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Géométrie des domaines propres dans les variétés de drapeaux

Geometry of proper domains in flag manifolds

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Mots clés : variétés de drapeaux, sous-groupes discrets des groupes de Lie, rigidité, espaces de Nagano, convexes divisibles, représentations anosoviennes.

Résumé : La géométrie projective convexe est l'étude des ouverts proprement convexes de l'espace projectif réel et de leurs quotients par des groupes discrets d'automorphismes projectifs. Elle contient la géométrie hyperbolique, en considérant le modèle de Klein de l'espace hyperbolique réel. Le cas où le quotient est compact s'inscrit dans la théorie des *convexes divisibles*, qui est développée depuis les années 1960 (par exemple par Benoist) et a produit de nombreux exemples, y compris non symétriques. En demandant que le groupe discret n'agisse plus cocompactement mais *convexe cocompactement*, on obtient des variétés projectives à cœur convexe compact. Danciger, Guéritaud et Kassel ont montré qu'une version forte de cette condition est équivalente au caractère P_1-anosovien (au sens de Labourie) du groupe discret considéré, offrant ainsi une caractérisation géométrique des représentations P_1-anosoviennes.

L'espace projectif réel est un exemple de *variété de drapeaux*, c'est-à-dire de quotient d'un groupe de Lie réel semi-simple G par un sous-groupe parabolique P de G. Dans cette thèse, poursuivant des travaux d'A. Zimmer, nous développons l'étude des domaines propres dans les variétés de drapeaux, en généralisant des outils de la géométrie projective convexe. Nous accordons une attention particulière aux *espaces de Nagano*, ou *espaces symétriques extrinsèques*, introduits par Nagano dans les années 1960. Par définition, un tel espace est une variété de drapeaux G/P qui s'identifie à un espace symétrique d'un sous-groupe compact maximal de G. Pour une large famille d'espaces de Nagano, dits *de type réel*, nous construisons une *distance de Kobayashi* géodésique, invariante et propre, sur tout domaine proprement convexe. Nous la comparons aux distances dites *de Carathéodory* introduites par Zimmer.

Selon une conjecture de rigidité de Limbeek et Zimmer, les domaines propres divisibles de la plupart des variétés de drapeaux différentes de l'espace projectif réel devraient être symétriques. La distance de Kobayashi nous permet d'étudier cette conjecture pour les espaces de Nagano de type réel. Par exemple, généralisant un résultat de Zimmer pour les grassmanniennes, nous montrons que lorsqu'un espace de Nagano de type réel est de rang supérieur, la distance de Kobayashi sur ses domaines propres divisibles (ou même *presque-homogènes*) ne peut pas être Gromov-hyperbolique. De plus, nous démontrons la conjecture pour les variétés de drapeaux admettant une structure causale et les univers d'Einstein (ce dernier cas en collaboration avec A. Chalumeau), où les domaines propres divisibles sont les *diamants*. Enfin, nous démontrons que le centralisateur d'un groupe discret projectif divisant un domaine propre à bord continu d'une grassmannienne différente de l'espace projectif est trivial. Ce dernier résultat met en évidence une perte de flexibilité par rapport au cas projectif réel, où le *joint* de deux convexes divisibles fournit un nouveau convexe, divisé par un groupe produit.

Si les groupes P_1-anosoviens préservant des domaines propres dans l'espace projectif réel sont bien compris grâce à la notion de convexe cocompacité projective, une caractérisation géométrique reste à établir pour les groupes P-anosoviens dans les variétés de drapeaux générales G/P. Nous déterminons des restrictions topologiques sur les groupes préservant un domaine propre dans une variété de drapeaux auto-opposée G/P, et construisons des exemples Zariski-denses P-anosoviens préservant des domaines propres. Dans certaines variétés de drapeaux à structure causale, nous introduisons une notion de *convexité causale*, inspirée de celle dans les espaces-temps conformes. Nous montrons que tout groupe P-transverse préservant un domaine propre de G/P agit cocompactement sur un fermé causalement convexe de ce domaine, à bord transverse ; autrement dit, ces groupes ont une dynamique essentiellement *spatiale*.

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Title : Geometry of proper domains in flag manifolds

Keywords : flag manifolds, discrete subgroups of Lie groups, rigidity, Nagano spaces, divisible convex sets, Anosov representations.

Abstract : Convex projective geometry is the study of properly convex open subsets of the real projective space, and of their quotients by discrete groups of projective automorphisms. It contains hyperbolic geometry, by considering the Klein model of real hyperbolic space. The case where the quotient is compact falls within the theory of *divisible convex sets*, which has been developed since the 1960s (for instance, by Benoist) and has produced numerous examples, including non-symmetric ones. By requiring the discrete group to no longer act cocompactly but *convex cocompactly*, one obtains projective manifolds with a compact convex core. Danciger, Guéritaud, and Kassel have shown that a strong version of this condition is equivalent to the P_1-Anosov property (in the sense of Labourie) for the given discrete group, thus providing a geometric characterization of P_1-Anosov representations.

Real projective space is an example of a *flag manifold*, i.e. of a quotient of a real semisimple Lie group G by a parabolic subgroup P of G. In this thesis, building on work of A. Zimmer, we develop the study of proper domains in flag manifolds, generalizing tools from convex projective geometry. We pay particular attention to *Nagano spaces*, also called *extrinsic symmetric spaces*, introduced by Nagano in the 1960s. By definition, such a space is a flag manifold G/P that identifies with a symmetric space of a maximal compact subgroup of G. For a large class of Nagano spaces, called *of real type*, we construct a geodesic, invariant, and proper *Kobayashi metric* on any properly convex domain. We compare this metric to the so-called *Caratheodory metrics* introduced by Zimmer.

According to a rigidity conjecture of Limbeek and Zimmer, proper divisible domains in most flag manifolds different from real projective space should be symmetric. Using the Kobayashi metric, we investigate this conjecture for Nagano spaces of real type. For instance, generalizing a result of Zimmer for Grassmannians, we show that when a Nagano space of real type has higher rank, the Kobayashi metric on its proper divisible (or even just *almost-homogeneous*) domains cannot be Gromov hyperbolic. Moreover, we prove the conjecture for flag manifolds admitting a causal structure, and for Einstein universes (the latter case in collaboration with A. Chalumeau), where the proper divisible domains are the *diamonds*. Finally, we prove that the centralizer of a discrete projective group dividing a proper domain with continuous boundary in a Grassmannian different from real projective space is trivial. This last result highlights a loss of flexibility compared to the real projective case, where the *join* of two divisible convex sets is again a divisible convex set, divided by a product group.

While P_1-Anosov groups preserving proper domains in real projective space are well understood through the notion of projective convex cocompactness, a geometric characterization remains to be established for P-Anosov groups in general flag manifolds G/P. We determine topological restrictions on groups preserving a proper domain in a self-opposite flag manifold G/P and construct Zariski-dense P-Anosov examples preserving proper domains. In certain flag manifolds with a causal structure, we introduce a notion of *causal convexity*, inspired by that in conformal spacetimes. We show that any P-transverse group preserving a proper domain of G/P acts cocompactly on a causally convex closed subset of this domain with transverse boundary ; in other words, the dynamics of these groups are essentially *spatial*.

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à Pascal Pronesti,

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Chapitre 1

Introduction

Le contexte général de cette thèse est celui des (G, X)-structures sur les variétés. Une variété M admet une (G, X)-*structure*, où G est un groupe de Lie et X un espace homogène de G, s'il existe un atlas de cartes sur M à valeurs dans X, dont les changements de cartes sont localement des éléments de G. Initiée par Klein dans son *programme d'Erlangen* en 1872, la théorie des (G, X)-structures a connu un développement considérable au vingtième siècle, grâce notamment aux travaux d'Ehresmann puis de Thurston (voir e.g. [Mos16, Mar91, BP89, Gro87, Per08, Tit74, Gol22]). Le groupe fondamental $\pi_1(M)$ d'une variété M admettant une (G, X)-structure admet alors une représentation dans G, appelée *holonomie* ; c'est pourquoi de nombreuses problématiques géométriques sont intrinsèquement liées à l'étude des représentations des groupes discrets dans des groupes de Lie.

Dans cette thèse, nous considérons le cas où G est un groupe de Lie semi-simple réel et X une *variété de drapeaux* de G. Nous nous intéressons en particulier aux (G, X)-variétés de la forme Ω/Γ, où Ω est un domaine « pas trop gros » (à savoir, *propre*) de la variété de drapeaux X, et Γ un sous-groupe discret « assez gros » de G, c'est-à-dire typiquement cocompact ou convexe cocompact. Les questions traitées sont à la fois d'ordre géométrique (existence de quotients compacts ou convexes cocompacts de Ω), algébrique et dynamique (étude des représentations de Γ). Le cas où X est l'espace projectif réel est bien connu et donne lieu à la *géométrie projective convexe*, voir le paragraphe 1.1.1 ci-dessous. Dans ce mémoire, nous développons de nouveaux outils et techniques, généralisant ceux de la géométrie projective convexe, pour étudier la géométrie des domaines propres des variétés de drapeaux, ainsi que la dynamique des groupes les préservant.

1.1 Géométrie et convexité des domaines propres dans les variétés de drapeaux

L'étude des variétés construites par quotients de domaines propres de variétés de drapeaux apparaît déjà dans des travaux d'A. Zimmer [Zim18a] ; nous en décrivons les principaux objets et outils dans le paragraphe 1.1.2. Ces derniers généralisent ceux, bien connus, de la géométrie projective convexe, que nous détaillons dans le paragraphe 1.1.1 ci-dessous.

1.1.1 Géométrie projective convexe

Un ouvert Ω de l'espace projectif réel $\mathbb{P}(\mathbb{R}^{n+1})$ est dit *propre* s'il est borné dans une carte affine, et *proprement convexe* s'il est de plus convexe dans cette carte affine. Le quotient d'un ouvert proprement convexe de $\mathbb{P}(\mathbb{R}^{n+1})$ par un sous-groupe discret de $\mathrm{PGL}(n+1,\mathbb{R})$ le préservant est une $(\mathrm{PGL}(n+1,\mathbb{R}),\mathbb{P}(\mathbb{R}^{n+1}))$-variété, appelée *variété projective convexe*. L'étude de telles variétés s'appelle la *géométrie projective convexe*, et généralise la *géométrie hyperbolique réelle*, puisque le modèle de Klein réalise l'espace hyperbolique réel $\mathbb{H}^n$ comme un ouvert proprement convexe de $\mathbb{P}(\mathbb{R}^{n+1})$.

1.1.1.1 Le groupe d'automorphismes. Le groupe $\mathsf{Aut}(\Omega)$ des éléments de $\mathrm{PGL}(n+1,\mathbb{R})$ préservant un ouvert propre Ω est appelé le *groupe d'automorphismes* de Ω. L'ouvert Ω est dit *symétrique* si tout point de Ω est un point fixe isolé d'un automorphisme involutif de Ω. Il est dit *divisible* s'il existe un sous-groupe discret de $\mathsf{Aut}(\Omega)$ qui agit de manière cocompacte sur Ω, *quasi-homogène* si $\mathsf{Aut}(\Omega)$ agit de manière cocompacte sur Ω, et *presque-homogène* si l'*ensemble limite orbital total*, c'est-à-dire l'ensemble des points d'accumulation de toutes les $\mathsf{Aut}(\Omega)$-orbites de points de Ω, est égal à son bord. La divibilité implique la quasi-homogénéité, qui elle-même implique la presque-homogénéité. En revanche, il existe des ouverts proprement convexes presque-homogènes et non quasi-homogènes, construits par exemple par pliage de variétés hyperboliques de volume fini [BM20], de même qu'il existe des ouverts proprement convexes quasi-homogènes (et même homogènes) non divisibles [Vin65]. L'abondance de tels exemples, ainsi que celle des ouverts convexes cocompacts et géométriquement fin (voir le paragraphe 1.3.2 plus bas), fait de la géométrie projective convexe une théorie riche, connectée à des domaines de recherche variés, comme les sous-groupes discrets des groupes de Lie, la dynamique, la théorie des représentations ou la théorie de Teichmüller supérieure par exemple.

1.1.1.2 La distance de Hilbert. Un outil primordial dans l'étude des ouverts proprement convexes de l'espace projectif est la distance de Hilbert. Cette dernière peut être définie de deux manières : étant donné un ouvert proprement convexe $\Omega \subset \mathbb{P}(\mathbb{R}^{n+1})$ et $x, y \in \Omega$, la distance de Hilbert $\mathsf{H}_\Omega(x,y)$ entre x et y est égale aux deux quantités suivantes :

1. $\inf \{\log(a : x : y : b) \mid a, b \in \Omega \cap \ell_{x,y},\ a, x, y, b \text{ alignés dans cet ordre}\}$, où $\ell_{x,y}$ est une droite projective passant par x et y, et $(\cdot : \cdot : \cdot : \cdot)$ est le *birapport* (voir le paragraphe 2.1.1.2).
2. $\sup \left\{\log \left|\frac{\widetilde{\xi}_1(\widetilde{x})\widetilde{\xi}_2(\widetilde{y})}{\widetilde{\xi}_1(\widetilde{y})\widetilde{\xi}_2(\widetilde{x})}\right| \mid \xi_1, \xi_2 \in \Omega^*\right\}$, où Ω^* est le *dual* de Ω, c'est-à-dire l'ensemble $\mathbb{P}\left(\{f \in ((\mathbb{R}^{n+1})^*) \mid f(\widetilde{z}) \neq 0 \quad \forall z \in \overline{\Omega}\}\right)$. Ici, on a utilisé les notations $\widetilde{z}, \widetilde{\xi}$ pour des relevés quelconques de points $z \in \overline{\Omega}$ et $\xi \in \Omega^*$.

Si Ω est un ouvert proprement convexe, alors H_Ω est une distance propre $\mathsf{Aut}(\Omega)$-invariante, et les segments sont des géodésiques.

Une autre notion primordiale de la géométrie projective convexe est celle de *facette*. La facette d'un point $x \in \partial\Omega$ est l'union des intervalles projectifs ouverts contenant x et contenus dans $\partial\Omega$ (où les singletons sont considérés comme des intervalles ouverts). Les notions de distance de Hilbert et de facette interagissent à travers des résultats asymptotiques : par exemple, deux suites d'éléments de Ω restant à distance de Hilbert bornée doivent converger vers deux points d'une même facette de $\partial\Omega$. Cette observation élémentaire a des conséquences dynamiques assez fortes, comme par exemple le lemme classique suivant :

Fait 1.1.1 (voir par ex. [Vey70, Lem. 4])**.** *Soit Ω un ouvert proprement convexe de $\mathbb{P}(\mathbb{R}^{n+1})$ et $a \in \partial\Omega$ un point extremal. Supposons qu'il existe une suite $(g_k) \in \mathsf{Aut}(\Omega)^{\mathbb{N}}$ et un point $x \in \Omega$ tels que $g_k \cdot x \to a$. Alors, la suite (g_k) est* contractante *en a, c'est-à-dire qu'il existe un hyperplan projectif $\mathbb{P}(H)$ tel que pour tout compact $\mathsf{K} \subset \mathbb{P}(\mathbb{R}^{n+1}) \smallsetminus \mathbb{P}(H)$, la limite de Hausdorff de $(g_k \cdot \mathsf{K})$ soit $\{a\}$.*

Au-delà du Fait 1.1.1, la distance de Hilbert H_Ω donne de nombreuses informations sur la dynamique du groupe d'automorphismes de Ω, ce qui en fait un outil fondamental en géométrie projective convexe.

1.1.2 Généralisation aux variétés de drapeaux

L'espace projectif est un exemple de *variété de drapeaux*, c'est-à-dire du quotient G/P d'un groupe de Lie semi-simple réel G (ici $G = \mathrm{PGL}(n+1,\mathbb{R})$) par un *sous-groupe parabolique* P de G (ici le stabilisateur d'une droite de $\mathbb{R}^{n+1}$). Un *sous-groupe parabolique* d'un groupe de Lie semi-simple réel G quelconque est le stabilisateur d'un point x dans le bord visuel de l'espace symétrique riemannien $\mathbb{X}_G$ de G. S'il existe une géodésique bi-infinie de $\mathbb{X}_G$ entre x et un autre point $y \in \partial\mathbb{X}_G$, alors le stabilisateur P^- de y dans G est un sous-groupe parabolique *opposé* à P, et la variété G/P^- est la variété de drapeaux *opposée* à G/P. Par exemple, l'espace projectif dual $\mathbb{P}((\mathbb{R}^{n+1})^*)$, qui s'identifie à l'espace des hyperplans projectifs de $\mathbb{R}^{n+1}$, est la variété de drapeaux opposée de $\mathbb{P}(\mathbb{R}^{n+1})$. Si P et P^- sont conjugués dans G, on dit que P et G/P sont auto-opposés, et on a l'égalité $G/P = G/P^-$ en tant que sous-ensembles de $\partial\mathbb{X}_G$.

1.1.2.1 Généralisation de la géométrie des domaines propres. Un *domaine* d'une variété de drapeaux est un ouvert connexe. Les définitions de groupe d'automorphismes, de symétrie, de divisibilité, de quasi-homogénéité et de presque-homogénéité pour les domaines d'une variété de drapeaux générale G/P sont les mêmes que dans l'espace projectif, en remplaçant $\mathrm{PGL}(n+1,\mathbb{R})$ par le groupe de Lie semi-simple G (voir la partie 3.1.1.2 et [Zim18a]).

Pour généraliser la propreté, l'idée est de remarquer que les objets de G/P qui vont jouer un rôle d'hyperplan projectif sont les sous-ensembles de G/P de la forme

$$\mathrm{Z}_z = \{x \in G/P \mid x \text{ n'est pas transverse à } z\}, \text{ où } z \in G/P^-. \tag{1.1.1}$$

Une *carte affine* est alors le complémentaire $\mathbb{A}$ d'un tel ensemble dans G/P ; elle admet une structure affine canonique. En revanche, l'adhérence d'une droite affine de $\mathbb{A}$ n'a en général pas une structure naturelle de droite projective. On ne peut donc pas dire qu'un domaine propre est convexe s'il l'est dans une carte affine. Pour généraliser la notion de convexité, l'approche d'A. Zimmer [Zim18a] est d'utiliser la caractérisation duale de la convexité dans l'espace projectif : un ouvert $\Omega \subset G/P$ est *dualement convexe* si pour tout $a \in \partial\Omega$, il existe $\xi \in G/P^-$ tel que $\Omega \cap \mathrm{Z}_\xi = \emptyset$ et $a \in \mathrm{Z}_\xi$. La propriété suivante, déjà vraie dans l'espace projectif (d'après [Sho84]), reste alors valable dans toutes les variétés de drapeaux : tout domaine propre quasi-homogène est (dualement) convexe [Zim18a].

Enfin, Zimmer définit les *distances de Carathéodory* par analogie avec la distance de Hilbert, en généralisant le point (2) de la définition donnée en paragraphe 1.1.1.2 (voir la partie 3.1.2), à nouveau en utilisant l'analogie entre les hyperplans projectifs et les variétés de Schubert propres maximales. Étant donné un domaine propre $\Omega \subset G/P$, il existe plusieurs distances de Carathéodory sur Ω, toutes induites par des représentations de G. Plus précisément, soit $\rho : G \to \mathrm{PGL}(V)$ une représentation réelle, irréductible, proximale par rapport au sous-groupe parabolique P (au sens de la partie 2.3.3.2), de dimension finie. Cette représentation induit deux plongements $\iota_\rho : G/P \to \mathbb{P}(V)$ et $\iota_\rho^- : G/P^- \to \mathbb{P}(V^*)$. Pour $x, y \in \Omega$, la *distance de Carathéodory induite par* (V, ρ) entre x et y est, par définition :

$$C_\Omega^\rho(x,y) := \sup_{\eta,\xi \in \Omega^*} \log \Big| \frac{f_\xi(\nu_x) f_\eta(\nu_y)}{f_\xi(\nu_y) f_\eta(\nu_x)} \Big|, \tag{1.1.2}$$

où ν_x (resp. ν_y) est un relevé de $\iota(x)$ (resp. $\iota(y)$) dans $V \smallsetminus \{0\}$ et f_ξ (resp. f_η) un relevé de $\iota^-(\xi)$ (resp. de $\iota^-(\eta)$). On a noté Ω^* le *dual* de Ω, c'est-à-dire l'ensemble des points de G/P^- qui sont transverses à tous les points de Ω.

La fonction C_Ω^ρ est une distance $\mathsf{Aut}(\Omega)$-invariante engendrant la topologie standard. Si Ω est de plus dualement convexe, alors C_Ω^ρ est propre et complète. Cependant, elle ne donne pas beaucoup d'informations sur la dynamique de $\mathsf{Aut}(\Omega)$, car l'existence de points « extremaux », pour une notion duale de facette qui interagirait asymptotiquement avec les distances de Carathéodory (comme c'est le cas pour la distance de Hilbert dans le lemme 1.1.1), n'est pas garantie. De plus, ces distances ne sont *a priori* pas géodésiques. On va voir dans les paragraphes suivants que pour certaines variétés de drapeaux (les *espaces de Nagano de type réel*), on peut définir une *distance de Kobayashi* $\mathsf{Aut}(\Omega)$-invariante, qui est géodésique dès que Ω est dualement convexe et qui vérifie des propriétés généralisant celles de la distance de Hilbert dans l'étude du bord et du groupe d'automorphismes de Ω.

Les exemples clés de variétés de drapeaux auxquels nous nous intéressons dans cette thèse sont listés dans les parties 1.1.2.2 à 1.1.2.5 suivantes.

1.1.2.2 Les grassmanniennes. Soient $p, q \geq 1$. Le groupe $G = \mathrm{PGL}(p+q, \mathbb{R})$ agit transitivement sur l'espace $\mathrm{Gr}_p(\mathbb{R}^{p+q})$ des p-plans de $\mathbb{R}^{p+q}$ et le stabilisateur d'un point est un sous-groupe parabolique P_p de G. Ainsi, on a l'identification naturelle $\mathrm{Gr}_p(\mathbb{R}^{p+q}) \simeq G/P_p$, qui munit $\mathrm{Gr}_p(\mathbb{R}^{p+q})$ d'une structure de variété de drapeaux. Dans le cas où $p = 1$, on retrouve l'espace projectif réel.

1.1.2.3 L'univers d'Einstein. Soient $p, q \geq 1$. L'*espace pseudo-euclidien* $\mathbb{R}^{p+1,q+1}$, est l'espace vectoriel $\mathbb{R}^{p+q+2}$ muni d'une forme bilinéaire symétrique $\mathbf{b}$ de signature $(p+1, q+1)$, où $p+1$ et $q+1$ désignent respectivement le nombre de signes positifs et négatifs. L'*univers d'Einstein* $\mathrm{Ein}^{p,q}$ est l'ensemble des droites isotropes de $\mathbb{R}^{p+1,q+1}$. C'est une variété de drapeaux auto-opposée du groupe $G = \mathrm{PO}(p+1, q+1)$ des éléments de $\mathrm{PGL}(p+q+2, \mathbb{R})$ dont un relevé dans $\mathrm{GL}(p+q+2, \mathbb{R})$ préserve $\mathbf{b}$. L'univers d'Einstein admet une structure conforme pseudo-riemannienne de signature (p, q), et d'après le théorème de Liouville (voir [Fra03] et le fait 2.4.1), le groupe d'automorphismes d'un ouvert $\Omega \subset \mathrm{Ein}^{p,q}$ coincide avec son *groupe conforme*, c'est-à-dire avec l'ensemble des difféomorphismes de Ω qui préservent la classe conforme de la métrique de $\mathrm{Ein}^{p,q}$ induite sur Ω.

1.1.2.4 Les variétés de drapeaux causales. Soit G un groupe de Lie hermitien de type tube — condition que nous noterons HTT pour alléger la rédaction — de rang réel $r \geq 1$, et $\mathfrak{g}$ son algèbre de Lie (la liste complète de telles algèbres de Lie $\mathfrak{g}$ est donnée dans le tableau 2.1). Soit Δ l'ensemble des racines simples restreintes de G. La seule racine longue $\alpha_r \in \Delta$ définit un sous-groupe parabolique $P_{\{\alpha_r\}}$ de G. La variété de drapeaux $\mathbf{Sb}(\mathfrak{g}) := G/P_{\{\alpha_r\}}$ est le *bord de Shilov* de l'espace symétrique riemannien $\mathbb{X}_G$ de G. Cette variété admet une *structure causale*, c'est-à-dire qu'il existe (à un sous-groupe d'indice deux de G près) une famille lisse G-équivariante $(c_x)_{x\in\mathbf{Sb}(\mathfrak{g})}$ de cônes ouverts proprement convexes dans le fibré tangent $T(\mathbf{Sb}(\mathfrak{g}))$; c'est pourquoi dans ce mémoire on les appellera également les *variétés de drapeaux causales*. Ces variétés de drapeaux apparaissent naturellement dans plusieurs contextes, tels que les algèbres de Jordan euclidiennes et l'analyse complexe (voir par exemple [FK94]), ou encore la Θ-positivité et la théorie de Teichmüller supérieure [GW18].

1.1.2.5 Les espaces de Nagano. Les trois familles d'exemples évoquées ci-dessus font partie d'une famille plus large de variétés de drapeaux G/P, celles qui sont aussi des espaces symétriques riemanniens compacts. L'étude de telles variétés a été initiée par Nagano [Nag65], qui observe que tout espace symétrique compact irréductible $\mathbb{X}$ admettant un groupe de transformations G plus grand que son groupe d'isométries, est en fait une variété de drapeaux de G. Ces espaces sont appelés *espaces de Nagano*, mais aussi *espaces symétriques extrinsèques* ou *R-espaces symétriques* (« R » pour « racine ») selon les auteurs, et leur liste complète est connue [Nag65] et donnée dans le tableau 8.1.

Plusieurs caractérisations algébriques des espaces de Nagano, parmi les variétés de drapeaux ou les espaces symétriques compacts, ont été étudiées, voir par exemple [KN64, KN65]. Leur groupe de transformations a également suscité l'intérêt : Peterson [Pet87] (pour le cas des grassmanniennes) et Takeuchi [Tak88] (pour le cas général) ont montré que G était en fait le groupe des difféomorphismes de $\mathbb{X}$ préservant une distance arithmétique définie par Chow [Cho49] (voir la partie 6.5.2), qui, dans le cas des grassmanniennes par exemple, est la codimension de l'intersection. Kaneyuki [Kan11] a donné une interprétation du groupe de transformations G en termes de fibrés principaux sur $\mathbb{X}$. Nous renvoyons par exemple à [Tak65, TK68, Loo71, Kan98, Kan06] pour plus de littérature à ce sujet.

Étant donné un espace de Nagano G/P, un résultat important [Nag65] est que l'espace symétrique dual non compact $\mathbb{X}(G/P)$ de G/P se plonge dans G/P comme un domaine propre, et le groupe d'automorphismes de son image est isomorphe à son groupe d'isométries $\mathrm{Isom}(\mathbb{X}(G/P))$. Son image est donc un domaine propre symétrique et divisible de G/P, au sens défini dans le paragraphe 1.1.2.1. Nous appelons *réalisation de* $\mathbb{X}(G/P)$ *dans* G/P l'image d'un tel plongement. Par exemple, tout ellipsoïde de $\mathbb{P}(\mathbb{R}^{n+1})$ est une réalisation de $\mathbb{X}(\mathbb{P}(\mathbb{R}^{n+1})) = \mathbb{H}^n$. Étant données deux réalisations Ω, Ω' de $\mathbb{X}(G/P)$ dans G/P, il existe $g \in G$ tel que $\Omega = g \cdot \Omega'$, c'est-à-dire que, à translation près, il n'existe qu'une seule réalisation de $\mathbb{X}(G/P)$ dans G/P.

Dans ce mémoire, on pourra toujours se ramener au cas où G/P est un espace de Nagano *irréductible*, c'est-à-dire que G est un groupe de Lie simple.

Makarevich a établi la liste des domaines (non nécessairement propres) symétriques des espaces de Nagano irréductibles dont le groupe d'automorphismes est réductif et transitif [Mak73]. L'espace projectif est le seul espace de Nagano irréductible qui contient des domaines propres symétriques à groupe d'automorphismes réductif qui ne sont pas des réalisations du dual non compact (voir le Lemme 5.4.1).

1.2 Le cas compact

Une partie importante de cette thèse est consacrée à l'étude des domaines propres divisibles des variétés de drapeaux G/P. Ces domaines produisent, par quotient, des $(G, G/P)$-variétés (ou orbi-variétés) compactes. Dans le cas où $G = \mathrm{PGL}(n+1, \mathbb{R})$ et $G/P = \mathbb{P}(\mathbb{R}^{n+1})$, on retrouve la théorie classique des *convexes divisibles*, dont les principaux résultats et idées sont exposés dans le paragraphe 1.2.1 ci-dessous. Dans le cas général, nous allons voir dans le paragraphe 1.2.2 que les domaines propres divisibles devraient, conjecturalement, être soumis à une forte rigidité (voir la question 1.2.1). Un des objectif de cette thèse est de mieux comprendre cette rigidité.

1.2.1 Convexes divisibles

Les domaines propres de l'espace projectif réel qui sont divisibles sont nécessairement proprement convexes [Sho84]. Il sont donc appelés *convexes divisibles*. Leur étude, initiée dans les années 1960 avec les travaux de Benzecri [Ben60], a depuis été développée par de nombreux auteurs (voir par exemple [Vin65, Gol90, CLT15]), en particulier par Benoist au début des années 2000 [Ben00, Ben03, Ben05, Ben06]. Le cas strictement convexe est bien compris, tandis que le cas non strictement convexe est encore en cours de développement [Isl25, CLM20, Zim23, Bla24]. Voir par exemple [Ben08, Mar14] pour des survols de cette théorie et d'autres références.

Un ouvert proprement convexe de $\mathbb{P}(\mathbb{R}^{n+1})$ est dit *irréductible* s'il ne peut pas s'écrire comme un joint d'ouverts proprement convexes d'espaces projectifs réels plus petits. D'après [Vey70, Ben03], si un convexe divisible n'est pas irréductible, il s'écrit en fait comme un joint de convexes divisibles. La théorie des convexes divisibles se ramène donc au cas où Ω est irréductible. Dans ce cas, soit Ω est symétrique (au sens défini dans le paragraphe 1.1.1.1), soit $\mathsf{Aut}(\Omega)$ est un sous-groupe discret Zariski-dense de $\mathrm{PGL}(n+1, \mathbb{R})$; voir [Vin65, Koe99, Ben03].

Il existe une liste exhaustive des domaines propres symétriques irréductibles en toute dimension [Koe99]. Tous les domaines de cette liste s'identifient à des espaces symétriques riemanniens, et leur groupe d'isométries coïncide avec leur groupe d'automorphismes (en tant qu'ouverts de l'espace projectif). L'exemple le plus simple est l'espace hyperbolique réel $\mathbb{H}^n$ plongé dans l'espace projectif $\mathbb{P}(\mathbb{R}^{n+1})$ via le modèle de Klein.

Il existe également des convexes divisibles non symétriques : certains réseaux cocompacts Γ de $\mathrm{PO}(n, 1)$ admettent des déformations Zariski-denses dans $\mathrm{PGL}(n+1, \mathbb{R})$ [JM87] ; d'après un théorème d'ouverture de Koszul [Kos68], l'image d'une telle petite déformation de Γ dans $\mathrm{PGL}(n+1, \mathbb{R})$ divise encore un convexe, non symétrique (par Zariski-densité de Γ). Il existe également plusieurs constructions explicites de convexes divisibles irréductibles non symétriques ; les premiers exemples ont été construits par Kac–Vinberg en dimension 2, en utilisant des groupes de Coxeter [KV67]. Kapovich a ensuite construit des exemples qui ne sont pas quasi-isométriques à des espaces symétriques, et dont le groupe d'automorphismes est discret et Gromov-hyperbolique, en toute dimension $n \geq 4$ [Kap07]. Des exemples avec un groupe d'automorphismes discret et non Gromov-hyperbolique ont été construits en dimensions projectives 3, 4, 5, 6 par Benoist [Ben06] et 4, 5, 6 par Choi–Lee–Marquis [CLM20] et en dimension 3 par Ballas–Danciger–Lee [BDL18], et plus récemment, en toute dimension projective $n \geq 3$ par Blayac–Viaggi [BV24].

A. Zimmer a récemment prouvé que les convexes divisibles de $\mathbb{P}(\mathbb{R}^{n+1})$ sont soumis à une certaine rigidité [Zim23] : ceux dont le groupe d'automorphismes ne contient pas d'isométrie *de rang 1* sont nécessairement symétriques de rang supérieur. La diversité des exemples non symétriques met cependant en lumière l'importance des résultats généraux les concernant, au-delà de ceux concernant les actions cocompactes sur les espaces symétriques riemanniens.

1.2.2 Domaines propres divisibles des variétés de drapeaux et rigidité

la question de savoir si la théorie des convexes divisibles se généralise à d'autres variétés de drapeaux que l'espace projectif réel a été posée par Limbeek–Zimmer. Les réalisations des duaux non compacts d'espaces de Nagano (voir la partie 1.1.2) sont des exemples de convexes divisibles dans des variétés de drapeaux différentes de l'espace projectif, mais ils sont symétriques. Le problème consiste à déterminer s'il existe des exemples non symétriques, comme dans le cas projectif :

Question 1.2.1 ([LZ19])**.** Étant donnés un groupe de Lie semi-simple réel non compact G et un sous-groupe parabolique P de G, tous les domaines propres divisibles de G/P sont-ils symétriques ?

On peut poser la même question en remplaçant « divisible » par « quasi-homogène » (et même par « presque-homogène », voir la question 8.9.1). Une réponse positive dans le cas quasi-homogène implique une réponse positive à la question 1.2.1. L'avantage est que l'étude des domaines propres quasi-homogènes se ramène au cas où G est simple, par le fait suivant :

Fait 1.2.2 ([Zim18a, Thm 1.7])**.** *Soit G un groupe de Lie semi-simple à centre trivial et sans facteur compact, de la forme $G = G_1 \times \cdots \times G_k$, où les G_i sont des groupes de Lie simples non compacts. Pour tout sous-groupe parabolique P de G, il existe des sous-groupes paraboliques $P_i \leq G_i$ tels que $P = P_1 \times \cdots \times P_k$. Soit maintenant $\Omega \subset G/P$ un domaine propre quasi-homogène. Alors, il existe des domaines propres quasi-homogènes $\Omega_i \subset G_i/P_i$ tels que $\Omega = \Omega_1 \times \cdots \times \Omega_k$.*

La question 1.2.1 admet une réponse négative dans le cas où $G = \mathrm{PO}(n,1)$ pour $n \geq 3$ et où P est l'unique (à conjugaison près) sous-groupe parabolique propre de G. Dans ce cas, la variété de drapeaux G/P est la sphère conforme, et admet des domaines propres divisibles non symétriques (ici « propre » signifie simplement que le complémentaire du domaine est d'intérieur non vide). Par exemple, l'ensemble limite d'une représentation obtenue par déformation de l'inclusion naturelle d'un réseau cocompact de $\mathrm{SO}(2,1)$ dans $\mathrm{SO}(3,1)$, appelée *quasi-fuchsienne*, est un quasi-cercle, qui sépare la sphère conforme en deux domaines propres divisibles. D'après le paragraphe 1.2.1, la question 1.2.1 admet également une réponse négative dans le cas où $G = \mathrm{PGL}(n+1,\mathbb{R})$ et $G/P = \mathbb{P}(\mathbb{R}^{n+1})$. Cependant, pour d'autres variétés de drapeaux G/P, on observe une plus forte rigidité, et A. Zimmer conjecture que tout domaine propre divisible de G/P est homogène [Zim18a, Conj. 2.6]. Suite aux observations faites dans cette thèse, nous pensons qu'il devrait même être symétrique (voir la partie 8.9).

La question 1.2.1 admet une réponse positive pour les variétés de drapeaux G/P où P est un sous-groupe parabolique propre non maximal :

Fait 1.2.3 ([Zim18a, Thm 1.5]). *Soient G un groupe de Lie simple réel non compact et $P \leq G$ un sous-groupe parabolique propre non maximal. Il n'existe aucun domaine propre quasi-homogène dans G/P.*

Les variétés de drapeaux définies par des sous-groupes paraboliques maximaux restent donc à étudier : la question 1.2.1 admet une réponse partielle positive pour les grassmanniennes complexes [Fra89], où les domaines proprement convexes divisibles sont biholomorphes à un domaine propre symétrique. A. Zimmer a renforcé ce résultat pour l'espace projectif complexe $\mathbb{P}(\mathbb{C}^{n+1})$ pour $n \geq 2$ [Zim18b], en montrant que les domaines proprement convexes divisibles y étaient symétriques. La question 1.2.1 admet également une réponse partielle positive pour les grassmanniennes auto-opposées $\mathrm{Gr}_p(\mathbb{R}^{2p})$ [LZ19]. Dans ces deux derniers cas, les auteurs montrent — sous des hypothèses additionnelles de convexité dans une carte affine — qu'il n'existe, à translation près par l'action de G, qu'un seul domaine propre divisible dans G/P, et que ce domaine est symétrique. Dans cette thèse, nous nous intéressons à ces questions pour plusieurs nouveaux cas de variétés de drapeaux. Nous y démontrons la rigidité (voir le chapitre 8) et, suivant une idée déjà présente dans [LZ19], nous l'interprétons comme un phénomène de rigidité de rang supérieur (voir la partie 8.9).

Remarque 1.2.4. Retirer l'hypothèse de propreté dans la question 1.2.1 permet davantage de flexibilité. En effet, comme mentionné en début de partie 1.2, il existe de nombreux exemples de domaines divisibles *non propres* et non symétriques dans des variétés de drapeaux G/P, construits, par exemple, comme domaines de discontinuité pour des représentations anosoviennes [Fra05, GW12, KLP18] (voir la partie 1.3.2).

