Proof. Necessity. To prove $(*)$ we may assume that X is irreducible with a generic point x and $p(x)$ is finite. Upon normalization $p(x) = 0$, $n(x) = 0$. Passing to an open dense subscheme we may assume that the dualizing complex is locally isomorphic to A. If the assumption on truncations holds then ${}^p\mathcal{H}^0(A_X)$ is a perverse sheaf in $D^b_c(X,A)$, and it is A at the generic point, hence on some open dense subscheme U. The depth of A_X at y is $2c(y,x)$ (8.1), but it should be $\geq p(y)$ (as $A_U \in {}^pD^{\geq 0}$), which gives $(*)$.

Sufficiency. We may assume the assertion holds on all proper closed subschemes of X. Let $i : F \hookrightarrow X$, $j : U \hookrightarrow X$ be a closed subscheme and the complementary open. The t-structure on $D^+(X,A)$ is the one obtained by "recollement", and one has ([BBD] 1.4.13.1) ${}^p\tau_{\leq 0} = {}^p\tau^F_{\leq 0}\, {}^p\tau^U_{\leq 0}$, where ${}^p\tau^U_{\leq 0}$ and ${}^p\tau^F_{\leq 0}$ are defined by distinguished triangles ${}^p\tau^U_{\leq 0}K \to K \to Rj_*\, {}^p\tau_{>0} j^*K \to$ and ${}^p\tau^F_{\leq 0}K \to K \to Ri_*\ {}^p\tau_{>0} i^*K \to$. We note also that when p is finite valued, $n - p$ is a weak perversity satisfying $(*)$ and the fact that the duality exchanges $i^*_{\overline{x}}$ with $i^!_{\overline{x}}$ implies that for $K \in D^b_c$, $K \in {}^pD^{\leq 0}$ iff $DK \in {}^{n-p}D^{\geq 0}$.

The above reduces the problem to showing that if $K \in D^b_c X$ then ${}^p\tau_{\leq 0}K|_U \in D^b_c(U)$ for some open dense U. We may assume that X is irreducible with a generic point x and that for every point y we have $\min(p(x), m) \leq p(y) \leq p(x) + 2c(y,x)$ where m is an integer s.t. $\mathcal{H}^iK = 0$ for $i > m$, and that A_X is dualizing and that the cohomology sheaves of K are locally constant. If $p(x) \geq m$ then $p \geq m$ and ${}^p\tau_{\leq 0}K = K$. If $p(x) < m$ it follows from the above that $\tau_{\leq p(x)}K$ is ${}^p\tau_{\leq 0}K$. $\square$

Theorem 8.3. *Let p be a weak perversity satisfying condition $(*)$ of 8.2. Then*

(1) *Every object in* $\mathrm{Perv}_p \cap D^b_c$ *is artinian iff p satisfies*

$(+)$ $\forall\, x \in X$ $\exists$ *open dense* $U \subset \overline{\{x\}}$ *s.t.* ($p(y) > p(x)$ *if* $p(x) < \infty$) $\forall\, y \in U - \{x\}$.

(2) *Every object in* $\mathrm{Perv}_p \cap D^b_c$ *is noetherian iff p satisfies*

$(++)$ $\forall\, x \in X$ $\exists$ *open dense* $U \subset \overline{\{x\}}$ *s.t.* ($p(y) < p(x) + 2c(y,x)$ *if* $p(x) < \infty$) $\forall\, y \in U - \{x\}$.

Proof. We indicate the proof for (1).

Necessity. Say $x \in X$ does not satisfy the condition in (+). We may assume $p(x) = 0$. Let $U \subset \overline{\{x\}}$ be open dense s.t. A_U is perverse. We have $p \geq 0$ on U, and by assumption one has an infinite sequence of non-generic points $y_1, y_2, \ldots$ of U s.t. $\forall i$ $p(y_i) = 0$ and $y_{i+1} \notin F_i = \overline{\{y_1, \ldots, y_i\}}$. Then ${}^p\mathcal{H}^0(A_{U \cap F_i})$ is the maximal perverse quotient of A_U and it has support exactly $U \cap F_i$. $\mathrm{Ker}({}^p\mathcal{H}^0(A_{\overline{\{x\}}}) \to {}^p\mathcal{H}^0(A_{F_n}))$ is a strictly decreasing sequence of perverse subobjects of ${}^p\mathcal{H}^0(A_{\overline{\{x\}}})$.

Sufficiency. We may assume the result is known on proper closed subschemes. Suppose that $F_0 \supset F_1 \supset \cdots$ is a decreasing sequence of perverse subobjects of F_0.

Claim. *F_n is stationary on some open dense $U \subset X$.*

Assuming the claim we may assume that $F_0|_U = F_n|_U \; \forall n$, so F_0/F_n corresponds to a quotient ${}^p\mathcal{H}^0(i^*F_0)/G_n$, $i : X - U \to X$. By assumption G_n is stationary.

To prove the claim we may assume that X is irreducible with generic point x. The sequence is stationary at the generic point. If $p(x) = \infty$ then F_0 is 0 in a neighborhood of x and there is nothing to show. We may assume that $p(x) = 0$ and $F_0 = F_n$ at $x \; \forall n$ and that F_0 is a locally constant sheaf (placed in degree 0) and that $p(y) > p(x)$ for every non generic point. Then for every proper closed subset $Z \subset X$, $F_0|_Z \in {}^pD^{<0}$ so F_0 has no nonzero perverse quotients supported on Z. Thus $F_0 = F_n \; \forall n$.