1.3 Le cas non compact

Dans cette section, nous discutons le cas des (G, X)-variétés non compactes, avec un accent sur celles qui sont *convexes cocompactes*. Une n-variété hyperbolique complète M est dite *convexe cocompacte* si son groupe fondamental $\Gamma \leq \mathrm{PO}(n, 1)$ agit cocompactement sur un convexe fermé non vide de l'espace hyperbolique réel $\mathbb{X}_{\mathrm{PO}(n,1)} = \mathbb{H}^n$. Dans ce cas M peut être identifiée au quotient $\mathbb{H}^n/\Gamma$, et son *cœur convexe*, à savoir, le plus petit sous-ensemble non vide géodésiquement convexe de M qui est bordé par des hypersurfaces totalement géodésiques, est compact. De nombreux exemples de variétés hyperboliques convexes cocompactes non compactes existent (voir, par exemple, [Kas18]).

1.3.1 Quotients d'espaces symétriques et rigidité

Tout sous-groupe discret d'un groupe de Lie simple réel G de rang réel $r \geq 2$ agissant de manière cocompacte sur un ensemble fermé (géodésiquement) convexe de l'espace symétrique riemannien $\mathbb{X}_G$ de G est un réseau cocompact de G [Qui05, KL06]. Ainsi, la généralisation géométrique intuitive de la convexe cocompacité devient rigide en rang supérieur. Plusieurs tentatives de définitions plus flexibles de la convexe cocompacité en rang supérieur ont émergé au cours des vingt dernières années, certaines ayant abouti avec succès, notamment celle des *représentations anosoviennes*.

1.3.2 Représentations anosoviennes et structures géométriques

Une généralisation de la convexe cocompacité basée sur les propriétés dynamiques des sous-groupes discrets convexes cocompacts de $\mathrm{PO}(n,1)$ a été développée au cours des vingt dernières années, à travers les *représentations anosoviennes* ; voir la partie 2.3 pour une définition. Ces représentations ont été introduites par Labourie [Lab06] dans son étude des représentations de Hitchin des groupes de surfaces, puis généralisées et approfondies par Guichard–Wienhard [GW12]. Elles sont discrètes, fidèles et structurellement stables, ce qui en fait un concept clé dans les récents développements de la théorie de Teichmüller supérieure [GW12, Wie18, GLW21, BK23, BGL+24] et des structures géométriques [GW12, KLP18, DGK18, DGK24].

La propriété d'être anosovienne dépend (à conjugaison près) du choix d'un sous-groupe parabolique propre P de G ; une représentation est dite *P-anosovienne* si elle est anosovienne par rapport à P. Lorsque $G = \mathrm{PO}(n,1)$, le parabolique P est nécessairement le seul — à conjugaison près — sous-groupe parabolique propre P_1 de G, et les représentations P_1-anosoviennes coïncident avec les représentations convexes cocompactes. Pour un groupe de Lie réductif réel G et un parabolique P quelconques, les propriétés dynamiques de l'action d'une représentation P-anosovienne sur la variété de drapeaux G/P sont désormais bien comprises (voir la partie 2.3.2).

Les images des représentations P-anosoviennes sont appelées *sous-groupes P-anosoviens* de G, et elles appartiennent à la vaste famille des sous-groupes *P-transverses* ou *P-antipodaux* (voir par exemple [KLP17, CZZ23] et la partie 2.3). Cette famille est caractérisée par des propriétés dynamiques plus faibles que celles des représentations anosoviennes et inclut, par exemple, les sous-groupes relativement anosoviens au sens de [ZZ22, KL23].

La question de savoir si de tels groupes possédant de fortes propriétés dynamiques fournissent des exemples de $(G, G/P)$-variétés M a été soulevée par de nombreux auteurs, en particulier dans le cas où M est un quotient Ω/Γ, où Ω est un ouvert, propre ou non, de G/P, et Γ est un sous-groupe P-transverse de G préservant Ω :

1. Dans [Fra05, GW12], pour certains groupes de Lie semi-simples G et certains sous-groupes paraboliques non nécessairement conjugués P, P', des exemples de variétés $(G, G/P')$-compactes sont construits comme quotients Ω/Γ d'un ouvert bien choisi $\Omega \subset G/P'$ préservé par un sous-groupe P-anosovien $\Gamma \leq G$. Une approche générale est développée dans [KLP17]. Ces ouverts sont en général non propres.
2. Danciger–Guéritaud–Kassel [DGK18, DGK24] et A. Zimmer [Zim18a] introduisent une notion de convexe cocompacité dans l'espace projectif : si Γ préserve un ouvert proprement convexe Ω de $\mathbb{P}(\mathbb{R}^{n+1})$ et agit cocompactement sur son cœur convexe $\mathscr{C} \subset \Omega$, et si le bord idéal de $\mathscr{C}$ ne contient pas de segment projectif, alors Γ est dit *fortement convexe cocompact dans* $\mathbb{P}(\mathbb{R}^{n+1})$. Un sous-groupe discret $\Gamma \leq \mathrm{PGL}(n+1, \mathbb{R})$ est fortement convexe cocompact si et seulement s'il est P_1-anosovien (où P_1 est le stabilisateur d'un droite de $\mathbb{R}^{n+1}$) et préserve un domaine propre de $\mathbb{P}(\mathbb{R}^{n+1})$ [DGK24] ; voir aussi [Zim21] pour le cas fortement irréductible.
3. Cooper–Long–Tillman définissent une variété projective *géométriquement finie* comme le quotient Ω/Γ d'un domaine strictement convexe $\Omega \subset \mathbb{P}(\mathbb{R}^{n+1})$ par un

sous-groupe discret $\Gamma \leq \mathrm{PGL}(n+1,\mathbb{R})$, tel que le cœur convexe de Ω/Γ soit l'union d'un ensemble compact et d'un nombre fini de bouts, appelés *cusps* [CLT15]. Fléchelles–Islam–Zhu démontrent qu'un sous-groupe discret $\Gamma \leq \mathrm{PGL}(n+1,\mathbb{R})$ préservant un ouvert proprement convexe dans $\mathbb{P}(\mathbb{R}^{n+1})$ est relativement P_1-anosovien si et seulement s'il existe un domaine proprement convexe, rond et Γ-invariant $\Omega \subset \mathbb{P}(\mathbb{R}^{n+1})$ tel que l'orbi-variété Ω/Γ est soit géométriquement finie [Flé24]. Ici « rond » signifie que l'ouvert proprement convexe est strictement convexe, et que tout point de son bord est dans un unique hyperplan d'appui.

En fait, d'après [CZZ23], tout sous-groupe discret P-transverse Γ de G préserve un domaine propre Ω dans un certain espace projectif $\mathbb{P}(V)$, donnant ainsi lieu à une variété projective Ω/Γ. Le point (2) (resp. (3)) ci-dessus, ainsi qu'une propriété de stabilité par composition des représentations anosoviennes [GW12], impliquent que si Γ est en outre P-anosovien (resp. relativement P-anosovien), alors il existe une telle variété qui soit de plus convexe cocompacte (resp. géométriquement finie). Cependant, cette variété est a priori modelée sur l'espace projectif, et non sur G/P. Il est donc naturel de chercher à construire des $(G, G/P)$-variétés de la forme Ω'/Γ, où Ω' est un domaine de G/P. On peut également se demander si ce domaine peut être choisi propre, comme dans les cas des points (3) et (4). Ceci conduit à la question suivante :

Question 1.3.1. Soient G un groupe de Lie semi-simple et P un sous-groupe parabolique de G, et soit $H \leq G$ un sous-groupe P-transverse.

1. Sous quelles conditions H préserve-t-il un domaine propre dans G/P ?
2. Si H est discret et préserve un domaine propre $\Omega \subset G/P$, quelles conditions géométriques supplémentaires sur l'action de H sur Ω sont nécessaires pour garantir que H est P-anosovien ?

1.4 Contenu du mémoire

L'objectif de cette thèse est de développer la théorie des domaines propres dans les variétés de drapeaux, initiée par A. Zimmer [Zim18a, Zim18b] et inspirée de la géométrie projective convexe. Dans cette partie, nous présentons le contenu du mémoire et les principaux résultats qui y sont établis.

1.4.1 Différentes notions de convexité

Bien que la convexité duale introduite dans le paragraphe 1.1.2.1 semble être une notion naturelle de convexité dans les variétés de drapeaux, il en existe d'autres. Une première chose à comprendre pour généraliser la géométrie projective convexe aux variétés de drapeaux est le lien entre les différentes notions de convexité. Contrairement au cas projectif, la convexité duale n'est pas équivalente à la notion plus naïve de convexité dans une carte affine. Dans les parties 3.2 et 3.3.2 et dans l'exemple 3.5.9, nous comparons ces deux notions, en particulier sur des exemples de variétés de drapeaux concrètes.

Nous définissons une troisième notion de convexité, spécifique aux les variétés de drapeaux causales (définies en partie 1.1.2.4). Si un point x est contenu dans une carte affine $\mathbb{A}$ de $\mathbf{Sb}(\mathfrak{g})$ (avec les notations de la partie 2.2.6.2), la causalité permet de définir le *futur* et le *passé* de x (voir la partie 2.4.4.3). Pour tout point y dans le futur de x, l'intersection du passé de y avec le futur de x est un *diamant*, noté $\mathbf{D}_{\mathbb{A}}(x,y)$. Autrement dit, il existe exactement deux diamants d'extrémités x et y : ce sont les deux seules composantes connexes propres de $\mathbf{Sb}(\mathfrak{g}) \smallsetminus (\mathrm{Z}_x \cup \mathrm{Z}_y)$ (voir la définition 3.5.2). Les diamants induisent une notion de *convexité causale* analogue à celle existant déjà en géométrie lorentzienne [MS08]. Un sous-ensemble connexe $X \subset \mathbf{Sb}(\mathfrak{g})$ est dit *causalement convexe* s'il est contenu dans une carte affine $\mathbb{A}$ et si pour tous $x, y \in X$ tels que y soit dans le futur large de x dans $\mathbb{A}$, le diamant fermé $\mathbf{D}^c_{\mathbb{A}}(x,y)$ est contenu dans X (voir la définition 3.5.14). On montre dans la proposition 3.5.22 que cette définition est indépendante du choix de la carte affine $\mathbb{A}$ contenant X. Nous la comparons avec la convexité duale :

Proposition 1.4.1 (voir la proposition 3.5.24)**.** *Soit G un groupe de Lie hermitien de type tube et soit $\Omega \subset \mathbf{Sb}(\mathfrak{g})$ un domaine dualement convexe. Si $\Omega \neq \mathbf{Sb}(\mathfrak{g})$, alors Ω est causalement convexe (en particulier Ω est contenu dans au moins une carte affine).*

1.4.2 Groupes préservant des domaines propres

Une partie de ce mémoire, bien que ne constituant pas son cœur, est consacrée à une étude préliminaire de la question 1.3.1, laquelle est développée dans le chapitre 4. Nous en présentons ici les principaux résultats. Pour résumer, la propriété pour un groupe de préserver un domaine propre s'avère assez restrictive (proposition 1.4.2 ci-dessous), tandis que, pour une notion naturelle de convexité dans certaines variétés de drapeaux (notamment la convexité causale dans les variétés de drapeaux causales), dès qu'un groupe préserve un domaine propre Ω, la propriété d'agir de manière cocompacte sur un sous-ensemble convexe fermé de Ω est, au contraire, très peu contraignante (voir la proposition 1.4.4 et la remarque 3.5.17).

1.4.2.1 Restrictions topologiques. La partie 1.1.2.4 est consacrée à l'étude de la question 1.3.1.(1). Dans [Ben00, Prop. 1.2], Benoist donne une condition nécessaire et suffisante pour qu'un sous-groupe fortement irréductible de $\mathrm{PGL}(n,\mathbb{R})$ préserve un ouvert proprement convexe de $\mathbb{P}(\mathbb{R}^n)$. La preuve fait intervenir certaines propriétés de base des convexes de l'espace projectif qui ne sont plus vraies dans le cas général des variétés de drapeaux G/P. On peut tout de même retrouver un analogue de sa condition nécessaire, exprimée dans la proposition 4.1.5 ci-dessous.

Soit G un groupe de Lie semi-simple réel et $P \leq G$ un sous-groupe parabolique auto-opposé. Soit P^- un conjugué de P qui soit transverse à P. Il existe une involution $\mathsf{s} : (G/P) \smallsetminus \mathrm{Z}_{P^-} \to (G/P) \smallsetminus \mathrm{Z}_{P^-}$ qui agit comme $-\mathrm{id}$ sur la carte affine $(G/P) \smallsetminus \mathrm{Z}_{P^-}$ contenant P, et permute les composantes connexes de $(G/P) \smallsetminus (\mathrm{Z}_P \cup \mathrm{Z}_{P^-})$.

Si $x \in G/P$ est transverse à P et P^-, alors le *type* $\mathsf{type}(P, x, P^-)$ du triplet (P, x, P^-) est l'orbite sous $P \cap P^-$ de la composante connexe de $(G/P) \smallsetminus (\mathrm{Z}_P \cup \mathrm{Z}_{P^-})$ qui contient x. Le type s'étend de manière G-invariante à tout triplet (a, b, c) de points deux à deux transverses de G/P et décrit les positions relatives de a, b et c. Lorsque G est un groupe de Lie HTT et $G/P = \mathbf{Sb}(\mathfrak{g})$, il est encodé par l'*indice de Maslov* classique $\mathsf{idx}(a, b, c)$ du triplet (a, b, c) (voir [LV80] et la partie 4.1.3).

Rappelons que l'ensemble P-limite $\Lambda_P(H)$ d'un sous-groupe $H \leq G$ est défini comme l'ensemble des points attractifs dans G/P de suites d'éléments de H. Nous démontrons :

Proposition 1.4.2 (voir la proposition 4.1.5 et le corollaire 4.1.7)**.** *Soient G un groupe de Lie semi-simple réel et $P \leq G$ un sous-groupe parabolique auto-opposé. Soit $H \leq G$ un sous-groupe préservant un domaine propre $\Omega \subset G/P$ tel que l'ensemble limite $\Lambda_P(H)$ contienne au moins trois points deux à deux transverses. Alors il existe une composante connexe* s*-invariante $\mathscr{O}$ de $(G/P) \smallsetminus (\mathrm{Z}_P \cup \mathrm{Z}_{P^-})$ telle que le type $\mathsf{type}(a,b,c)$ d'un triplet $(a,b,c) \in \Lambda_P(H)^3$ de points deux à deux transverses soit égal à la $(P \cap P^-)$-orbite de $\mathscr{O}$.*

Dans le cas où G est un groupe de Lie HTT *de rang réel $r \geq 2$ et $G/P = \mathbf{Sb}(\mathfrak{g})$, alors r est pair et $\mathsf{idx}(x,y,z) = 0$ pour tout triplet de points deux à deux transverses $(x,y,z) \in \Lambda_P(H)^3$.*

La proposition 1.4.2 fournit directement de fortes restrictions sur les domaines propres presque-homogènes des variétés de drapeaux auto-opposées G/P telles que $(G/P) \smallsetminus (\mathrm{Z}_P \cup \mathrm{Z}_{P^-})$ n'ait pas de composante connexe s-invariante, comme illustré dans la proposition 8.6.1.

Remarque 1.4.3. La proposition 1.4.2 est reliée à la notion de *Propriété I* définie dans [DGR24]. Une variété de drapeaux G/P satisfait à la *Propriété I* si aucune composante connexe de $(G/P) \smallsetminus (\mathrm{Z}_P \cup \mathrm{Z}_{P^-})$ n'est invariante par s ; voir la partie 2.2.6 pour plus de détails. D'après la proposition 1.4.2, s'il existe $H \leq G$ préservant un domaine propre dans G/P, alors G/P n'a *pas* la Propriété I. La proposition 1.4.2 implique donc que la question 1.3.1.(1) est étroitement liée à la question suivante, posée par Dey–Greenberg–Riestenberg et étudiée par Dey [Dey22], Dey–Greenberg–Riestenberg [DGR24] et Kineider–Troubat [KT24] (voir la Remarque 4.1.1) : quelles sont variétés de drapeaux auto-opposées G/P qui satisfont la Propriété I ?

Nous construisons des exemples Zariski-denses P-anosoviens de G préservant un domaine propre dans G/P dans la partie 4.4, comme nous allons le voir dans le paragraphe 1.4.2.3.

1.4.2.2 Convexité et groupes transverses. Dans la partie 4.3, initialement motivée par la question 1.3.1.(2), nous étudions la géométrie des variétés de la forme Ω/Γ, où Γ est un sous-groupe discret P-transverse d'un groupe de Lie HTT G, préservant un domaine propre $\Omega \subset G/P = \mathbf{Sb}(\mathfrak{g})$ (voir la proposition 1.4.4). Étant donné un domaine propre $\Omega \subset \mathbf{Sb}(\mathfrak{g})$ préservé par un sous-groupe discret $\Gamma \leq G$, l'*ensemble limite orbital total* $\Lambda_\Omega^{\mathrm{orb}}(\Gamma)$ de (Ω, Γ) est l'ensemble des points d'accumulation des orbites des éléments de Ω sous l'action de Γ (voir [DGK24]). Un *cœur convexe* de (Ω, Γ) est un sous-ensemble fermé (dans Ω), connexe, causalement convexe et Γ-invariant $\mathscr{C}$ de Ω tel que le bord idéal $\partial_i \mathscr{C} := \overline{\mathscr{C}} \smallsetminus \mathscr{C}$ de $\mathscr{C}$ contienne $\Lambda_\Omega^{\mathrm{orb}}(\Gamma)$. Nous démontrons :

Proposition 1.4.4 (voir la proposition 4.3.2)**.** *Soient G un groupe de Lie* HTT *et $\Gamma \leq G$ un sous-groupe discret. Soit $P \leq G$ un sous-groupe parabolique tel que $G/P = \mathbf{Sb}(\mathfrak{g})$. Les assertions suivantes sont équivalentes :*

1. *Le groupe Γ est de type fini, P-transverse, préserve un domaine propre $\Omega \subset \mathbf{Sb}(\mathfrak{g})$, et $\Lambda_P(\Gamma)$ contient au moins 3 points.*
2. *Il existe un domaine propre causalement convexe Γ-invariant $\Omega \subset \mathbf{Sb}(\mathfrak{g})$ tel que Γ agisse de manière cocompacte sur un cœur convexe $\mathscr{C}$ de (Ω, Γ) dont le bord idéal est transverse, et contient au moins trois points.*

3. Il existe un domaine propre dualement convexe Γ-invariant $\Omega' \subset \mathbf{Sb}(\mathfrak{g})$ tel que Γ agisse de manière cocompacte sur un cœur convexe $\mathscr{C}'$ de (Ω', Γ) dont le bord idéal est transverse, et contient au moins trois points.

Si ces assertions sont vérifiées, on a $\partial_i \mathscr{C} = \Lambda_P(\Gamma) = \Lambda^{\mathrm{orb}}_{\Omega}(\Gamma) = \Lambda^{\mathrm{orb}}_{\Omega'}(\Gamma) = \partial_i \mathscr{C}'$.

La proposition 1.4.4 montre qu'une définition en apparence naturelle de convexe cocompacité dans $\mathbf{Sb}(\mathfrak{g}) = G/P$ — point (2) de la proposition 1.4.4 — ne distingue pas les sous-groupes P-anosoviens des autres sous-groupes discrets P-transverses de type fini de G parmi les groupes préservant un domaine propre. Ce phénomène est dû à la nature intrinsèquement « temporelle » de la convexité considérée (convexité causale) et à la nature « spatiale » du comportement dynamique de Γ, déjà observée dans la proposition 1.4.2 ; voir la remarque 4.3.4 et l'exemple 4.3.3.

1.4.2.3 Exemples. Dans la partie 4.4, nous construisons des sous-groupes P-anosoviens Zariski-denses de groupes de Lie HTT G préservant un domaine propre dans $\mathbf{Sb}(\mathfrak{g}) = G/P$, par déformations de sous-groupes préservant des diamants (voir la proposition 4.4.3) :

Proposition 1.4.5 (voir la proposition 4.4.2 et l'exemple 4.4.6)**.** *Soit $r = 2p$, avec $p \in \mathbb{N}^*$. Si G est un groupe de Lie* HTT *de rang réel r et si $P \leq G$ est un sous-groupe parabolique tel que $G/P = \mathbf{Sb}(\mathfrak{g})$, alors il existe des groupes de surfaces P-anosoviens Zariski-denses dans G préservant un domaine propre dans $\mathbf{Sb}(\mathfrak{g})$. Si p est pair, alors il existe aussi de tels exemples qui ne sont ni virtuellement libres, ni des groupes de surface.*

Les exemples construits dans la démonstration de la proposition 4.4.3 sont eux-mêmes soumis à des restrictions dynamiques et topologiques, voir la proposition 4.4.5 et l'exemple 4.4.6. Dans le cas où $G = \mathrm{SO}(n, 2)$, avec $\mathbf{Sb}(\mathfrak{g}) = \mathrm{Ein}^{n-1,1}$, d'autres exemples issus de [DGK24, Sma22] apparaissent ; voir le corollaire 4.5.4 et l'exemple 4.5.5.

1.4.3 Espaces de Nagano

Les espaces de Nagano irréductibles définis dans le paragraphe 1.1.2.5 ont une structure particulière par rapport aux autres variétés de drapeaux, qui permet de construire de nouveaux objets pour l'étude de leurs domaines propres. Par exemple, si la racine simple α définissant l'espace de Nagano G/P (c'est-à-dire telle que $G/P = G/P_{\{\alpha\}}$) est de multiplicité 1, alors les *photons* tels que définis dans le paragraphe suivant ont des propriétés d'invariance analogues à celles des droites projectives dans l'espace projectif réel. Cela permet de construire une *distance de Kobayashi* (voir la partie 1.4.3.2) et de développer une notion de *facette*, généralisant ainsi des notions de géométrie projective convexe.

La condition sur la multiplicité de la racine α est nécessaire pour assurer un bon comportement aux photons, voir la remarque 6.3.4. Une partie significative de cette thèse, contenue dans les chapitres 5 à 7, est consacrée au développement d'une théorie des domaines propres dans les espaces de Nagano vérifiant cette propriété, dits *de type réel*. Dans le reste de ce paragraphe, les variétés de drapeaux considérées seront toujours des espaces de Nagano irréductibles de type réel.

Dans le tableau 8.1, la dimension de $\mathfrak{g}_\alpha$ est égale à 1 exactement lorsque G/P est un espace de Nagano irréductible de type réel ; c'est le cas des trois familles clés introduites dans les paragraphes 1.1.2.2 à 1.1.2.4. Un espace de Nagano de type réel G/P est dit *de rang supérieur* si son rang en tant qu'espace symétrique riemannien compact est ≥ 2, ou de manière équivalente, si ce n'est ni l'espace projectif réel, ni son dual (voir le tableau 8.2). On va voir que le rang d'un espace de Nagano irréductible joue un rôle fondamental dans les propriétés géométriques de ses domaines propres presque-homogènes, voir la partie 1.4.4.

1.4.3.1 Photons. Une action de $\mathrm{SL}(2,\mathbb{R})$ sur G/P est dite *photon-génératrice* si elle est conjuguée à celle induite par le $\mathfrak{sl}_2$-triplet associé à α. Un *photon* est une orbite fermée dans G/P d'une action photon-génératrice. Il s'agit d'un cercle topologique dont les propriétés sont similaires à celles des droites projectives dans $\mathbb{P}(\mathbb{R}^n)$ (voir les lemmes 6.3.6 et 6.3.7). Les photons ont été définis et étudiés pour les grassmanniennes dans [LZ19], pour les variétés de drapeaux causales dans [Gal24] et dans toutes variétés de drapeaux G/P dans [BGL+24].

1.4.3.2 La distance de Kobayashi. Si $\Omega \subset G/P$ est un domaine propre, les distances de Carathéodory ne fournissent pas suffisamment d'informations sur $\partial\Omega$ pour étudier la dynamique du groupe d'automorphismes de Ω. À la place, nous définissons la *distance de Kobayashi*, notée K_Ω dans ce mémoire, qui généralise la définition (1) de la distance de Hilbert donnée dans le paragraphe 1.1.1. Étant donnés deux points $x, y \in \Omega$, une *chaîne de segments de photons* est un chemin continu défini par concaténation de segments contenus dans des photons. Comme chacun des photons en question est une droite projective, on peut calculer la longueur d'un tel chemin comme la somme des longueurs de Hilbert de chacun de ses segments de photons. La distance de Kobayashi $K_\Omega(x,y)$ entre x et y est l'infimum des longueurs de chaînes de segments de photons reliant x à y dans Ω (voir la partie 6.4 pour plus de détails). Contrairement aux distances de Carathéodory, la définition de la distance de Kobayashi est spécifique aux espaces de Nagano de type réel, puisqu'elle requiert l'existence de photons satisfaisant certaines conditions d'invariance (lemmes 6.3.6 et 6.3.7) et d'abondance (observation 6.4.2). Nous démontrons alors le résultat suivant :

Théorème 1.4.6 (voir la proposition 6.4.8 et le corollaire 6.4.12)**.** *Soit G/P un espace de Nagano irréductible de type réel et soit $\Omega \subset G/P$ un domaine propre. Alors K_Ω est une distance $\mathsf{Aut}(\Omega)$-invariante qui induit la topologie standard sur Ω. Si Ω est en outre dualement convexe, alors K_Ω est une distance propre et géodésique.*

Pour démontrer la seconde assertion du théorème 1.4.6, nous comparons la distance de Kobayashi aux distances de Carathéodory. Avec les notations de la partie 2.3.3, nous obtenons la proposition suivante, dont la preuve est contenue dans la proposition 6.4.10 et le corollaire 6.4.12 :

Proposition 1.4.7. *[voir la proposition 6.4.10] Soit G/P un espace de Nagano irréductible de type réel et soit (V,ρ) une représentation linéaire réelle irréductible, proximale, de dimension finie de G de plus haut poids $\chi = N\omega_\alpha$, où $N \in \mathbb{N}^*$ et ω_α est le poids fondamental associé à α. Soit $\Omega \subset G/P$ un domaine propre dualement convexe et soit C^ρ_Ω la distance de Carathéodory sur Ω induite par (V,ρ) (voir l'équation (1.1.2)). Alors on a :*

$$K_\Omega \geq \frac{1}{N} C^\rho_\Omega.$$

En particulier, la distance K_Ω est propre.

Dans la partie 6.4.7, nous montrons que dans une réalisation Ω de $\mathbb{X}(\mathrm{Ein}^{p,q})$ dans $\mathrm{Ein}^{p,q}$ (resp. de $\mathbb{X}(\mathrm{Gr}_p(\mathbb{R}^{p+q}))$ dans $\mathrm{Gr}_p(\mathbb{R}^{p+q})$), tout couple de points peut être relié par une 2-chaîne (resp. une $\min(p,q)$-chaîne) géodésique pour K_Ω (voir les propositions 6.4.13 et 6.4.15) et on a alors égalité entre la distance de Kobayashi et toutes les distances de Carathéodory sur Ω. Ce phénomène est commun aux espaces de Nagano de type réel, voir [Gal25]. Nous pensons qu'il caractérise les réalisations du dual non compact dès que l'espace symétrique riemannien compact G/P est de rang supérieur :

Conjecture 1.4.8. Soit G/P un espace de Nagano irréductible de type réel et de rang $s \geq 2$. Soit $\Omega \subset G/P$ une réalisation de $\mathbb{X}(G/P)$ et (V,ρ) une représentation linéaire réelle irréductible proximale de dimension finie de G de plus haut poids $\chi = N\omega_\alpha$. Si $\Omega' \subset \mathscr{F}(\mathfrak{g},\alpha)$ est un domaine propre dualement convexe tel que $K_{\Omega'} = \frac{1}{N}C^\rho_{\Omega'}$, alors Ω' est une réalisation de $\mathbb{X}(G/P)$.

1.4.4 Rigidité des convexes divisibles dans les variétés de drapeaux

Le chapitre 8 est consacré à la question 1.2.1. On remarque d'abord que les faits 1.2.2 et 1.2.3 peuvent être étendus aux domaines propres presque-homogènes, dans les lemmes 8.1.1 et 8.1.2. Ensuite, on s'attarde sur les espaces de Nagano de type réel, puis sur nos trois exemples clés de familles d'espaces de Nagano de type réel : les grassmanniennes, les variétés de drapeaux causales et les univers d'Einstein. Dans les théorèmes 1.4.14 et 1.4.17, que l'on va énoncer dans les prochains paragraphes 1.4.4.3 et 1.4.4.4, on montre que tout domaine propre presque-homogène de G/P est symétrique, pour G/P une variété causale ou un univers d'Einstein de signature supérieure. Puisque tout domaine propre divisible dans une variété de drapeaux est presque-homogène, les théorèmes 1.4.14 et 1.4.14 fournissent une réponse positive à la question 1.2.1 pour les variétés de drapeaux considérées. Ils impliquent également que, réciproquement, tout domaine propre presque-homogène de G/P ($= \mathbf{Sb}(\mathfrak{g})$ ou $\mathrm{Ein}^{p,q}$) est divisible, ce qui est faux dans l'espace projectif réel, comme mentionné dans le paragraphe 1.1.1, où les trois notions de divisibilité, de quasi-homogénéité et de presque-homogénéité ne sont pas équivalentes.

Dans la partie 8.9, on donne une interprétation des théorèmes 1.4.9, 1.4.14 et 1.4.17 en termes de rigidité de rang supérieur, ce qui nous pousse à préciser la question 1.2.1 (voir la Conjecture 8.9.1).

1.4.4.1 Non-hyperbolicité. Un célèbre résultat de géométrie projective convexe, dû à Benoist [Ben01], est le suivant : si $\Gamma \leq \mathrm{PGL}(n,\mathbb{R})$ divise un ouvert proprement convexe $\Omega \subset \mathbb{P}(\mathbb{R}^n)$, alors les trois assertions suivantes sont équivalentes :

1. Le groupe Γ est Gromov-hyperbolique.
2. L'ouvert Ω est strictement convexe.
3. La distance de Hilbert sur Ω est Gromov-hyperbolique.

Ce comportement hyperbolique est typiquement de rang un, au sens où Ω hérite de certaines propriétés de l'espace hyperbolique réel $\mathbb{H}^n$, qui est la réalisation du dual non compact de $\mathbb{P}(\mathbb{R}^{n+1})$. Or on sait que les seuls espaces de Nagano de type réel dont la réalisation du dual non compact est de rang réel un sont les espaces projectifs réels et leurs duaux ; voir le tableau 8.2. On peut donc s'attendre, au vu de la question 1.2.1, à ce que ce comportement ne soit plus possible pour les espaces de Nagano de type réel de rang supérieur. Dans [Zim18b], A. Zimmer démontre que si l'on remplace l'espace projectif réel par la grassmannienne $\mathrm{Gr}_p(\mathbb{R}^{p+q})$ avec $2 \leq p \leq n-2$, l'espace géodésique (Ω, K_Ω) ne peut pas être Gromov-hyperbolique. Avec le formalisme introduit sur les espaces de Nagano, on peut généraliser son résultat aux espaces de Nagano de type réel :

Théorème 1.4.9 (voir le théorème 8.2.2)**.** *Soit G/P un espace de Nagano irréductible de type réel de rang supérieur. Si $\Omega \subset G/P$ un domaine propre presque-homogène muni de sa distance de Kobayashi K_Ω, alors l'espace métrique géodésique (Ω, K_Ω) n'est pas Gromov-hyperbolique.*

Un corollaire du théorème 1.4.9, qui va dans le sens d'une réponse affirmative à la question 1.2.1, est le suivant :

Corollaire 1.4.10 (voir le corollaire 8.2.3)**.** *Soient G/P un espace de Nagano irréductible de type réel de rang supérieur et $\Gamma \leq G$ un sous-groupe discret. Supposons que Γ divise un domaine propre de G/P. Alors Γ n'est pas Gromov-hyperbolique.*

1.4.4.2 Rigidité dans les grassmanniennes. Soit $\phi_{p,q}$ la forme quadratique standard de signature (p, q) sur $\mathbb{R}^{p+q}$ et soit $\mathbb{B}$ l'ensemble des p-plans de $\mathbb{R}^{p+q}$ qui sont définis positifs pour $\phi_{p,q}$. Alors $\mathbb{B}$ est une réalisation de $\mathbb{X}(\mathrm{Gr}_p(\mathbb{R}^{p+q})) = \mathrm{PO}(p, q)/\,\mathrm{P}(\mathrm{O}(p) \times \mathrm{O}(q))$ dans $\mathrm{Gr}_p(\mathbb{R}^{p+q})$ (voir la partie 3.3.1). Limbeek–Zimmer ont montré :

Fait 1.4.11 ([LZ19])**.** *Tout domaine divisible, convexe et borné dans une carte affine de $\mathrm{Gr}_p(\mathbb{R}^{2p})$, est une réalisation de $\mathbb{X}(\mathrm{Gr}_p(\mathbb{R}^{2p}))$, c'est-à-dire s'écrit $g \cdot \mathbb{B}_{p,p}$, avec $g \in \mathrm{PGL}(2p, \mathbb{R})$.*

Au cours de leur démonstration, ils démontrent que l'algèbre engendrée par le centralisateur dans $\mathrm{PGL}(p + q, \mathbb{R})$ d'un groupe divisant un domaine propre de $\mathrm{Gr}_p(\mathbb{R}^{p+q})$ (avec $p, q \in \mathbb{N}$) se décompose en une somme de sous-algèbres de dimension 1 [LZ19, Thm 9.3]. Ce résultat est bien connu dans le cas projectif [Vey70]. En nous basant sur ce résultat, nous menons l'étude du centralisateur d'un groupe discret divisant un domaine propre de $\mathrm{Gr}_p(\mathbb{R}^{p+q})$:

Théorème 1.4.12 (voir le théorème 8.5.1)**.** *Soit $2 \leq p \leq q$. Soit $\Gamma \leq \mathrm{PGL}(p + q, \mathbb{R})$ un sous-groupe discret, agissant cocompactement sur un domaine propre $\Omega \subset \mathrm{Gr}_p(\mathbb{R}^{p+q})$ dont le bord est une hypersurface topologique de $\mathrm{Gr}_p(\mathbb{R}^{p+q})$. Alors toute décomposition Γ-invariante de $\mathbb{R}^{p+q}$ est triviale.*

Le théorème 1.4.12 s'applique en particulier lorsque le domaine Ω est proprement convexe dans une carte affine.

Lorsque $p = 1$, on retrouve $\mathrm{Gr}_p(\mathbb{R}^{p+q}) = \mathbb{P}(\mathbb{R}^{q+1})$. Si $\mathbb{R}^{q+1} = V_1 \oplus V_2$ est une décomposition non triviale et si $\Omega_1 \subset \mathbb{P}(V_1)$ et $\Omega_2 \subset \mathbb{P}(V_2)$ sont deux domaines proprement convexes dans une même carte affine, alors on peut construire un nouvel ouvert proprement convexe $\mathsf{joint}(\Omega_1, \Omega_2)$ de $\mathbb{P}(\mathbb{R}^{q+1})$, appelé *joint* de Ω_1 et Ω_2 : on relève Ω_1 et Ω_2 en deux cônes ouverts proprement convexes C_1, C_2 d'un même demi-espace de $\mathbb{R}^{q+1}$. Alors $\mathsf{joint}(\Omega_1, \Omega_2) := \mathbb{P}(C_1 + C_2)$. Topologiquement, il s'agit du produit $\Omega_1 \times \Omega_2 \times \mathbb{R}$. Si Ω_1 et Ω_2 sont divisibles, alors $\mathsf{joint}(\Omega_1, \Omega_2)$ l'est aussi, par un groupe qui préserve la décomposition $\mathbb{R}^{q+1} = V_1 \oplus V_2$.

Dans le cas où $p \geq 2$, une opération analogue est impossible. Ceci est dû au fait que, lorsque $p, q \geq 2$ et $\dim(V_1), \dim(V_2) > p$, on a

$$\dim(\mathrm{Gr}_p(\mathbb{R}^{p+q})) > \dim(\mathrm{Gr}_p(V_1)) + \dim(\mathrm{Gr}_p(V_2)) + 1. \tag{1.4.1}$$

Ainsi, le joint $\Omega_1 \times \Omega_2 \times \mathbb{R}$ ne peut pas être un ouvert de $\mathrm{Gr}_p(\mathbb{R}^{p+q})$.

Dans le cadre de la preuve du théorème 1.4.12, cette contradiction peut être formalisée par la *dimension cohomologique* : si Γ est un sous-groupe discret de $\mathrm{PGL}(p+q, \mathbb{R})$ agissant cocompactement sur un domaine propre $\Omega \subset \mathrm{Gr}_p(\mathbb{R}^{p+q})$, alors il ne peut pas agir proprement discontinûment sur un *joint* de deux convexes $\Omega_1 \subset \mathrm{Gr}_p(V_1)$ et de $\Omega_2 \subset \mathrm{Gr}_p(V_2)$, c'est-à-dire, topologiquement, un produit $\Omega_1 \times \Omega_2 \times \mathbb{R}$. La preuve du théorème 1.4.12 consiste donc à construire, étant donnée une décomposition Γ-invariante non triviale de $\mathbb{R}^{p+q}$, deux domaines propres de grassmanniennes tels que Γ agisse proprement discontinument sur $\Omega_1 \times \Omega_2 \times \mathbb{R}$, pour obtenir $\dim(\Omega) = \dim(\Omega_1) + \dim(\Omega_2) + 1$, en contradiction avec l'équation (1.4.1).