9. Now suppose X is of finite Krull dimension and admits a dualizing complex ([Ha]) and $R = \mathcal{O}_X$. There is an associated codimension function c such that $c(x) = c(y) + 1$ when x is an immediate specialization of y, and the dualizing complex is realized by a residual complex $R^\bullet$ such that

$$R^i \cong \oplus_{c(x)=i} j_{x*} I(x)$$

where $j_x : x \to X$ is the inclusion and $I(x)$ is an injective hull of the residue field $\kappa(x)$ as an $\mathcal{O}_{X,x}$-module.

We say that a weak perversity function p satisfies (c) iff $p(y) \leq p(x)+1$ whenever y is an immediate specialization of x.

Theorem 9.1.

(1) *If p is a finite valued strong perversity function satisfying (c) then the t-structure defined above induces a t-structure on $D^*_{\mathrm{coh}}(X)$, $* = \phi, +, -,$ or b.*

(2) *If p is as in* (1) *and $q = c-p$ then the duality functor $D = R\mathcal{H}om(\_, R^\bullet)$ induces an anti-equivalence* $\mathrm{coh}\,\mathrm{Perv}_p^{\mathrm{op}} \xrightarrow{\sim} \mathrm{coh}\,\mathrm{Perv}_q$ *on perverse objects with coherent cohomology.*

Proof. The last statement follows from the definitions and the local duality theorem. To show the first statement one uses the construction of truncation. Note that p is bounded, and $D^{\leq \min(p)} \subset D^{\leq p}$ and $D^{\geq \max(p)} \subset D^{\geq p}$. We have distinguished triangles

$$\tau_{\leq \min(p)} K \to \tau_{\leq p} K \to \tau_{\leq p} \tau_{> \min(p)} \tau_{\leq \max(p)} K \to .$$

So it suffices to work with complexes with bounded coherent cohomology. To do the passage from p_d to p_{d+1} one has to verify that if $K \in D^b_{\mathrm{coh}}$, $K \in D^{\geq p_d + 1}$ and $\Phi = \{x \in X | p_{d+1}(x) = d+1\}$, then $\mathcal{H}^{d+1}_\Phi(K)$ is coherent. This follows from a version of the finiteness theorem of [SGA 2] VIII given below.

Proposition 9.2. *If Φ is a union of closed subsets of X, and $K \in D^b_{\mathrm{coh}}$ (for the Zariski topology) satisfies the condition*

$$\forall x \in \Phi \text{ and } \forall y \notin \Phi \text{ such that } x \in \overline{\{y\}},\ H_y^{i-c(x)+c(y)}(K) = 0. \tag{9.2.1}$$

Then $\mathcal{H}^i_\Phi(K)$ is coherent.

Proof. Recall that for $Z \subset X$ closed, $F \in D^-(X,R)$ and $G \in D^+(X,R)$ we have

$$R\underline{\Gamma}_Z R\mathcal{H}om(F,G) \xrightarrow{\sim} R\mathcal{H}om(F, R\underline{\Gamma}_Z(G)).$$

This is a generality on any topos and a closed subtopos. In our case $\underline{\Gamma}_\Phi = \varinjlim \underline{\Gamma}_Z$ where Z runs over closed subsets of X contained in Φ. If F is pseudo-coherent (for $R = \mathcal{O}_X$ this means that F is in D^-_{coh}) one can pass to the limit and obtain

$$R\underline{\Gamma}_\Phi R\mathcal{H}om(F,G) \xrightarrow{\sim} R\mathcal{H}om(F, R\underline{\Gamma}_\Phi(G)).$$

If D is the duality functor we have

$$\mathcal{H}^i_\Phi(K) = \mathcal{H}^i_\Phi(DDK) \xrightarrow{\sim} \mathcal{E}xt^i_\Phi(DK, R^\bullet) \xrightarrow{\sim} \mathcal{E}xt^i(DK, \underline{\Gamma}_\Phi(R^\bullet)).$$

There is a spectral sequence converging to the rightmost term above with

$$E_2^{p,q} = \mathcal{E}xt^p(\mathcal{H}^{-q}(DK), \underline{\Gamma}_\Phi(R^\bullet));$$

its terms are quasi-coherent and it suffices to show that for $p+q=i$, $E_2^{p,q}$ is coherent. Let $\mathcal{H}_0^{-q}$ be $\underline{\Gamma}_\Phi \mathcal{H}^{-q}(DK)$ and $\overline{\mathcal{H}}^{-q} = \mathcal{H}^{-q}/\mathcal{H}_0^{-q}$.

$$R\mathcal{H}om(\mathcal{H}_0^{-q}, \underline{\Gamma}_\Phi(R^\bullet)) = R\mathcal{H}om(\mathcal{H}_0^{-q}, R^\bullet)$$

has coherent cohomology so it suffices to show that $\mathcal{E}xt^p(\overline{\mathcal{H}}^{-q}, \underline{\Gamma}_\Phi(R^\bullet)) = 0$. Since $\underline{\Gamma}_\Phi(R^\bullet)$ is a complex of injectives it suffices to show that $\mathcal{H}om(\overline{\mathcal{H}}^{-q}, \underline{\Gamma}_\Phi(R^p)) = 0$.