Le corollaire suivant découle alors du théorème 1.4.12 et de [LZ19, Thm 9.3] :

Corollaire 1.4.13 (voir le corollaire 8.5.2)**.** *Soit $2 \leq p \leq q$. Soit $\Omega \subset \mathrm{Gr}_p(\mathbb{R}^{p+q})$ un domaine propre dont le bord est une hypersurface topologique de $\mathrm{Gr}_p(\mathbb{R}^{p+q})$. Supposons qu'il existe un sous-groupe discret $\Gamma \leq \mathrm{PGL}(p+q, \mathbb{R})$ agissant cocompactement sur Ω. Alors le centralisateur de Γ dans $\mathrm{PGL}(p+q, \mathbb{R})$ est fini.*

Le théorème 1.4.12 et le corollaire 1.4.13 vont dans le sens d'une réponse affirmative à la question 1.2.1, puisqu'ils expriment une perte de flexibilité : si l'on peut joindre deux convexes divisibles dans l'espace projectif réel pour en obtenir un nouveau dans un espace projectif réel plus grand, ce procédé n'est plus possible dans les grassmanniennes supérieures.

1.4.4.3 Rigidité dans les variétés Θ-positives. Les résultats principaux énoncés dans cette partie sont le théorème 1.4.14 et le corollaire 1.4.16, dont les preuves sont données dans la partie 8.3.

1.4.4.3.1 Rigidité dans les variétés causales. Les diamants dans les variétés causales ont été définis dans le paragraphe 1.4.1. Ce sont en fait des réalisations de $\mathbb{X}(\mathbf{Sb}(\mathfrak{g}))$ dans l'espace de Nagano de type réel $\mathbf{Sb}(\mathfrak{g})$. On répond positivement à la question 1.2.1 pour $G/P = \mathbf{Sb}(\mathfrak{g})$:

Théorème 1.4.14 (voir le théorème 8.3.1)**.** *Soit G un groupe de Lie simple de hermitien de type tube. Alors tout domaine propre presque-homogène de $\mathbf{Sb}(\mathfrak{g})$ est un diamant.*

Tout domaine propre $\Omega \subset \mathbf{Sb}(\mathfrak{g})$ hérite d'une structure causale issue de celle de $\mathbf{Sb}(\mathfrak{g})$. Une généralisation du théorème classique de Liouville implique que $\mathsf{Aut}(\Omega)$ est commensurable au *groupe conforme* de Ω, c'est-à-dire au groupe des difféomorphismes $f : \Omega \to \Omega$ tels que $d_x f(c_x) = c_{f(x)}$ pour tout $x \in \Omega$ [Kan11] (avec les notations du paragraphe 1.1.2.4). Le théorème 1.4.14 affirme donc que la presque-homogénéité du groupe conforme caractérise les diamants parmi les domaines propres de $\mathbf{Sb}(\mathfrak{g})$.

Plus généralement, le resultat suivant découle directement du Lemme 8.1.1 et du théorème 1.4.14 :

Corollaire 1.4.15. *Soit G un groupe de Lie semi-simple de type Hermitien de type tube, avec centre trivial et sans facteur compact. Écrivons $G = G_1 \times \cdots \times G_k$, où chaque G_i est un groupe de Lie simple non compact de type Hermitien de type tube pour tout $1 \leq i \leq k$. Alors, pour tout domaine propre presque homogène $\Omega \subset \mathbf{Sb}(\mathfrak{g})$, il existe des diamants $D_i \subset \mathbf{Sb}(\mathfrak{g}_i)$ pour $1 \leq i \leq k$ tels que $\Omega = D_1 \times \cdots \times D_k \subset \mathbf{Sb}(G_1) \times \cdots \times \mathbf{Sb}(G_k)$.*

1.4.4.3.2 Structures Θ-positives. La positivité totale est connue et étudiée depuis le début du XXe siècle pour $\mathrm{SL}(N, \mathbb{R})$. Elle a été généralisée aux groupes de Lie semi-simples réels déployés par Lusztig [Lus94]. Par ailleurs, il était connu que les groupes d'isométries des espaces symétriques hermitiens de type tube admettent une structure causale (voir par exemple [Kan06]).

Guichard–Wienhard ont généralisé ces deux notions de positivité totale et de causalité avec leur notion de *structure Θ-positive*, où Θ est un ensemble de racines simples restreintes d'un groupe de Lie semi-simple réel G. Ils ont classifié tous les couples (G, Θ) tels que G admette une structure Θ-positive [GW18, GW25]. Dans leur liste, les couples $(G, \{\alpha_r\})$, où G est un groupe de Lie HTT de rang r, constituent la seule famille où Θ est un singleton (c'est-à-dire où le sous-groupe parabolique propre défini par Θ est maximal). Ainsi, le théorème 1.4.14 et le Lemme 8.1.2 complètent la classification des domaines propres presque-homogènes dans les variétés de drapeaux Θ-positives :

Corollaire 1.4.16 (voir le corollaire 8.3.4)**.** *Soit G un groupe de Lie simple réel non compact et soit Θ un sous-ensemble des racines simples restreintes de G tel que G admette une structure Θ-positive. Alors on a la dichotomie suivante :*

1. Si $|\Theta| = 1$, alors G est hermitien de type tube et $G/P_\Theta = \mathbf{Sb}(\mathfrak{g})$ admet exactement un domaine propre presque-homogène à conjugaison par G près, qui est un diamant.

2. Si $|\Theta| \geq 2$, alors il n'existe aucun domaine propre presque-homogène dans G/P_Θ.

Ici, on a noté P_Θ le sous-groupe parabolique de G défini par Θ, avec la convention que P_Θ est minimal si et seulement si Θ est l'ensemble de toutes les racines simples de G. Par le Lemme 8.1.1, la question 1.2.1 admet une réponse positive pour les variétés de drapeaux G/P_Θ munies d'une structure Θ-positive, où G est un groupe de Lie semi-simple (pas nécessairement simple) non compact et Θ un sous-ensemble des racines simples de G.

1.4.4.4 L'univers d'Einstein. Un *diamant* dans $\mathrm{Ein}^{p,q}$ est une réalisation de $\mathbb{X}(\mathrm{Ein}^{p,q}) = \mathbb{H}^p \times \mathbb{H}^q$ dans $\mathrm{Ein}^{p,q}$. Nous donnons une construction de ces domaines dans la partie 3.4.2 (voir aussi [Tro24]). Dans un travail en collaboration avec Adam Chalumeau [CG24], nous donnons une réponse positive à la question 1.2.1 pour $G/P = \mathrm{Ein}^{p,q}$:

Théorème 1.4.17 (avec Chalumeau, voir le théorème 8.4.1)**.** *Tout domaine propre presque-homogène de $\mathrm{Ein}^{p,q}$ est un diamant.*

L'isomorphisme exceptionnel $\mathfrak{so}(3,3) \simeq \mathfrak{sl}(4,\mathbb{R})$ et la trialité dans $\mathfrak{so}(4,4)$ donnent alors :

Corollaire 1.4.18 (avec Chalumeau, voir le corollaire 8.4.4)**.** (1) *Soit $\Omega \subset \mathrm{Gr}_2(\mathbb{R}^4)$ un domaine propre presque-homogène. Alors Ω est une réalisation de $\mathbb{X}(\mathrm{Gr}_2(\mathbb{R}^4))$. Autrement dit, il existe $g \in \mathrm{PGL}(4,\mathbb{R})$ tel que $\Omega = g \cdot \mathbb{B}_{2,2}$.*

(2) *Soit $\mathscr{F}$ l'une des deux composantes connexes de l'espace des sous-espaces totalement isotropes maximaux de $\mathbb{R}^{4,4}$. Soit $\Omega \subset \mathscr{F}$ un domaine propre presque homogène. Alors Ω est une réalisation de $\mathbb{X}(\mathscr{F}(\mathfrak{g}, \alpha))$. En particulier,* $\mathsf{Aut}(\Omega)$ *est conjugué à* $\mathrm{SO}(3,1) \times \mathrm{SO}(1,3)$ *dans* $\mathrm{SO}(4,4)$.

Le corollaire 1.4.18.(1) renforce le fait 1.4.11 pour $p = 2$, en remplaçant l'hypothèse de divisibilité par celle, plus faible, de presque-homogénéité, et en supprimant l'hypothèse de convexité dans une carte affine.

1.4.4.5 Une application aux $(G, G/P)$-structures. Étant donnée une (G, X)-variété M, il existe une application du revêtement universel $\widetilde{M}$ de M dans X, appelée *développante*, construite par recollement d'images de cartes de M. Cette développante est unique à multiplication par un élément de G près.

Une $(G, G/P)$-variété M est dite *propre* si l'image de sa développante est propre dans G/P. Dans l'espace projectif réel $\mathbb{P}(\mathbb{R}^n)$, le quotient d'un convexe divisible non symétrique Ω par un sous-groupe discret Γ de $\mathrm{PGL}(n, \mathbb{R})$ le divisant fournit une $(\mathrm{PGL}(n, \mathbb{R}), \mathbb{P}(\mathbb{R}^n))$-variété compacte propre Ω/Γ qui n'est pas difféomorphe de manière G-équivariante à un quotient compact de l'espace symétrique riemannien d'un groupe de Lie non compact. Dans les variétés de drapeaux où la rigidité a pu être observée, la situation est différente : A. Zimmer démontre, en conséquence de son théorème énoncé dans le fait 1.2.3, que si G est un groupe de Lie semi-simple réel et P est un sous-groupe parabolique non maximal, alors il n'existe pas de $(G, G/P)$-variétés propre [Zim18a]. Les cas énoncé dans cette partie fournissent aussi des résultats de classification sur les $(G, G/P)$-variétés propres. Le résultat suivant est un corollaire du théorème 1.4.14 :

Corollaire 1.4.19 (voir le corollaire 8.8.5)**.** *Soit G un groupe de Lie simple de type Hermitien de type tube et soit M une $(G, \mathbf{Sb}(\mathfrak{g}))$-variété compacte connexe propre. Alors, la variété M s'identifie, en tant que $(G, \mathbf{Sb}(\mathfrak{g}))$-variété, à un quotient D/Γ, où D est un diamant de* $\mathbf{Sb}(\mathfrak{g})$ *et Γ est un réseau cocompact de* $\mathsf{Aut}(D)$. *Ainsi, la variété M est un revêtement fini de*

$$(\mathbb{X}_{L_s}/\Gamma') \times \mathbb{S}^1,$$

où $\mathbb{X}_{L_s}$ est l'espace symétrique riemannien de la partie semi-simple L_s d'un sous-groupe de Levi L de $P_{\{\alpha_r\}}$ et Γ' est un réseau cocompact de L_s.

Le résultat suivant est un corollaire du théorème 1.4.17 (voir la partie 2.4.3.3 pour des définitions) :

Corollaire 1.4.20 (avec Chalumeau, voir le corollaire 8.8.4)**.** *Soient $p, q \geq 2$ deux entiers et M une variété pseudo-riemannienne conformément plate de signature (p, q) (où p est le nombre de $+$ et q le nombre de $-$), propre, compacte et connexe. Alors M est conformément équivalente à un quotient D/Γ, où D est un diamant de* $\mathrm{Ein}^{p,q}$ *et* $\Gamma \leq \mathsf{Aut}(D)$ *est un réseau cocompact. Si de plus $1 \leq p < q$ avec $(p, q) \neq (2, 3)$, alors à revêtement fini près, la variété M est conformément équivalente à*

$$\Sigma^p \times (-\Sigma^q),$$

où Σ^p et Σ^q sont des variétés hyperboliques compactes de dimensions respectives p et q. En signature lorentzienne, c'est-à-dire pour $q = 1$, la variété M est (à un revêtement fini près) conformément équivalente au produit $\Sigma \times (-\mathbb{S}^1)$, où Σ est une variété hyperbolique compacte.

En particulier, toute $(G, G/P)$-variété compacte propre (avec G un HTT et $G/P = \mathbf{Sb}(\mathfrak{g})$, ou $G = \mathrm{SO}(p+1, q+1)$ et $G/P = \mathrm{Ein}^{p,q}$) est *kleinienne*, autrement dit, sa développante est un difféomorphisme sur son image. En général, il n'est même pas assuré que cette application soit un revêtement sur son image : le principe de déformation d'Ehresmann–Thurston fournit des structures lorentziennes conformes sur $M = \Sigma_g \times \mathbb{S}^1$, où Σ_g est une surface hyperbolique compacte de genre $g \geq 2$ et l'holonomie de M est non discrète dans $\mathrm{SO}(3, 2)$ (voir [Fra05, Sect. 10.3.4]).

1.4.5 Perspectives sur les espaces de Nagano

Les parties 6.5 et 8.9, concluant chacune un chapitre clé du mémoire, sont consacrées à des discussions sur les résultats du mémoire. En particulier, nous y abordons de possibles interprétations de la rigidité et généralisations de la distance de Kobayashi.

1.4.5.1 Rang et structure du bord. Dans le chapitre 5 et la partie 6.5, nous énonçons des résultats fondamentaux et bien connus sur les espaces de Nagano, dont plusieurs mettent en évidence un lien profond entre le rang de G/P et la structure des variétés de Schubert propres maximales (définies en (1.1.1)), voir par exemple l'observation 5.1.10 et le théorème 6.5.8.

Si G/P est un espace de Nagnano irréductible de type réel, alors, comme déjà mentionné, ce rang est 1 si et seulement si G/P est l'espace projectif réel ou son dual. Lorsque ce rang est ≥ 2, une contrainte géométrique forte apparaît sur la structure du bord d'un domaine propre presque-homogène (voir le théorème 7.2.6). Dans les cas examinés dans ce mémoire — c'est-à-dire ceux abordés dans les théorèmes 1.4.9, 1.4.12, 1.4.14 et 1.4.17 — cette contrainte induit de la rigidité pour ces domaines. Nous discutons cette observation dans la partie 8.9 et l'interprétons comme un phénomène de rang supérieur.

1.4.5.2 Sphères d'Helgason. Comme évoqué dans le paragraphe 1.4.3.1, les photons ne sont des objets naturels que dans les espaces de Nagano irréductibles *de type réel*, c'est-à-dire définis par une racine simple α de multiplicité 1 ; c'est uniquement dans ce cadre que leurs propriétés d'invariance sont satisfaites. Lorsque α est de multiplicité $k \geq 2$, cette invariance peut être retrouvée en considérant, à la place, des sphères de dimension k, connues sous le nom de *sphères d'Helgason* et introduites dans [Pet87]. Dans la partie 6.5.3, nous discutons une potentielle généralisation de la distance de Kobayashi aux espaces de Nagano (non nécessairement de type réels), en remplaçant les chaînes de photons par des chaînes de sphères d'Helgason. Une telle pseudo-distance devrait fournir des résultats semblables à ceux obtenus avec la distance de Kobayashi ; ces considérations font partie d'un projet en cours.

Chapter 2

Preliminaries

In this chapter, we set notation that will be used throughout this memoir. We recall some structural properties of semisimple Lie groups and flag manifolds. We illustrate these concepts through the three main examples of this thesis, namely Grassmannians, Einstein universes, and causal flag manifolds; see Section 2.4.

2.1 Some general reminders and notation

In this section, we provide some basic reminders on objects that will be used throughout this thesis and establish the corresponding notations.

2.1.1 Projective geometry

We start with reminders on real and complex projective spaces, cross ratios, and the Hilbert metric.

2.1.1.1 Real and complex projective space. Given a finite-dimensional real vector space V, we will denote by $[v]$ the projection in $\mathbb{P}(V)$ of a vector $v \in V \smallsetminus \{0\}$. In the case where $V = \mathbb{R}^2$, we denote by $[t_1 : t_2]$ the projection in $\mathbb{P}(\mathbb{R}^2)$ of a vector $(t_1, t_2) \in \mathbb{R}^2 \smallsetminus \{0\}$.

We denote by V^* the space of all linear forms on V. The space $\mathbb{P}(V^*)$ can be identified with the space of hyperplanes of V, or equivalently, the space of projective hyperplanes of $\mathbb{P}(V)$, via the maps

$$[f] \mapsto \ker(f) \text{ and } [f] \mapsto \mathbb{P}(\ker(f)). \tag{2.1.1}$$

An open subset of the form $\mathbb{P}(V) \smallsetminus \mathbb{P}(\ker(f))$ for some $[f] \in \mathbb{P}(V^*)$ is called an *affine chart* of $\mathbb{P}(V)$, and admits a canonical affine structure.

In Section 6.4.5, we will also be led to consider the complex projective spaces. Since we will consider both real and complex vector spaces, to avoid the confusion we denote by $\mathbb{P}_c(W)$ the complex projective space of a finite-dimensional complex vector space W, and by $[v]_c$ the projection in $\mathbb{P}_c(W)$ of a vector $v \in W \smallsetminus \{0\}$. We use the notation $[z_1 : z_2]_c$ for the projection in $\mathbb{P}_c(\mathbb{C}^2)$ of a vector $(z_1, z_2) \in \mathbb{C}^2 \smallsetminus \{0\}$.

2.1.1.2 Cross ratio and Hilbert metric of an interval. We denote by $(\cdot : \cdot : \cdot : \cdot)$ the classical cross ratio on $\mathbb{P}(\mathbb{R}^2)$. Recall that it is $\mathrm{SL}(2,\mathbb{R})$-invariant and satisfies $([1:0]:[1:1]:[1:t]:[0:1]) = t$.

If $I \subset \mathbb{P}(\mathbb{R}^2)$ is a proper open interval with (possibly equal) endpoints t_1 and t_2, then the *Hilbert pseudo-metric* on I is denoted by H_I and defined as follows: for any pair $s_1, s_2 \in I$ such that t_1, s_1, s_2, t_2 are aligned in this order (taking any order if $s_1 = s_2$ or $t_1 = t_2$), one has $H_I(s_1, s_2) := \log(t_1 : s_1 : s_2 : t_2)$. If $I = \mathbb{P}(\mathbb{R}^2)$, then H_I is by convention the constant map equal to 0 on I^2.

2.1.1.3 Convexity and the Hilbert metric. An open set $\Omega \subset \mathbb{P}(\mathbb{R}^n)$ is said to be *properly convex* if there exists an affine chart of $\mathbb{P}(\mathbb{R}^n)$ containing Ω as a bounded convex set. The *Hilbert metric* on Ω is then defined as follows: given two points $x, y \in \Omega$, there exists a projective line ℓ through x and y. The projective interval $I := \Omega \cap \ell$ admits a Hilbert metric, according to the previous section. Then, we define $\mathsf{H}_\Omega(x,y) := H_I(x,y)$.

The Hilbert metric on a properly convex open set of $\mathbb{P}(\mathbb{R}^n)$ is a proper, geodesic metric, with projective segments being geodesics, and it is invariant under the automorphism group

$$\{g \in \mathrm{PGL}(n,\mathbb{R}) \mid g \cdot \Omega = \Omega\}$$

of Ω. For a more in-depth description of the Hilbert metric, see for instance [DLH93, PT14, Gol22].

Remark 2.1.1. Similarly, one can define a Hilbert metric on a properly convex open set of any affine space $\mathbb{A}$, in an analogous way. If $\Omega \subset \mathbb{P}(\mathbb{R}^n)$ is properly convex, then its projective Hilbert metric coincides with its Hilbert metric in any affine chart that contains it as a bounded subset.

2.1.2 Signature

Let $n \in \mathbb{N}_{>0}$ and $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Given $v \in \mathbb{K}^n$, we will denote by $\overline{v}$ the vector whose i-th entry is the conjugate (in $\mathbb{K}$) of the i-th entry of v. Let $\mathbf{b}$ be a $\mathbb{K}$-hermitian form on $\mathbb{K}^n$, i.e.

$$\begin{aligned} \mathbf{b}(u,v) &= \overline{\mathbf{b}(v,u)} \quad \forall u,v \in \mathbb{K}^n \\ \mathbf{b}(\lambda u, v) &= \lambda \mathbf{b}(v,u) \quad \forall u,v \in \mathbb{K}^n,\ \lambda \in \mathbb{K}. \end{aligned}$$

If $\mathrm{Sub}(\mathbb{K}^n)$ is the set of all $\mathbb{K}$-vector subspaces of $\mathbb{K}^n$, then the quantities

$$\begin{aligned} p &:= \max\{\dim(V) \mid\ V \in \mathrm{Sub}(\mathbb{K}^n),\ \mathbf{b}(v,v) > 0 \quad \forall v \in V \setminus \{0\}\}; \\ q &:= \max\{\dim(V) \mid\ V \in \mathrm{Sub}(\mathbb{K}^n),\ \mathbf{b}(v,v) < 0 \quad \forall v \in V \setminus \{0\}\}, \end{aligned}$$

are well defined and satisfy $p+q \leq n$. *The signature* of $\mathbf{b}$ is the triple $\mathrm{sgn}(\mathbf{b}) = (p, q, n-p-q)$. Whenever $p + q = n$, we will simply denote this signature by (p,q) and say that $\mathbf{b}$ is *nondegenerate*.

If $\mathbb{K} = \mathbb{R}$ and $\mathbf{b}$ is nondegenerate, then the group $\mathrm{O}(\mathbf{b})$ (resp. $\mathrm{PO}(\mathbf{b})$) will be the subgroup of the elements of $\mathrm{GL}(n,\mathbb{R})$ (resp. $\mathrm{PGL}(\mathbf{b})$) preserving $\mathbf{b}$. The abstract Lie group isomorphic to $\mathrm{SO}(\mathbf{b})$ (resp. $\mathrm{PO}(\mathbf{b})$) is $\mathrm{SO}(p,q)$ (resp. $\mathrm{PO}(p,q)$). When there is no ambiguity, the space $\mathbb{R}^n$ endowed with $\mathbf{b}$ will be denoted by $\mathbb{R}^{p,q}$.

Given a matrix $X \in \mathrm{Mat}_n(\mathbb{K})$, we denote by $\overline{X}$ the matrix whose (i,j)-th entry is the conjugate (in $\mathbb{K}$) of the (i,j)-th entry of X. Let $X \in \mathrm{Mat}_n(\mathbb{K})$ be such that ${}^t\overline{X} = X$. Then the $\mathbb{K}$-hermitian form on $\mathbb{K}^n$ defined by

$$\mathbf{b}_X(u,v) = {}^t\overline{v}Xu$$

for all $u, v \in \mathbb{K}^n$ is uniquely defined by X, and the *signature* $\mathrm{sgn}(X)$ *of* X is by definition the signature of $\mathbf{b}_X$.

2.2 Preliminaries on Lie theory

In this section, we recall some well-known facts about semisimple Lie groups and fix notations that will hold for the rest of the memoir. Illustrative examples are given and described in detail in Section 2.4.

All the Lie groups and Lie algebras in this memoir are supposed to be linear. Given a semisimple Lie group G, we will always denote by $\mathfrak{g}$ its Lie algebra in this memoir. In this section, we fix a noncompact real semisimple Lie group G.

2.2.1 $\mathfrak{sl}_2$-triples

A triple $\mathsf{t} = (e,h,f)$ of nonzero elements of $\mathfrak{g}$ satisfying the equalities $[h,e] = 2e$, $[h,f] = -2f$ and $[e,f] = h$ is called an $\mathfrak{sl}_2$*-triple.* There is a Lie algebras embedding $\mathrm{j}_{\mathsf{t}} : \mathfrak{sl}_2(\mathbb{R}) \hookrightarrow \mathfrak{g}$ such that $\mathrm{j}_{\mathsf{t}}(\mathrm{E}) = e$, $\mathrm{j}_{\mathsf{t}}(\mathrm{H}) = h$ and $\mathrm{j}_{\mathsf{t}}(\mathrm{F}) = f$, where

$$\mathrm{E} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad \mathrm{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \mathrm{F} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

2.2.2 Cartan decomposition

Let B be the Killing form on $\mathfrak{g}$. Let $K \leq G$ be a maximal compact subgroup and $\mathfrak{h}$ be the B-orthogonal of the Lie algebra $\mathfrak{k}$ of K in $\mathfrak{g}$. Then one has $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{h}$. The *Cartan involution* of $\mathfrak{g}$ (with respect to K) is then the Lie algebra automorphism $\sigma_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g}$ defined by $(\sigma_{\mathfrak{g}})_{|\mathfrak{k}} = \mathrm{id}_{\mathfrak{k}}$ and $(\sigma_{\mathfrak{g}})_{|\mathfrak{h}} = -\mathrm{id}_{\mathfrak{h}}$. It induces a Lie group automorphism of G, denoted by σ_G and called the *Cartan involution of G*.

2.2.3 Restricted root system

Let $\mathfrak{a} \subset \mathfrak{h}$ be a maximal abelian subspace, and $\mathfrak{g}_0$ the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$. We denote by $\mathfrak{a}^*$ the space of all linear forms on $\mathfrak{a}$. For $\alpha \in \mathfrak{a}^*$, we define

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} \mid [H,X] = \alpha(H)X \quad \forall H \in \mathfrak{a}\}.$$

One has $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ for any $\alpha, \beta \in \mathfrak{a}^*$. If $\alpha \in \mathfrak{a}^* \smallsetminus \{0\}$ satisfies $\mathfrak{g}_\alpha \neq \{0\}$, then we say that α is *a restricted root* of $(\mathfrak{g}, \mathfrak{a})$. We denote by $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ the set of all restricted roots of $(\mathfrak{g}, \mathfrak{a})$. One has $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha$. We fix a *fundamental system* $\Delta = \{\alpha_1, \ldots, \alpha_N\} \subset \Sigma$, i.e. a family of restricted roots such that any root of $\mathfrak{g}$ can be uniquely written as $\alpha = \sum_{i=1}^{N} n_i \alpha_i$, where the n_i all have same sign for $1 \leq i \leq N$. The elements of Δ are called *simple restricted roots*. From now on, whenever we fix a real semisimple Lie algebra of noncompact $\mathfrak{g}$, it will always implicitly be endowed a fixed set Δ of simple restricted roots.

The choice of a fundamental system determines a set of *positive roots* Σ^+, i.e. those roots α where the n_i are all nonnegative.

For any $\alpha \in \Sigma$ and $X \in \mathfrak{g}_\alpha \smallsetminus \{0\}$, there exists a unique scalar multiple X' of X such that $(X', [\sigma_{\mathfrak{g}}(X'), X'], \sigma_{\mathfrak{g}}(-X'))$ is an $\mathfrak{sl}_2$-triple. The element $[\sigma_{\mathfrak{g}}(X'), X']$ does not depend on the choice of $X \in \mathfrak{g}_\alpha$, and is denoted by h_α. The family $(h_\alpha)_{\alpha \in \Delta}$, whose elements are called the *coroots of* $\mathfrak{g}$, forms a basis of $\mathfrak{a}$, whose dual basis in $\mathfrak{a}^*$ is denoted by $(\omega_\alpha)_{\alpha \in \Delta}$.

The *nonnegative Weyl chamber associated with* Δ is

$$\overline{\mathfrak{a}}^+ = \{X \in \mathfrak{a} \mid \alpha(X) \geq 0 \quad \forall \alpha \in \Delta\}.$$

For all $g \in G$, there exist $k, \ell \in K$, and a unique $\mu(g) \in \overline{\mathfrak{a}}^+$ such that $g = k \exp(\mu(g)) \ell$. This defines the *Cartan projection* $\mu : G \to \overline{\mathfrak{a}}^+$.

2.2.4 The restricted Weyl group

The *restricted Weyl group* W of G is the quotient $N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ of the normalizer of $\mathfrak{a}$ in K (for the adjoint action) by the centralizer of $\mathfrak{a}$ in K. For its natural embedding in $\mathrm{GL}(\mathfrak{a})$, it is a finite group generated by the B-orthogonal reflexions in $\mathfrak{a}$ with respect to the kernels of the simple restricted roots. By duality with respect to B (which induces a scalar product on $\mathfrak{a}$), the action of W on $\mathfrak{a}$ induces an action on $\mathfrak{a}^*$ preserving Σ. There exists a unique $w_0 \in W$, called the *longest element*, such that $w_0 \cdot \Sigma^+ = -\Sigma^+$. The element $\mathsf{i} : \mathfrak{a}^* \to \mathfrak{a}^*$ defined as $\mathsf{i} = -w_0$ is called the *opposition involution*, and satisfies $\mathsf{i}(\Delta) = \Delta$.

2.2.5 Parabolic subgroups

Let $\Theta \subset \Delta$ be a subset of the simple restricted roots. *The standard parabolic subgroup* P_Θ^+ (resp. the *standard opposite parabolic subgroup* P_Θ^-) is defined as the normalizer in G of the Lie algebra

$$\mathfrak{u}_\Theta^+ := \bigoplus_{\alpha \in \Sigma_\Theta^+} \mathfrak{g}_\alpha \quad \left(\text{resp. } \mathfrak{u}_\Theta^- := \bigoplus_{\alpha \in \Sigma_\Theta^+} \mathfrak{g}_{-\alpha}\right), \tag{2.2.1}$$

where $\Sigma_\Theta^+ := \Sigma^+ \smallsetminus \mathrm{Span}(\Delta \smallsetminus \Theta)$. By "standard", we mean with respect to the above choices. One has

$$\mathfrak{p}_\Theta^+ := \mathrm{Lie}(P_\Theta^+) = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \Sigma \smallsetminus \Sigma_\Theta^+} \mathfrak{g}_{-\alpha}. \tag{2.2.2}$$

The Lie algebra of P_Θ^- is denoted by $\mathfrak{p}_\Theta^-$.

For any representative $k_0 \in N_K(\mathfrak{a})$ of w_0, one has $k_0 P_\Theta^- k_0 = P_{\mathsf{i}(\Theta)}^+$.

More generally, a *parabolic subgroup of type* Θ of G is a conjugate of P_Θ^+ in G. A *Borel subgroup* is a conjugate of P_Δ^+ in G.

The *Levi subgroup* associated with Θ is the reductive Lie group defined as the intersection $L_\Theta := P_\Theta^+ \cap P_\Theta^-$. The unipotent radical of P_Θ^+ (resp. P_Θ^-) is $U_\Theta^+ := \exp(\mathfrak{u}_\Theta^+)$ (resp. $U_\Theta^- := \exp(\mathfrak{u}_\Theta^-)$). One then has $P_\Theta^+ = U_\Theta^+ \rtimes L_\Theta$ (resp. $P_\Theta^- = U_\Theta^- \rtimes L_\Theta$).

The group $S_\Theta := [L_\Theta, L_\Theta]$ is a real semisimple Lie group, not necessarily connected, and a Cartan subspace of S_Θ is $\mathfrak{a}_\Theta := \bigoplus_{\alpha \in \Delta \smallsetminus \Theta} \mathbb{R}\, h_\alpha$.

The *Weyl group with respect to* Θ, denoted by W_Θ, is the subgroup of W generated by the reflections s_α, with $\alpha \in \Theta$. Note that $W_{\Delta \smallsetminus \Theta}$ is the Weyl group of S_Θ.

If $\Theta = \Delta$, then P_Δ^+ is the standard Borel subgroup of G, and is contained in $P_{\Theta'}^+$ for all $\Theta' \subset \Delta$. The following equality then holds:

$$G = \bigsqcup_{w \in W} P_\Delta^+ w P_\Delta^+. \tag{2.2.3}$$

It is classically called the *Bruhat decomposition* of G.

2.2.6 Flag manifolds

A *flag manifold* is a smooth compact manifold M endowed with a transitive action of a noncompact semisimple Lie group G, such that the stabilizer of a point of M stabilizer of a point a parabolic subgroup P of G. In this case, there exists a subset Θ of the simpe restricted roots of G such that M is G-equivariantly diffeomorphic to G/P_Θ^+.

Two flag manifolds M, M' under semisimple Lie groups G, G' are said to be *equivalent* if there exists a Lie algebra isomorphism $\rho : \mathfrak{g} \to \mathfrak{g}'$ and a diffeomorphism $f : M \to M'$ such that $df \circ \mathrm{ad}(X) = \mathrm{ad}(\rho(X)) \circ df$ for all $X \in \mathfrak{g}$.

The coset space $\mathscr{F}(\mathfrak{g}, \Theta) := \mathfrak{g}/\mathfrak{p}_\Theta^+$, endowed with the natural action of G induced by Ad, is a flag manifold, called the *standard flag manifold* associated with $\mathfrak{g}$ and Θ. We then have a flag manifold equivalence

$$G/P_\Theta \simeq \mathscr{F}(\mathfrak{g}, \Theta) \quad (\text{resp. } G/P_\Theta^- \simeq \mathscr{F}(\mathfrak{g}, \Theta)^-). \tag{2.2.4}$$

We will simply denote by $g \cdot x$ the action of an element $g \in G$ on $x \in \mathscr{F}(\mathfrak{g}, \Theta)$ (instead of $\mathrm{Ad}(g) \cdot x$).

Equation (2.2.4) implies in particular that any flag manifold M is equivalent to the standard flag manifold of associated with some $\mathfrak{g}$ and Θ. We will thus always be able to assume that $M = \mathscr{F}(\mathfrak{g}, \Theta)$.

If $\mathsf{i}(\Theta) = \Theta$, then we say that Θ and $\mathscr{F}(\mathfrak{g}, \Theta)$ are *self-opposite*. For any representative $k_0 \in N_K(\mathfrak{a})$ of the longest element w_0, one has $\mathfrak{p}_\Theta^+ = k_0 \cdot \mathfrak{p}_\Theta^-$, and one has $\mathscr{F}(\mathfrak{g}, \Theta) = \mathscr{F}(\mathfrak{g}, \Theta)^-$.

Example 2.2.1. In this example, given some integer $N \in \mathbb{N}_{>0}$, we denote by ε_i the map $\varepsilon_i : \mathrm{diag}(\lambda_1, \ldots, \lambda_N) \mapsto \lambda_i$, for $1 \leq i \leq N$.

1. For $\mathfrak{g} = \mathfrak{sl}(p+q, \mathbb{R}) = \mathrm{Mat}_{p+q}(\mathbb{R})$, we fix the Cartan subspace

$$\mathfrak{a} := \Big\{ \mathrm{diag}(\lambda_1, \ldots, \lambda_{p+q}) \mid \lambda_i \in \mathbb{R} \quad \forall 1 \leq i \leq p+q,\ \sum_{i=1}^{p+q} \lambda_i = 0 \Big\}.$$

Then the associate root system of $\mathfrak{sl}(p+q,\mathbb{R})$ is $\Sigma = \{\pm(\varepsilon_i - \varepsilon_j) \mid 1 \leq i < j \leq p+q\}$. A fundamental system is then

$$\Delta := \{\alpha_i := \varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i \leq p+q-1\}.$$

If $P_p := P_{\{\alpha_p\}}$, then the flag manifold $\mathscr{F}(\mathfrak{g}, \alpha_p)$ identifies with the space $\mathrm{Gr}_p(\mathbb{R}^{p+q})$ of p-planes of $\mathbb{R}^{p+q}$, called the *Grassmannian of p-planes of* $\mathbb{R}^{p+q}$. The opposite flag manifold identifies with the Grassmannian $\mathscr{F}(\mathfrak{g}, \mathsf{i}(\alpha_p)) = \mathscr{F}(\mathfrak{g}, \alpha_q) = \mathrm{Gr}_q(\mathbb{R}^{p+q})$ of q-planes of $\mathbb{R}^{p+q}$. We give more details on these identifications in Section 2.4.2.

2. For $\mathfrak{g} = \mathfrak{so}(p+1, q+1) = \{X \in \mathrm{Mat}_{p+q+2}(\mathbb{R}) \mid {}^t XS + SX = 0\}$ with $0 \leq q \leq p$ (the case where $0 \leq p \leq q$ is symmetric) and

$$S = \begin{pmatrix} 0 & 0 & J_{p+1} \\ 0 & I_{q-p} & 0 \\ J_{p+1} & 0 & 0 \end{pmatrix},$$

where $J_k = (a_{ij})$ satisfies

$$a_{ij} = \begin{cases} 1 \text{ if } j = p+q+2-i+1; \\ 0 \text{ otherwise.} \end{cases}$$

We fix the Cartan subspace

$$\mathfrak{a} := \{\mathrm{diag}(\lambda_1, \ldots, \lambda_{p+1}, 0, \ldots, 0, -\lambda_{p+1}, \ldots, -\lambda_1) \mid \lambda_i \in \mathbb{R}\ \forall 1 \leq i \leq p+1\}.$$

The associate root system is

$$\Sigma := \begin{cases} \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq p+1\} \cup \{\pm\varepsilon_i \mid 1 \leq i \leq p\} & \text{if } p < q \\ \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq p+1\} & \text{if } p = q \end{cases}.$$

A fundamental system is $\Delta := \{\alpha_1, \ldots, \alpha_{p+1}\}$, with $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq p$, and

$$\alpha_{p+1} = \begin{cases} \varepsilon_{p+1} \text{ if } p < q; \\ \varepsilon_p + \varepsilon_{p+1} \text{ if } p = q. \end{cases}$$

The flag manifold $\mathscr{F}(\mathfrak{g}, \alpha_1)$ identifies with the space $\mathrm{Ein}^{p,q}$ of isotropic lines of $\mathbb{R}^{p+1,q+1}$, called the *the Einstein universe*, as we will see in Section 2.4.1.

2.2.6.1 The automorphism group. The group of all Lie algebra automorphisms of $\mathfrak{g}$ is called the *automorphism group* of $\mathfrak{g}$ and denoted by $\mathsf{Aut}(\mathfrak{g})$. It is a Lie group with Lie algebra $\mathfrak{g}$. When G is semisimple, the map $\mathrm{Ad} : G \to \mathsf{Aut}(\mathfrak{g})$ has finite kernel.

In general, the group $\mathsf{Aut}(\mathfrak{g})$ does not act on $\mathscr{F}(\mathfrak{g}, \Theta)$. However, it admits a finite-index subgroup that does: indeed, any $g \in \mathsf{Aut}(\mathfrak{g})$ induces an automorphism ψ_g of the fundamental system Δ. This defines a group homomorphism $\mathsf{Aut}(\mathfrak{g}) \to \mathsf{Aut}(\Delta)$. For $\Theta \subset \Delta$, we denote by $\mathsf{Aut}_\Theta(\mathfrak{g})$ the subgroup of $\mathsf{Aut}(\mathfrak{g})$ of all Lie algebra automorphisms g such that ψ_g fixes Θ. This group contains the kernel $\mathsf{Aut}_1(\mathfrak{g})$ of f, which itself contains $\mathrm{Ad}(G)$. The group $\mathsf{Aut}_\Theta(\mathfrak{g})$ acts on $\mathscr{F}(\mathfrak{g}, \Theta)$. In particular $\ker(\mathrm{Ad})$ acts trivially on $\mathscr{F}(\mathfrak{g}, \Theta)$.

Definition 2.2.2. We denote by $\mathscr{G}_\Theta(\mathfrak{g})$ the set of finite-index subgroups of $\mathsf{Aut}_\Theta(\mathfrak{g})$.

Since $\ker(\mathrm{Ad})$ acts trivially on $\mathscr{F}(\mathfrak{g}, \Theta)$, we will always be able to assume that $G \in \mathscr{G}_\Theta(\mathfrak{g})$, identifying it with its image under Ad.

2.2.6.2 Transversality. The action of G on $\mathscr{F}(\mathfrak{g},\Theta)\times\mathscr{F}(\mathfrak{g},\Theta)^-$ by left translations has exactly one open orbit $\mathscr{O}$, which is the orbit of $(\mathfrak{p}_\Theta^+,\mathfrak{p}_\Theta^-)$ and is dense. Two elements $x\in\mathscr{F}(\mathfrak{g},\Theta)$ and $y\in\mathscr{F}(\mathfrak{g},\Theta)^-$ are said to be *transverse* if $(x,y)\in\mathscr{O}$.

If Θ is self-opposite, then a subset F of $\mathscr{F}(\mathfrak{g},\Theta)$ is said to be *transverse* if any pair of distinct points of F is transverse.

Given a point $y\in\mathscr{F}(\mathfrak{g},\Theta)^-$ (resp. $x\in\mathscr{F}(\mathfrak{g},\Theta)$), we let Z_y (resp. Z_x) be the set of all elements of $\mathscr{F}(\mathfrak{g},\Theta)$ (resp. $\mathscr{F}(\mathfrak{g},\Theta)^-$) that are not transverse to y (resp. to x):

$$\begin{aligned}\mathrm{Z}_y &:= \{z\in\mathscr{F}(\mathfrak{g},\Theta)\mid (z,y)\notin\mathscr{O}\};\\ \mathrm{Z}_x &:= \{z'\in\mathscr{F}(\mathfrak{g},\Theta)\mid (x,z')\notin\mathscr{O}\}.\end{aligned}\tag{2.2.5}$$

It defines an algebraic hypersurface of $\mathscr{F}(\mathfrak{g},\Theta)$ (resp. of $\mathscr{F}(\mathfrak{g},\Theta)^-$), called a *maximal proper Schubert subvariety*. The space $\mathscr{F}(\mathfrak{g},\Theta)\smallsetminus\mathrm{Z}_y$ is called an *affine chart* (or more classically a *big Schubert cell*) and is an open dense subset of $\mathscr{F}(\mathfrak{g},\Theta)$. The affine chart

$$\mathbb{A}_{\mathsf{std}} := \mathscr{F}(\mathfrak{g},\Theta)\smallsetminus\mathrm{Z}_{\mathfrak{p}_\Theta^-}\tag{2.2.6}$$

is called the *standard affine chart*. The bijection

$$\begin{array}{rccc}\varphi_{\mathsf{std}}: & \mathfrak{u}_\Theta^- & \xrightarrow{\sim} & \mathbb{A}_{\mathsf{std}}\\ & X & \longmapsto & \exp(X)\cdot\mathfrak{p}_\Theta^+\end{array}\tag{2.2.7}$$

induces an affine structure on $\mathbb{A}_{\mathsf{std}}$. Since G acts transitively on $\mathscr{F}(\mathfrak{g},\Theta)^-$, any affine chart $\mathscr{F}(\mathfrak{g},\Theta)\smallsetminus\mathrm{Z}_y$ with $y\in\mathscr{F}(\mathfrak{g},\Theta)^-$ admits an affine structure, which moreover depends only on y (and not on the choice of $g\in G$ such that $y=g\cdot\mathfrak{p}_\Theta^-$).

Example 2.2.3. Take the notation of Example 2.2.1.(1). If $y\in\mathrm{Gr}_q(\mathbb{R}^{p+q})$ is a q-plane of $\mathbb{R}^{p+q}$, then in the identification $\mathscr{F}(\mathfrak{sl}(p+q,\mathbb{R}),\alpha_p)\simeq\mathrm{Gr}_p(\mathbb{R}^{p+q})$, we have

$$\mathrm{Z}_y = \{x\in\mathrm{Gr}_p(\mathbb{R}^{p+q})\mid x\cap y\neq\{0\}\}.$$

2.2.6.3 Incidence. Let $\Theta,\Theta'\subset\Delta$. Any element $(x,y)\in\mathscr{F}(\mathfrak{g},\Theta)\times\mathscr{F}(\mathfrak{g},\Theta')$ can be written $(x,y)=(g\cdot\mathfrak{p}_\Theta^+,\ gw\cdot\mathfrak{p}_{\Theta'})$, with $g\in G$ and $w\in W$. Let

$$\mathsf{pos}^{(\Theta,\Theta')}:\mathscr{F}(\mathfrak{g},\Theta)\times\mathscr{F}(\mathfrak{g},\Theta')\to W_{\Delta\smallsetminus\Theta}\backslash W/W_{\Delta\smallsetminus\Theta'}$$

be the diag(G)-invariant map such that $\mathsf{pos}^{(\Theta,\Theta')}(\mathfrak{p}_\Theta^+,w\cdot\mathfrak{p}_{\Theta'}^+)=\overline{w}$ for all $w\in W$. This map only depends on the Lie algebra $\mathfrak{g}$ of G. Given a point $x\in\mathscr{F}(\mathfrak{g},\Theta)$ and an element $\overline{w}\in W_{\Delta\smallsetminus\Theta}\backslash W/W_{\Delta\smallsetminus\Theta'}$, let

$$\mathsf{C}_{\overline{w}}^{(\Theta,\Theta')}(x)=\{x'\in\mathscr{F}(\mathfrak{g},\Theta')\mid \mathsf{pos}^{(\Theta,\Theta')}(x,x')=\overline{w}\}.$$

The following fact follows from the definitions:

Fact 2.2.4. *Let $w,w'\in W$. If $\mathsf{C}_{\overline{w}}^{(\Theta,\Theta')}(x)\cap\mathsf{C}_{\overline{w}'}^{(\Theta,\Theta')}(x)\neq\emptyset$, then $\overline{w}=\overline{w}'$.*

There is a partial order on $W_{\Delta\smallsetminus\Theta}\backslash W/W_{\Delta\smallsetminus\Theta'}$, defined in the following way: given two elements $w, w' \in W$, we have

$$\overline{w} \leq \overline{w}' \iff \mathsf{C}_{\overline{w}}^{(\Theta,\Theta')}(x) \subset \overline{\mathsf{C}_{\overline{w}'}^{(\Theta,\Theta')}(x)} \text{ for some (hence any) } x \in \mathscr{F}(\mathfrak{g},\Theta).$$

Note that this partial order admits a maximum, which is $\overline{w_0}$.

In the case where $\Theta' = \mathsf{i}(\Theta)$, and $y \in \mathscr{F}(\mathfrak{g},\Theta)^-$, the sets $\overline{\mathsf{C}_{\overline{w}}^{(\Theta,\mathsf{i}(\Theta))}(y)}$, for $w \in W$, are called *Schubert cells defined by* y. Note that the maximal proper Schubert subvariety defined by y (in the notation of Section 2.2.6.2) is then the union of the closures of all the Schubert cells of the form $\overline{\mathsf{C}_{\overline{w}}^{(\Theta,\mathsf{i}(\Theta))}(y)}$, where $w \in W$ satisfies $\overline{w} \neq \overline{w_0}$. A point $x \in \mathscr{F}(\mathfrak{g},\Theta)$ is transverse to y if and only if one has $w_0 \in \mathsf{pos}^{(\Theta,\mathsf{i}(\Theta))}(x,y)$.

Example 2.2.5. With the notation of Example 2.2.1.(1), if $\mathfrak{g} = \mathfrak{sl}(p+q,\mathbb{R})$, if $1 \leq p' \leq p+q$, then for all $x \in \mathrm{Gr}_p(\mathbb{R}^{p+q})$ and $w \in W$, the integer

$$p - \dim(y \cap x), \quad y \in \mathsf{C}_{\overline{w}}^{(\{\alpha_p\},\{\alpha_{p'}\})}(x)$$

is constant on $\mathsf{C}_{\overline{w}}^{(\{\alpha_p\},\{\alpha_{p'}\})}(x)$. It is minimal if and only if $\overline{w} = \overline{w_0}$, and maximal if and only if $\overline{w} = \overline{\mathrm{id}}$.

Lemma 2.2.6. *1. Let $x, y \in \mathscr{F}(\mathfrak{g},\Theta)^-$ be a triple such that $w_0 \in \mathsf{pos}^{(\mathsf{i}(\Theta),\mathsf{i}(\Theta))}(x,y)$. Then there exists $z \in \mathscr{F}(\mathfrak{g},\Theta) \smallsetminus \mathrm{Z}_y$ such that $\mathrm{id} \in \mathsf{pos}^{(\Theta,\mathsf{i}(\Theta))}(z,x)$.*

2. Let (x,y,z) be a triple of $\mathscr{F}(\mathfrak{g},\Theta)^2 \times \mathscr{F}(\mathfrak{g},\Theta)^-$ such that $\mathrm{id} \in \mathsf{pos}^{(\Theta,\mathsf{i}(\Theta))}(x,z)$ and $w_0 \in \mathsf{pos}^{(\Theta,\mathsf{i}(\Theta))}(y,z)$. Then $w_0 \in \mathsf{pos}^{(\Theta,\Theta)}(x,y)$.

Proof. 1) Since $\mathsf{pos}^{(\mathsf{i}(\Theta),\mathsf{i}(\Theta))}(x,y) = \overline{w_0}$, we may assume that $x = \mathfrak{p}_\Theta^-$ and $y = w_0 \cdot \mathfrak{p}_\Theta^-$. Let us set $z := w_0 \cdot \mathfrak{p}_\Theta^+$. Then z is transverse to y, and by G-invariance of pos, we have

$$\mathsf{pos}^{(\Theta,\mathsf{i}(\Theta))}(z,x) = \mathsf{pos}^{(\Theta,\mathsf{i}(\Theta))}(\mathfrak{p}_\Theta^+, w_0 \cdot \mathfrak{p}_\Theta^-) = \mathsf{pos}^{(\Theta,\mathsf{i}(\Theta))}(\mathfrak{p}_\Theta^+, \mathfrak{p}_{\mathsf{i}(\Theta)}^+) = \overline{\mathrm{id}}.$$

2) Since $\mathrm{id} \in \mathsf{pos}^{(\Theta,\mathsf{i}(\Theta))}(x,z)$, we may assume that $(x,z) = (\mathfrak{p}_\Theta^+, \mathfrak{p}_{\mathsf{i}(\Theta)}^+)$. By the Bruhat decomposition (recall Equation (2.2.3)), there exist $p \in P_\Delta^+$ and $w \in W$ such that $y = pw \cdot \mathfrak{p}_\Theta^+$. One has $w_0 \in \mathsf{pos}^{(\Theta,\mathsf{i}(\Theta))}(y, \mathfrak{p}_{\mathsf{i}(\Theta)}^+) = \mathsf{pos}^{(\Theta,\mathsf{i}(\Theta))}(\mathfrak{p}_\Theta^+, w^{-1} \cdot \mathfrak{p}_{\mathsf{i}(\Theta)}^+)$, which implies that $\overline{w^{-1}} = \overline{w_0}$. Thus there exist $(a,b) \in W_{\Delta\smallsetminus\Theta} \times W_{\Delta\smallsetminus\mathsf{i}(\Theta)}$ such that $w^{-1} = aw_0b = ab'w_0$, with $b' \in W_{\Delta\smallsetminus\Theta}$.

On the other hand, we have

$$\mathsf{pos}^{(\Theta,\Theta)}(y, \mathfrak{p}_\Theta^+) = \mathsf{pos}^{(\Theta,\Theta)}(w_0 b'^{-1} a^{-1} \cdot \mathfrak{p}_\Theta^+, \mathfrak{p}_\Theta^+) = \mathsf{pos}^{(\Theta,\Theta)}(\mathfrak{p}_\Theta^+, w_0 \cdot \mathfrak{p}_\Theta^+),$$

since $W_{\Delta\smallsetminus\Theta} \subset S_\Theta \subset L_\Theta$. Thus $w_0 \in \mathsf{pos}^{(\Theta,\Theta)}(y, \mathfrak{p}_\Theta^+)$. ☐

Remark 2.2.7. The cardinality of $W_{\Delta\smallsetminus\Theta}\backslash W/W_{\Delta\smallsetminus\Theta'}$, which corresponds to the number of G-orbits for its action on $\mathscr{F}(\mathfrak{g},\Theta) \times \mathscr{F}(\mathfrak{g},\Theta')$ (and only depends on the Lie algebra $\mathfrak{g}$ of G) will sometimes be referred to as the *number of incidence degrees* between a pair of points in $\mathscr{F}(\mathfrak{g},\Theta) \times \mathscr{F}(\mathfrak{g},\Theta')$.

2.3 Dynamics of groups acting on flag manifolds

In this section, we recall important definitions concerning actions of Lie groups on flag manifolds, and their properties. We fix a semisimple Lie group G with Cartan projection $\mu : G \to \mathfrak{a}^+$, and a subset Θ of simple restricted roots of G. We recall notions on the dynamics of the elements of G and prove lemmas that we will use in this memoir.

2.3.1 Divergent groups

Divergent groups are groups with strong dynamical properties for their action on flag manifolds.

A sequence $(g_k) \in G^{\mathbb{N}}$ is Θ-*divergent* if $\alpha(\mu(g_k)) \to +\infty$ for every $\alpha \in \Theta$. It is Θ-*contracting* if there exists $(x, \xi) \in \mathscr{F}(\mathfrak{g}, \Theta) \times \mathscr{F}(\mathfrak{g}, \Theta)^-$ such that $g_k \cdot y \to x$ uniformly on compact subsets of $\mathscr{F}(\mathfrak{g}, \Theta) \smallsetminus \mathrm{Z}_\xi$; the pair (x, ξ) is then uniquely determined by (g_k), and we say that (g_k) is Θ-contracting with respect to (x, ξ), and that x is *the* Θ-*limit* of (g_k). Let us recall the following fact, which is an immediate consequence of the Cartan decomposition of G (see e.g. [KLP17]):

Fact 2.3.1. *Let $(g_k) \in G^{\mathbb{N}}$.*

1. *Assume that that there exist an open subset $\mathscr{U} \subset \mathscr{F}(\mathfrak{g}, \Theta)$ and a point $x \in \mathscr{F}(\mathfrak{g}, \Theta)$ such that $g_k \cdot \mathscr{U} \to \{x\}$ for the Hausdorff topology. Then (g_k) admits a subsequence which is Θ-contracting with Θ-limit x.*
2. *A sequence (g_k) is Θ-divergent if and only if every subsequence of (g_k) admits a Θ-contracting subsequence.*
3. *The sequence (g_k) admits a subsequence which is Θ-contracting with respect to $(x, \xi) \in \mathscr{F}(\mathfrak{g}, \Theta) \times \mathscr{F}(\mathfrak{g}, \Theta)^-$ if and only if the sequence (g_k^{-1}) admits a subsequence which is $\mathsf{i}(\Theta)$-contracting with respect to (ξ, x).*

Thus, given a Θ-divergent sequence $(g_k) \in G^{\mathbb{N}}$, one can define the Θ-*limit set of* (g_k) as the set, denoted by $\Lambda_\Theta(g_k)$, of all Θ-limits of Θ-contracting subsequences of (g_k).

Let $H \leq G$ be a subgroup. If we denote by $H_\Theta^{\mathbb{N}}$ the set of Θ-divergent sequences of elements of H, then define the Θ-*limit set of* H as

$$\Lambda_\Theta(H) = \bigcup_{(g_k) \in H_\Theta^{\mathbb{N}}} \Lambda_\Theta(g_k).$$

We say that H is Θ-*divergent* if every sequence of pairwise distinct elements of H is Θ-divergent. If $\Theta \subset \Delta$ is self-opposite, a subgroup $H \leq G$ is called Θ-*transverse* if it is Θ-divergent and its Θ-limit set $\Lambda_\Theta(H)$ is transverse, in the sense of Section 2.2.6.2.

2.3.2 Anosov representations

Anosov representations are generalizations to arbitrary reductive Lie groups G of rank-one convex cocompact representations of Gromov-hyperbolic groups. Labourie initially introduced them in his work on the Hitchin component [Lab06] in 2006, and they were then generalized and further investigated by Guichard–Wienhard [GW12]. The original definition is essentially dynamical, but in this memoir, we use as a definition the following characterization of Θ-Anosov representations proven in [GGKW17]:

Definition 2.3.2. Let Γ be a discrete word-hyperbolic group and let $\rho : \Gamma \to G$ be a representation. We say that ρ is Θ-*Anosov* if the following properties are satisfied:

1. For every sequence of elements $(g_k) \in \Gamma^{\mathbb{N}}$ diverging in Γ, the sequence $(\rho(g_k))$ is Θ-divergent.
2. There exist continuous, ρ-equivariant maps $\xi_\rho : \partial_\infty \Gamma \to \mathscr{F}(\mathfrak{g}, \Theta)$ and $\xi_\rho^- : \partial_\infty \Gamma \to \mathscr{F}(\mathfrak{g}, \Theta)^-$ which are:
   a) *transverse*, i.e. for all $x, y \in \partial_\infty \Gamma$, we have:
   $$x \neq y \Longrightarrow \xi_\rho(x) \text{ and } \xi_\rho^-(y) \text{ are transverse;}$$
   b) *dynamics-preserving*, that is, for any infinite-order element $g \in \Gamma$ with attracting fixed point $a \in \partial_\infty \Gamma$, the sequence $(\rho(g)^k)$ is Θ-contracting (resp. $P_{\mathsf{i}(\Theta)}$-contracting) with limit $\xi_\rho(a)$ (resp. $\xi_\rho^-(a)$).

The maps ξ_ρ and ξ_ρ^- are unique and are called respectively the *boundary map* and the *dual boundary map* of ρ. If Θ is self-opposite, then $\xi_\rho = \xi_\rho^-$. In this memoir we will denote by $\mathrm{Hom}_{\Theta-\mathsf{An}}(\Gamma, G)$ the space of all Θ-Anosov representations of Γ into G. A Θ-*Anosov subgroup* of G is by definition the image of a Θ-Anosov representation.

Anosov representations are discrete and faithful, and *structurally stable*, i.e. the set $\mathrm{Hom}_{\Theta-\mathsf{An}}(\Gamma, G)$ is open in the set $\mathrm{Hom}(\Gamma, G)$ of all representations of Γ into G [Lab06, GW12]. This last property allows to deform well-known Θ-Anosov groups, sometimes to get new Zariski-dense subgroups in G. The first two properties ensure that, for sufficiently small deformations, the group obtained remains discrete and is commensurable with the original subgroup. We will use these properties in Section 4.4.2 to construct Θ-Anosov Zariski-dense groups preserving proper domains in some flag manifolds.

2.3.3 Irreducible representations of semisimple Lie groups

In this section, we recall some fundamental results on (irreducible, proximal) representations of reductive Lie groups G (in the sense of [Kna96]). Some of these representations induce an equivariant embedding of a flag manifold of G into a projective space, which, in this memoir, will sometimes allow us to reduce general proofs on flag manifolds to proofs in projective space. A first example will be given in Section 2.3.3.3, where we will prove a continuity property for the orbits of Anosov representations, reducing to projective arguments.

Let (V, ρ) be a finite-dimensional real linear (resp. projective) representation of G, i.e. a group homomorphism $G \to \mathrm{GL}(V)$ (resp. $G \to \mathrm{PGL}(V)$). We will denote by $\rho_* : \mathfrak{g} \to \mathrm{End}(V)$ the differential of ρ at id.

2.3.3.1 Restricted weights. For any $\lambda \in \mathfrak{a}^*$, the *weight space* defined by λ is

$$V^\lambda := \{v \in V \mid \rho_*(h) \cdot v = \lambda(h) v \quad \forall h \in \mathfrak{a}\}.$$

If $V^\lambda \neq \{0\}$ we say that λ is a *restricted weight* of (V,ρ). Given $\alpha, \lambda \in \mathfrak{a}^*$, one has $\rho_*(X) \cdot V^\lambda \subset V^{\lambda+\alpha}$ for all $X \in \mathfrak{g}_\alpha$. For each $\alpha \in \Delta$, the element $\omega_\alpha \in \mathfrak{a}^*$ introduced in Section 2.2.3 is called the *fundamental weight* associated with α. The cone generated by the simple restricted roots determines a partial ordering on $\mathfrak{a}^*$ given by

$$\lambda \leq \lambda' \Longleftrightarrow \lambda' - \lambda \in \sum_{\alpha\in\Delta} \mathbb{R}_+\alpha.$$

If (V,ρ) is a finite-dimensional real irreducible linear or projective representation of G, then the set of restricted weights of (V,ρ) admits a unique maximal element for that ordering (see [GW09, Cor. 3.2.3]). This element is called the *highest weight* of ρ, and denoted by χ_ρ or χ.

2.3.3.2 Proximality and Θ-proximal representations. An automorphism g of $\mathrm{GL}(V)$ (resp. $\mathrm{PGL}(V)$) is said to be *proximal* in $\mathbb{P}(V)$ if it has (resp. any lift of g in $\mathrm{GL}(V)$ has) a unique eigenvalue of maximal modulus and if the corresponding eigenspace is one-dimensional.

Let $\Theta \subset \Delta$ be a nonempty subset of the simple restricted roots. We say that a linear (resp. projective) representation (V,ρ) is Θ-*proximal* if $\rho(G)$ contains a proximal element and $\{\alpha \in \Delta \mid \langle \chi_\rho, \alpha\rangle > 0\} = \Theta$ (see [GGKW17]). Note that the last condition is equivalent to saying that $\chi_\rho \in \sum_{\alpha\in\Theta} \mathbb{N}\omega_\alpha$. If (V,ρ) is proximal, then we denote by $V^{<\chi_\rho}$ the sum of all weight spaces of weights $\lambda \neq \chi_\rho$ of (V,ρ).

We will often use irreducible Θ-proximal representations in this memoir. Thus we will use the following terminology:

Definition 2.3.3. Let $\mathfrak{g}$ be a real semisimple Lie algebra and Θ be a subset of the simple restricted roots of $\mathfrak{g}$. A triple (G,ρ,V) is a *linear (resp. projective)* Θ-*proximal triple* if $G \in \mathscr{G}_\Theta(\mathfrak{g})$ and if (V,ρ) is a finite-dimensional, real, irreducible, linear (resp. projective), Θ-proximal representation of G.

We will often use the following result, which allows some arguments in general flag manifolds to reduce to arguments in real projective space:

Fact 2.3.4 ([GGKW17, Prop. 3.3])**.** *Let (G,ρ,V) be a linear (resp. projective) Θ-proximal triple of $\mathfrak{g}$.*

1. *The stabilizer of V^{χ_ρ} in G (resp. $V^{<\chi_\rho}$) is P_Θ^+ (resp. P_Θ^-).*
2. *The maps $g \mapsto \rho(g)\cdot V^{\chi_\rho}$ and $g \mapsto \rho(g)\cdot V^{<\chi_\rho}$ induce two ρ-equivariant embeddings:*

$$\iota_\rho : \mathscr{F}(\mathfrak{g},\Theta) \longrightarrow \mathbb{P}(V) \text{ and } \iota_\rho^- : \mathscr{F}(\mathfrak{g},\Theta)^- \longrightarrow \mathbb{P}(V^*).$$

   Two elements $x \in \mathscr{F}(\mathfrak{g},\Theta)$ and $\xi \in \mathscr{F}(\mathfrak{g},\Theta)^-$ are transverse if and only if their images $\iota_\rho(x)$ and $\iota_\rho^-(\xi)$ are.

3. *For any sequence $(g_k) \in G^{\mathbb{N}}$, the sequence (g_k) is Θ-divergent (resp. Θ-contracting) if and only if the sequence $(\rho(g_k)) \in \mathrm{PGL}(V)^{\mathbb{N}}$ is $\{\alpha_1\}$-divergent (resp. $\{\alpha_1\}$-contracting). If (g_k) is Θ-contracting with Θ-limit $x \in \mathscr{F}(\mathfrak{g},\Theta)$, then the $\{\alpha_1\}$-limit of $(\rho(g_k))$ is $\iota_\rho(x)$.*

4. *Let Γ be a word hyperbolic group and let $\rho' : \Gamma \to G$ be a representation. The representation ρ' is Θ-Anosov if and only if the representation $\rho \circ \rho' : \Gamma \to \mathrm{PGL}(V)$ is $\{\alpha_1\}$-Anosov.*

Note that for $\nu \in V \smallsetminus \{0\}$ and $f \in V^* \smallsetminus \{0\}$, the transversality of $[\nu] \in \mathbb{P}(V)$ and $[f] \in \mathbb{P}(V^*)$ is equivalent to $f(\nu) \neq 0$.

With the identification (2.1.1) and the notations of Fact 2.3.4, a pair $(x,\xi) \in \mathscr{F}(\mathfrak{g},\Theta)\times\mathscr{F}(\mathfrak{g},\Theta)^-$ are transverse if and only if one has $\iota_\rho(x) \notin \iota_\rho^-(\xi)$.

In the notation of Fact 2.3.4, we have the following:

Fact 2.3.5. *[Zim18a] Let $\Omega \subset \mathscr{F}(\mathfrak{g},\Theta)$ be a subset with nonempty interior. There exist $\xi_1,\dots,\xi_n \in \Omega$ such that $\mathbb{P}(V) = \iota_\rho(\xi_1) \oplus \dots \oplus \iota_\rho(\xi_n)$.*

2.3.3.3 Continuity of the orbit for Anosov representations. The map $\rho \mapsto \xi_\rho$, associating to a Θ-Anosov representation its limit map, is continuous on $\mathrm{Hom}_{P_\Theta-\mathsf{An}}(\Gamma,G)$. In the next lemma, using Fact 2.3.4, we strengthen this continuity property in the case where the hypersurfaces $\mathrm{Z}_{\xi_\rho^-(\eta)}$, with $\eta \in \partial_\infty\Gamma$, do not cover $\mathscr{F}(\mathfrak{g},\Theta)$.

Lemma 2.3.6. *Let G be a real semisimple Lie group and Θ be a subset of the simple restricted roots of G. Let Γ be a word hyperbolic group with boundary $\partial_\infty\Gamma$. Let $\rho : \Gamma \to G$ be a Θ-Anosov representation such that the set*

$$\mathscr{O}_\rho := \mathscr{F}(\mathfrak{g},\Theta) \smallsetminus \bigcup_{\eta\in\partial_\infty\Gamma} \mathrm{Z}_{\xi_\rho^-(\eta)}$$

is nonempty. Let $x_0 \in \mathscr{O}_\rho$. Then the map

$$\Psi : \mathrm{Hom}_{\Theta-\mathsf{An}}(\Gamma,G) \longrightarrow \big\{\textit{closed subsets of } \mathscr{F}(\mathfrak{g},\Theta)\big\};\ \rho' \longmapsto \overline{\rho'(\Gamma)\cdot x_0}$$

is continuous at ρ for the Hausdorff topology.

Proof. For any infinite-order element $g \in \Gamma$, we denote by g^+ and g^- the attracting and repelling fixed point of g in $\partial_\infty\Gamma$.

By Fact 2.3.4.(4), it suffices to prove the lemma for $G = \mathrm{PGL}(n,\mathbb{R})$ and $\Theta = \{\alpha_1\}$ the first simple root of G, i.e. $\mathscr{F}(\mathfrak{g},\Theta) = \mathbb{P}(\mathbb{R}^n)$ and $\mathscr{F}(\mathfrak{g},\Theta)^- = \mathbb{P}((\mathbb{R}^n)^*)$, where $n \in \mathbb{N}_{\geq 2}$. We fix the *angle metric* on $\mathbb{P}(\mathbb{R}^n)$, given by

$$d(u,v) = |\sin\angle(\tilde{u},\tilde{v})| \quad \forall u,v \in \mathbb{P}(\mathbb{R}^n),$$

where we have denoted by $\tilde{u}$ (resp. $\tilde{v}$) any lift of u (resp. of v) in $\mathbb{P}(\mathbb{R}^n)$.

To prove the lemma, it suffices to prove that for any sequence of representations $(\rho_k) \in \mathrm{Hom}_{\{\alpha_1\}-\mathsf{An}}(\Gamma,\mathrm{PGL}(n,\mathbb{R}))^{\mathbb{N}}$ and for any diverging sequence $(g_k) \in \Gamma^{\mathbb{N}}$, any limit point of the sequence $(\rho_k(g_k)\cdot x_0)$ converges to an element of $\xi_\rho(\partial_\infty\Gamma)$. To this end, since $\mathbb{P}(\mathbb{R}^n)$ is compact, it suffices to prove that $(\rho_k(g_k)\cdot x_0)$ admits a subsequence that converges to an element of $\xi_\rho(\partial_\infty\Gamma)$.

By [AMS95, Thm 4.1], which can be applied to (not necessarily irreducible) Anosov representations, there exists a finite subset $F \subset \Gamma$ and some $\varepsilon > 0$ such that for any $g \in \Gamma$, there exists $f \in F$ such that the element $\rho(fg)$ is proximal and we have $d(x^+, H^-) \geq \varepsilon$, where x^+ and H^- are respectively the attracting fixed point and repelling fixed hyperplane of $\rho(fg)$. In particular, the element fg has infinite order, and we have $x^+ = \xi_\rho((fg)^+)$ and $H^- = \xi_\rho^-((fg)^-)$. Hence, up to extracting, we may assume that there exists $f \in F$ such that for all $k \in \mathbb{N}$, the element $fg_k \in \Gamma$ has infinite order, the element $\rho(fg_k) \in \mathrm{PGL}(n, \mathbb{R})$ is proximal, and

$$d\big(\xi_\rho((fg_k)^+), \xi_\rho^-((fg_k)^-)\big) \geq \varepsilon.$$

Since ξ_ρ is ρ-equivariant, we may actually replace g_k with fg_k for all $k \in \mathbb{N}$. Then g_k has infinite order, and $\rho_k(g_k)$ is proximal with attracting fixed point $\xi_{\rho_k}(g_k^+)$ and repelling fixed hyperplane $\xi_{\rho_k}^-(g_k^-)$.

Up to further extracting, we may assume that there exist $a, b \in \partial_\infty\Gamma$ such that $g_k \to a$ and $g_k^{-1} \to b$ in $\Gamma \sqcup \partial_\infty\Gamma$. Note that we have $a = \lim_{k\to+\infty} g_k^+$ and $b = \lim_{k\to+\infty} g_k^-$.

We know that each of the elements $\rho_k(g_k)$ is proximal, and we will actually prove that the sequence $(\rho_k(g_k))_k$ is $\{\alpha_1\}$-divergent (see Equation (2.3.1) below).

By [GW12], there exist a neighborhood $\mathscr{U}$ of ρ in $\mathrm{Hom}_{\{\alpha_1\}-\mathsf{An}}(\Gamma, \mathrm{PGL}(n, \mathbb{R}))$ and two constants $D > 1, L > 0$ such that for all $\rho' \in \mathscr{U}$ and $g \in \Gamma$:

$$\alpha_1\big(\mu(\rho'(g))\big) \geq \frac{1}{D}|g| - L,$$

where μ is the Cartan projection of $\mathrm{PGL}(V)$ (see Section 2.2.2) and $|\cdot|$ is the word length on Γ, determined by a finite generating set of Γ. Since for k large enough we have $\rho_k \in \mathscr{U}$, this proves that

$$\alpha_1\big(\mu(\rho_k(g_k))\big) \underset{k\to+\infty}{\longrightarrow} +\infty. \tag{2.3.1}$$

This gives, for all $k \in \mathbb{N}$:

$$d(\xi_\rho((g_k)^+), \xi_\rho^-((g_k)^-)) \geq \varepsilon.$$

Taking the limit as $k \to +\infty$, we get

$$d(\xi_\rho(a), \xi_\rho^-(b)) \geq \varepsilon. \tag{2.3.2}$$

On the other hand, we have

$$|||D_{\xi_{\rho_k}((g_k)^+)}\rho_k(g_k)|||_k \leq e^{-\alpha_1\left(\mu(\rho_k(g_k))\right)}, \tag{2.3.3}$$

where $|||\cdot|||_k$ is the operator norm associated with the norm $||\cdot||_k$ induced by the Riemannian metric d of $\mathbb{P}(\mathbb{R}^n)$ on $T_{\xi_{\rho_k}((g_k)^+)}\mathbb{P}(\mathbb{R}^n)$. Let us identify $T_{\xi_{\rho_k}((g_k)^+)}\mathbb{P}(\mathbb{R}^n)$ with the affine chart $\mathbb{A}_k := \mathbb{P}(\mathbb{R}^n) \smallsetminus \xi_{\rho_k}^-((g_k)^-)$ via a stereographic projection at the basepoint $\xi_{\rho_k}((g_k)^+)$. The norm $||\cdot||_k$ then induces a metric d_k on $\mathbb{A}_k$. The same procedure, taking $\xi_\rho(a)$ as a basepoint, gives us a metric d_∞ on $\mathbb{A} := \mathbb{P}(\mathbb{R}^n) \smallsetminus \xi_\rho^-(a)$. Since all of our choices are continuous, and since we have $\xi_{\rho_k}(g_k^+) \to \xi_\rho(a)$ and $\xi_{\rho_k}^-(g_k^-) \to \xi_\rho(b)$ (by continuity of $\rho' \mapsto \xi_{\rho'}$ for the uniform convergence [GW12]), we deduce that $d_k \to d_\infty$ uniformly on compact subsets of $\mathbb{A}^2$.

By the Mean value inequality, for k large enough and for all $x, y \in \mathbb{A}$, Equation (2.3.4) gives:

$$d_k(\rho_k(g_k)\cdot x, \rho_k(g_k)\cdot y) \leq e^{-\alpha_1\left(\mu(\rho_k(g_k))\right)} d_k(x, y). \tag{2.3.4}$$

By assumption on x_0, we know that for k large enough, we have $x_0 \in \mathbb{A}_k$. Hence one also has $\rho_k(g_k) \cdot x_0 \in \mathbb{A}_k$. On the other hand, since $g_k^+ \neq g_k^-$, by transversality of the limit maps of ρ_k, we also have $\xi_{\rho_k}((g_k)^+) \in \mathbb{A}_k$. Hence Equation (2.3.4) gives:

$$\begin{aligned} d_k(\rho_k(g_k) \cdot x_0, \xi_{\rho_k}((g_k)^+)) &= d_k(\rho_k(g_k) \cdot x_0, \rho_k(g_k) \cdot \xi_{\rho_k}((g_k)^+)) \\ &\leq e^{-\alpha_1\left(\mu(\rho_k(g_k))\right)} d_k(x_0, \xi_{\rho_k}((g_k)^+)). \end{aligned} \tag{2.3.5}$$

But according to (2.3.2), and since $\xi_{\rho_k}((g_k)^+) \to \xi_\rho(a) \in \mathbb{A}$, for k large enough the points $\xi_{\rho_k}((g_k)^+)$ remain in a bounded subset of $\mathbb{A}$. Thus by uniform convergence of (d_k) to d_∞ on compact subsets of $\mathbb{A}^2$, we have $d_k(x_0, \xi_{\rho_k}((g_k)^+)) \to d_\infty(x_0, \xi_\rho(a)) < +\infty$. Equations (2.3.1) and (2.3.5) then implies that $d_k(\rho_k(g_k) \cdot x_0, \xi_{\rho_k}((g_k)^+)) \to 0$. This implies that the points $\rho_k(g_k) \cdot x_0$, for k large enough, remain in a bounded subset of $\mathbb{A}$. Hence up to extracting, it converges to some point $y \in \mathbb{A}$. Since $\xi_{\rho_k}((g_k)^+) \to \xi_\rho(a)$, we have

$$d_\infty(y, \xi_\rho(a)) = \lim_{k \to +\infty} d_k(\rho_k(g_k) \cdot x_0, \xi_{\rho_k}((g_k)^+)) = 0.$$

Hence $y = \xi_\rho(a)$. Hence, up to extracting, we have $\rho_k(g_k) \cdot x_0 \to \xi_\rho(a)$. $\square$

Remark 2.3.7. In Lemma 2.3.6, we made the assumption that G is semsimple. This assumption is here in order to use Fact 2.3.4. But in [GGKW17], this fact is actually stated for reductive Lie groups G such that have finitely many connected components (for the real topology) of the set of real points $\mathbf{G}(\mathbb{R})$ for some algebraic group $\mathbf{G}$. Hence Lemma 2.3.6 is actually still true if G satisfies this weaker condition, with the exact same proof.

2.4 Key Examples

In this section, we introduce several families of flag manifolds, which will be extensively studied in this thesis. We describe their explicit construction, not only to illustrate the concepts already introduced, but also to enable a detailed analysis of the objects that will be introduced later in this thesis (the *photons* in Section 6.3, the *Kobayashi metric* in Section 6.4, and the *Plücker triples* in Section 7.1). The notations introduced here will be used and referred to throughout the memoir.

2.4.1 The Lorentzian Einstein universe

Let $n \geq 2$. The *Einstein universe of signature* $(n-1, 1)$ is the space of isotropic lines of $\mathbb{R}^{n,2}$. Although it does not depend on the chosen bilinear form of signature $(n, 2)$ on $\mathbb{R}^{n+2}$, in this section, we choose the one introduced in Example 2.2.1.(2), in order to explicitly describe the parabolic subgroup $P^+_{\{\alpha_1\}}$.

Let $(e_1, \ldots, e_{n+2})$ be the canonical basis of $\mathbb{R}^{n+2}$. We take the notations of Example 2.2.1.(2), in the case where $p = n-1$ and $q = 1$.