Indeed, if it were non zero, we would have $x \in \Phi$ with $c(x) = p$ and $x \in \operatorname{supp} \overline{\mathcal{H}}^{-q}$. x is a specialization of a maximal point y of $\operatorname{supp} \overline{\mathcal{H}}^{-q}$ and $y \notin \Phi$. By local duality $\mathcal{H}^{-q}(DK)_y\hat{}$ and $H_y^{c(y)+q}(K)$ are dual to each other (the pairing takes values in $H_y^{c(y)}(R^\bullet) \xrightarrow{\sim} I(y)$). Hence $H_y^{c(y)+q}(K) \neq 0$ contradicting (9.2.1). □

Remark 9.2. If p is as in 9.1 we have an equivalence of categories $\mathrm{Ind}(\mathrm{coh}\,\mathrm{Perv}_p) \to$ quasicoh Perv_p.

Indeed, by [Ha] II 7.19 every object of $D^+_{\mathrm{qcoh}}(X)$ can be represented by a complex of quasicoherent sheaves, which is a filtered colimit of its bounded coherent subcomplexes, so we may argue as in remark 7.2.

Remark 9.3. If p is a weak perversity function on X, then $\tau_{\leq p}$ preserves $D^*_{\mathrm{coh}}(X)$ ($* = \phi, +, -,$ or b) iff p is strong and satisfies (c).

The condition is sufficient by Th. 9.1(1). (On every connected component of X p is ∞ or finite valued.) For the necessity, the fact that p is strong is shown as in remark 6.1(8). To show that p satisfies (c), note that if X is the spectrum of a one dimensional local domain, with closed point y and generic point x, and p is s.t. $p(x) = 0$ and $p(y) \geq 2$, then ${}^p\tau_{>0}\mathcal{O}_X$ is the sheaf of meromorphic functions.

Remark 9.4. When X is of infinite Krull dimension and equipped with a strongly pointwise dualizing complex ([Co] p. 120), a perversity function as in Th. 9.1 (1) defines a t-structure on D^b_{coh}. Indeed, it suffices to show that if $K \in D^b_{\mathrm{coh}}(X)$ and $x \in X$ then $\tau_{\leq p}K$ has bounded coherent cohomology on some open neighbourhood U of x. By 9.1 there is a distinguished triangle on $X(x)$: $M \to K|_{X(x)} \to L \to$ with $M \in D^{b,\leq p}_{\mathrm{coh}}$, $L \in D^{b,>p}_{\mathrm{coh}}$. This spreads out to a distinguished triangle $\tilde{M} \to K|_U \to \tilde{L} \to$ in $D^b_{\mathrm{coh}}(U)$ for some U. Shrink U so that all the irreducible components of the supports of the cohomology sheaves of $\tilde{M}$ and $D\tilde{L}$ meet $X(x)$. Then $\tilde{M} \in D^{\leq p}$, $\tilde{L} \in D^{>p}$, so $\tilde{M} \cong (\tau_{\leq p}K)|_U$.

10. One can use Artin-Schreier theory as in [SGA $4\frac{1}{2}$, p. 120] and the above strategy for studying t-structures on $D^b_c(X_{\acute{e}t}, \mathbb{Z}/p\mathbb{Z})$. Recall

Proposition 10.1.

(1) *Let R be an $\mathbb{F}_p$-algebra, M a finitely generated R-module, $\sigma : M \to M$ a p-linear map. Then for every $m \in M$, $\sum_i R\sigma^i(m)$ is a finitely generated R-module.*
(2) *If R is strictly henselian $1 - \sigma : M \to M$ is a surjection.*
(3) *If in (2) $N \subset \mathfrak{m}M$ ($\mathfrak{m}$ being the maximal ideal of R) is σ-stable then $1 - \sigma : N \to N$ is bijective.*

Proof. (1) Take an epimorphism $R^n \to M$ and lift σ to $\overline{\sigma} : R^n \to R^n$. For $x \in R^n$ the submodule generated by $x, \overline{\sigma}(x), \overline{\sigma}^2(x), \ldots$ is shown to be finitely generated by reduction to the case of a finitely generated ring, which is noetherian. (Another argument shows that this submodule is generated by $\{\overline{\sigma}^i(x) \mid i < n\}$.)

(2) Write $M = F/F_1$, with F free and finitely generated, lift σ to F, and use that $1 - \sigma$ on F comes from a surjective étale map on Spec(Sym($F\check{}$)) [SGA $4\frac{1}{2}$, p. 121].

(3) By (1) we may assume N is finitely generated. By (2) $1 - \sigma$ is surjective on N. For injectivity it is enough to consider $\mathfrak{m}M$. write $M = F/F_1$ as before, with F finite and free over R, and $F_1 \subset \mathfrak{m}F$, and lift σ to F. Then $\mathfrak{m}M = \mathfrak{m}F/F_1$. The map $1 - \sigma$ is bijective on $\mathfrak{m}F$ by étaleness of the map considered in (2) and surjective on F_1 by (2). Hence it is injective on $\mathfrak{m}M$. □

Proposition 10.2. *If X is a noetherian $\mathbb{F}_p$-scheme, $\mathcal{F}$ a coherent $\mathcal{O}_X$-module (for the étale topology), and $\sigma : \mathcal{F} \to \mathcal{F}$ a p-linear map, then $1 - \sigma : \mathcal{F} \to \mathcal{F}$ is an epimorphism and* $\ker(1-\sigma)$ *is a constructible sheaf of $\mathbb{Z}/p$-modules.*

Proof. X has a finite covering by locally closed subschemes Z_i such that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{Z_i}$ is locally free on Z_i. The map $1 - \sigma$ is surjective by Proposition 10.1(2) and by 10.1(3)

$$\ker(1-\sigma)|_{Z_i} \xrightarrow{\sim} \ker(1 - \sigma : \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{Z_i} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{Z_i}).$$