For any vector $v \in \mathbb{R}^{n+2}$, we denote by v_i the i-th coordinate of v, that is $v = \sum_{i=1}^{n+2} v_i e_i$. Let $\mathbf{b}$ be the quadratic form of signature $(n, 2)$ on $\mathbb{R}^{n+2}$ defined as:

$$\mathbf{b}(v, w) = (v_1 w_{n+2} + v_{n+2} w_1) + (v_2 w_{n+1} + v_{n+1} w_2) + \sum_{i=3}^{n} v_i w_i \quad \forall v, w \in \mathbb{R}^{n+2}.$$

The Einstein universe $\mathrm{Ein}^{n-1,1}$ of signature $(n-1,1)$, also called the *Lorentzian Einstein universe*, is then the space of isotropic lines of $(\mathbb{R}^{n,2}, \mathbf{b})$. The group $\mathrm{PO}(\mathbf{b}) \simeq \mathrm{PO}(n,2)$ acts transitively on $\mathrm{Ein}^{n-1,1}$, and the stabilizer of $[e_1]$ in $\mathrm{PO}(n,2)$ is $P_{\{\alpha_1\}}$. Then (2.2.4) gives a $\mathrm{PO}(n,2)$-equivariant identification

$$\mathrm{Ein}^{n-1,1} \simeq \mathscr{F}(\mathfrak{so}(n,2), \alpha_1).$$

The flag manifold $\mathscr{F}(\mathfrak{so}(n,2), \alpha_1)$ is self-opposite, and in the above identification, if $x \in \mathrm{Ein}^{n-1,1}$, then one has $\mathrm{Z}_x = \mathbb{P}(x^\perp) \cap \mathrm{Ein}^{n-1,1}$. The standard affine chart as defined in Equation (2.2.6) can thus be written explicitly:

$$\begin{aligned}\mathbb{A}_{\mathsf{std}} &= \mathrm{Ein}^{n-1,1} \smallsetminus \mathrm{Z}_{[e_1]} = \mathrm{Ein}^{n-1,1} \smallsetminus \mathbb{P}\left((\mathbb{R}e_1))^{\perp_{\mathbf{b}}}\right) \\ &= \mathbb{P}\Big\{e_1 + \sum_{i=2}^{n+1} v_i e_i - \psi\Big(\sum_{i=2}^{n+1} v_i e_i\Big) e_{n+2}\Big\},\end{aligned}$$

where ψ is the quadratic form of signature $(n-1,1)$ on $V := \mathrm{Span}(e_2, \dots, e_{n+1})$ defined by $\psi\left(\sum_{i=2}^{n+1} v_i e_i\right) = \frac{1}{2}\sum_{i=3}^{n} v_i^2 + v_2 v_{n+1}$. The identification $V \simeq \mathbb{A}_{\mathsf{std}}$ given by

$$\sum_{i=2}^{n+1} v_i e_i \longmapsto \Big[e_1 + \sum_{i=2}^{n+1} v_i e_i - \psi\Big(\sum_{i=2}^{n+1} v_i e_i\Big) e_{n+2}\Big] \tag{2.4.1}$$

endows $\mathbb{A}_{\mathsf{std}}$ with the structure of a Minkowski space. Still denoting by ψ the quadratic form induced on $\mathbb{A}_{\mathsf{std}}$ by this identification, one has (see Figure 3.1):

$$\mathbb{A}_{\mathsf{std}} \cap \mathrm{Z}_{y_0} = \{y \in \mathbb{A}_{\mathsf{std}} \mid \psi(y - y_0) = 0\} \quad \forall y_0 \in \mathbb{A}_{\mathsf{std}}.$$

2.4.2 Grassmannians

Let $p, q \geq 1$. The *Grassmannian of p-planes of* $\mathbb{R}^{p+q}$, denoted by $\mathrm{Gr}_p(\mathbb{R}^{p+q})$, is the set of p-planes of $\mathbb{R}^{p+q}$. Let us describe its flag-manifold structure explicitly. We denote by $(e_1, \dots, e_{p+q})$ the canonical basis of $\mathbb{R}^{p+q}$. Let

$$x_0 := \mathrm{Span}(e_1, \dots, e_p) \in \mathrm{Gr}_p(\mathbb{R}^{p+q}) \text{ and } \xi_\infty := \mathrm{Span}(e_{p+1}, \dots, e_{p+q}) \in \mathrm{Gr}_q(\mathbb{R}^{p+q}). \tag{2.4.2}$$

We take the notation of Example 2.2.1.(1). The group $\mathrm{PGL}(p+q, \mathbb{R})$ acts transitively on $\mathrm{Gr}_p(\mathbb{R}^{p+q})$, and the stabilizer of x_0 is $P_p := P_{\{\alpha_p\}}$. Thus Equation (2.2.4) gives a $\mathrm{PGL}(p+q, \mathbb{R})$-equivariant identification

$$\mathrm{Gr}_p(\mathbb{R}^{p+q}) \simeq \mathscr{F}(\mathfrak{sl}(p+q, \mathbb{R}), \alpha_p).$$

The flag manifold opposite to $\mathscr{F}(\mathfrak{sl}(p+q, \mathbb{R}), \alpha_p)$ then admits a similar identification with $\mathrm{Gr}_q(\mathbb{R}^{p+q})$ — note that it is self-opposite if, and only if, $p = q$. We make these identifications for the rest of this section.

In this setting, we have

$$\begin{aligned}\mathfrak{u}^-_{\{\alpha_p\}} &= \left\{ \begin{bmatrix} 0_p & 0_{p,q} \\ X & 0_q \end{bmatrix} \mid X \in \mathrm{Mat}_{q,p}(\mathbb{R}) \right\} \\ L_{\{\alpha_p\}} &= \left\{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \mid A \in \mathrm{GL}_p(\mathbb{R}),\ B \in \mathrm{GL}_q(\mathbb{R}) \right\}.\end{aligned} \tag{2.4.3}$$

2.4.2.1 Models for the Grassmannians. The set

$$\widetilde{\mathscr{U}} := \{M \in \mathrm{Mat}_{p+q,p}(\mathbb{R}) \mid \mathrm{rk}(M) = p\}$$

is an open subset of $\mathrm{Mat}_{p+q,p}(\mathbb{R})$, on which the group $\mathrm{GL}(p,\mathbb{R})$ acts by right multiplication. Two matrices $M, M' \in \widetilde{\mathscr{U}}$ are in the $\mathrm{GL}(p,\mathbb{R})$-same orbit if and only if they have same image. For the rest of this memoir, we make the following $\mathrm{GL}(p+q,\mathbb{R})$-equivariant identification

$$\widetilde{\mathscr{U}} / \mathrm{GL}(p,\mathbb{R}) \simeq \mathrm{Gr}_p(\mathbb{R}^{p+q});\ [M] \longmapsto \mathrm{Im}(M). \tag{2.4.4}$$

2.4.2.2 The Plücker embeddings. We consider the projective $\{\alpha_p\}$-proximal triple $(\mathrm{PGL}(p+q,\mathbb{R}), \rho_0, \bigwedge^p \mathbb{R}^{p+q})$ of $\mathfrak{sl}(p+q,\mathbb{R})$ with highest weight ω_{α_p}; explicitly, it is defined by the natural action of $\mathrm{PGL}(p+q,\mathbb{R})$ on $\mathbb{P}(\bigwedge^p \mathbb{R}^{p+q})$:

$$\rho_0(g) \cdot [v_1 \wedge \cdots \wedge v_p] = [(\widetilde{g} \cdot v_1) \wedge \cdots \wedge \widetilde{g} \cdot v_p)] \quad \forall (v_1, \dots, v_p) \text{ basis of } \mathbb{R}^{p+q},$$

where $\widetilde{g}$ is any lift of g in $\mathrm{GL}(p+q,\mathbb{R})$. The associated embeddings via Fact 2.3.4 are the classical *Plücker embeddings*:

$$\begin{array}{rccc} \iota_{\rho_0} : & \mathrm{Gr}_p(\mathbb{R}^{p+q}) & \longrightarrow & \mathbb{P}(\bigwedge^p \mathbb{R}^{p+q}) \\ & \mathrm{Span}(v_1, \dots, v_p) & \longmapsto & [v_1 \wedge \cdots \wedge v_p]; \end{array} \tag{2.4.5}$$

$$\begin{array}{rccc} \iota^-_{\rho_0} : & \mathrm{Gr}_q(\mathbb{R}^{p+q}) & \longrightarrow & \mathbb{P}(\bigwedge^p \mathbb{R}^{p+q})^* \\ & \mathrm{Span}(v_1, \dots, v_q) & \longmapsto & [x \mapsto x \wedge v_1 \wedge \cdots \wedge v_q] \end{array}$$

2.4.2.3 The standard affine chart. In the identification (8.5.1), the standard affine chart (as defined in Section 2.2.6) is

$$\mathbb{A}_{\mathsf{std}} := \mathrm{Gr}_p(\mathbb{R}^{p+q}) \smallsetminus \mathrm{Z}_{\xi_\infty} = \left\{ \begin{bmatrix} I_p \\ X \end{bmatrix} \mid X \in \mathrm{Mat}_{q,p}(\mathbb{R}) \right\}. \tag{2.4.6}$$

Thus there is a diffeomorphism

$$\begin{array}{rccc} \varphi_p : & \mathrm{Mat}_{q,p}(\mathbb{R}) & \longrightarrow & \mathbb{A}_{\mathsf{std}} \\ & X & \longmapsto & \begin{bmatrix} I_p \\ X \end{bmatrix}. \end{array} \tag{2.4.7}$$

Note that, with the natural identification of $\mathrm{Mat}_{q,p}(\mathbb{R})$ with $\mathfrak{u}_{\{\alpha_p\}}$ given by Equation (2.4.3), the map φ_p identifies with φ_{std}.

Given a point $\xi := \begin{bmatrix} A \\ B \end{bmatrix} \in \mathrm{Gr}_q(\mathbb{R}^{p+q})$, with $A \in \mathrm{Mat}_{p,q}(\mathbb{R})$ and $B \in \mathrm{Mat}_q(\mathbb{R})$, a computation gives:

$$\mathbb{A}_{\mathsf{std}} \cap \mathrm{Z}_\xi = \left\{ \begin{bmatrix} I_p \\ X \end{bmatrix} \mid \det(B - XA) = 0 \right\} = \varphi_p(\{X \in \mathrm{Mat}_{q,p}(\mathbb{R}) \mid \det(B - XA) = 0\}). \tag{2.4.8}$$

2.4.3 The Einstein universe of signature (p,q)

In Section 2.4.1, we have defined the Lorentzian Einstein universe. In this section, we generalize the construction, in order to define the Einstein universes of any signature.

Let $p,q \in \mathbb{N}$ and let $\mathbf{b}$ a bilinear form of signature $(p+1,q+1)$ on $\mathbb{R}^{p+q+2}$. The *Einstein universe* of signature (p,q) is the space of isotropic lines of $(\mathbb{R}^{p+q+2},\mathbf{b})$:

$$\mathrm{Ein}^{p,q} = \{[v] \in \mathbb{P}(\mathbb{R}^{p+1,q+1}) \mid v \in \mathbb{R}^{p+1,q+1} \smallsetminus \{0\},\ \mathbf{b}(v,v)=0\}.$$

The group $G := \mathrm{PO}(\mathbf{b}) \simeq \mathrm{PO}(p+1,q+1)$ acts transitively on $\mathrm{Ein}^{p,q}$. If P_x is the stabilizer of a point $x \in \mathrm{Ein}^{p,q}$, then P_x is conjugate to the parabolic subgroup $P_{\{\alpha_1\}}$ of G (in the notation of Example 2.2.1.(2)). Thus Equation (2.2.4) gives a $\mathrm{PO}(p+1,q+1)$-equivariant identification

$$\mathrm{Ein}^{p,q} \simeq \mathscr{F}(\mathfrak{so}(p+1,q+1),\alpha_1).$$

We make this identification for the rest of this section.

When $q=0$ (resp. $p=0$), then the corresponding Einstein universe $\mathrm{Ein}^{p,q}$ is the *conformal sphere*, denoted by $\mathbb{S}^p := \mathrm{PO}(p+1,1)/P_{\{\alpha_1\}}$ (resp. $-\mathbb{S}^q := \mathrm{PO}(1,q+1)/P_{\{\alpha_1\}}$).

2.4.3.1 Lightcones and photons. We briefly recall the basic tools for studying $\mathrm{Ein}^{p,q}$. For a more general overview, see [Fra05] and [BCD+08]. The flag manifold $\mathrm{Ein}^{p,q}$ is self-opposite in the sense of Section 2.2.6. Given a point $x \in \mathrm{Ein}^{p,q}$, the set Z_x can be geometrically described as the following:

$$\mathrm{Z}_x = \mathbb{P}(x^{\perp}) \cap \mathrm{Ein}^{p,q}.$$

It is called the *lightcone of* x. Depending on the values of p,q, there are two possibilities, described in the following Section 2.4.3.1.1 and 2.4.3.1.2:

2.4.3.1.1 The sphere case. If $p=0$ or $q=0$, then $\mathrm{Ein}^{p,q} = \mathbb{S}^{\max(p,q)}$, and the lightcone of x is equal to $\{x\}$. In the notation of Section 2.2.6.3, when $\Theta = \Theta' = \{\alpha_1\}$, one has $|W_{\Delta\smallsetminus\Theta}\backslash W/W_{\Delta\smallsetminus\Theta'}| = 2$: the action of $\mathrm{PO}(p+1,q+1)$ on $\mathbb{S}^{\max(p,q)} \times \mathbb{S}^{\max(p,q)}$ has 2 orbits, which are the two sets

$$\begin{aligned}&\{(x,y) \mid\ x \neq y\};\\ &\{(x,y) \mid x = y\}.\end{aligned} \tag{2.4.9}$$

2.4.3.1.2 The higher-signature case. If $p,q \geq 1$, then there exist 2-dimensional totally isotropic subspaces $V \subset \mathbb{R}^{p+1,q+1}$. In this case, a *photon* of $\mathrm{Ein}^{p,q}$ is the projectivization $\mathbb{P}(V) \subset \mathbb{P}(\mathbb{R}^{p+q+2})$ of such a subspace. By definition, such a projective line is always contained in $\mathrm{Ein}^{p,q}$. The union of all photons through a point $x \in \mathrm{Ein}^{p,q}$ coincides with its lightcone Z_x. In the notation of Section 2.2.6.3, when $\Theta = \Theta' = \{\alpha_1\}$, one has $|W_{\Delta\smallsetminus\Theta}\backslash W/W_{\Delta\smallsetminus\Theta'}| = 3$: the action of $\mathrm{PO}(p+1,q+1)$ on $\mathrm{Ein}^{p,q} \times \mathrm{Ein}^{p,q}$ has 3 orbits, which are the three sets

$$\begin{aligned}&\{(x,y) \mid\ x \text{ and } y \text{ are tranverse}\};\\ &\{(x,y) \mid\ x \text{ and } y \text{ are on a common photon but are different}\};\\ &\{(x,y) \mid x = y\}.\end{aligned} \tag{2.4.10}$$

2.4.3.2 Plücker embedding. The natural inclusion $\mathfrak{so}(p+1, q+1) \subset \mathfrak{sl}(p+q+2, \mathbb{R})$ given by the description of $\mathfrak{so}(p+1, q+1)$ in Example 2.2.1.(2) induces the natural embedding

$$\rho_1 : \mathrm{PO}(p+1, q+1) \hookrightarrow \mathrm{PGL}(p+q+2, \mathbb{R}), \tag{2.4.11}$$

which is a projective $\{\alpha_1\}$-proximal triple of $\mathfrak{so}(p+1, q+1)$, with highest weight ω_{α_1}. The associated embeddings $\iota_{\rho_1}, \iota^-_{\rho_1}$ via Fact 2.3.4 coincide with the natural embeddings

$$\begin{aligned} \iota_{\rho_1} &: \mathrm{Ein}^{p,q} \subset \mathbb{P}(\mathbb{R}^{p+q+2}); [v] \mapsto [v], \\ \iota^-_{\rho_1} &: \mathrm{Ein}^{p,q} \subset \mathbb{P}((\mathbb{R}^{p+q+2})^*); [v] \mapsto \mathbb{P}([v]^\perp). \end{aligned} \tag{2.4.12}$$

2.4.3.3 Reminders on conformal manifolds. In this section, we give more geometric features of $\mathrm{Ein}^{p,q}$, coming from pseudo-Riemannian geometry.

A *pseudo-Riemannian conformal manifold* is a manifold M equipped with a *conformal class* $[g]$, i.e. a set of the form

$$[g] = \left\{ e^f \cdot g \mid f \in C^\infty(M) \right\},$$

where g is a pseudo-Riemannian metric on M.

The signature of g will be denoted by (p, q), i.e. for all $x \in M$ the quadratic form g_x on T_xM has signature (p, q) in the notation of Section 2.1.2. This signature only depends on the conformal class of g. We say that $(M, [g])$ is a *conformal manifold of signature* (p, q). If $(p, q) = (n-1, 1)$ for some $n \in \mathbb{N}_{\geq 2}$, then we say that $(M, [g])$ is a *conformal spacetime.*

A tangent vector $v \in TM$ is *timelike* (resp. *lightlike, spacelike*) if $g(v, v)$ is negative (resp. null, positive). The tangent vector v is said to be *causal* if it is either timelike or lightlike. This enables us to talk about *timelike curves* in M (resp. *causal, lightlike, spacelike*).

A smooth map $\varphi : (M, [g_M]) \to (N, [g_N])$ is *conformal* if $\varphi^* g_N \in [g_M]$, which is equivalent to saying that φ sends causal curves to causal curves and spacelike curves to spacelike curves. We denote by $\mathrm{Conf}(M)$ the group of conformal automorphisms of M, and call it the *conformal group* of M.

Two different metrics in the same conformal class $[g]$ define in general different geodesics; however the image of a lightlike geodesic only depends on the conformal class of metric $[g]$ (see e.g. [Mar81]). It is called an *unparametrized lightlike geodesic.*

2.4.3.3.1 The conformal structure of the Einstein universe. For all $x \in \mathrm{Ein}^{p,q}$, the signature of $\mathbf{b}$ restricted to $\mathbb{P}(x^\perp) \simeq T_x \mathrm{Ein}^{p,q}$ has signature (p, q). Hence it induces a pseudo-Riemanniann metric on $\mathrm{Ein}^{p,q}$ with signature (p, q), still denoted by $\mathbf{b}$. The manifold $(\mathrm{Ein}^{p,q}, [\mathbf{b}])$ is a compact conformal manifold of signature (p, q). It admits a 2-sheeted conformal cover by $(\mathbb{S}^p \times \mathbb{S}^q, [g_{\mathbb{S}^p} \oplus (-g_{\mathbb{S}^q})])$, with nontrivial deck transformation $(x, y) \mapsto (-x, -y)$.

The unparametrized ligthlike geodesics of $(\mathrm{Ein}^{p,q}, [\mathbf{b}])$ are exactly the photons as defined in Section 2.4.3.1.

The natural action of $\mathrm{PO}(p+1, q+1)$ on $\mathrm{Ein}^{p,q}$ is by conformal automorphisms, and the conformal group of $\mathrm{Ein}^{p,q}$ coincides with $\mathrm{PO}(p+1, q+1)$. More generally, the following fundamental result, attributed to Liouville, holds (see [Fra03]):

Fact 2.4.1. *Let $p, q \in \mathbb{N}$ such that $p + q \geq 3$ and let $\mathscr{U}, \mathscr{V}$ be two connected open subsets of* $\mathrm{Ein}^{p,q}$*. If $\varphi : \mathscr{U} \to \mathscr{V}$ is a smooth conformal map, then there exists a unique $g \in \mathrm{PO}(p+1, q+1)$ such that $g|_{\mathscr{U}} = \varphi$.*

In particular, for a connected open subset $\Omega \subset \mathrm{Ein}^{p,q}$, the conformal group of Ω is precisely the subgroup of $\mathrm{PO}(p+1, q+1)$ of all transformations g preserving Ω (that is $g \cdot \Omega = \Omega$), see Remark 3.1.4.

2.4.3.3.2 Affine charts. Recall from Section 2.1.2 that for $p, q \in \mathbb{N}$, we denote by $\mathbb{R}^{p,q}$ the vector space $\mathbb{R}^{p+q}$ endowed with a bilinear form of signature (p, q).

Let $\mathbb{A}$ be an affine chart of $\mathrm{Ein}^{p,q}$, and fix an origin $0 \in \mathbb{A}$. As for the Lorentzian case (see Section 2.4.1), the restriction of any metric in $[\mathbf{b}]$ to $\mathbb{A}$ induces a bilinear form $\mathbf{b}_{p,q}$ on the vector space $(\mathbb{A}, 0)$ of signature (p, q). If $x \in \mathbb{A}$, then we have

$$\mathrm{Z}_x \cap \mathbb{A} = \{y \in \mathbb{A} \mid \mathbf{b}_{p,q}(y - x) = 0\}. \tag{2.4.13}$$

This set does not depend on the choice of an origin $0 \in \mathbb{A}$ or of the metric in $[\mathbf{b}]$.

2.4.4 Causal flag manifolds

If G is a simple Lie group of Hermitian tube type, that is, if the symmetric space $\mathbb{X}_G$ of G is irreducible and Hermitian of tube type, then we will say that G is a HTT *Lie group*, and $\mathfrak{g}$ a HTT *Lie algebra*. In this section, we fix an HTT Lie group G with Lie algebra $\mathfrak{g}$ and prove useful preliminary results.

2.4.4.1 Strongly orthogonal roots and root system. Two roots $\alpha, \beta \in \Sigma$ are called *strongly orthogonal* if neither $\alpha + \beta$ nor $\alpha - \beta$ is a restricted root. Since G is of tube type, there exists a (maximal) set $\{2\varepsilon_1, \ldots, 2\varepsilon_r\} \subset \Sigma$ of strongly orthogonal roots, such that the set $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ is a fundamental system of Σ, where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i < r$ and $\alpha_r = 2\varepsilon_r$. The system Σ is then of type C_r (see e.g. [FK94]):

$$\begin{aligned} &\Sigma = \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i \leq j \leq r\}; \quad \Sigma^+ = \{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq r\} \cup \{2\varepsilon_i \mid 1 \leq i \leq r\}; \\ &\Delta = \{\alpha_i := \varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i \leq r-1\} \cup \{\alpha_r := 2\varepsilon_r\}. \end{aligned} \tag{2.4.14}$$

If $\Theta = \{\alpha_r\}$, then the flag manifold $\mathscr{F}(\mathfrak{g}, \Theta)$ is called the *Shilov boundary* of $\mathbb{X}_G$ and we will denote it by $\mathbf{Sb}(\mathfrak{g})$. These flag manifolds are all listed in Table 2.1.

Since Δ is of type C_r, its automorphism group is trivial (see e.g. [Kna96]). Hence the opposition involution is trivial and the flag manifold $\mathbf{Sb}(\mathfrak{g})$ is self-opposite. Moreover, the groups $\mathsf{Aut}_\Theta(\mathfrak{g})$ and $\mathsf{Aut}(\mathfrak{g})$ coincide, hence the group $\mathsf{Aut}(\mathfrak{g})$ acts on $\mathbf{Sb}(\mathfrak{g})$. Thus the set $\mathscr{G}_{\{\alpha_r\}}(\mathfrak{g})$ is here just the set of all finite index subgroups of $\mathsf{Aut}(\mathfrak{g})$.

Notation 2.4.2. When $\mathfrak{g}$ is a HTT Lie algebra, $\Theta = \{\alpha_r\}$ and $G \in \mathscr{G}_{\{\alpha_r\}}(\mathfrak{g})$, we will always use the following simplified notation:

$$\begin{gathered} \mathfrak{u}^{\pm} = \mathfrak{u}^{\pm}_{\{\alpha_r\}},\ U^{\pm} = U^{\pm}_{\{\alpha_r\}},\ \mathfrak{l} = \mathfrak{l}_{\{\alpha_r\}},\ L = L_{\{\alpha_r\}}, \\ \mathfrak{p}^{+} = \mathfrak{p}^{+}_{\{\alpha_r\}},\ P^{+} = P^{+}_{\{\alpha_r\}},\ \mathfrak{p}^{-} = \mathfrak{p}^{-}_{\{\alpha_r\}},\ P^{-} = P^{-}_{\{\alpha_r\}}. \end{gathered}$$

Since $\mathfrak{p}^+$ is a maximal proper parabolic subalgebra of $\mathfrak{g}$, the center of $\mathfrak{l}$ is one-dimensional, and one can write $\mathfrak{l} = \mathfrak{l}_s \oplus \mathbb{R}H_0$, where H_0 is in the center of $\mathfrak{l}$ and $\mathfrak{l}_s$ is the semisimple part of $\mathfrak{l}$. The possible values of $\mathfrak{l}_s$ are listed in Table 2.1. Note that the Lie algebras $\mathfrak{u}^\pm$ are abelian.

2.4.4.2 Dilations and translations. Let $\mathfrak{g}$ be a HTT Lie algebra. There exists H_0 in the center of $\mathfrak{l}$ such that $\mathfrak{u}^\pm$ is the root space of $\mathrm{ad}(H_0)$ for the eigenvalue ± 1 (see e.g. [Kan98]). If $G \in \mathscr{G}_{\{\alpha_r\}}(\mathfrak{g})$, then for all $t \in \mathbb{R}_{>0}$ we define $\ell_0(t) = \exp\left(-\log(t)\, H_0\right) \in L$. The element $\mathrm{Ad}(\ell_0(t))$ acts on $\mathfrak{u}^\pm$ by

$$\mathrm{Ad}(\ell_0(t))X = \begin{cases} tX & \forall X \in \mathfrak{u}^-; \\ \frac{1}{t}X & \forall X \in \mathfrak{u}^+. \end{cases} \tag{2.4.15}$$

Hence any positive dilation of $\mathbb{A}_{\mathsf{std}}$ (see Equation (2.2.6)) at $\mathfrak{p}^+ = \varphi_{\mathsf{std}}(0)$ can be realized as the restriction to $\mathbb{A}_{\mathsf{std}}$ of a map of the form $x \mapsto \ell_0(t) \cdot x$ of $\mathbf{Sb}(\mathfrak{g})$ for some $t \in \mathbb{R}_{>0}$.

Moreover, since $\mathfrak{u}^-$ is abelian, any translation in $\mathbb{A}_{\mathsf{std}}$ is realized as left multiplication by an element of $U^- \leq G$. Each time we will talk about *a translation* in $\mathbb{A}_{\mathsf{std}}$, it will mean that we apply a multiplication by an element of U^-.

According to the two previous paragraphs, for any affine dilation d with center a point $x_0 \in \mathbb{A}_{\mathsf{std}}$, there exists $g \in G$ such that d coincides with the restriction of the map $x \mapsto g \cdot x$ of $\mathbf{Sb}(\mathfrak{g})$ to $\mathbb{A}_{\mathsf{std}}$. Each time we will talk about *dilating at* x_0 *in* $\mathbb{A}_{\mathsf{std}}$, it will mean that we apply such a map.

Remark 2.4.3. It is not true for a general simple Lie algebra $\mathfrak{g}$ and subset of the simple restricted roots Θ that there exists $H_0 \in \mathfrak{l}_\Theta$ such that $\mathrm{ad}(H_0)X^\pm = \pm X^\pm$ for all $X^\pm \in \mathfrak{u}^\pm_\Theta$. This property is equivalent to $\mathfrak{u}^\pm_\Theta$ being abelian (and to $(\mathfrak{g}, \Theta)$ being a *Nagano pair* in the sense of Section 5.1). When this is the case, the algebra $\mathfrak{g}$ admits a decomposition $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, with $\mathfrak{g}_{-1} = \mathfrak{u}^-_\Theta$, $\mathfrak{g}_0 = \mathfrak{l}_\Theta$ and $\mathfrak{g}_1 = \mathfrak{u}^+_\Theta$, and $[\mathfrak{g}_k, \mathfrak{g}_{k'}] \subset \mathfrak{g}_{k+k'}$ for $k, k' \in \{-1, 0, 1\}$, with $\mathfrak{g}_m := \{0\}$ if $m \notin \{-1, 0, 1\}$, see Section 5.1. The element H_0 is then called the characteristic element of the graded Lie algebra $\mathfrak{g}$, that is, each space $\mathfrak{g}_k$ with $k \in \{-1, 0, 1\}$ is the eigenspace of $\mathrm{ad}(H_0)$ for the eigenvalue k.

2.4.4.3 An invariant cone and causality. It is a classical fact from [Kos10] (see also Fact 5.1.3) that the identity component L^0 of L acts irreducibly on $\mathfrak{u}^-$. By [Ben00, Prop. 4.7] applied to this action, there exists an open L^0-invariant properly convex cone c^0 in $\mathfrak{u}^-$ (see e.g. [GW25]). This cone is defined as the interior of the convex hull in $\mathfrak{u}^-$ of the orbit $\mathrm{Ad}(L^0) \cdot v^-$, where v^- is a nonzero vector of $\mathfrak{g}_{-\alpha_r}$.

Let $\mathbb{A}$ be an affine chart of $\mathbf{Sb}(\mathfrak{g})$. There exists $g \in G$ such that $\mathbb{A} = g \cdot \mathbb{A}_{\mathsf{std}}$ (recall Equation (2.2.6)). Given some point $x \in \mathbb{A}$, there exists a unique $X \in \mathfrak{u}^-$ such that $x = g \exp(X) \cdot \mathfrak{p}^+$. The set

$$\mathbf{I}_{\mathbb{A}}(x) := (g \exp(X + c^0) \cdot \mathfrak{p}^+) \cup (g \exp(X - c^0) \cdot \mathfrak{p}^+)$$

only depends on $\mathbb{A}$ and x, not on g. It has two connected components, denoted by $\mathbf{I}^+_{\mathbb{A}}(x)$, called the *future* of x in $\mathbb{A}$, and by $\mathbf{I}^-_{\mathbb{A}}(x)$, called the *past* of x in $\mathbb{A}$. The choice of these components depends on g. However, we can chose them in a continuous way, i.e. we can chose $\mathbf{I}_{\mathbb{A}}^{+}(x)$ (resp. $\mathbf{I}_{\mathbb{A}}^{-}(x)$) for all $x \in \mathbb{A}$ so that the map $x \mapsto \mathbf{I}_{\mathbb{A}}^{+}(x)$ (resp. $x \mapsto \mathbf{I}_{\mathbb{A}}^{-}(x)$) is continuous on $\mathbb{A}$ for the Hausdorff topology. We will implicitly make such a continuous choice (called a choice of *time orientation*) each time we fix an affine chart $\mathbb{A}$ of $\mathbf{Sb}(\mathfrak{g})$.

Remark 2.4.4. We have just endowed the manifold $M := \mathbf{Sb}(\mathfrak{g})$ with an invariant causal structure (see [Kan06]), i.e. a smooth G-equivariant (up to opposition) family of properly convex open cones $(c_x)_{x\in M}$ in TM. By [Nee25], Shilov boundaries associated with HTT Lie groups are the only flag manifolds admitting a causal structure, which is why we will sometimes refer to them as *causal flag manifolds*.

We can now define

$\mathbf{J}_{\mathbb{A}}^{\pm}(x) := \overline{\mathbf{I}_{\mathbb{A}}^{\pm}(x)}$, the *large future* (resp. *large past*) of x in $\mathbb{A}$;
$\mathbf{C}_{\mathbb{A}}^{\pm}(x) := \partial \mathbf{I}_{\mathbb{A}}^{\pm}(x)$, the *future lightcone* (resp. *past lightcone*) of x in $\mathbb{A}$;
$\mathbf{C}_{\mathbb{A}}(x) := \mathbf{C}_{\mathbb{A}}^{+}(x) \cup \mathbf{C}_{\mathbb{A}}^{-}(x)$, the *lightcone* of x in $\mathbb{A}$.

These sets satisfy the following straightforward properties:

Fact 2.4.5. *Let $\mathbb{A}$ be an affine chart.*

1. *The lightcone $\mathbf{C}_{\mathbb{A}}(x)$ of $x \in \mathbb{A}$ is always contained in $\mathrm{Z}_x \cap \mathbb{A}$, and $\mathbf{I}_{\mathbb{A}}^{\pm}(x)$ are connected components of $\mathbb{A} \smallsetminus \mathrm{Z}_x$.*
2. *For all $x, y, z \in \mathbb{A}$, one has:*
   * (reflexivity) $x \in \mathbf{J}_{\mathbb{A}}^{+}(y) \Leftrightarrow y \in \mathbf{J}_{\mathbb{A}}^{-}(x)$*;*
   * (antisymmetry) $\mathbf{J}_{\mathbb{A}}^{+}(y) \cap \mathbf{J}_{\mathbb{A}}^{-}(y) = \{y\}$*;*
   * (transitivity) $[x \in \mathbf{J}_{\mathbb{A}}^{\pm}(y)$ *and* $y \in \mathbf{J}_{\mathbb{A}}^{\pm}(z)\ ] \Rightarrow x \in \mathbf{J}_{\mathbb{A}}^{\pm}(z)$

Reflexivity and antisymmetry are also true replacing "$\mathbf{J}$" with "$\mathbf{C}$". Reflexivity and transitivity are also true replacing "$\mathbf{J}$" with "$\mathbf{I}$".

When $\mathbb{A} = \mathbb{A}_{\mathsf{std}}$, we will ommit the "$\mathbb{A}$" in subscript.

In general, we do not have the equality $\mathbf{C}_{\mathbb{A}}(x) = \mathrm{Z}_x \cap \mathbb{A}$. This equality is specific to the case where $\mathfrak{g} = \mathfrak{so}(n-1, 1)$, see Section 2.4.4.4.2.

The past, the future and the lightcone of a point $x \in \mathbb{A}_{\mathsf{std}}$ are not invariant under the stabilizer $\mathrm{Stab}_G(x)$ of x in G. However, they are locally invariant:

Lemma 2.4.6. *Let $x \in \mathbb{A}_{\mathsf{std}}$ and $g \in G$ be such that $g \cdot x \in \mathbb{A}_{\mathsf{std}}$. Then for any $\delta_1 \in \{-, +\}$, there exist $\delta_2 \in \{-, +\}$ and a neighborhood $\mathscr{U}$ of x such that $g \cdot (\mathscr{U} \cap \mathbf{I}^{\delta_1}(x)) \subset \mathbf{I}^{\delta_2}(g \cdot x)$.*

Proof. Noticing that $\exp(X) \cdot \mathbf{I}^{\delta_1}(\mathfrak{p}^+) = \mathbf{I}^{\delta_1}(\exp(X) \cdot \mathfrak{p}^+)$ for all $X \in \mathfrak{u}^-$, we may assume that $x = \mathfrak{p}^+$. Let us prove the lemma for $\delta_1 = +$, the proof being the same for $\delta_1 = -$.

Since $g \cdot \mathfrak{p}^+ \in \mathbb{A}_{\mathsf{std}}$, by Equation (2.2.7) one can write $g = g' \exp(Y)$, with $Y \in \mathfrak{u}^+$ and $g' \in P^-$. There exists a neighborhood $\mathscr{U}$ of P, convex in $\mathbb{A}_{\mathsf{std}}$, such that $g \cdot \mathscr{U} \subset \mathbb{A}_{\mathsf{std}}$. Hence we have $\exp(Y) \cdot \mathscr{U} \subset (g')^{-1} \cdot \mathbb{A}_{\mathsf{std}} = \mathbb{A}_{\mathsf{std}}$. Recall the map $\ell_0 : \mathbb{R}_{>0} \to L$ defined in Section 2.4.4.2. Since $\mathscr{U}$ is convex, by Equation (2.4.15), one has $\ell_0(t) \cdot \mathscr{U} \subset \mathscr{U}$ for all $t \in]0, 1]$. Then:

$$\exp(tY)\cdot\mathscr{U} = \ell_0(t)^{-1} \exp(Y)\ell_0(t)\cdot\mathscr{U} \subset \ell_0(t)^{-1} \exp(Y)\cdot\mathscr{U} \subset \ell_0(t)^{-1}\cdot\mathbb{A}_{\mathsf{std}} = \mathbb{A}_{\mathsf{std}}. \quad (2.4.16)$$

Since $\mathscr{U}$ and $\mathbf{I}^+(\mathfrak{p}^+)$ are both convex, the set $\mathscr{U} \cap \mathbf{I}^+(\mathfrak{p}^+)$ is connected. For this reason, by Equation (2.4.16) and since U^+ stabilizes $\mathrm{Z}_{\mathfrak{p}^+}$, the set $\exp(tY) \cdot (\mathscr{U} \cap \mathbf{I}^+(\mathfrak{p}^+))$ is contained in a connected component of $\mathbb{A}_{\mathsf{std}} \smallsetminus \mathrm{Z}_{\mathfrak{p}}$ for all $t \in]0,1]$, let us denote this component by $\mathscr{V}$.

By continuity, this component $\mathscr{V}$ does not depend on t. Moreover, for t small, we have

$$\exp(tY) \cdot (\mathscr{U} \cap \mathbf{I}^+(\mathfrak{p}^+)) \cap (\mathscr{U} \cap \mathbf{I}^+(\mathfrak{p}^+)) \neq \emptyset.$$

Thus $\mathscr{V} = \mathbf{I}^+(\mathfrak{p}^+)$. Hence $\exp(tY) \cdot (\mathscr{U} \cap \mathbf{I}^+(\mathfrak{p}^+)) \subset \mathbf{I}^+(\mathfrak{p}^+)$.