The latter sheaf is represented by an étale group scheme, which is affine over Z_i, and hence constructible. □

It follows that if $K^\bullet$ is a complex of $\mathcal{O}_X$-modules with bounded coherent cohomology and $\sigma : K^\bullet \to K^\bullet$ a p-linear map, then

$$\mathrm{Fib}(1-\sigma) = \mathrm{Cone}(1-\sigma)[-1] \in D^b_c(X, \mathbb{Z}/p\mathbb{Z}).$$

Let X be as in §9. Fix a perversity function q as in 9.1. Then $c-q$ is also such a perversity function. One checks that if $K^\bullet \in {}^qD^{\leq n}$ (resp. $K^\bullet \in {}^qD^{\geq n}$) then $\mathrm{Fib}(1-\sigma) \in {}^qD^{\leq n}$ (this uses 10.1(2)) (resp. $\mathrm{Fib}(1-\sigma) \in {}^qD^{\geq n}$).

We recall that we have a functorial construction of ${}^q\tau_{\leq n}$ and ${}^q\tau_{\geq n}$ on complexes of abelian sheaves. It sends p-linear maps to p-linear maps.

Consider ${}^q\tau_{\leq n}K \subset C^{(\omega)}K$. We have an exact sequence of complexes

$$0 \to \mathrm{Fib}(1-\sigma|{}^q\tau_{\leq n}K) \to \mathrm{Fib}(1-\sigma|C^{(\omega)}K) \to \mathrm{Fib}(1-\sigma|C^{(\omega)}K/{}^q\tau_{\leq n}K) \to 0$$

One deduces that $\mathrm{Fib}(1-\sigma|{}^q\tau_{\leq n}K)$ is isomorphic to ${}^q\tau_{\leq n}(\mathrm{Fib}(1-\sigma))$ in the derived category, and in particular the latter complex has bounded constructible cohomology and ${}^q\mathcal{H}^n(\mathrm{Fib}(1-\sigma)) \in D^b_c(X,\mathbb{Z}/p)$.

Theorem 10.3. *If X is a noetherian $\mathbb{F}_p$-scheme of finite dimension, having a dualizing complex, and q a perversity function as in Theorem 9.1, then the t-structure on $D(X,\mathbb{Z}/p)$ associated to q induces a t-structure on $D^*_c(X,\mathbb{Z}/p)$ ($*=\phi,+,-,$ or b).*

Proof. It suffices to treat $*=b$. Every $K \in D^b_c(X,\mathbb{Z}/p)$ lies in some ${}^qD^{\leq n}$ and in some ${}^qD^{\geq m}$. It suffices to show that the ${}^q\mathcal{H}^n(K)$ are in D^b_c. By the above it suffices to realize K as $\mathrm{Fib}(1-\sigma)$ for a p-linear endomorphism of a bounded coherent complex. This will be done in §11. □

Theorem 10.4. *Then conclusion of Theorem 10.3 holds when q is a weak perversity function satisfying (c).*

Proof (for $*=b$). Suppose $K \in D^b_c(X,\mathbb{Z}/p)$ and m is an integer s.t. $K \in D^{\leq m}$. Let U be the maximal open in X s.t. $q \geq m$ on U, and $F = X-U$. Then $q<m$ at the maximal points of F, so (c) implies that q is finite on F. We have $\tau_{\leq q}K = \tau^F_{\leq q}K$, so we may assume that q is finite, in which case the assertion follows from the following lemma.

Lemma 10.4.1. *Let $K \in D^b_c(X_{\acute{e}t},\mathbb{Z}/p)$, $i: F \to X$ an irreducible closed subscheme, $U = X-F$. Suppose $\tau_{\leq q}K$ has constructible cohomology sheaves on U. Then there is an open neighbourhood V of the generic point ξ of F such that $(\tau_{\leq q}K)|_V$ has constructible cohomology sheaves.*

Proof. We may assume $q(\xi)=c(\xi)=0$. By condition (c), $q \geq c$ on $X(\xi)$. Hence $(\tau_{>q}K)|_{X(\xi)}$ is in $D^{>c}$. Hence $\varphi: \tau_{>q}(K|_U) \to \tau_{>c}\tau_{>q}(K|_U)$ is an isomorphism on $U \cap X(\xi)$. But φ is a map of objects in D^b_c (by assumption and 10.3 for c). So there is V s.t. φ is an isomorphism on $V \cap U$. We have ([BBD] (1.4.13.1))

(1) $\tau_{>q}K = \tau^F_{>q}\tau^U_{>q}K$
(2) $\tau_{>c}L = \tau^F_{>c}\tau^U_{>c}L$ where $L = \tau^U_{>q}K$.

By assumption L is in D^b_c and by the above $L = \tau^U_{>c}L$ on some L. Since $\tau_{>c}L$ is in D^b_c (10.3 for c), (2) and the description of $\tau^F_{>c}$ in terms of a distinguished triangle

$$i_*\tau_{\leq c}Ri^!L \to L \to \tau^F_{>c}L \to$$

give that $M = \tau_{\leq c}Ri^!L|_{V\cap F} \in D^b_c(V\cap F)$ for some V. Shrink V s.t. the cohomology sheaves of M are locally constant. Then $M \in D^{\leq 0}(V\cap F) \subset D^{\leq q}(V\cap F)$ (if $V\cap F$ is made small enough so that $q \geq q(\xi)$ on $V\cap F$). By condition (c), $q \leq c$ on F.