Now, since $g' \in \mathfrak{p}^-$, there exist $X \in \mathfrak{u}^-$ and $\ell \in L$, such that $g' = \exp(X)\ell$. But the element $\ell \in L$ either preserves $\mathbf{I}^+(\mathfrak{p}^+)$ or maps it to $\mathbf{I}^-(\mathfrak{p}^+)$ (see e.g. [GW25, Cor. 5.3]). On the other hand, since U^- is abelian, one has $\exp(X) \cdot \mathbf{I}^{\pm}(\mathfrak{p}^+) = \mathbf{I}^+(\exp(X) \cdot \mathfrak{p}^+)$. Then:

$$\begin{aligned} g \cdot (\mathscr{U} \cap \mathbf{I}^+(\mathfrak{p}^+)) &\subset g' \cdot \mathbf{I}^+(\mathfrak{p}^+) \\ &= \exp(X)\ell \cdot \mathbf{I}^+(\mathfrak{p}^+) \\ &= \begin{cases} \mathbf{I}^+(\exp(X) \cdot \mathfrak{p}^+) = \mathbf{I}^+(g \cdot \mathfrak{p}^+) & \text{if } \ell \cdot \mathbf{I}^+(\mathfrak{p}^+) = \mathbf{I}^+(\mathfrak{p}^+); \\ \mathbf{I}^-(\exp(X) \cdot \mathfrak{p}^+) = \mathbf{I}^-(g \cdot \mathfrak{p}^+) & \text{if } \ell \cdot \mathbf{I}^+(\mathfrak{p}^+) = \mathbf{I}^-(\mathfrak{p}^+). \end{cases} \end{aligned} \qquad \square$$

2.4.4.4 Examples. The complete list of Shilov boundaries associated with HTT Lie algebras is given in Table 2.1 below.

$\mathfrak{g}$	$\mathbf{Sb}(\mathfrak{g})$	$\mathfrak{l}_s$
$\mathfrak{so}(2,n)$, $n \geq 3$	$\mathrm{Ein}^{n-1,1}$	$\mathfrak{so}(n-1,1)$
$\mathfrak{sp}(2r,\mathbb{R})$	$\mathrm{Lag}_r(\mathbb{R}^{2r})$	$\mathfrak{sl}(r,\mathbb{R})$
$\mathfrak{u}(r,r)$	$\mathrm{Lag}_r(\mathbb{C}^{2r})$	$\mathfrak{sl}(r,\mathbb{C})$
$\mathfrak{so}^*(4r)$	$\mathrm{Lag}_r(\mathbb{H}^{2r})$	$\mathfrak{sl}(r,\mathbb{H})$
$\mathfrak{e}_{7(-25)}$	$(E_{6(-26)}/F_4) \times \mathbb{R}$	$\mathfrak{e}_{6(-26)}$

Table 2.1 – Shilov boundaries associated with all HTT Lie algebras.

Let us explain the notations in the table. For the notation $\mathfrak{e}_{7(-25)}$ and $\mathfrak{e}_{6(-26)}$, see [FK94] or [OV12].

2.4.4.4.1 The Lagrangians. Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, and let $r \geq 2$. Let

$$J_{\mathbb{K}} = \begin{pmatrix} 0 & -I_r \\ I_r & 0 \end{pmatrix} \in \mathrm{GL}(2r, \mathbb{K}),$$

where I_r is the identity matrix of size r. Given a matrix $g \in \mathrm{Mat}_{2r}(\mathbb{K})$, recall form Section 2.1.2 that we denote by $\overline{g}$ the matrix whose (i,j)-th entry is the conjugate (in $\mathbb{K}$) of the (i,j)-th entry of g. Let $G_{\mathbb{K}} := \left\{ g \in \mathrm{SL}(2r, \mathbb{K}) \mid {}^t\overline{g} J_{\mathbb{K}} g = J_{\mathbb{K}} \right\}$. Then $G_{\mathbb{K}}$ is a HTT Lie group, and one has $G_{\mathbb{R}} = \mathrm{Sp}(2r,\mathbb{R})$, $G_{\mathbb{C}} = \mathrm{SU}(r,r)$ and $G_{\mathbb{H}} = \mathrm{SO}^*(4r)$. Let us describe its root system explicitly. The space

$$\mathfrak{a} := \{\mathrm{diag}(\lambda_1, \ldots, \lambda_r, -\lambda_1, \ldots, -\lambda_r) \mid \lambda_i \in \mathbb{R} \ \forall 1 \leq i \leq r\}$$

is a Cartan subspace of $\mathfrak{g}_{\mathbb{K}}$. If we define

$$\varepsilon_i : \mathrm{diag}(\lambda_1, \ldots, \lambda_r, -\lambda_1, \ldots, -\lambda_r) \longmapsto \lambda_i. \tag{2.4.17}$$

then the strongly orthogonal roots defined in Section 2.4.4.1 in can be taken to be $(2\varepsilon_i)_{1\leq i\leq r}$. Let Δ be the associate fundamental system simple restricted roots of $\mathfrak{g}_{\mathbb{K}}$ (by Equation (2.4.14)).

Now let $\mathbf{b}$ be the bilinear form whose matrix in the canonical basis of $\mathbb{K}^{2r}$ is $J_{\mathbb{K}}$. The space $\mathrm{Lag}_r(\mathbb{K}^{2r})$ is the space of Lagrangians of $(\mathbb{K}, \mathbf{b})$, i.e. the space of totally isotropic r-planes of $\mathbb{K}^{2r}$ (on the left, for $\mathbb{K} = \mathbb{H}$).

The group $G_{\mathbb{K}}$ acts transitively on $\mathrm{Lag}_r(\mathbb{K}^{2r})$. If $(e_1, \dots e_{2r})$ is the canonical basis of $\mathbb{K}^{2r}$, then the parabolic P of Notation 2.4.2 is the stabilizer in $G_{\mathbb{K}}$ of $\xi_0 := \mathrm{Span}(e_1, \dots, e_r)$, and $P^-_{\{\alpha_r\}}$ is the stabilizer in $G_{\mathbb{K}}$ of $\xi_\infty := \mathrm{Span}(e_{r+1}, \dots, e_{2r})$. Thus Equation (2.2.4) gives a $G_{\mathbb{K}}$-identification

$$\mathrm{Lag}_r(\mathbb{K}^{2r}) \simeq \mathbf{Sb}(\mathfrak{g}_{\mathbb{K}}).$$

We make this identification till the end of this section. This model gives the following descriptions of $\mathfrak{u}^\pm$ and L:

$$\begin{aligned}
\mathfrak{u}^- &= \Big\{ \begin{pmatrix} 0_r & 0 \\ X & 0_r \end{pmatrix} \mid X \in \mathrm{Mat}_r(\mathbb{K}),\ {}^t\overline{X} = X \Big\};\\
\mathfrak{u}^+ &= \{ \begin{pmatrix} 0_r & X \\ 0 & 0_r \end{pmatrix} \mid X \in \mathrm{Mat}_r(\mathbb{K}),\ {}^t\overline{X} = X \Big\};\\
L &= \big\{ \mathrm{diag}(A,\ {}^t\overline{A}^{-1}) \mid A \in \mathrm{GL}(r, \mathbb{K}) \big\}.
\end{aligned}$$

The action of an element $\ell = \mathrm{diag}(A,\ {}^t\overline{A}^{-1}) \in L$ on $v = \begin{pmatrix} 0_r & 0 \\ X & 0_r \end{pmatrix} \in \mathfrak{u}^-$ is given by

$$\mathrm{Ad}(\ell) \cdot v = \begin{pmatrix} 0_r & 0 \\ {}^t\overline{A}^{-1} X A^{-1} & 0_r \end{pmatrix} \tag{2.4.18}$$

The standard affine chart $\mathbb{A}_{\mathsf{std}} = \mathrm{Lag}_r(\mathbb{K}^{2r}) \smallsetminus \mathrm{Z}_{\mathfrak{p}^-}$ defined in Equation (2.2.6) can be described as follows:

$$\mathbb{A}_{\mathsf{std}} = \Big\{ \mathrm{Im} \begin{pmatrix} I_r \\ X \end{pmatrix} \ \Big|\ X \in \mathrm{Mat}_r(\mathbb{K}),\ {}^t\overline{X} = X \Big\}.$$

Then a computation gives:

$$\begin{aligned}
\mathbb{A}_{\mathsf{std}} \cap \mathrm{Z}_{\mathfrak{p}^+} &= \Big\{ \mathrm{Im} \begin{pmatrix} I_r \\ X \end{pmatrix} \ \Big|\ {}^t\overline{X} = X,\ \det(X) = 0 \Big\};\\
\mathbf{C}(\mathfrak{p}^+) &= \Big\{ \mathrm{Im} \begin{pmatrix} I_r \\ X \end{pmatrix} \ \Big|\ {}^t\overline{X} = X,\ \det(X) = 0 \text{ and } {}^t\overline{x} X x \in \mathbb{R}_{\geq 0} \quad \forall x \in \mathbb{K}^r \Big\}.
\end{aligned}$$

Note that if $r \geq 3$, then the inclusion $\mathbf{C}(\mathfrak{p}^+) \subset \mathbb{A}_{\mathsf{std}} \cap \mathrm{Z}_{\mathfrak{p}^+}$ is strict.

2.4.4.4.2 Einstein universe. The Einstein universe of signature $(n-1, 1)$ has been defined in Section 2.4.1. It has been defined as the flag manifold $\mathscr{F}(\mathfrak{so}(n, 2), \alpha_1)$. However, when we consider it as the Shilov boundary associated with the HTT Lie algebra $\mathfrak{so}(n, 2)$, given the setting introduced in this section, it is more relevent to reverse the ordering of the two simple restricted roots of $\mathfrak{so}(n,2)$, to get $\mathrm{Ein}^{n-1,1} = \mathscr{F}(\mathfrak{so}(n,2), \alpha_2)$.

The causal structure of $\mathrm{Ein}^{n-1,1}$ as a Shilov boundary is exactly the same as its causal structure as a Lorentzian manifold: with the notation of Section 2.4.1, for any affine chart $\mathbb{A}$ of $\mathrm{Ein}^{n-1,1}$ and $y_0 \in \mathbb{A}$, we have $\mathbb{A} \cap \mathrm{Z}_{y_0} = \mathbf{C}_{\mathbb{A}}(y_0)$. Hence in this case, the inclusion $\mathbf{C}_{\mathbb{A}}(y_0) \subset \mathrm{Z}_{y_0} \cap \mathbb{A}$ is an equality. By Lemma 6.3.9, which will be stated and proven in Section 6.3, this equality is due to the fact that

$$|W_{\Delta \smallsetminus \{\alpha_1\}} \backslash W / W_{\Delta \smallsetminus \{\alpha_1\}}| = 3,$$

as already mentioned in Section 2.4.3.1, and is specific to the case of $\mathrm{Ein}^{n-1,1}$.

2.5 Equivalence between flag manifolds

Before moving on to the next chapter, we must emphasize a subtlety in the use of certain notations for flag manifolds.

Let $n \in \mathbb{N}_{>0}$. In the notation of Example 2.2.1.(1), the real projective space $\mathbb{P}(\mathbb{R}^{2n})$ is a flag manifold of $\mathfrak{sl}(2n, \mathbb{R})$ associated with $\mathfrak{sl}(2n, \mathbb{R})$ and $\{\alpha_1\}$, see Section 2.4.2.

On the other hand, if $(\mathfrak{g}, \alpha) = (\mathfrak{sp}(2n, \mathbb{R}), \alpha_1)$ (where the restricted root system of $\mathfrak{sp}(2n, \mathbb{R})$ has been defined in Section 2.4.4.1), then any group $G \in \mathscr{G}_{\{\alpha_1\}}(\mathfrak{g})$ acts transitively on $\mathbb{P}(\mathbb{R}^{2n})$, and the stabilizer of a point is conjugate to the parabolic subgroup $P_{\{\alpha_1\}}$ of G. Thus the real projective space $\mathbb{P}(\mathbb{R}^{2n})$ can also be endowed with the structure of a flag manifold associated with $\mathfrak{sp}(2n, \mathbb{R})$ and $\{\alpha_1\}$. The two flag manifolds $\mathscr{F}(\mathfrak{sp}(2n, \mathbb{R}), \alpha_1)$ and $\mathscr{F}(\mathfrak{sl}(2n, \mathbb{R}), \alpha_1)$ here are equal *as G-homogeneous manifolds*, but are not equivalent as flag manifolds.

From now on, whenever we talk about the *real projective space* or write $\mathbb{P}(\mathbb{R}^n)$, it will only be endowed with its structure of flag manifold associated with $\mathfrak{sl}(n, \mathbb{R})$ and $\{\alpha_1\}$.

Similarly, whenever we talk about the *Grassmannian of p-planes of* $\mathbb{R}^{p+q}$ (resp. the *Einstein universe of signature* (p,q)) or write $\mathrm{Gr}_p(\mathbb{R}^{p+q})$ (resp. $\mathrm{Ein}^{p,q}$), they will be endowed with their structure of flag manifold associated with $\mathfrak{sl}(p+q, \mathbb{R})$ and $\{\alpha_p\}$ (resp. with $\mathfrak{so}(p+1, q+1)$ and $\{\alpha_1\}$).

Chapter 3

Proper domains and convexity in flag manifolds

The aim of this chapter is, first, to review the tools and notions that are already known and useful for studying *proper dually convex domains* in flag manifolds. These tools are built in a way that generalizes classical convex projective geometry. For this reason, the study of proper domains is close to those of convex projective geometry.

However, several important results holding in convex projective geometry fail in the general case, which can make the study of proper domains more challenging. We conduct an in-depth analysis of the different possible notions of convexity and their analogies with the projective case, focusing on our three key examples of flag manifolds: the Grassmannians (Section 3.3), the Einstein universes (Section 3.4), and causal flag manifolds (Section 3.5). Through these three families of examples, we highlight the subtleties that arise when attempting to define convexity in flag manifolds.

3.1 Reminders: proper domains in flag manifolds

In this section, we recall some definitions and properties of domains in a flag manifold, generalizing those of classical convex projective geometry. Most of them were introduced in [Zim18a].

3.1.1 Generalities on proper domains

Let $\mathfrak{g}$ be a real semisimple Lie algebra of noncompact type and $\Theta \subset \Delta$ a subset of the simple restricted roots.

Definition 3.1.1. Let $\Omega \subset \mathscr{F}(\mathfrak{g}, \Theta)$ be a subset. We say that Ω is:

1. a *domain* if Ω is open, nonempty and connected;
2. *proper* if there exists $\xi \in \mathscr{F}(\mathfrak{g}, \Theta)^-$ such that $\overline{\Omega} \cap \mathrm{Z}_\xi = \emptyset$. In particular, if $\xi = \mathfrak{p}_\Theta^-$, then we will say that Ω is *proper in* $\mathbb{A}_{\mathsf{std}}$. This is equivalent to saying that $\overline{\Omega} \subset \mathbb{A}_{\mathsf{std}}$.

Remark 3.1.2. Given a proper domain Ω of $\mathscr{F}(\mathfrak{g}, \Theta)$, we will always be able to assume that Ω is proper in $\mathbb{A}_{\mathsf{std}}$. Indeed, since any $G \in \mathscr{G}_\Theta(\mathfrak{g})$ acts transitively on $\mathscr{F}(\mathfrak{g}, \Theta)^-$, there exists $g \in G$ such that $\overline{g \cdot \Omega} \subset \mathbb{A}_{\mathsf{std}}$, and the properties we will investigate on Ω will be invariant under the action of G on $\mathscr{F}(\mathfrak{g}, \Theta)$. In this case, it will be possible to see Ω as a bounded domain of the affine space $\mathbb{A}_{\mathsf{std}}$.

3.1.1.1 The automorphism group. Given an open subset $\Omega \subset \mathscr{F}(\mathfrak{g}, \Theta)$ and a Lie group $G \in \mathscr{G}_\Theta(\mathfrak{g})$, the *automorphism group* of Ω with respect to G is

$$\mathsf{Aut}_G(\Omega) = \{g \in G \mid g \cdot \Omega = \Omega\}.$$

One has:

Fact 3.1.3. *[Zim18a] The group $\mathsf{Aut}_G(\Omega)$ is a Lie subgroup of G. Moreover, it acts properly on Ω as soon as Ω is a proper domain.*

Since G has finite index in $\mathsf{Aut}_\Theta(\mathfrak{g})$, the group $\mathrm{Ad}(\mathsf{Aut}_G(\Omega))$ has finite index in $\mathsf{Aut}_{\mathsf{Aut}_\Theta(\mathfrak{g})}(\Omega)$.

Remark 3.1.4.
1. Let $p, q \in \mathbb{N}_{>0}$. Given an open subset $\Omega \subset \mathrm{Ein}^{p,q}$, by Fact 2.4.1, the automorphism group $\mathsf{Aut}_{\mathrm{PO}(p+1,q+1)}(\Omega)$ of Ω is equal to its *conformal group* $\mathrm{Conf}(\Omega)$.
2. In the case where $\mathfrak{g}$ is a HTT Lie algebra and $\mathscr{F}(\mathfrak{g}, \Theta) = \mathbf{Sb}(\mathfrak{g})$, recall that we have $\mathsf{Aut}_\Theta(\mathfrak{g}) = \mathsf{Aut}(\mathfrak{g})$ and the group $\mathsf{Aut}_{\mathsf{Aut}(\mathfrak{g})}(\Omega)$ is commensurable to the *conformal group* of Ω, that is, the group of all invertible maps from Ω to itself that preserve the causal structure of Ω [Kan11, Thm 2.3].

The domain $\Omega \subset \mathscr{F}(\mathfrak{g}, \Theta)$ is said to be *homogeneous* if *there exists* $G \in \mathscr{G}_\Theta(\mathfrak{g})$, such that $\mathsf{Aut}_G(\Omega)$ acts transitively on Ω.

The domain Ω is said to be *quasi-homogeneous* if there exists a compact subset $\mathsf{K} \subset \Omega$ such that $\Omega = \mathsf{Aut}_G(\Omega) \cdot \mathsf{K}$. It is said to be *divisible* if there exists a discrete subgroup $\Gamma \leq \mathsf{Aut}_G(\Omega)$ and a compact subset $\mathsf{K} \subset \Omega$ such that $\Omega = \Gamma \cdot \mathsf{K}$. Since $\mathrm{Ad}(\mathsf{Aut}_G(\Omega))$ has finite index in $\mathsf{Aut}_{\mathsf{Aut}_\Theta(\mathfrak{g})}(\Omega)$, these two properties do not depend on $G \in \mathscr{G}_\Theta(\mathfrak{g})$.

The *full orbital limit set* of Ω is the set

$$\Lambda_\Omega^{\mathrm{orb}} := \bigcup_{x \in \Omega} (\overline{\mathsf{Aut}_G(\Omega) \cdot x}) \smallsetminus (\mathsf{Aut}_G(\Omega) \cdot x),$$

see [DGK24]. This set does not depend on $G \in \mathscr{G}_\Theta(\mathfrak{g})$, since $\mathrm{Ad}(\mathsf{Aut}_G(\Omega)) \leq \mathsf{Aut}_{\mathsf{Aut}_\Theta(\mathfrak{g})}(\Omega)$ has finite index. If Ω is proper, since $\mathsf{Aut}_G(\Omega)$ acts properly on Ω by Fact 3.1.3, we have $\Lambda_\Omega^{\mathrm{orb}} \subset \partial\Omega$.

Definition 3.1.5. A proper domain Ω is said to be *almost-homogeneous* if $\Lambda_\Omega^{\mathrm{orb}} = \partial\Omega$.

A proper domain $\Omega \subset \mathscr{F}(\mathfrak{g}, \Theta)$ is almost-homogeneous if and only if for all $a \in \partial\Omega$, there exist $x \in \Omega$ and $(g_k) \in \mathsf{Aut}_G(\Omega)^{\mathbb{N}}$ such that $g_k \cdot x \to a$. For any proper domain $\Omega \subset \mathscr{F}(\mathfrak{g}, \Theta)$, we have:

$$\Omega \text{ divisible } \Longrightarrow \Omega \text{ quasi-homogeneous } \Longrightarrow \Omega \text{ almost-homogeneous.}$$

Remark 3.1.6. 1. The three properties above are in general not equivalent. For instance, when $(\mathfrak{g}, \Theta) = (\mathfrak{sl}(n, \mathbb{R}), \{\alpha_1\})$ with $n \geq 3$ (in the notation of Example 2.2.1.(1)), there exist proper domains in $\mathscr{F}(\mathfrak{g}, \Theta) = \mathbb{P}(\mathbb{R}^n)$ that are almost-homogeneous but not quasi-homogeneous [BM20]. There also exist proper homogeneous (and thus quasi-homogeneous) domains of $\mathbb{P}(\mathbb{R}^n)$ that are not divisible [Vin65]. A consequence of Lemma 8.1.2 and Theorems 8.3.1 and 8.4.1 will be that these three properties are equivalent when $\mathscr{F}(\mathfrak{g}, \Theta)$ is the Einstein Universe of signature (p, q) $(p, q \geq 1)$, a causal flag manifold or when $|\Theta| \geq 2$.

2. Since the group G does not play a role in the notions of divisibility, quasi-homogeneity and almost-homogeneity, these properties are invariant under equivalence of flag manifolds. However, they are not invariant under diffeomorphisms between flag manifolds. For instance, we saw in Section 2.5 that $\mathbb{P}(\mathbb{R}^{2n}) = \mathscr{F}(\mathfrak{sl}(2n, \mathbb{R}), \alpha_1)$ identifies $\mathrm{Sp}(2n, \mathbb{R})$-equivariantly with $\mathscr{F}(\mathfrak{sp}(2n, \mathbb{R}), \alpha_1)$. However, given a domain $\Omega \subset \mathscr{F}(\mathfrak{sp}(2n, \mathbb{R}), \alpha_1)$, the automorphism group $\mathsf{Aut}_{\mathrm{Sp}(2n,\mathbb{R})}(\Omega)$ is in general smaller (even up to finite index) than its automorphism group $\mathsf{Aut}_{\mathrm{SL}(2n,\mathbb{R})}(\Omega)$ if we see Ω as a domain in $\mathbb{P}(\mathbb{R}^n)$, see e.g. Section 8.7. One must always be careful, as the notions of divisibility, quasi-homogeneity, and almost-homogeneity are not preserved under diffeomorphisms between flag manifolds.

The domain Ω is said to be *symmetric* if for any $x \in \Omega$ there exists an order-two element $s_x \in \mathsf{Aut}_{\mathsf{Aut}_\Theta(\mathfrak{g})}(\Omega)$ such that x is the only fixed point of s_x in Ω.

Note that $\mathsf{Aut}_G(g \cdot \Omega) = g\mathsf{Aut}_G(\Omega)g^{-1}$ for all $g \in G$; therefore the property of being almost-homogeneous, resp. symmetric, is invariant under the action of G on $\mathscr{F}(\mathfrak{g}, \Theta)$. Thus by Remark 3.1.2, it will always be possible to assume that Ω is proper in $\mathbb{A}_{\mathsf{std}}$, and given a point $x \in \overline{\Omega}$, we can always assume furthermore that $x = \mathfrak{p}_\Theta^+$, up to translating Ω by an element of U^-.

We will make use of the following lemma:

Lemma 3.1.7. *Let $G \in \mathscr{G}_\Theta(\mathfrak{g})$, and let Ω, Ω' be two proper domains of $\mathscr{F}(\mathfrak{g}, \Theta)$ such that $\Omega \subset \Omega'$. Assume that $\mathsf{Aut}_G(\Omega) \subset \mathsf{Aut}_G(\Omega')$ and that Ω is almost-homogeneous. Then $\Omega = \Omega'$.*

Proof. Let $a \in \partial\Omega$. There exist $x \in \Omega$ and $(g_k) \in \mathsf{Aut}_G(\Omega)^{\mathbb{N}}$ such that $g_k \cdot x \to a$. Thus (g_k) is unbounded in G. Since $\mathsf{Aut}_G(\Omega) \subset \mathsf{Aut}_G(\Omega')$, the group $\mathsf{Aut}_G(\Omega)$ acts properly on the proper domain Ω'. Thus $a \in \partial\Omega'$. We have proven that Ω is closed in Ω'; since it is also open, and Ω' is connected, we have $\Omega = \Omega'$. ☐

3.1.1.2 The dual . Let $\Omega \subset \mathscr{F}(\mathfrak{g}, \Theta)$ be a subset. The *dual of* Ω is the set

$$\Omega^* := \{\xi \in \mathscr{F}(\mathfrak{g}, \Theta)^- \mid \mathrm{Z}_\xi \cap \Omega = \emptyset\} \subset \mathscr{F}(\mathfrak{g}, \Theta)^-.$$

Let us recall some properties of this set (see [Zim18a]):

1. For all for all $G \in \mathscr{G}_\Theta(\mathfrak{g})$, the set Ω^* is $\mathsf{Aut}_G(\Omega)$-invariant. Equivalently, one has $\mathsf{Aut}_G(\Omega) \subset \mathsf{Aut}_G(\Omega^*)$.
2. If Ω is open, then Ω^* is compact.
3. The domain Ω is proper if and only if its dual Ω^* has nonempty interior.

Remark 3.1.8. If $(\mathfrak{g}, \Theta) = (\mathfrak{sl}(n, \mathbb{R}), \alpha_1)$, then for any proper domain

$$\Omega \subset \mathscr{F}(\mathfrak{g}, \Theta) = \mathbb{P}(\mathbb{R}^n),$$

the dual Ω^* is properly convex, and $\Omega^* = \overline{\{\xi \in \mathscr{F}(\mathfrak{g}, \Theta) \mid \mathrm{Z}_\xi \cap \overline{\Omega} = \emptyset\}}$ This is not true for a general flag manifold $\mathscr{F}(\mathfrak{g}, \Theta)$; see Example 3.5.9.

In [Zim18a], A. Zimmer defines the following notion of convexity:

Definition 3.1.9. An open subset $\Omega \subset \mathscr{F}(\mathfrak{g}, \Theta)$ is *dually convex* if for all $a \in \partial\Omega$, there exists $\xi \in \Omega^*$ such that $a \in \mathrm{Z}_\xi$.

This notion of convexity generalizes the one in real projective space. Indeed, if $\Omega \subset \mathbb{P}(\mathbb{R}^n)$ is a proper domain, then it is a classical fact that Ω is properly convex — in the sense of Section 2.1.1.3 — if and only if for all $a \in \partial\Omega$, there exists a projective hyperplane $H \in \mathbb{P}(\mathbb{R}^n)$ such that $a \in H$ and $\Omega \cap H = \emptyset$. As we will see in Example 3.1.12 and in the rest of this chapter, it is in general not true in a flag manifold $\mathscr{F}(\mathfrak{g}, \Theta)$ that dual convexity is equivalent to convexity in an affine chart; this fact seems to be specific to the real projective case.

Given a proper domain $\Omega \subset \mathscr{F}(\mathfrak{g}, \Theta)$, its *bidual* Ω^{**} is a proper open set containing Ω, not necessarily connected (see e.g. Remark 5.3.3). It is however dually convex, and each of its connected components are also dually convex.

Definition 3.1.10. Let $\Omega \subset \mathscr{F}(\mathfrak{g}, \Theta)$ be a proper domain. We denote by Ω_0^{**} the connected component of Ω^{**} which contains Ω, and call it the *dual convex hull* of Ω.

The dual convex hull of a proper domain is a proper dually convex domain containing Ω.

In [Zim18a], Zimmer proves that quasi-homogeneous domains are dually convex. In this memoir, we will use the slightly stronger following result, whose proof relies on the one of [Zim18a, Cor. 9.3]:

Proposition 3.1.11. *Any proper almost-homogeneous domain of $\mathscr{F}(\mathfrak{g}, \Theta)$ is dually convex.*

Proof. Since the proper domain Ω_0^{**} given in Definition 3.1.10 is $\mathsf{Aut}_G(\Omega)$-invariant (for any $G \in \mathscr{G}_\Theta(\mathfrak{g})$) and contains Ω, by Lemma 3.1.7 we have $\Omega = \Omega_0^{**}$. The Proposition then follows from the dual convexity of Ω_0^{**}. ☐

Example 3.1.12. If $(\mathfrak{g}, \Theta) = (\mathfrak{so}(n, 1), \alpha_1)$ with $n \geq 1$ in the notation of Section 2.4.3, then by Equation (2.4.9), a domain $\Omega \subset \mathbb{S}^{n-1} := \mathscr{F}(\mathfrak{g}, \alpha_1)$ is proper if and only if we have $\overline{\Omega} \neq \mathbb{S}^{n-1}$. Moreover, again by Equation (2.4.9), any open subset of $\mathbb{S}^{n-1}$ is dually convex. Thus we cannot expect a structural result for dually convex domain of a general flag manifold $\mathscr{F}(\mathfrak{g}, \Theta)$, as this property is empty for the conformal sphere $\mathbb{S}^{n-1}$. However, proper dually convex domains of real projective space have very nice properties: they are contractible, equal to the interior of their closure and to their bidual, and their dual is convex. We could thus ask for which families of flag manifolds these properties, or weaker versions of them, are satisfied. We will see in Sections from 3.3 up to 3.5, with our three key families of flag manifolds, that most of these properties are not satisfied. However, as we will see in Proposition 7.1.9 of Section 7.1, that for a certain family of flag manifolds (namely, *Nagano spaces of real type*, see Section 7.1), we can recover the property that proper dually convex domains are equal to the interior of their closure, and we have a sufficient condition for the dual to be connected.

3.1.1.3 Dual faces. Let $\Omega \subset \mathscr{F}(\mathfrak{g}, \Theta)$ be a proper domain. Given some point $x \in \partial\Omega$, the *dual support to* Ω *at* x is the set $\mathrm{Supp}_\Omega(x) := \{\xi \in \Omega^* \mid x \in \mathrm{Z}_\xi\}$. This set is nonempty whenever Ω is dually convex. The *dual face of* x is then the subset

$$\mathscr{F}^d_\Omega(x) := \bigcap_{\xi \in \mathrm{Supp}_\Omega(x)} \partial\Omega \cap \mathrm{Z}_\xi$$

of $\partial\Omega$. The dual faces of Ω are always closed. If $\mathscr{F}(\mathfrak{g}, \Theta)$ is the real projective space and Ω is convex, then we recover the classical closed faces of Ω.

3.1.2 The Caratheodory metrics

Let $\mathfrak{g}$ be a real semisimple Lie algebra of noncompact type and $\Theta \subset \Delta$ be a subset of the simple restricted roots of $\mathfrak{g}$. Let $G \in \mathscr{G}_\Theta(\mathfrak{g})$. If $\Omega \subset \mathscr{F}(\mathfrak{g}, \Theta)$ is a domain, then we will say that a metric d on Ω is $\mathsf{Aut}_G(\Omega)$-*invariant* if $d(g \cdot x, g \cdot y) = d(x, y)$ for all $x, y \in \Omega$ and $g \in \mathsf{Aut}_G(\Omega)$. In this section, we recall A. Zimmer's construction of $\mathsf{Aut}_G(\Omega)$-invariant metrics on proper domains Ω, called *Caratheodory metrics*.

Let (G, ρ, V) be a linear or projective Θ-proximal triple of $\mathfrak{g}$, in the sense of Definition 2.3.3. Let $\iota_\rho : \mathscr{F}(\mathfrak{g}, \Theta) \hookrightarrow \mathbb{P}(V)$ and $\iota_\rho^- : \mathscr{F}(\mathfrak{g}, \Theta)^- \hookrightarrow \mathbb{P}(V^*)$ be the two embeddings induced by ρ, see Fact 2.3.4. Given $x, y \in \mathscr{F}(\mathfrak{g}, \Theta)$ and $\xi, \eta \in \mathscr{F}(\mathfrak{g}, \Theta)^-$, we choose lifts

$$\nu_x \in \iota_\rho(x) \smallsetminus \{0\}; \quad \nu_y \in \iota_\rho(y) \smallsetminus \{0\}; \quad f_\xi \in \iota_\rho^-(\xi) \smallsetminus \{0\}; \quad f_\eta \in \iota_\rho^-(\eta) \smallsetminus \{0\}.$$

We define the *cross ratio* of ξ, x, y, η *relative to* (V, ρ) as follows:

$$[\xi : x : y : \eta]_\rho := \frac{f_\xi(\nu_x) f_\eta(\nu_y)}{f_\xi(\nu_y) f_\eta(\nu_x)}. \tag{3.1.1}$$

This quantity does not depend on the choice of representatives $\nu_x, \nu_y, f_\xi, f_\eta$. In [Zim18a], Zimmer introduces the following map C^ρ_Ω associated with a domain $\Omega \subset \mathscr{F}(\mathfrak{g}, \Theta)$:

$$\begin{array}{rccc} C^\rho_\Omega : & \Omega \times \Omega & \longrightarrow & \mathbb{R}_+ \\ & (x, y) & \longmapsto & \sup_{\xi, \eta \in \Omega^*} \log \left| [\xi : x : y : \eta]_\rho \right|. \end{array}$$

By [Zim18a, Theorems 5.2 and 9.1], as soon as Ω is a proper domain of $\mathscr{F}(\mathfrak{g}, \Theta)$, the map C^ρ_Ω is an $\mathsf{Aut}_G(\Omega)$-invariant metric for any $G \in \mathscr{G}_\Theta(\mathfrak{g})$, generating the standard topology. Whenever this is the case, we will say that C^ρ_Ω is the *Caratheodory metric* on Ω *induced by* ρ.

The following fact follows from the definition of the Caratheodory metrics:

Fact 3.1.13. *Let (V, ρ) be a finite-dimensional representation of G such that (G, ρ, V) is linear (resp. projective) Θ-proximal triple of the Lie algebra $\mathfrak{g}$ of G, in the sense of Definition 2.3.3. Let $\Omega \subset \mathscr{F}(\mathfrak{g}, \Theta)$ be a proper domain. For any two sequences $(x_k), (y_k) \in \Omega^\mathbb{N}$, we have:*

1. *If $C^\rho_\Omega(x_k, y_k) \to 0$, then $y_k \to x$ whenever $x_k \to x$ (even if $x \in \partial\Omega$).*
2. *If $\sup_{k \in \mathbb{N}} C^\rho_\Omega(x_k, y_k) < +\infty$ and $x_k, y_k \to x, y \in \overline{\Omega}$, then $y \in \mathscr{F}^d_\Omega(x)$.*

3.2 Dual convexity and convexity in an affine chart

In this section, we fix a real semisimple Lie algebra of noncompact type $\mathfrak{g}$ and a subset Θ of the simple restricted roots of $\mathfrak{g}$. Let $\mathbb{A} \subset \mathscr{F}(\mathfrak{g}, \Theta)$ be an affine chart. A domain Ω that is properly convex in $\mathbb{A}$ — with respect to the canonical affine structure on $\mathbb{A}$ — admits, at every boundary point $a \in \partial\Omega$, a supporting affine hyperplane (in $\mathbb{A}$). If Ω is moreover dually convex, it also admits a supporting maximal proper Schubert variety at a. In this section, we highlight the connection between these two supporting varieties.

Given an algebraic variety Z, we recall that a *regular point* of Z is a point $x \in \mathrm{Z}$ such that Z is a smooth variety in a neighborhood of x.

Proposition 3.2.1. *Assume that the varieties of the form Z_ξ, with $\xi \in \mathscr{F}(\mathfrak{g}, \Theta)^-$, are hypersurfaces of $\mathscr{F}(\mathfrak{g}, \Theta)$ (i.e. are of codimension 1). Suppose that Ω is bounded and convex in an affine chart $\mathbb{A}$, and dually convex. Let $x \in \partial\Omega \cap \mathbb{A}$ and $\eta \in \mathscr{F}(\mathfrak{g}, \Theta)^-$ such that $x \in \mathrm{Z}_\eta$ and $\Omega \cap \mathrm{Z}_\eta = \emptyset$. If x is a regular point of Z_η, then $T_x(\mathrm{Z}_\eta \cap \mathbb{A})$ is a supporting hyperplane of Ω in $\mathbb{A}$.*

Proof. We may assume that $\mathbb{A} = \mathbb{A}_{\mathsf{std}}$ is the standard affine chart defined in Equation (2.4.6), and that $x = \varphi_{\mathsf{std}}(0)$. Since x is a regular point of the algebraic hypersurface $\mathrm{Z}_\eta \cap \mathbb{A}_{\mathsf{std}}$ of $\mathbb{A}$, there exist a neighborhood $\mathscr{U}$ of 0 in $\mathfrak{u}_\Theta^-$ and a smooth map $f : \mathscr{U} \to \mathbb{R}$ such that $\varphi_{\mathsf{std}}^{-1}(\mathrm{Z}_\eta) \cap \mathscr{U} = f^{-1}(\{0\})$. For all $Y \in \varphi_{\mathsf{std}}^{-1}(\Omega) \cap \mathscr{U}$, one has $f(Y) \neq 0$. Since Ω is convex in $\mathbb{A}$, we may take $\mathscr{U}$ so that $\Omega \cap \mathscr{U}$ is connected. Then we may assume that $f(\varphi_{\mathsf{std}}^{-1}(\Omega) \cap \mathscr{U}) \subset \mathbb{R}_{>0}$. We have:

$$f(Y) = f(0) + d_0 f(Y) + o(||Y||) = d_0 f(Y) + o(||Y||)$$

in a neighborhood of 0. Since $f(Y) > 0$ for $Y \in \varphi_{\mathsf{std}}^{-1}(\Omega) \cap \mathscr{U}$, there exists a neighborhood $\mathscr{V} \subset \mathscr{U}$ of 0 such that $d_0 f(Y) \geq 0$ for all $Y \in \varphi_{\mathsf{std}}^{-1}(\Omega) \cap \mathscr{V}$. Thus $d_0 f(\mathscr{V} \cap \varphi_{\mathsf{std}}^{-1}(\Omega)) \subset \mathbb{R}_{\geq 0}$.