So $\tau_{\le q} Ri^! L = \tau_{\le q}\tau_{\le c} Ri^! L$; on some $V \cap F$ this is $\tau_{\le q} M = M$; now (1) and the description of $\tau^F_{>q}$ imply that for some V, $\tau_{>q} K|_V \in D^b_c(V)$ which is equivalent to the required conclusion. □

In the situation of Th. 10.4, if $K \in D^b_c(X, \mathbb{Z}/p)$ is q-perverse then q is finite on $\mathrm{supp}(K)$.

11. Let $Y \xrightarrow{f} X$ be a finite morphism. One has an operation

$$f^\flat : \mathcal{O}_X - \text{modules} \to \mathcal{O}_Y - \text{modules}$$

defined by $f^\flat M = \mathcal{O}_Y \otimes_{f^{-1} f_* \mathcal{O}_Y} f^{-1}\mathcal{H}om(f_* \mathcal{O}_Y, M)$. This is compatible with ϵ^* and extends to quasi-finite maps and $f^\flat M \xrightarrow{\sim} f^* M$ for f étale.

Let $f : U \to X$ be a separated étale morphism of finite type. Using Zariski's Main Theorem, embed U as an open in a finite X-scheme Y. Let I be a coherent sheaf on Y defining $Y - U$. Write $j : U \hookrightarrow Y$ and $\overline{f} : Y \to X$ for the given maps. By [SGA $4\frac{1}{2}$, p. 120] if $\mathcal{L}$ is a locally constant constructible $\mathbb{F}_p$-sheaf on U, then there is a coherent $\mathcal{O}_Y$-module $\mathcal{G}$ and a p-linear map $\sigma : \mathcal{G} \to \mathcal{G}$ such that $\mathcal{O}_U \otimes_{\mathbb{F}_p} \mathcal{L} \xrightarrow{\sim} \mathcal{G}_{|U}$ and $j_! \mathcal{L} \xrightarrow{\sim} \ker(1-\sigma)$. Then $\ker(1 - \sigma : \overline{f}_* \mathcal{G} \to \overline{f}_* \mathcal{G}) \xrightarrow{\sim} f_! \mathcal{L}$. The sheaf $f_! \mathcal{L}$ is also realized using σ on $\overline{f}_* I^n \mathcal{G}$ $\forall n \in \mathbb{N}$.

Lemma. *If $\mathcal{F}$ is a quasi-coherent sheaf with a p-linear map φ then every map of étale sheaves $a : f_! \mathcal{L} \to \ker(1-\varphi)$ comes from some map $b : \overline{f}_* I^n \mathcal{G} \to \mathcal{F}$ compatible with the p-linear maps.*

Proof. Note that b should correspond to a map $c : I^n \mathcal{G} \to \overline{f}^\flat \mathcal{F}$ whose restriction to U is defined by the map $\mathcal{L} \to f^* \ker(1-\varphi)$ corresponding to a by adjunction. Once $c_{|U}$ is known, c exists for some n by [Ha] p. 410, but we do not know that the resulting b is compatible with the p-linear maps. For $m \ge n$, b restricts to $b_m : \overline{f}_* I^m \mathcal{G} \to \mathcal{F}$ and the loci $\mathrm{Bad}(b_m)$ where the stalk of b_m is not compatible with the p-linear maps form a descending sequence of closed subsets. By the noetherian property it suffices to show that if $\mathrm{Bad}(b_n)$ is non-empty then there is $m > n$ such that $\mathrm{Bad}(b_m) \ne \mathrm{Bad}(b_n)$, and this is reduced to the case where X is strictly local and $\mathrm{Bad}(b_n)$ is the closed point $\{x\}$. Moreover we can decompose Y and assume Y is strictly local. If $Y = U$ then f is an isomorphism and b_n is good. If $Y \ne U$ let $d, e > 0$ be such that I^d is contained in the extension of the maximal ideal $\mathfrak{m}$ of $\mathcal{O}_{X,x}$ and $\mathfrak{m}^{pe}$ kills the discrepancy for b_n. Then b_{n+de} is good. □

If $K^\bullet$ is a complex of étale $\mathbb{F}_p$-Modules, with bounded above constructible cohomology, one finds (similarly to SGA6 II) a quasi-isomorphism $L^\bullet \to K^\bullet$, with $L^\bullet$ bounded above, and each L^i of the form $j_{i!}\mathbb{Z}/p\mathbb{Z}$, and $j_i : U_i \to X$ étale, seperated, and of finite type. Now, using successively the Lemma, realize $L^\bullet$ as $\ker(1-\sigma)$, σ a p-linear endomorphism of a bounded above coherent complex. When $QK^\bullet \in D^b_c$ one can apply a truncation to realize it using a bounded coherent complex.

12. It follows from the above that in the situation of 10.3 an object with constructible cohomology in $\mathrm{Perv}_q(X, \mathbb{Z}/p\mathbb{Z})$ can be realized using a p-linear map on a perverse complex with bounded coherent cohomology. It is not clear whether this can be done with a complex of coherent sheaves.

We recall that a cofinite module over a local noetherian ring A is an $\mathfrak{m}$-power torsion module N such that $\dim_k \mathrm{Hom}(k, N) < \infty$, where $k = A/\mathfrak{m}$ denotes the residue field of A. A quotient of a cofinite module is cofinite.

Lemma 12.1. *If A is a local noetherian $\mathbb{F}_p$-algebra, N a cofinite A-module, and $\varphi : N \to N$ a p-linear map, then $\ker(1-\varphi)$ is finite.*

Proof. Let $N' = \cup_n \ker(\varphi^n)$. Then $1-\varphi$ is bijective on N' so we can pass to N/N'. That is we assume that φ is injective. If $n = \varphi(n)$ then for $f \in \mathfrak{m}$

$$\varphi^k(fn) = f^{p^k}\varphi^k(n) = f^{p^k} n = 0 \text{ for large } k\,,$$

so $fn = 0$. Hence n lies in the largest φ-stable subspace, V, of $\mathrm{Ann}_N(\mathfrak{m})$. We have $\dim_k V < \infty$, and in this case the result is known. $\square$

Lemma 12.2. *If X is the spectrum of a local ring and X, q satisfy the conditions of Proposition 10.3 or 10.4, and $F \in \mathrm{Perv}_q(X, \mathbb{Z}/p)$, then the set of perverse subobjects of F with support in the closed point satisfies the ascending chain condition and the set of perverse subobjects F' of F such that F/F' is supported in the closed point satisfies the descending chain condition. (Here F and its subobjects are assumed to be in D^b_c.)*

Proof. If x is the closed point we may assume $q(x) = 0$. Then the perverse sheaves supported on $\{x\}$ are just constructible sheaves on x. The assertion for quotient objects follows from finiteness of $\mathcal{H}^0(i^*_{\overline{x}}F)$. For the assertion on subobjects one has to show constructibility of $\mathcal{H}^0(Ri^!_{\overline{x}}F)$. That is $H^0_x(F)$ is finite when X is strictly local. Under the conditions of 10.3 we can realize F as $\mathrm{Fib}(\sigma - 1 : K \to K)$, K a perverse complex with bounded coherent cohomology. As $H^{-1}_x(K) = 0$,

$$H^0_x(F) = \ker(1-\varphi : H^0_x(K) \to H^0_x(K)), \quad \varphi = H^0_x(\sigma).$$

$H^0_x(K)$ is cofinite [Ha2, Cor 1.4], so the assertion follows from Lemma 12.1. To treat the case of 10.4 we may assume $c(x) = 0$ and then $q \geq c$. Thus $F \in {}^cD^{\geq 0} \cap D^b_c$. There is a spectral sequence (coming from the filtration of F by the ${}^c\tau_{\leq n}F$)

$$E_2^{rs} = H^r_x({}^c\mathcal{H}^s(F)) \Rightarrow H^{r+s}_x(F)\,,$$

which is concentrated in the first quadrant and implies the result using the case of the perversity function c. $\square$

Lemma 12.3. *Let X, q be as in 10.4.*

1. *If q satisfies (+) (see 8.3), and K is q-perverse in $D^b_c(X, \mathbb{Z}/p)$, then there is a finite subset $S \subset X$ such that for every quotient object $K \twoheadrightarrow K'$ the generic points of the irreducible components of the support of K' lie in S.*
2. *If the absolute Frobenius $F : X \to X$ is finite and K is q-perverse in D^b_c, then there exists $S \subset X$ finite such that for every subobject $K' \subset K$ the generic points of the irreducible components of the support of K' lie in S.*

Proof. (1) Take $S = \{x \in X \mid q(x) < \infty \text{ and } \mathcal{H}^{q(x)}(K)_{\overline{x}} \neq 0\}$. We have to show that S is finite, equivalently that for every $n \in \mathbb{Z}$ the set $S_n = \{x \in X \mid q(x) = n$ and $\mathcal{H}^n(K)_{\overline{x}} \neq 0\}$ is finite. $T = \{x \in X \mid \mathcal{H}^n(K)_{\overline{x}} \neq 0\}$ is constructible, hence compact for the constructible topology. So it suffices to show that for every $x \in T$ there is a constructible set Z with $x \in Z$ and $Z \cap S_n$ finite. As $q(x) \geq n$ it suffices to take as Z the open in $\{\overline{x}\}$ provided by condition (+).

(2) will be proved in §13.

Corollary 12.4.

(1) *If q satisfies (+) then every q-perverse object in D^b_c is artinian.*
(2) *If $F : X \to X$ is finite then every q-perverse object K in D^b_c is noetherian and moreover every perverse subobject of K in the category $D(X, \mathbb{Z}/p)$ is in D^b_c.*

Proof. (1) Let $K \in \mathrm{Perv}_q \cap D^b_c$ and $K_i \subset K$, $i \geq 1$, a descending sequence of subobjects (in $\mathrm{Perv}_q \cap D^b_c$). $\Phi_i = \overline{\cup_{j \geq i} \mathrm{supp}(K_j/K_{j+1})}$ is a descending sequence of closed subsets of X, so we may assume it is constant (omit finitely many K_i). Suppose Φ_i is non-empty. Restricting to an open we can assume that there is a closed irreducible $\Phi \subset X$ such that $\Phi_i = \Phi\ \forall i$. Let ξ be the generic point of Φ.

The pull-backs of K_1/K_n to $\mathrm{Spec}(\mathcal{O}_{X,\xi})$ have support in $\{\xi\}$, so by 12.2 we may assume that $n \mapsto K_n$ is constant on $\mathrm{Spec}(\mathcal{O}_{X,\xi})$, hence each $\mathrm{supp}(K_j/K_{j+1})$ is a proper closed subset of Φ. Applying (12.3)(1) to K_1 we get a finite subset S of X such that

$$\mathrm{supp}(K_1/K_n) \subset \bigcup_{s \in S \cap \Phi, s \neq \xi} \overline{\{s\}}.$$

This contradicts the definition of Φ_1.