Now let us prove that $d_0 f(\varphi_{\mathsf{std}}^{-1}(\Omega)) \subset \mathbb{R}_{\geq 0}$. Let us assume by contradiction that there exists $Y \in \varphi_{\mathsf{std}}^{-1}(\Omega)$ such that $d_0 f(Y) < 0$. For all $k \in \mathbb{N}_{>0}$, we set $Y_k := \frac{1}{k} Y$. Then by convexity of Ω, we have $Y_k \in \overline{\varphi_{\mathsf{std}}^{-1}(\Omega)}$. Moreover one has $Y_k \to 0$ as $k \to +\infty$. Let $k_0 \in \mathbb{N}_{>0}$ such that $\varphi_{\mathsf{std}}(Y_k) \in \Omega \cap \mathscr{V}$ for all $k \geq k_0$. Then for $k \geq k_0$, one has

$$0 \geq d_0 f(Y_k) = \frac{1}{k} d_0 f(Y) < 0.$$

This is a contradiction. Thus $d_0 f(\varphi_{\mathsf{std}}^{-1}(\Omega)) \subset \mathbb{R}_{\geq 0}$ and $T_0(\varphi_{\mathsf{std}}^{-1}(\mathrm{Z}_\eta))$ is a supporting hyperplane of $\varphi_{\mathsf{std}}^{-1}(\Omega)$. □

Remark 3.2.2. In the notation of Section 7.1, the assumptions of Proposition 3.2.1 will be satisfied for every *Nagano space $\mathscr{F}(\mathfrak{g}, \alpha)$ of real type*: the maximal proper Schubert subvarieties — i.e. the varieties of the form Z_ξ, with $\xi \in \mathscr{F}(\mathfrak{g}, \Theta)^-$ — will be hypersurfaces of $\mathscr{F}(\mathfrak{g}, \Theta)$.

3.3 Proper domains in the Grassmannians

In this section, we investigate proper domains of Grassmannians, i.e. the flag manifolds $\mathrm{Gr}_p(\mathbb{R}^{p+q}) = \mathscr{F}(\mathfrak{sl}(p+q,\mathbb{R}),\alpha_p)$. We will intensively use the notation introduced in Section 2.4.2 for the Grassmannians, in particular the map φ_p defined in Equation (2.4.7).

3.3.1 The symmetric domain

In this section, we investigate a model of a Riemannian symmetric space embedded in Grassmannian. This model is a proper symmetric domain of $\mathrm{Gr}_p(\mathbb{R}^{p+q})$. In the notation of Chapter 5, it will actually be a realization of the non-compact dual of the Nagano space $\mathrm{Gr}_p(\mathbb{R}^{p+q})$.

Let $1 \leq p \leq q$ and let $\mathbf{b}$ be a bilinear form of signature (p,q) on $\mathbb{R}^{p+q}$. Recall the notation introduced in Section 2.1.2. We define

$$\mathbb{B}(\mathbf{b}) := \big\{V \in \mathrm{Gr}_p(\mathbb{R}^{p+q}) \mid \mathbf{b}_{|V\times V} \text{ is positive-definite}\big\}$$
$$\mathbb{B}(\mathbf{b})^- := \big\{W \in \mathrm{Gr}_q(\mathbb{R}^{p+q}) \mid \mathbf{b}_{|W\times W} \text{ is negative-definite}\big\}$$

If $\mathbf{b}$ is the standard bilinear form on $\mathbb{R}^{p+q}$ of signature (p,q), i.e. if

$$\mathbf{b}_{\mathsf{std}}(x,x) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 \quad \forall x \in \mathbb{R}^{p+q}, \tag{3.3.1}$$

then we will write $\mathbb{B}_{p,q} := \mathbb{B}(\mathbf{b}_{\mathsf{std}})$. The following Propositions 3.3.1 and 3.3.2 are well known, but we give their proof for completeness.

Let us denote by $||\cdot||_2$ the L^2-norm on $\mathbb{R}^p$ and $\mathbb{R}^q$, and still denote by $||\cdot||_2$ the associated operator norm on $\mathrm{Mat}_{q,p}(\mathbb{R})$.

Proposition 3.3.1. *One has*

$$\mathbb{B}_{p,q} = \varphi_p\big(\{X \in \mathrm{Mat}_{q,p}(\mathbb{R}) \mid ||X||_2 < 1\}\big),$$

and its dual is the closure of $\mathbb{B}(\mathbf{b}_{\mathsf{std}})^-$.

Proof. Let us first prove the second assertion. Let $(V,W) \in \mathbb{B}_{p,q} \times \overline{\mathbb{B}(\mathbf{b}_{\mathsf{std}})^-}$. Then by definition, the form $\mathbf{b}_{\mathsf{std}}$ is both positive-definite and negative on $V \cap W$. Thus $V\cap W = \{0\}$. This proves that $\overline{\mathbb{B}(\mathbf{b}_{\mathsf{std}})^-} \subset \mathbb{B}^*_{p,q}$. Conversely, let $W \in \mathbb{B}^*_{p,q}$. If there exists $w \in W$ such that $\mathbf{b}_{\mathsf{std}}(v,v) > 0$, then one can complete a p-dimensional positive-definite vector space V containing v. Then we have $V \in \mathbb{B}_{p,q}$, but $v \in (W\cap V) \smallsetminus \{0\}$, contradicting the fact that $W \in \mathbb{B}^*_{p,q}$. Thus $\mathbf{b}_{\mathsf{std}}(v,v) \leq 0$ for all $v \in W$, and $W \in \overline{\mathbb{B}(\mathbf{b}_{\mathsf{std}})^-}$. We have proven the equality $\overline{\mathbb{B}(\mathbf{b}_{\mathsf{std}})^-} = \mathbb{B}^*_{p,q}$.

Now let us prove the first assertion. Let

$$\mathscr{B} := \varphi_p\big(\{X \in \mathrm{Mat}_{q,p}(\mathbb{R}) \mid ||X||_2 < 1\}\big).$$

Let $V \in \mathbb{B}_{p,q}$. According to the previous paragraph, one has $V \cap \xi_\infty = \{0\}$, so $V \in \mathbb{A}_{\mathsf{std}}$ (recall Equation (2.4.6)) and there exists $X \in \mathrm{Mat}_{q,p}(\mathbb{R})$ such that $V = \varphi_p(X)$. This says that

$$V = \Big\{ \begin{pmatrix} v \\ Xv \end{pmatrix} \mid v \in \mathbb{R}^p \Big\}.$$

Since $\mathbf{b}_{\mathsf{std}}$ is positive-definite on V, one has

$$||v||_2^2 - ||Xv||_2^2 = \mathbf{b}_{\mathsf{std}}\Big(\begin{pmatrix} v \\ Xv \end{pmatrix}, \begin{pmatrix} v \\ Xv \end{pmatrix} \Big) > 0 \quad \forall v \in \mathbb{R}^p \smallsetminus \{0\}. \tag{3.3.2}$$

Thus $||X||_2 < 1$. This proves $\mathbb{B}_{p,q} \subset \mathscr{B}$. Conversely, if $V' := \phi_p(X) \in \mathscr{B}$, then Equation (3.3.2) is valid for V'. Thus $(\mathbf{b}_{\mathsf{std}})_{|V' \times V'}$ is positive-definite, and $V' \in \mathbb{B}_{p,q}$. This proves the equality $\mathbb{B}_{p,q} = \mathscr{B}$. □

Proposition 3.3.1 implies in particular that the domain $\mathbb{B}_{p,q}$ is a proper domain of $\mathrm{Gr}_p(\mathbb{R}^{p+q})$, since its dual has nonempty interior.

Proposition 3.3.2. *Let $p, q \in \mathbb{N}$. The domains $\mathbb{B}(\mathbf{b})$ are:*

1. *all translates of each other by elements of $\mathrm{PGL}(p+q,\mathbb{R})$;*
2. *proper, with dual $\overline{\mathbb{B}(\mathbf{b})^-}$;*
3. *symmetric, with transitive automorphism group with respect to $\mathrm{PGL}(p+q,\mathbb{R})$, equal to $\mathrm{PO}(\mathbf{b})$; they are thus also divisible and dually convex;*
4. *equivariantly diffeomorphic to the symmetric space of $\mathrm{PO}(p,q)$.*

Proof. Point (1) directly follows from the fact that all bilinear forms of signature (p,q) on $\mathbb{R}^{p+q}$ are conjugate in $\mathrm{GL}(p+q,\mathbb{R})$. Point (2) is then then a consequence of Point (1) and Proposition 3.3.1.

Since $\mathrm{PO}(\mathbf{b})$ preserves $\mathbf{b}$, one has $\mathrm{PO}(\mathbf{b}) \leq \mathsf{Aut}_{\mathrm{PGL}(p+q,\mathbb{R})}(\mathbb{B}(\mathbf{b}))$. Conversely, let $g \in \mathrm{GL}(p+q,\mathbb{R})$ satisfying $[g] \in \mathsf{Aut}_{\mathrm{PGL}(p+q,\mathbb{R})}(\mathbb{B}(\mathbf{b}))$. Let $v \in \mathbb{R}^{p+q}$ with $\mathbf{b}(v,v) > 0$. For any $V \in \mathbb{B}(\mathbf{b})$ such that $v \in V$, one has $g \cdot v \in g \cdot V$, and $g \cdot V \in \mathbb{B}(\mathbf{b})$. Thus $\mathbf{b}(g \cdot v, g \cdot v) > 0$. Since $[g] \in \mathsf{Aut}_{\mathrm{PGL}(p+q,\mathbb{R})}(\mathbb{B}(\mathbf{b})) = \mathsf{Aut}_G(\mathbb{B}(\mathbf{b})^*)$, by Point (2) and the same argument, one has $\mathbf{b}(g \cdot v, g \cdot v) < 0$ for all $v \in \mathbb{R}^{p+q}$ such that $\mathbf{b}(v,v) < 0$. Thus g also preserves the isotropic cone

$$\{v \in \mathbb{R}^{p+q} \mid \mathbf{b}(v,v) = 0\}.$$

It is then a classical fact that there exists $\lambda \in \mathbb{R}$ such that $\mathbf{b}(g\ \cdot, g\ \cdot) = \lambda \mathbf{b}$. Thus $[g] \in \mathrm{PO}(\mathbf{b})$. This proves Point (3).

By Point (2), proving Point (4) reduces to the case where $\mathbf{b} = \mathbf{b}_{\mathsf{std}}$. In this case, the group $\mathrm{PO}(\mathbf{b}_{\mathsf{std}})$ acts transitively on $\mathbb{B}_{p,q}$. An explicit computation gives that the stabilizer of $x_0 = \mathrm{Span}(e_1, \ldots, e_p)$ — defined in Equation (2.4.2) — in $\mathrm{PO}(\mathbf{b}_{\mathsf{std}})$ is

$$\mathrm{PO}(\mathbf{b}_{\mathsf{std}}) \cap P = \mathrm{P}\Big(\begin{pmatrix} \mathrm{O}(p) & 0 \\ 0 & \mathrm{O}(q) \end{pmatrix} \Big),$$

which is a maximal compact subgroup of $\mathrm{PO}(\mathbf{b}_{\mathsf{std}})$ isomorphic to $\mathrm{P}(\mathrm{O}(p) \times \mathrm{O}(q))$. Thus we have a $\mathrm{PO}(\mathbf{b}_{\mathsf{std}})$-equivariant diffeomorphism

$$\mathrm{PO}(\mathbf{b}_{\mathsf{std}}) / \mathrm{P}(\mathrm{O}(p) \times \mathrm{O}(q)) \simeq \mathbb{B}_{p,q},$$

and Point (4) follows by definition of $\mathrm{PO}(p,q)$, since $\mathrm{PO}(\mathbf{b}_{\mathsf{std}})$ has signature (p,q). □

3.3.2 Dual convexity versus convexity in an affine chart

In this section, we give explicit examples of proper domains of $\mathrm{Gr}_2(\mathbb{R}^4)$ which are dually convex and convex in no affine chart, and reciprocally. In the case of causal flag manifolds, it will be much clearer that these two notions are not equivalent, see Example 3.5.9.

By Equation (2.4.8), the algebraic subvarieties maximal proper Schubert subvarieties of $\mathrm{Gr}_p(\mathbb{R}^{p+q})$ are hypersurfaces. Thus Proposition 3.2.1 holds in $\mathrm{Gr}_p(\mathbb{R}^{p+q})$.

Given some point $\xi := \begin{bmatrix} A \\ B \end{bmatrix} \in \mathrm{Gr}_q(\mathbb{R}^{p+q})$, with $A \in M_{p,q}(\mathbb{R})$ and $B \in \mathrm{Mat}_q(\mathbb{R})$, by (2.4.8), one has $\mathrm{Z}_\xi \cap \mathbb{A}_{\mathsf{std}} = \varphi_p\big(\{X \in \mathrm{Mat}_{q,p}(\mathbb{R}) \mid \det(XA - B) = 0\}\big)$. We define the two open sets

$$(\mathbb{A}_{\mathsf{std}} \smallsetminus \mathrm{Z}_\xi)^+ := \varphi_p\big(\{X \in \mathrm{Mat}_{q,p}(\mathbb{R}) \mid \det(XA - B) > 0\}\big);$$
$$(\mathbb{A}_{\mathsf{std}} \smallsetminus \mathrm{Z}_\xi)^- := \varphi_p\big(\{X \in \mathrm{Mat}_{q,p}(\mathbb{R}) \mid \det(XA - B) < 0\}\big).$$

Whenever $\mathbb{A}_{\mathsf{std}} \cap \mathrm{Z}_\xi \neq \emptyset$, the open set $(\mathbb{A}_{\mathsf{std}} \smallsetminus \mathrm{Z}_\xi)^+$ (resp. $(\mathbb{A}_{\mathsf{std}} \smallsetminus \mathrm{Z}_\xi)^-$) is dense in $(\mathbb{A}_{\mathsf{std}} \smallsetminus \mathrm{Z}_\xi)^+ \sqcup \mathrm{Z}_\xi$ (resp. $(\mathbb{A}_{\mathsf{std}} \smallsetminus \mathrm{Z}_\xi)^- \sqcup \mathrm{Z}_\xi$). In particular, in this case, it is nonempty.

If $p = q$ and $\xi := \varphi_p(B) \in \mathrm{Gr}_p(\mathbb{R}^{2p})$ then we have

$$(\mathbb{A}_{\mathsf{std}} \smallsetminus \mathrm{Z}_\xi)^+ = \varphi_p\big(\{X \in \mathrm{Mat}_p(\mathbb{R}) \mid \det(X - B) > 0\}\big)$$
$$(\mathbb{A}_{\mathsf{std}} \smallsetminus \mathrm{Z}_\xi)^- = \varphi_p\big(\{X \in \mathrm{Mat}_p(\mathbb{R}) \mid \det(X - B) < 0\}\big).$$

Remark 3.3.3. Even if dual convexity and convexity in an affine chart are different, dually convex domains of $\mathrm{Gr}_p(\mathbb{R}^{p+q})$ share an important property with domains that are properly convex in an affine chart: if $\Omega \subset \mathrm{Gr}_p(\mathbb{R}^{p+q})$ is a proper domain which is dually convex, then $\mathrm{int}(\overline{\Omega}) = \Omega$. This is a particular case of Proposition 7.1.9, which will be proven in Section 7.1 for any *Nagano space of real type*.

3.3.2.1 Supporting hyperplanes and proper Schubert subvarieties. In this section, we assume that $p = q$. Lemma 3.3.4 below describes the local behavior of Schubert hypersurfaces of $\mathrm{Gr}_p(\mathbb{R}^{2p})$ near a regular point. It will allow us to construct proper domains that are convex in an affine chart but not dually convex, and vice versa.

Recall the point $x_0 := \varphi_2(0)$ defined in Equation (2.4.2).

Lemma 3.3.4. *Let $x := \varphi_p(X) \in \mathbb{A}_{\mathsf{std}} \cap \mathrm{Z}_{x_0}$ be such that $\mathrm{rk}(X) = p - 1$ — i.e. x is a regular point of the algebraic hypersurface Z_{x_0}. Then for any neighborhood $\mathscr{V}$ of X in $\mathrm{Mat}_p(\mathbb{R})$, there exists $X^\pm \in T_X(\varphi_p^{-1}(\mathrm{Z}_{x_0}))$ such that $\varphi_p(X^\pm) \in (\mathbb{A}_{\mathsf{std}} \smallsetminus \mathrm{Z}_{x_0})^\pm \cap \mathscr{V}$.*

Note that $\varphi_p(X^+) \in (\mathbb{A}_{\mathsf{std}} \smallsetminus \mathrm{Z}_{\xi_0})^+$ (resp. $\varphi(X^-) \in (\mathbb{A}_{\mathsf{std}} \smallsetminus \mathrm{Z}_{\xi_0})^-$) is equivalent to $\det(X^+) > 0$ (resp. $\det(X^-) < 0$).

Proof of Lemma 3.3.4. Since X has rank $p - 1$, by Gauss' reduction there exist two matrices $P, Q \in \mathrm{GL}_p(\mathbb{R})$ such that

$$X = Q\widetilde{I}_p P^{-1},$$

where $\widetilde{I}_p = \begin{pmatrix} I_{p-1} & 0 \\ 0 & 0 \end{pmatrix}$. Thus $\xi = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \cdot \begin{bmatrix} I_p \\ \widetilde{I}_p \end{bmatrix}$. Hence, up to translating x by the element of the Levi subgroup

$$\begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \in L := \left\{ \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix} \mid A, C \in \mathrm{GL}_p(\mathbb{R}) \right\}$$

(see Equation (2.4.3)), we may assume that $X = \widetilde{I}_p$. For all $H = (h_{i,j}) \in \mathrm{Mat}_p(\mathbb{R})$, we have

$$d_{\widetilde{I}_p} \det(H) = E^*_{p,p}(H) = h_{p,p}.$$

This map is nonzero, which means that det is a submersion at $\widetilde{I}_p$ and that

$$\begin{aligned} T_{\widetilde{I}_p}(\{X \in \mathrm{Mat}_p(\mathbb{R}) \mid \det(X) = 0\}) &= \ker(d_{\widetilde{I}_p} \det) + \widetilde{I}_p \\ &= \{H = (h_{i,j}) \in \mathrm{Mat}_p(\mathbb{R}) \mid h_{p,p} = 0\} + \widetilde{I}_p. \end{aligned}$$

Let $\varepsilon > 0$ and

$$Y^{\pm}_{\varepsilon} = \left(\begin{array}{c|cc} 0_{p-2} & \multicolumn{2}{c}{0_{p-2,2}} \\ \hline \multirow{2}{*}{$0_{2,p-2}$} & 0 & \varepsilon \\ & \mp\varepsilon & 0 \end{array} \right) \in \ker(d_{\widetilde{I}_p} \det).$$

Then $Y^{\pm}_{\varepsilon} \in \ker(d_{\widetilde{I}_p} \det)$, and $\det(Y^{\pm}_{\varepsilon} + \widetilde{I}_p) = \pm\varepsilon^2$. For ε small enough, one has $Y^{\pm}_{\varepsilon} + \widetilde{I}_p \in \mathscr{V}$. Thus for ε small enough, the matrix $X^{\pm} := \widetilde{I}_p + Y^{\pm}_{\varepsilon}$ works. $\square$

The case where $p = q = 2$ is particularly interesting, as the regular point of $\mathbb{A}_{\mathsf{std}} \cap \mathrm{Z}_{x_0}$ are exactly the elements of $\mathbb{A}_{\mathsf{std}} \cap \mathrm{Z}_{x_0}$ that are different from x_0: a point $x := \varphi_2(X)$ is regular in $\varphi_2^{-1}(\mathrm{Z}_{x_0}) = \{Y \mid \det(Y) = 0\}$ if and only if its rank is 1.

3.3.2.1.1 Example: A domain which is both dually convex and properly convex. Let us denote by $||\cdot||$ the infinite norm on $\mathrm{Mat}_2(\mathbb{R})$, i.e. $||(x_{ij})|| = \max_{i,j} |x_{ij}|$. Let $B(r)$ be the ball of center 0 and of radius r for this norm, with $r > 0$. Then $\varphi_2(B(r))$ is properly convex in $\mathbb{A}_{\mathsf{std}}$. It is also dually convex, as it is equal to the intersection

$$\begin{aligned} &(\mathbb{A}_{\mathsf{std}} \setminus \mathrm{Z}_{\xi_1^+})^+ \cap (\mathbb{A}_{\mathsf{std}} \setminus \mathrm{Z}_{\xi_2^+})^+ \cap (\mathbb{A}_{\mathsf{std}} \setminus \mathrm{Z}_{\xi_3^+})^+ \cap (\mathbb{A}_{\mathsf{std}} \setminus \mathrm{Z}_{\xi_4^+})^+ \cap \\ &(\mathbb{A}_{\mathsf{std}} \setminus \mathrm{Z}_{\xi_1^-})^+ \cap (\mathbb{A}_{\mathsf{std}} \setminus \mathrm{Z}_{\xi_2^-})^+ \cap (\mathbb{A}_{\mathsf{std}} \setminus \mathrm{Z}_{\xi_3^-})^+ \cap (\mathbb{A}_{\mathsf{std}} \setminus \mathrm{Z}_{\xi_4^-})^+, \end{aligned}$$

with

$$\begin{aligned} &\xi_1^+ = \begin{bmatrix} \frac{1}{r} & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}; \xi_2^+ = \begin{bmatrix} 0 & \frac{1}{r} \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}; \xi_3^+ = \begin{bmatrix} 0 & 0 \\ \frac{1}{r} & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}; \xi_4^+ = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{r} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}; \\ &\xi_1^- = \begin{bmatrix} -\frac{1}{r} & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}; \xi_2^- = \begin{bmatrix} 0 & -\frac{1}{r} \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}; \xi_3^- = \begin{bmatrix} 0 & 0 \\ -\frac{1}{r} & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}; \xi_4^- = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{r} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned} \tag{3.3.3}$$

Note that we already know another example of both dually convex and properly convex domain, which is $\mathbb{B}_{2,2}$, by Propositions 3.3.1 and 3.3.2.

3.3.2.1.2 Example: a properly convex domain which is not dually convex. As we have seen in Section 3.3.2.1.1, the domain $\varphi_2(B(2))$ is both properly convex in $\mathbb{A}_{\mathsf{std}}$ and dually convex. However, if we intersect it with a well-chosen hyperplane of $\mathbb{A}_{\mathsf{std}}$, we can get a new properly convex domain of $\mathbb{A}_{\mathsf{std}}$ which is not dually convex anymore. We set:

$$\Omega := \varphi_2(B(2) \cap \{X \mid \operatorname{tr}(X) < 1\}).$$

Lemma 3.3.5. *The domain Ω is properly convex in $\mathbb{A}_{\mathsf{std}}$, but it is not dually convex.*

Proof. It is clear by definition that the domain Ω is properly convex in the affine chart $\mathbb{A}_{\mathsf{std}}$. Let $X := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $x := \varphi_{\mathsf{std}}(X) \in \partial\Omega$. There exists a neighborhood $\mathscr{V}$ of X such that $\varphi_2(\mathscr{V} \cap \{\operatorname{tr} = 1\})$ is a nonempty open subset of $\partial\Omega$.

Assume for a contradiction that Ω is dually convex. By dual convexity, there exists $\xi \in \Omega^*$ such that $x \in \mathrm{Z}_\xi$. By connectedness of Ω, there exists $\varepsilon \in \{+,-\}$ such that $\Omega \subset (\mathbb{A}_{\mathsf{std}} \smallsetminus \mathrm{Z}_\xi)^\varepsilon$. In particular, one has

$$\varphi_2\big(\mathscr{V} \cap \{\operatorname{tr} = 1\}\big) \subset \overline{(\mathbb{A}_{\mathsf{std}} \smallsetminus \mathrm{Z}_\xi)^\varepsilon} \tag{3.3.4}$$

Since $x_0 \in \Omega$, we have $x_0 \cap \xi = \{0\}$, and thus we can write $\xi = \begin{bmatrix} Y \\ I_2 \end{bmatrix}$, with $Y = \begin{pmatrix} y_{1,1} & y_{1,2} \\ y_{2,1} & y_{2,2} \end{pmatrix}$ and $y_{1,1}, y_{1,2}, y_{2,1}, y_{2,2} \in \mathbb{R}$. Let us distinguish the cases where Y is invertible or not.

1. If Y is invertible, then $\xi \in \mathbb{A}_{\mathsf{std}}$. If $x \in \mathrm{Z}_\xi$ is regular, then by Proposition 3.2.1, the affine hyperplane $\varphi_2(T_X(\varphi_2^{-1}(\mathrm{Z}_\xi)))$ of $\mathbb{A}_{\mathsf{std}}$ is a supporting hyperplane of Ω at x. Since the unique supporting hyperplane of Ω at x is $H_0 := \varphi_2(\{\operatorname{tr} = 1\})$, we have $T_X(\varphi_2^{-1}(\mathrm{Z}_\xi)) = \{\operatorname{tr} = 1\}$. But Lemma 3.3.4, there exists

$$X^{-\varepsilon} \in \mathscr{V} \cap \{\operatorname{tr} = 1\} \cap \varphi_2^{-1}\big((\mathbb{A}_{\mathsf{std}} \smallsetminus \mathrm{Z}_\xi)^{-\varepsilon}\big).$$

   This is in contradiction with Equation (3.3.4). Thus x cannot be regular in Z_ξ. Thus since $p = 2$, we have $x = \xi$, so $X = Y^{-1}$. This is absurd because X is not invertible.

2. Now assume that Y is not invertible. Since $Y \neq 0$, we have $\operatorname{rk}(Y) = 1$. Moreover, since $x \in \mathrm{Z}_\xi$, we have $\det(I_2 - XY) = 0$, which implies that $y_{1,1} = 1$. Thus there exist $\lambda, \mu \in \mathbb{R}$ such that $Y = \begin{pmatrix} 1 & \lambda \\ \mu & \lambda\mu \end{pmatrix}$. On the other hand, by Equation (3.3.4) one has

$$\varepsilon \det(I_2 - ZY) > 0 \quad \forall z = \varphi_2(Z) \in \Omega. \tag{3.3.5}$$

   Write $Z = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}$. Equation (3.3.5) is equivalent to:

$$\varepsilon(1 - z_1 - \mu z_2 - \lambda z_3 - \lambda\mu z_4) > 0. \tag{3.3.6}$$

   Assume that $\varepsilon = +$. Let $Z = \begin{pmatrix} 1 & \delta \\ 0 & -\delta^2 \end{pmatrix}$, with $0 < \delta < 1$. Since $\varphi_2(Z) \in \Omega$, by Equation (3.3.6), we have $-\delta\mu + \delta^2\mu > 0$ for all $0 < \delta < 1$. This implies that $\mu \leq 0$. Noticing that $\varphi_2\Big(\begin{pmatrix} 1 & -\delta \\ 0 & -\delta^2 \end{pmatrix}\Big) \in \Omega$ for all $0 < \delta < 1$, we also get $\mu \geq 0$. Thus $\mu = 0$.

But $\varphi_2\Big(\begin{pmatrix}\frac{3}{2} & 0\\ 0 & -1\end{pmatrix}\Big) \in \Omega$, which, by Equation (3.3.6), implies that $-\frac{1}{2} > 0$, contradiction.

Thus $\varepsilon = -$. We similarly get $\mu = 0$, and since $\varphi_2\Big(\begin{pmatrix}\frac{1}{3} & 0\\ 0 & \frac{1}{3}\end{pmatrix}\Big) \in \Omega$, we get $\frac{1}{3} < 0$, contradiction.

In any case, we get a contradiction, so Ω is not dually convex. □

3.3.2.1.3 Example: a dually convex domain which is convex in no affine chart. As in Section 3.3.2.1.2, we will shrink a domain of the form $B(r)$ for some $r > 0$ (in the notation of Section 3.3.2.1.1), but this time, we want to keep the dual convexity property and lose the "convexity in an affine chart" propery. Thus we will shrink it with a well-chosen maximal proper Shubert subvariety. Let

$$\Omega := \varphi_2\big(B(1)\big) \cap (\mathbb{A}_{\mathsf{std}} \smallsetminus \mathrm{Z}_{x_0})^+.$$

Recall the elements defined in (3.3.3) for $r = 1$. An explicit computation gives:

$$\bigwedge^2 \mathbb{R}^4 = \iota_{\rho_0}(\xi_1^+) \oplus \iota_{\rho_0}(\xi_3^+) \oplus \iota_{\rho_0}(\xi_4^+) \oplus \iota_{\rho_0}(\xi_2^+) \oplus \iota_{\rho_0}(\xi_1^-) \oplus \iota_{\rho_0}(x_0), \tag{3.3.7}$$

where ι_{ρ_0} is the Plücker embedding defined in Equation (2.4.5). Note moreover that $\partial\Omega$ contains an open subset of $\mathrm{Z}_{\xi_i^+}$ for all $1 \leq i \leq 4$, of $\mathrm{Z}_{\xi_1^-}$ and of Z_{x_0}.

Lemma 3.3.6. *The open set Ω is contractible and dually convex, but there exists no affine chart in which it is properly convex.*

Proof. The open set Ω is dually convex, as it is the intersection of two dually convex domains. Moreover, a computation using the explicit description of Ω gives that Ω is starshaped at $\varphi_2\Big(\begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix}\Big)$ in $\mathbb{A}_{\mathsf{std}}$, and hence contractible. Assume for a contradiction that Ω is properly convex in some affine chart of $\mathrm{Gr}_2(\mathbb{R}^4)$. Then there exists $g \in \mathrm{PGL}(4,\mathbb{R})$ such that $g^{-1} \cdot \Omega$ is properly convex in $\mathbb{A}_{\mathsf{std}}$.

By Equation (3.3.7), there exists $\xi \in \{\xi_1^+, \xi_3^+, \xi_4^+, \xi_2^+, \xi_1^-, x_0\}$ such that $g^{-1} \cdot \xi$ is transverse to $\xi_\infty := \begin{bmatrix}0\\ I_2\end{bmatrix}$. We can thus write $g^{-1} \cdot \xi = \varphi_2(B)$, where $B \in \mathrm{Mat}_2(\mathbb{R})$. Let $\mathscr{V} \subset \mathrm{Gr}_2(\mathbb{R}^4)$ be an open subset such that $\mathscr{V} \cap \partial\Omega \subset \mathrm{Z}_\xi$ is nonempty. By connectedness of Ω, we may assume that $\Omega \subset (\mathbb{A}_{\mathsf{std}} \smallsetminus \mathrm{Z}_{g^{-1}\cdot\xi})^+$. We may moreover assume that $\mathscr{V} \cap (\mathbb{A}_{\mathsf{std}} \smallsetminus \mathrm{Z}_{g^{-1}\cdot\xi})^+ \subset \Omega$. By density of regular points in $\mathrm{Z}_{g^{-1}\cdot\xi}$, there exists a regular point $x = \varphi_2(X) \in \mathrm{Z}_{g^{-1}\cdot\xi} \cap \mathscr{V}$. By Proposition 3.2.1, the affine hyperplane $\varphi_2(H)$, where $H := T_X(\varphi_2^{-1}(\mathrm{Z}_{g^{-1}\cdot\xi}))$, is a supporting hyperplane at x to $g^{-1} \cdot \Omega$. By Lemma 3.3.4, there exists

$$X^+ \in H \cap \varphi_2^{-1}\big((\mathscr{V} \cap \mathrm{Z}_{g^{-1}\cdot\xi})^+\big) \subset H \cap \varphi_2^{-1}(\Omega),$$

which contradicts the fact that $\varphi_2(H) \cap \Omega = \emptyset$.

We have proven that there exists no affine chart containing Ω as a properly convex domain. □

3.4 Proper domains in the Einstein Universe

We define and investigate the properties of certain proper symmetric domains of $\mathrm{Ein}^{p,q}$ for $p, q \geq 1$, all conformally equivalent to each other, called *diamonds*. We start with the Lorentzian case, that is, when $(p, q) = (n-1, 1)$, in Section 3.4.1, where the diamonds are well-known and admit an explicit description in terms of causality. We generalize their construction to higher signatures in Section 3.4.2.

The diamonds are models of the symmetric space of $\mathrm{PO}(p, 1)\times\mathrm{PO}(1, q)$, this symmetric space being, with the notation of Chapter 5, the non-compact dual of the Nagano space $\mathrm{Ein}^{p,q}$. By Theorem 8.4.1, they will be the only proper almost-homogeneous domains in $\mathrm{Ein}^{p,q}$.

Let us first fix some notations for this section:

Notation 3.4.1. Let V be a finite-dimensional real vector space and $\mathbf{b}$ be a quadratic form of signature (p, q) on V. The sheet $\mathbb{H}^{p,q-1} \subset V$ is defined by

$$\mathbb{H}^{p,q-1} = \{x \in V \mid \mathbf{b}(x, x) = -1\}\,.$$

The metric $\mathbf{b}$ restricts to a complete pseudo-Riemannian metric of signature $(p, q-1)$ and of constant negative curvature on $\mathbb{H}^{p,q-1}$. The two connected components of $\mathbb{H}^{n,0}$ are standard models for the real n-dimensional hyperbolic space, and we denote them by $\mathbb{H}^n$. The space $\mathbb{H}^{n,1}$ is usually referred to as the *anti de Sitter space* in Lorentzian geometry. Similarly, we define

$$\mathrm{dS}^{p-1,q} = \{x \in V \mid \mathbf{b}(x, x) = +1\}\,,$$

so that the metric $\mathbf{b}$ restricts to a complete pseudo-Riemannian metric of signature $(p-1, q)$ and of constant positive curvature on $\mathrm{dS}^{p-1,q}$. The space $\mathrm{dS}^{n,1}$ is called the *de Sitter space* in Lorentzian geometry. We will use the notation $-\mathbb{H}^n$ to denote $\mathrm{dS}^{0,n}$.

3.4.1 Reminders on the Lorentzian diamond and its conformal structure

Recall the setting of Section 2.4.1 for the definition of the Lorentzian Einstein universe, in particular the quadratic form ψ on $V \simeq \mathbb{A}_{\mathsf{std}}$. We still denote by ψ the symmetric bilinear form associate to ψ. Given a point $x \in \mathbb{A}_{\mathsf{std}} \simeq \mathbb{R}^{n-1,1}$, its future and its past are the sets

$$\mathbf{I}^+(x) = \{a \in \mathbb{R}^{n-1,1} \mid \psi(a-x) < 0 \text{ and } \psi(a-x, e_{n+1}) < 0\};$$
$$\mathbf{I}^-(x) = \{a \in \mathbb{R}^{n-1,1} \mid \psi(a-x) < 0 \text{ and } \psi(a-x, e_{n+1}) > 0\}.$$

Now let $x, y \in \mathbb{A}_{\mathsf{std}} \simeq \mathbb{R}^{1,n-1}$ such that $y \in \mathbf{I}^+(x)$. We denote by $\mathbf{D}(x, y)$ the domain defined by

$$\mathbf{D}(x, y) = \mathbf{I}^+(x) \cap \mathbf{I}^-(y),$$

see Figure 3.1. In the notation of Section 2.4.3.3, the set $\mathbf{D}(x, y)$ is the union of all timelike curves of $\mathbb{A}_{\mathsf{std}} \simeq \mathbb{R}^{n-1,1}$ joining x to y. More generally, if $x \in \mathrm{Ein}^{n-1,1}$ and $y \notin \mathrm{Z}_x$, a *diamond with extremities x and y* is a connected component of $\mathrm{Ein}^{n-1,1} \smallsetminus (\mathrm{Z}_x \cup \mathrm{Z}_y)$ which is proper. Note that, since $\mathrm{PO}(n, 2)$ acts transitively on the pairs of transverse points of $\mathrm{Ein}^{n-1,1}$, it acts transitively on the set of diamonds, so they are all conformally equivalent.

If $x \in \mathbb{A}_{\mathsf{std}}$, then the future $\mathbf{I}^+(x)$ is a diamond, as it is a proper connected component of $\mathrm{Ein}^{n-1,1} \smallsetminus (\mathrm{Z}_x \cup \mathrm{Z}_{[e_1]})$.

In the notation of Section 2.4.3.3, we have the following well-known result:

Proposition 3.4.2. *Any diamond of* $\mathrm{Ein}^{n-1,1}$ *is conformally equivalent to*

$$(\mathbb{H}^{n-1} \times \mathbb{R}, [g_{\mathbb{H}^{n-1}} \oplus (-dt^2)]).$$

Proof. Since all diamonds are conformally equivalent, we may consider the diamond $\mathbf{I}^+(0) \subset \mathbb{A}_{\mathsf{std}}$, where $0 \in \mathbb{A}_{\mathsf{std}}$ is a fixed origin. We have a conformal identification

$$\begin{array}{ccc} \mathbf{I}^+(0) & \longrightarrow & \mathbb{H}^{n-1} \times \mathbb{R} \\ x & \longmapsto & \Big(\frac{x}{\psi(x)}, \log\big(-\psi(x)\big)\Big), \end{array}$$

where we have conformally identified $\mathbb{H}^{n-1}$ with $\{z \in \mathbf{I}^+(0) \,|\, \psi(z) = -1\}$ by Notation 3.4.1. $\square$

Now by Fact 2.4.1, for any diamond $D \subset \mathrm{Ein}^{n-1,1}$, the group $\mathsf{Aut}_{\mathrm{PO}(n,2)}(D)$ is isomorphic to $\mathrm{Isom}(\mathbb{H}^{n-1} \times \mathbb{R}) = \mathrm{PO}(1, n-1) \times (\mathbb{Z}_2 \ltimes \mathbb{R})$. In particular, the domain D is divisible, homogeneous and symmetric.

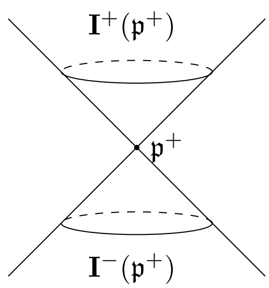

Figure 3.1 – Past and future of $\mathfrak{p}^+$ in $\mathbb{A}_{\mathsf{std}}$ in the case where $(p,q) = (2,1)$

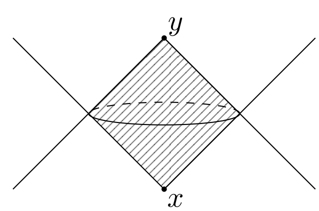

Figure 3.2 – The diamond $\mathbf{D}(x,y)$ for $x, y \in \mathbb{A}_{\mathsf{std}}$ and $y \in \mathbf{I}^+(x)$ (greyed-out area), seen in $\mathbb{A}_{\mathsf{std}} \simeq \mathbb{R}^{2,1}$ for $(p,q) = (2,1)$.