(2) The noetherian property is shown in the same way. By remark 7.2 and Prop. 10.4, $\mathrm{Perv}_q \cap D^b_c$ generates Perv_q. Hence for $K \in \mathrm{Perv}_q \cap D^b_c$ every Perv_q subobject of K is the supremum of a set of $\mathrm{Perv}_q \cap D^b_c$ subobjects and by 12.4(2) a finite sum serves as a supremum. □

13. Let X be an $\mathbb{F}_p$-scheme. A p^{-1}-linear map $M \to N$ of $\mathcal{O}_X$-modules is an $\mathcal{O}_X$-linear map $F_*M \to N$.

Lemma 13.1. *If X is a noetherian $\mathbb{F}_p$-scheme and M is a coherent $\mathcal{O}_X$-module and $\lambda : M \to M$ a p^{-1}-linear map, then there exists n such that $\forall m \geq n$, $\lambda^n(M) = \lambda^m(M)$.*

Proof. The $\lambda^n(M)$ are $\mathcal{O}_X$-submodules of M and $\mathrm{supp}(\lambda^n(M)/\lambda^{n+1}(M))$ is a descending sequence of closed subsets. Hence, replacing M by some $\lambda^n(M)$ we may assume it is constant. Localizing, we may assume $X = \mathrm{Spec}(A)$ is local and $\forall n \geq 0$, $\mathrm{supp}(\lambda^n(M)/\lambda^{n+1}(M)) = \{\mathfrak{m}\}$, the closed point. Then one derives a contradiction: Say $\mathfrak{m}^n M \subset \lambda(M)$. Then for $f \in \mathfrak{m}^n$

$$f^2 M \subset f\mathfrak{m}^n M \subset f\lambda(M) = \lambda(f^p M) \subset \lambda(f^2 M),$$

so $f^2 M \subset \lambda^k(f^2 M) \subset \lambda^k(M)\ \forall k$. But $M/\sum_{f \in \mathfrak{m}^n} f^2 M$ is of finite length. □

Lemma 13.2. *If A is a local $\mathbb{F}_p$-algebra, M an A-module of finite length, $\lambda : M \to M$, p^{-1}-linear and surjective, then M is annihilated by the maximal ideal.*

Proof. $(\forall f \in \mathfrak{m})$ $fM = \lambda(f^p M)$, so $f^p M = 0 \implies fM = 0$. □

Lemma 13.3. *Let X be a noetherian $\mathbb{F}_p$-scheme such that $F : X \to X$ is finite, M a coherent $\mathcal{O}_X$-module, $\lambda : M \to M$ p^{-1}-linear. Then there is a finite subset $S \subset X$ such that every $x \in X \setminus S$ has the following property:*

(13.3.1) *If $M' \subset M_x$ is a λ-stable $\mathcal{O}_{X,x}$-submodule with* $\mathrm{length}(M_x/M') < \infty$ *then λ is nilpotent on M_x/M'.*

Proof. We can replace M by some $\lambda^n(M)$ by 13.1 and assume $\lambda : M \to M$ is surjective. Hence by 13.2, in the situation of (13.3.1) $M' \supset \mathfrak{m}M_x$ ($\mathfrak{m}$ the maximal ideal of $\mathcal{O}_{X,x}$).

It suffices to show that for every irreducible closed set $Z \subset X$, there is a non-empty open $U \subset Z$, such that (13.3.1) holds at all non-generic points of U. Take Z with the reduced scheme structure. Replace M by $M/\sum_n \lambda^n(IM)$, where I is the ideal of Z in X. Thus, we may assume $X = Z$.

$\lambda^n : M \to M$ defines by adjointness $\mu_n : M \to (F^n)^\flat M$, $\ker(\mu_n)$ is an increasing sequence of submodules, hence stationary, and λ preserves and is nilpotent on $\varinjlim_n \ker(\mu_n)$. Let $\bar{M} = M/\varinjlim_n \ker(\mu_n)$. Then $\bar{M} \to F^\flat \bar{M}$ is a monomorphism.

In the situation of (13.3.1) we get similarly a quotient $\overline{M_x/M'}$ of $\bar{M}_x$, and we have to show it is 0.

Since $\bar{M}$ and $F^\flat \bar{M}$ have the same rank at the generic point we may assume, restricting to an open, that $\bar{M} \xrightarrow{\sim} F^\flat \bar{M}$. Also we may assume F is finite flat, which implies that X is regular by a theorem of Kunz ([Ma] Th. 107). Then $F^\flat$ is exact. We prove that (13.3.1) holds for x non-generic. Indeed, if it fails we get a non-zero quotient $N = \overline{M_x/M'}$ of $\bar{M}_x$ such that $N \hookrightarrow F^\flat N$, and $N \to F^\flat N$ is surjective since it is so for M. Hence $N \xrightarrow{\sim} F^\flat N$, but for $\mathcal{O}_{X,x}$-modules of finite length $F^\flat$ multiplies the length by $p^{\dim(\mathcal{O}_{X,x})}$. $\square$

Theorem 13.4. *Let X be a noetherian $\mathbb{F}_p$-scheme with finite absolute Frobenius and admitting a dualizing complex. Let $K \in D^b_{\mathrm{coh}}(X, \mathcal{O}_X)$, $\sigma : K \to F_*K$ a map in the derived category. Then*

$$\{(x,n) \in X \times \mathbb{Z} | \ker(\sigma - 1 | H^n_{\bar{x}}(K)) \neq 0\}$$

is finite. Here $\bar{x}$ is a geometric point above x, and $H^n_{\bar{x}}(K)$ is local cohomology computed at the strict henselization.

Corollary. *For X as above and q as in 9.1, if $K \in \mathrm{Perv}_q \cap D^b_c(X,\ \mathbb{Z}/p)$ then*

$$\{x \in X | H^{q(x)}_{\bar{x}}(K) \neq 0\}$$

is finite.

Proof of 13.4. Note that X is finite dimensional (cf. [Ma] §42 Lemma 7). Choose a residual complex $R^\bullet$. Then $F^\flat R^\bullet$ is also a residual complex with the same codimension function. By a unicity statement $F^\flat R^\bullet \xrightarrow{\sim} R^\bullet$ Zariski locally. So we may assume we have such an isomorphism. This gives p^{-1}-linear maps on the components of $R^\bullet$. In particular, for every $x \in X$ we have a p^{-1}-linear map on the injective hull of the residue field $I(x) = H^{c(x)}_x(R^\bullet)$ that induces by adjunction $I(x) \xrightarrow{\sim} F^\flat I(x)$.

If $D = R\mathcal{H}om(\_, R^\bullet)$ on D^b_{coh} then $F_*D \xrightarrow{\sim} DF_*$ using the choice made, so σ gives $\lambda : F_*DK \to DK$, and hence p^{-1}-linear maps $\mathcal{H}^i(\lambda)$ on $\mathcal{H}^i(DK)$.

If $\mathcal{O}_{\bar{x}}$ is the strict henselization, $I(\bar{x}) = I(x) \otimes_{\mathcal{O}_x} \mathcal{O}_{\bar{x}}$, the local duality theorem gives

$$H^n_{\bar{x}}(K) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_{\bar{x}}}(\mathcal{H}^{c(x)-n}(DK)_{\bar{x}}, I(\bar{x})).$$

One verifies that the operator σ on $H^n_{\bar{x}}(K)$ corresponds via local duality to

$$(f : H = \mathcal{H}^{c(x)-n}(DK)_{\overline{x}} \to I(\bar{x})) \rightsquigarrow (H \to F^\flat H \overset{F^\flat f}{\to} F^\flat I(\bar{x}) \xrightarrow{\sim} I(\bar{x})).$$

So

$$\ker(1-\sigma) = \{f\text{'s that are compatible with the given } p^{-1}\text{-linear self maps on both sides}\}.$$

For a non-zero element of $H^n_{\bar{x}}(K)$ we get a non-zero quotient $\mathrm{im}(f)$ with an induced non-nilpotent p^{-1}-linear map. (Indeed for an $\mathcal{O}_x$-submodule $N \subset I(\bar{x})$ stable under the p^{-1}-linear map, $N \hookrightarrow F^\flat N$ so $N \hookrightarrow (F^k)^\flat N \; \forall k$, so $N \neq 0 \implies \lambda$ on N is not nilpotent.) So the assertion follows from 13.3. □

Theorem 13.5. *For X as in 13.4 and q a finite weak perversity function satisfying (c) and $K \in \mathrm{Perv}_q \cap D^b_c(X, \mathbb{Z}/p)$, the set $S = \{x \in X \mid H^{q(x)}_{\bar{x}}(K) \neq 0\}$ is finite. This gives 12.3(2).*

Proof. Since X is compact for the constructible topology it suffices to show that if $x \in X$ there is an open neighbourhood U of x in $\overline{\{x\}}$ s.t. $U \cap S \subset \{x\}$. We may assume $q(x) = c(x) = 0$. We have $q \geq c$ on $X(x)$, so $K \to \tau_{\geq c} K$ is an isomorphism on $X(x)$, hence by constructibility it is an isomorphism on some open neighbourhood U of x. For $y \in \overline{\{x\}} \cap U$ we have a spectral sequence

$$E_2^{rs} = H^r_{\bar{y}}({}^c\mathcal{H}^s K) \Rightarrow H^{r+s}_{\bar{y}}(K)\,,$$

with $E_2^{rs} \neq 0$ only if $s \geq 0$ and $r \geq c(y) \geq q(y)$. The assertion follows from this and the corollary to Theorem 13.4. □

Remark 13.6. In Theorem 13.4 the assumption that X admits a dualizing complex follows from the other assumptions. We explain this for X affine. It suffices to know that $A := \mathcal{O}(X)$ is a quotient of a regular ring. By assumption there are finitely many elements a_i $(1 \leq i \leq n)$ of A such that $A = F(A)[a_1, \ldots, a_n]$. Define

$$A_i := A[z_1, \ldots, z_n]/(z_j^{p^i} - a_j, 1 \leq j \leq n).$$

Let $A_{i+1} \to A_i$ be the homomorphism that is F on A and sends z_j to z_j; it is surjective with nilpotent kernel killed by F. Set $A_\infty = \varprojlim(A_i)$. One verifies that $\mathrm{Ker}(A_{n+m} \to A_n)$ is generated by $F^n \, \mathrm{Ker}(A_{n+m} \to A)$ and deduces that the closures of the powers of $\mathrm{Ker}(A_\infty \to A)$ form a fundamental system of neighborhoods of 0, so ([AC] Chap. III, § 2, n° 11) gives that A_∞ is adic and noetherian. Also A_∞ admits a (strong) p-basis $[z_1], \ldots, [z_n]$ so it is regular by [Ma] Th. 107.

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