3.4.2 Diamonds and other homogeneous domains of $\mathrm{Ein}^{p,q}$

The content of this section comes from joint work with Adam Chalumeau [CG24], with some adjustments to align with the present manuscript. We generalize the construction of Section 3.4.1 to the higher-signature case. Let $p, q \geq 1$ and $\mathbf{b}$ be a bilinear form of signature $(p+1, q+1)$ on $\mathbb{R}^{p+q+2}$. Let us write $\mathbb{R}^{p+1,q+1} = V_+ \oplus V_-$, where V_+ and V_- are two orthogonal vector subspaces of $\mathbb{R}^{p+q+2}$. For $i \in \{+,-\}$, let (p_i, q_i) be the signature of V_i and we assume for instance that $q_+ \leq q_-$. We denote by F_i the (possibly empty) intersection $F_i = \mathrm{Ein}^{p,q} \cap \mathbb{P}(V_i)$. Let $J \subset \mathrm{Ein}^{p,q}$ be the *joint* of F_+ and F_-, that is, the union of all photons intersecting F_+ and F_-, and let $U := \mathrm{Ein}^{p,q} \smallsetminus J$. We then have:

Proposition 3.4.3 (with Chalumeau, see [CG24])**.** *One has the following 4 possible cases:*

* *If $p_i = 0$ or $q_i = 0$ for some $i \in \{+,-\}$, then the domain U is connected, homogeneous, symmetric and dense in* $\mathrm{Ein}^{p,q}$.
* *If $p_+ = p_- = q_+ = q_- = 1$, then the domain U has 4 connected components, all of which are Lorentzian diamonds.*
* *If $q_+p_- = 1$ and $q_-p_+ \geq 2$, then the domain U has 3 connected components, all of which are symmetric and two of which are proper and isomorphic to each other. The same conclusion holds if $q_+p_- \geq 2$ and $q_-p_+ = 1$.*
* *If $q_+p_- \geq 2$ and $q_-p_+ \geq 2$, then the domain U has 2 connected components, which are both symmetric and nonproper.*

Proof. First assume that $p_i, q_i \geq 1$ for $i \in \{+,-\}$. Then

$$F_i = \mathbb{P}\{v_i \in V_i \mid v_i \neq 0,\ \mathbf{b}(v_i, v_i) = 0\},$$

and $J = \mathbb{P}\{v_+ + v_- \in V_+ \oplus V_- \mid \mathbf{b}(v_i, v_i) = 0 \text{ for } i \in \{+,-\}\}$. Thus we have $U = U_+ \sqcup U_-$, where

$$U_i = \mathbb{P}\{v_+ + v_- \in V_+ \oplus V_- \mid -\mathbf{b}(v_+, v_+) = \mathbf{b}(v_-, v_-) = i\}.$$

Now the map

$$\begin{array}{rccc} \pi: & \mathbb{H}^{p_+, q_+ - 1} \times \mathrm{dS}^{p_- - 1, q_-} & \longrightarrow & U_+ \\ & (v_+, v_-) & \longmapsto & \mathbb{P}(v_+ + v_-). \end{array}$$

is a conformal 2-sheeted covering. The nontrivial deck transformation φ defined in Section 2.4.3.3.1 centralizes $\mathrm{PO}(p+1, q+1)$, so U_+ is conformally equivalent to $\mathbb{H}^{p_+, q_+ - 1} \times \mathrm{dS}^{p_- - 1, q_-} /_{x \sim \varphi(x)}$. It is in particular symmetric, with isometry group $\mathrm{SO}(p_+, q_+ - 1) \times \mathrm{SO}(p_- - 1, q_-)$. If $q_+ \neq 1$ or $p_- \neq 1$, then at least one of the two factors $\mathbb{H}^{p_+, q_+ - 1}$ or $\mathrm{dS}^{p_- - 1, q_-}$ contains a lightlike geodesic γ defined over $\mathbb{R}$ (in the sense of Section 2.4.3.3). To find such a geodesic, intersect any degenerate 2-plane of signature $(0,1,0)$ or $(1,0,0)$ with $\mathbb{H}^{p_+, q_+ - 1}$ or $\mathrm{dS}^{p_- - 1, q_-}$, respectively. In particular, the domain U_+ contains a photon minus a point. Hence U_+ is not proper. If $q_+ = p_- = 1$, then the total space $\mathbb{H}^{p_+, 0} \times \mathrm{dS}^{0, q_-}$ has 4 connected components and U_+ is the union of two connected components, both conformal to $(-\mathbb{H}^{p_+}) \times \mathbb{H}^{q_-}$. In order to write the components of U_+ explicitly, let $e_+ \in \mathbb{H}^{p_+, 0}$ and $e_- \in \mathrm{dS}^{0, q_-}$. Then

$$U_+ = D_+ \sqcup D_-,$$

where

$$D_\delta = \mathbb{P}\{v_+ + v_- \in U_+ \mid \delta \mathbf{b}(v_+, e_+)\mathbf{b}(v_-, e_-) > 0\}, \quad \delta \in \{-,+\}.$$

For $x = \mathbb{P}(v_+ + v_-) \in D_+$ and $y = \mathbb{P}(w_+ + w_-) \in D_-$, the signs of $\mathbf{b}(v_+, w_+)$ and $\mathbf{b}(v_-, w_-)$ are the same. In particular the value of $\mathbf{b}(u, w)$ cannot be zero. This means that the lightcone of every element of D_+ (resp. D_-) does not intersect D_- (resp. D_+). Hence $D_- \subset D_+^*$. Since D_- is open, the domain D_+ is proper. Since $D_+ \subset D_-^*$, the domain D_- is also proper.

Similarly, the open set U_- is symmetric and has one or two connected components depending on the values of q_- and p_+. In the case where (for instance) $q_+ = 0$, the subset F_+ is empty, and $J = F_-$ has an empty interior in $\mathrm{Ein}^{p,q}$. Thus U is dense in $\mathrm{Ein}^{p,q}$. A map similar to π shows that U is connected and symmetric. ∎

A *diamond* is then a proper connected component of a set U constructed above from a decomposition $\mathbb{R}^{p+q+2} = V_+ \oplus V_-$, with $\mathrm{sgn}(\mathbf{b}_{|V_+\times V_+}) = (p,1)$ and $\mathrm{sgn}(\mathbf{b}_{|V_+\times V_+}) = (1,q)$. By the proof of Proposition 3.4.3, any diamond of $\mathrm{Ein}^{p,q}$ is conformally equivalent to

$$(\mathbb{H}^p \times \mathbb{H}^q, [g_{\mathbb{H}^p} \oplus -g_{\mathbb{H}^q}]).$$

Thus, by Fact 2.4.1, all diamonds are $\mathrm{PO}(p+1,q+1)$-translates of each other in $\mathrm{Ein}^{p,q}$, and the automorphism group of any diamond $D \subset \mathrm{Ein}^{p,q}$ is

$$\mathsf{Aut}_{\mathrm{PO}(p+1,q+1)}(D) \simeq \mathrm{PO}(p,1) \times \mathrm{PO}(1,q).$$

Since this group is semisimple and any diamond identifies equivariantly with its symmetric space, we have:

Fact 3.4.4. *All diamonds of* $\mathrm{Ein}^{p,q}$ *are symmetric, divisible and homogeneous domains.*

Let us give an explicit construction of diamonds in the standard affine chart $\mathbb{A}_{\mathsf{std}}$:

Construction 3.4.5 (with Chalumeau, see [CG24])**.** First, we chose an origin 0 of $\mathbb{A}_{\mathsf{std}}$, and denote by $\mathbf{b}_{p,q}$ the bilinear form on the vector space $\mathbb{A}_{\mathsf{std}}$ of signature (p,q) induced by the metric $\mathbf{b}$ on $\mathbb{R}^{p+1,q+1}$ (as in Section 2.4.3.3.2).

Let H_p be a positive-definite p-plane in $\mathbb{A}_{\mathsf{std}}$, and let H_q be a negative-definite q-plane orthogonal to H_p. We write S_{p-1} and S_{q-1} for the balls of center 0 and radius 1 in H_p and H_q, respectively. By Equation (2.4.13), given a point $a \in S^{p-1}$ and $b \in S^{q-1}$, the affine segment $[a,b]$ is a segment of photon. Then the union $\mathscr{S}$ of all such segments is a topological sphere (it is a topological join of S_{p-1} and S_{q-1}) which separates $\mathbb{R}^{p,q}$ into two connected components, one of which is bounded and convex. This component is a diamond, which we denote by D here. Let us now write D as a unit ball of $\mathbb{A}_{\mathsf{std}}$ for a suitable norm. This norm is defined by

$$|x|_{p,q} = \sqrt{\mathbf{b}_{p,q}(x_p,x_p)} + \sqrt{-\mathbf{b}_{p,q}(x_q,x_q)} \quad \forall x \in \mathbb{A}_{\mathsf{std}}, \tag{3.4.1}$$

where $x = x_p + x_q \in H_p \oplus H_q$. Then $|\cdot|_{p,q}$ defines a norm on $\mathbb{A}_{\mathsf{std}}$ which depends on H_p and H_q, and we have:

$$D = \{x \in \mathbb{R}^{p,q} \mid |x-c|_{p,q} < 1\}.$$

In particular, the domain D is a convex domain of $\mathbb{A}_{\mathsf{std}}$ for the canonical affine structure; see Figure 3.3.

Remark 3.4.6. There is a purely causal way to define diamonds of $\mathrm{Ein}^{p,q}$. In signature (p,q), one can define the future of an inextensible $(p-1)$-timelike curve (see [Tro24]). In this setting, the diamond is the future of the timelike sphere S^{p-1}.

The following lemma is intrinsically related to the fact that the number of incidence degrees between two points of $\mathrm{Ein}^{p,q}$ $(p,q \geq 1)$ is 3 (see Observation 5.1.10) as it is actually a particular case of Remark 6.4.3 which will be stated later with the formalism of *Nagano spaces* (see Section 5):

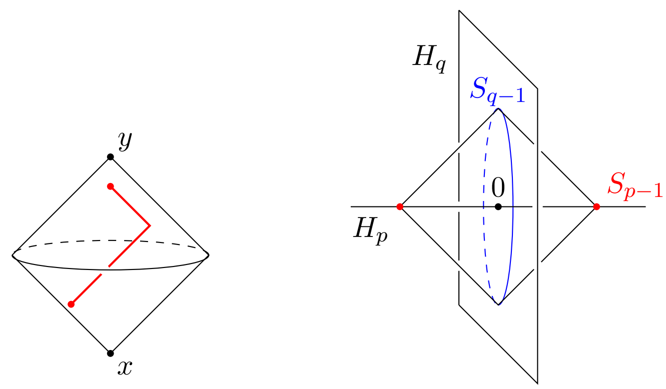

Figure 3.3 – On the left: the Lorentzian diamond in $\mathrm{Ein}^{2,1}$. Two points in the diamond are joined by a sequence of two segments of photons, see Lemma 3.4.7. On the right: a diamond in $\mathbb{R}^{p,q}$. The spheres S_{p-1} and S_{q-1} correspond to the subsets F_+ and F_- in Proposition 3.4.3. The boundary $\mathscr{S}$ of D is a subset of the set J in Proposition 3.4.3.

Lemma 3.4.7 (with Chalumeau, see [CG24])**.** *Let D be a diamond of $\mathrm{Ein}^{p,q}$. Any two points $x, y \in D$ can be joined by a sequence of at most two segments of photons contained in D, see Figure 3.3.*

Proof. We consider any conformal identification $D \simeq \mathbb{H}^p \times (-\mathbb{H}^q)$, and see $\mathbb{H}^p \times \mathbb{H}^q$ as a Riemannian symmetric space. Write $x = (\mathbf{x}_p, \mathbf{x}_q)$ and $y = (\mathbf{y}_p, \mathbf{y}_q)$ in the model $\mathbb{H}^p \times \mathbb{H}^q$. We let d_p, resp. d_q be the hyperbolic metric on $\mathbb{H}^p$, resp. $\mathbb{H}^q$, and $[\mathbf{x}_p, \mathbf{y}_p]$ (resp. $[\mathbf{x}_q, \mathbf{y}_q]$) the geodesic segment between $\mathbf{x}_p$ and $\mathbf{y}_p$ for d_p (resp. between $\mathbf{x}_q$ and $\mathbf{y}_q$ for d_q), and γ_p (resp. γ_q) the associate bi-infinite geodesic. Assume for instance that $d_p(\mathbf{x}_p, \mathbf{y}_p) < d_q(\mathbf{x}_q, \mathbf{y}_q)$. Let $\mathbf{z}_q \in \mathbb{H}^q$ be such that $d_q(\mathbf{x}_q, \mathbf{z}_q) + d_q(\mathbf{x}_q, \mathbf{y}_q) = d_q(\mathbf{z}_q, \mathbf{y}_q)$, and let $\mathbf{z}_p \in [\mathbf{x}_p, \mathbf{y}_p] \subset \mathbb{H}^p$ be such that $d_p(\mathbf{x}_p, \mathbf{z}_p) = d_p(\mathbf{x}_p, \mathbf{z}_p)$. Then the lightcone of $z = (\mathbf{z}_p, \mathbf{z}_q)$ contains both x and y. ☐

Remark 3.4.8. In the setting of Lemma 3.4.7, we may even chose the path between x and y contained in a flat of D (for any conformal identification with the symmetric space $\mathbb{H}^p \times \mathbb{H}^q$): in the proof, if we chose $\mathbf{z}_q \in \gamma_q$, then z lies in the flat of $\mathbb{H}^p \times \mathbb{H}^q$ generated by γ_p and γ_q. We will see in Section 6.4.7.1 that we can also construct such paths using Construction 3.4.5; see in particular Remark 6.4.14.

3.5 Proper domains and convexity in causal flag manifolds

In this section, we first introduce, in Section 3.5.1, some proper symmetric domains in causal flag manifolds $\mathbf{Sb}(\mathfrak{g})$, called *diamonds*, generalizing Lorentzian diamonds introduced in Section 3.4.1. These domains will, in fact, be the only proper almost-homogeneous domains in causal flag manifolds, by Theorem 8.3.1. In the notation of Chapter 5, given a HTT Lie algebra $\mathfrak{g}$, each diamond in $\mathbf{Sb}(\mathfrak{g})$ is a realization of the non-compact dual of the Nagano space $\mathbf{Sb}(\mathfrak{g})$.

The construction of these diamonds relies on the causal structure on $\mathbf{Sb}(\mathfrak{g})$. Diamonds allow us to define, in analogy with the case of *conformal spacetimes* [MS08], a notion of *causal convexity*, the properties of which we study in Section 3.5.2.5. We compare it with dual convexity in Proposition 3.5.24. This will enable us, in Section 4.3, to better understand the geometric properties of transverse groups preserving proper domains in $\mathbf{Sb}(\mathfrak{g})$.

Notation 3.5.1. For all this section, we take Notation 2.4.2.

3.5.1 Diamonds

In this section, we generalize the construction of Section 3.4.1, but here in another direction than the one of Section 3.4.2, the one of causal flag manifolds.

Fix a HTT Lie algebra $\mathfrak{g}$ and $G \in \mathscr{G}_{\{\alpha_r\}}(\mathfrak{g})$. Given two transverse points $x, y \in \mathbf{Sb}(\mathfrak{g})$, the set $\mathbf{Sb}(\mathfrak{g}) \smallsetminus (\mathrm{Z}_x \cup \mathrm{Z}_y)$ admits several connected components, exactly two of which are proper.

Definition 3.5.2. A subset Ω of $\mathbf{Sb}(\mathfrak{g})$ is called a *diamond* if there exist a (unique) pair of transverse points $x, y \in \mathbf{Sb}(\mathfrak{g})$ such that Ω is one of the two proper connected components of $\mathbf{Sb}(\mathfrak{g}) \smallsetminus (\mathrm{Z}_x \cup \mathrm{Z}_y)$. The two points x, y are then called the *endpoints of* Ω.

Let $\mathbf{D}_{\mathsf{std}} := \mathbf{I}^+(\mathfrak{p}^+)$. Recall from Section 2.2.6 that there exists an order-two element $k_0 \in G$ such that $k_0 \cdot \mathfrak{p}^+ = \mathfrak{p}^-$. Then $\mathbf{D}'_{\mathsf{std}} := \mathbf{I}^-(\mathfrak{p}^+) = k_0 \cdot \mathbf{D}_{\mathsf{std}}$ is the interior of the dual of $\mathbf{D}_{\mathsf{std}}$ (see e.g. [GW25, Lem. 13.11]), and the domains $\mathbf{D}_{\mathsf{std}}$ and $\mathbf{D}'_{\mathsf{std}}$ are exactly the two diamonds with endpoints $\mathfrak{p}^+$ and $\mathfrak{p}^-$. They are proper in $\mathbf{Sb}(\mathfrak{g})$ — although they are not proper in $\mathbb{A}_{\mathsf{std}}$.

Given two transverse points $x, y \in \mathbf{Sb}(\mathfrak{g})$, one has $(x, y) = g \cdot (\mathfrak{p}^+, \mathfrak{p}^-)$ for some $g \in G$. The two diamonds $g \cdot \mathbf{D}_{\mathsf{std}}$ and $g \cdot \mathbf{D}'_{\mathsf{std}}$ are the diamonds with endpoints x and y.

By the two previous paragraphs, any diamond is a G-translate of $\mathbf{D}_{\mathsf{std}}$. In particular, up to the action of G on $\mathbf{Sb}(\mathfrak{g})$, there is only one model of diamond in $\mathbf{Sb}(\mathfrak{g})$. It is convenient to consider models of diamonds that are proper in affine charts:

Definition 3.5.3. Let $\mathbb{A} \subset \mathbf{Sb}(\mathfrak{g})$ be an affine chart. If $x, y \in \mathbb{A}$ and $y \in \mathbf{I}^+(x)$, we define $\mathbf{D}_{\mathbb{A}}(x, y)$ as the set $\mathbf{I}^+(x) \cap \mathbf{I}^-_{\mathbb{A}}(y)$. It is one of the two diamonds with endpoints x and y.

For $x, y \in \mathbb{A}$ and $y \in \mathbf{I}^+_{\mathbb{A}}(x)$, the diamond $\mathbf{D}_{\mathbb{A}}(x, y)$ is the only one of the two diamonds with endpoints x and y that is proper in $\mathbb{A}$; see Figure 3.2. When $\mathbb{A} = \mathbb{A}_{\mathsf{std}}$, we will ommit the "$\mathbb{A}_{\mathsf{std}}$" in subscript.

Remark 3.5.4. When $\mathfrak{g} = \mathfrak{so}(n, 2)$, with $n \geq 3$, given two transverse points $x, y \in \mathrm{Ein}^{n-1,1}$, the set $\mathbf{Sb}(\mathfrak{g}) \smallsetminus (\mathrm{Z}_x \cup \mathrm{Z}_y)$ has exactly three connected components (see Figure 3.1 for $x = \mathfrak{p}^+$ and $y = \mathfrak{p}^-$). For a general HTT Lie group G, the set $\mathbf{Sb}(\mathfrak{g}) \smallsetminus (\mathrm{Z}_x \cup \mathrm{Z}_y)$, where p, q are two transverse points, may have more connected components. If $\dim(\mathbf{Sb}(\mathfrak{g})) \geq 3$, then there are exactly $(r + 1)$ connected components, where r is the real rank of G; see Section 3.5.2.2.

The following fact is well known (see e.g. [GW18, Prop. 3.7, 5.2 and Remark 5.4] and [Kan11, Thm 2.3 and 3.5]):

Fact 3.5.5. *The diamond $\mathbf{D}_{\mathsf{std}}$ is a symmetric domain of $\mathbf{Sb}(\mathfrak{g})$.*

Moreover, if $G \in \mathscr{G}_{\{\alpha_1\}}(\mathfrak{g})$, then the action of the identity component L^0 of L on $\mathbf{D}_{\mathsf{std}}$ is transitive and the stabilizer of a point in $\mathbf{D}_{\mathsf{std}}$ is a maximal compact subgroup of L^0, so that any diamond is a model for the symmetric space of L^0.

Since $L \simeq L_s \times \mathbb{R}$, the diamond $\mathbf{D}_{\mathsf{std}}$ is L-equivariantly diffeomorphic to $\mathbb{X}_{L_s} \times \mathbb{R}$, where $\mathbb{X}_{L_s}$ is the symmetric space of L_s. The corresponding identifications, for $\mathfrak{g}$ ranging in HTT Lie algebras, are listed in Table 3.1.

Remark 3.5.6.

1. The family of diamonds $\mathbf{D}(x, y)$, where $x, y \in \mathbb{A}_{\mathsf{std}}$ and $y \in \mathbf{I}^+(x)$, forms a basis of neighborhoods of $\mathbb{A}_{\mathsf{std}}$.
2. Since L_s admits cocompact lattices [Bor63], any diamond is divisible.
3. Diamonds are defined more generally in any flag manifold $\mathscr{F}(\mathfrak{g}, \Theta)$ admitting a Θ-positive structure [GLW21]. By Zimmer's theorem (Fact 1.2.3) and Fact 3.5.5, these diamonds are quasi-homogeneous (resp. divisible) if and only if the flag manifold $\mathscr{F}(\mathfrak{g}, \Theta)$ is the Shilov boundary associated with a HTT Lie algebra.

Example 3.5.7 (Explicit construction of diamonds)**.** Let us see what diamonds look like, for different values of G.

(1) $\mathfrak{g} = \mathfrak{so}(n, 2)$**, with** $n \geq 2$**.** The diamonds are exactly those defined in Section 3.4.1.

(2) $\mathfrak{g}$ **is neither** $\mathfrak{so}(n, 2)$ **for some** $n \geq 2$**, nor** $\mathfrak{e}_{7(-25)}$**.** In the notation of Section 2.4.4.4, we have

$$\mathbf{D}_{\mathsf{std}} = \left\{ \operatorname{Im} \begin{pmatrix} I_r \\ X \end{pmatrix} \mid X \in \mathrm{H}_r^{++}(\mathbb{K}) \right\};\ \mathbf{D}'_{\mathsf{std}} = \left\{ \operatorname{Im} \begin{pmatrix} I_r \\ -X \end{pmatrix} \mid X \in \mathrm{H}_r^{++}(\mathbb{K}) \right\},$$

where $\mathrm{H}_r^{++}(\mathbb{K}) = \{X \in \mathrm{Mat}_r(\mathbb{K}) \mid {}^t\overline{X} = X,\ \mathrm{sgn}(X) = (r, 0)\}$ is the set of positive-definite $\mathbb{K}$-hermitian matrices of size r. In particular, we recover that the diamond $\mathbf{D}_{\mathsf{std}}$ is L-equivariantly diffeomorphic to the symmetric space $\mathbb{X}_{L_s} \times \mathbb{R} = (\mathrm{SL}(r, \mathbb{K})/K) \times \mathbb{R}$, where K is a maximal compact subgroup of $\mathrm{SL}(r, \mathbb{K})$, see Table 3.1.

$\mathfrak{g}$	$\mathbf{D}_{\mathsf{std}} \simeq$
$\mathfrak{so}(n, 2)$	$\mathbb{H}^n \times \mathbb{R}$
$\mathfrak{sp}(2r, \mathbb{R})$	$(\mathrm{SL}(r, \mathbb{R})/\mathrm{SO}(r)) \times \mathbb{R}$
$\mathfrak{su}(r, r)$	$(\mathrm{SL}(r, \mathbb{C})/\mathrm{SU}(r)) \times \mathbb{R}$
$\mathfrak{so}^*(4r)$	$(\mathrm{SL}(r, \mathbb{H})/\mathrm{Sp}(r)) \times \mathbb{R}$
$\mathfrak{e}_{7(-25)}$	$(E_{6(-26)}/F_4) \times \mathbb{R}$

Table 3.1 – The diamonds in $\mathbf{Sb}(\mathfrak{g})$ for every HTT Lie algebra.

Given an affine chart $\mathbb{A}$ and $x, y \in \mathbb{A}$ such that $y \in \mathbf{I}^+_{\mathbb{A}}(x)$, the diamond $\mathbf{D}_{\mathbb{A}}(x, y)$ is the only one of the two diamonds with endpoints x and y that is proper in $\mathbb{A}$. The converse is true:

Lemma 3.5.8. *Let $\mathbb{A}$ be an affine chart of $\mathbf{Sb}(\mathfrak{g})$. Let $x, y \in \mathbb{A}$ be two transverse points. Assume that there exists a diamond D with endpoints x and y such that $\overline{D} \subset \mathbb{A}$. Then one of the following is satisfied:*

1. *One has* $y \in \mathbf{I}^+_{\mathbb{A}}(x)$ *and* $D = \mathbf{D}_{\mathbb{A}}(x, y)$.
2. *One has* $y \in \mathbf{I}^-_{\mathbb{A}}(x)$ *and* $D = \mathbf{D}_{\mathbb{A}}(y, x)$.

Proof. Let $z \in \mathbf{Sb}(\mathfrak{g})$ be such that $\mathbb{A} = \mathbb{A}_z$. Since $\overline{D} \subset \mathbb{A}_z$, the point z lies in the interior of D^*. Then by [GW18], the point x belongs to one of the two diamonds with endpoints y, z. These two diamonds are exactly $\mathbf{I}^+_{\mathbb{A}}(y)$ and $\mathbf{I}^-_{\mathbb{A}}(y)$, and the lemma follows. □

Example 3.5.9. We can now give an example to Remark 3.1.8, i.e. a proper domain of a flag manifold $\mathscr{F}(\mathfrak{g}, \Theta)$ whose dual is not the closure of the open set $\{\xi \in \mathscr{F}(\mathfrak{g}, \Theta) \mid \mathrm{Z}_\xi \cap \overline{\Omega} = \emptyset\}$. Let $D \subset \mathrm{Ein}^{n-1,1}$ be a diamond, and let us consider it as a proper domain in an affine chart $\mathbb{A}$. Let $y \in D$ and let us define $\Omega := D \smallsetminus (\mathbf{J}^+_{\mathbb{A}}(y) \cup \mathbf{J}^-_{\mathbb{A}}(y))$. Then Ω is a proper domain of $\mathrm{Ein}^{n-1,1}$. Its dual is $\Omega^* = D^* \cup \{y\}$, but

$$\overline{\{\xi \in \mathrm{Ein}^{n-1,1} \mid \mathrm{Z}_\xi \cap \overline{\Omega} = \emptyset\}} = D^* \neq \Omega^*.$$

Moreover, the dual Ω^* of Ω is not connected.

Finally, the domain Ω is dually convex, but there exists no affine chart in which Ω is contained as a convex domain (for the affine structure on the affine chart), since Ω is not simply connected.

If $x, y \in \mathbb{A}$ and $y \in \mathbf{J}^+_{\mathbb{A}}(x)$, then we will denote by $\mathbf{D}^c_{\mathbb{A}}(x, y)$ the Hausdorff limit of the sequence of $\overline{\mathbf{D}_{\mathbb{A}}(x, y_k)}$, where $(y_k) \in \mathbf{I}^+(x)$ and $y_k \to y$; this limit does not depend on the choice of the sequence (y_k). If $y \in \mathbf{I}^+(x)$, then one has $\mathbf{D}^c_{\mathbb{A}}(x, y) = \overline{\mathbf{D}_{\mathbb{A}}(x, y)}$.Again, we will ommit the "$\mathbb{A}_{\mathsf{std}}$" subscript if $\mathbb{A} = \mathbb{A}_{\mathsf{std}}$.

3.5.2 Another notion of convexity: causal convexity

Let $\mathfrak{g}$ be a HTT Lie algebra. There exists a notion of convexity in $\mathbf{Sb}(\mathfrak{g})$ called *dual convexity*, as mentioned in Section 3.1.1.2. In this section, we introduce another (weaker) notion of convexity, called *causal convexity*, inspired from causal convexity in conformal space-times (see [MS08]). Contrary to dual convexity, causal convexity is specific to flag manifolds admitting a causal structure. We investigate the properties of causally convex domains and relate the two notions of convexity.

3.5.2.1 Reminders on causal convexity in conformal spacetimes. A domain Ω of a conformal spacetime $(M, [g])$ is said to be *causally convex* if every causal curve of M joining two points of Ω is contained in Ω; see e.g. [MS08, pp 8].

Given two transverse points $x, y \in \mathrm{Ein}^{n-1,1}$, there always exists a timelike curve joining x to y: take for instance any smooth curve contained in one of the two diamonds with extremities x and y. Thus, the only causally convex domains of $\mathrm{Ein}^{n-1,1}$ are either empty or $\mathrm{Ein}^{n-1,1}$: the notion of causal convexity does not make sense. However, this notion makes sense in affine charts of $\mathrm{Ein}^{n-1,1}$, since those are identified with Minkowski space (by Equation (2.4.1)), and it is then easy to check that a domain $\Omega \subset \mathbb{A}$ is causally convex if and only if for every pair $a, b \in \Omega$, the diamond $\mathbf{D}_{\mathbb{A}}(a, b)$ is contained in Ω.

This observation is the inspiration of the notion of causal convexity we will introduce in Definition 3.5.14. Before giving this definition, we first investigate, in next Section 3.5.2.2, the connected components of $\mathbf{Sb}(\mathfrak{g}) \smallsetminus (\mathrm{Z}_x \cup \mathrm{Z}_y)$ that are not diamonds, where $\mathfrak{g}$ is a HTT Lie algebra and $x, y \in \mathbf{Sb}(\mathfrak{g})$ are transverse.

3.5.2.2 Other L^0-orbits. In Section 3.5.1, we have defined *diamonds* in $\mathbf{Sb}(\mathfrak{g})$. By Fact 3.5.5, we know that the standard diamond $\mathbf{D}_{\mathsf{std}}$ is an L^0-orbit. In this section, we investigate the other L^0-orbits in $\mathbf{Sb}(\mathfrak{g}) \smallsetminus (\mathrm{Z}_{\mathfrak{p}^+} \cup \mathrm{Z}_{\mathfrak{p}^-})$. Recall the strongly orthogonal roots $2\varepsilon_1, \ldots, 2\varepsilon_r$ defined in Section 2.4.4.1. For all $1 \leq i \leq r$, let $v_i \in \mathfrak{g}_{-2\varepsilon_i}$ be such that $[v_i, h_{-2\varepsilon_i}, \sigma_{\mathfrak{g}}(v_i)]$ is an $\mathfrak{sl}_2$-triple, where $h_{-2\varepsilon_i}$ is defined in Section 2.2.3. For all $1 \leq i, j \leq r$ such that $i + j \leq r$, we define

$$X_i := v_1 + \cdots + v_i - v_{i+1} - \cdots - v_{i+j} \in \mathfrak{u}^-.$$

Let $V_{i,j}$ be the L^0-orbit of $X_{i,j}$. We write $\mathscr{O}_i := V_{i,r-i}$.

Using the terminology of Jordan algebras and generalizing classical Sylvester's law of inertia, Kaneyuki proves that the set $\mathscr{O}_i$ is open and is the connected component of $\mathfrak{u}^- \smallsetminus \varphi_{\mathsf{std}}^{-1}(\mathrm{Z}_{\mathfrak{p}^+})$ containing $X_{i,r-i}$ [KAN88]. Kaneyuki also proves that $\varphi_{\mathsf{std}}^{-1}(\mathrm{Z}_{\mathfrak{p}^+}) = \bigsqcup_{i+j \leq r-1} V_{i,j}$ and

$$\overline{\mathscr{O}}_i = \bigsqcup_{k \leq i,\ \ell \leq r-i} V_{k,l} \quad \forall 1 \leq i \leq r. \tag{3.5.1}$$

Finally, one has $c^0 = \mathscr{O}_r$.

Example 3.5.10. 1. When $\mathfrak{g}$ is neither $\mathfrak{so}(n,2)$ for some $n \geq 2$, nor $\mathfrak{e}_{7(-25)}$, using the notation of Section 2.4.4.4.1, we can describe the domains $\mathscr{O}_i$, for $0 \leq i \leq r$. By Equation (2.4.17), one has $v_i = \begin{pmatrix} 0_r & 0 \\ E_{i,i} & 0_r \end{pmatrix}$, where $E_{i,i}$ is the $(r \times r)$-matrix with every coefficient equal to 0 except the one on the i-th row and i-th column. Then, by Equation (2.4.18), we have:

$$\mathscr{O}_i = \left\{ \begin{pmatrix} 0_r & 0 \\ X & 0_r \end{pmatrix} \mid X \in \mathrm{Mat}_r(\mathbb{K}),\ {}^t\overline{X} = X \text{ and } \mathrm{sgn}(X) = (i, r-i, 0) \right\}. \tag{3.5.2}$$

When $i = r$, we recover the cone c^0, and $\varphi_{\mathsf{std}}(\mathscr{O}_r)$ is exactly the diamond $\mathbf{D}_{\mathsf{std}}$ determined in Example 3.5.7. See [KAN88, Kan98] for more details.

2. When $\mathfrak{g} = \mathfrak{so}(n,2)$ for some $n \geq 2$, in the notation of Section 2.4.1 and the identification $\mathfrak{u}^- \simeq V$, one has $\mathfrak{u}^- = \mathscr{O}_0 \cup \mathscr{O}_1 \cup \mathscr{O}_2$, where:

$$\begin{aligned} \mathscr{O}_0 &= \{v \in \mathfrak{u}^- \mid \psi(v) < 0 \text{ and } v_n < 0\}; \\ \mathscr{O}_1 &= \{v \in \mathfrak{u}^- \mid \psi(v) > 0\}; \\ \mathscr{O}_2 &= \{v \in \mathfrak{u}^- \mid \psi(v) < 0 \text{ and } v_n > 0\}. \end{aligned}$$

□

The goal of this section is to prove Lemma 3.5.12 below. To this end, let us introduce some notation. Since $\dim(\mathfrak{g}_{-\alpha_r}) = 1$, any element $X \in \mathfrak{u}^-$ can be uniquely written $X = \lambda_r(X)v_r + X'$, with $\lambda_r(X) \in \mathbb{R}$ and $X' \in \sum_{\beta \in \Sigma^+_{\{\alpha_r\}} \smallsetminus \{\alpha_r\}} \mathfrak{g}_{-\beta}$. The map λ_r is then a linear form of $\mathfrak{u}^-$. The following holds:

Lemma 3.5.11. *One has $c^0 \subset \{\lambda_r > 0\}$.*

Proof. Recall that $c^0 = \mathrm{Ad}(L^0) \cdot X_{r,0}$. In the notation of Section 2.4.4.1, we have $\mathfrak{l} = \mathfrak{l}_s \oplus \mathbb{R}H_0$, where $\exp(tH_0)$ acts by positive dilations on $\mathfrak{u}^-$ for all $t \in \mathbb{R}$ (see e.g. [Gal24, Sect. 3.2]). Hence it suffices to prove that $\lambda_r(\ell \cdot X_{r,0}) > 0$ for all $\ell \in L_s$, where L_s is the semisimple part of L^0. The Lie algebra $\mathfrak{l}_s$ admits the following decomposition

$$\mathfrak{l}_s = \mathfrak{g}_0 \oplus \bigoplus_{\beta \in \Sigma^+ \cap \mathrm{Span}(\Delta \smallsetminus \{\alpha_r\})} (\mathfrak{g}_\beta \oplus \mathfrak{g}_{-\beta}) = \mathfrak{g}_0 \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-, \tag{3.5.3}$$

with

$$\mathfrak{n}^+ = \bigoplus_{\beta \in \Sigma^+ \cap \mathrm{Span}(\Delta \smallsetminus \{\alpha_r\})} \mathfrak{g}_\beta, \quad \mathfrak{n}^- = \bigoplus_{\beta \in \Sigma^+ \cap \mathrm{Span}(\Delta \smallsetminus \{\alpha_r\})} \mathfrak{g}_{-\beta},$$

and where $\mathfrak{g}_0$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$.

Let us set $N^\pm = \exp(\mathfrak{n}^\pm)$ and $G^0 = \exp(\mathfrak{g}_0)$. A direct computation gives that $\lambda_r(\mathrm{Ad}(\ell) \cdot X_{r,0}) = \lambda_r(X_{r,0}) = 1$ for all $\ell \in N^+$. On the other hand, one has $[\mathfrak{g}_0 \oplus \mathfrak{n}^-, \ker(\lambda_r)] \subset \ker(\lambda_r)$, where

$$\ker(\lambda_r) = \sum_{\beta \in \Sigma^+_{\{\alpha_r\}} \smallsetminus \{\alpha_r\}} \mathfrak{g}_{-\beta},$$

which implies that the group $\mathrm{Ad}(N^- G_0)$ preserves the kernel $\ker(\lambda_r)$. It follows that $\lambda_r(\mathrm{Ad}(\ell) \cdot X_{r,0}) \neq 0$, for all $\ell \in N^- G_0 N^+$. But the set $N^- G_0 N^+$ is dense in L_s, by (3.5.3). Since $N^- G_0 N^+$ is connected and $\lambda_r(X_{r,0}) > 0$, we have $\lambda_r(\ell \cdot X_{r,0}) > 0$ for all $\ell \in N^- G_0 N^+$. But the set $N^- G_0 N^+$ is dense in L_s, so $\lambda_r(\ell \cdot X_{r,0}) \geq 0$ for all $\ell \in L_s$. Since c^0 is open, we then have $c^0 \subset \{\lambda_r > 0\}$. $\square$

The orbits $\mathscr{O}_i$, for $0 \leq i \leq r$, satisfy the following elementary properties:

Lemma 3.5.12. *1. For all i, one has $\mathscr{O}_i + c^0 \subset \bigcup_{j \geq i} \overline{\mathscr{O}}_j$, and $\mathscr{O}_i - c^0 \subset \bigcup_{j \leq i} \overline{\mathscr{O}}_j$.*

2. For all i, the set $\mathscr{O}_i$ is equal to the interior of its closure.

3. For $1 \leq i \leq k \leq j \leq r$, one has

$$\overline{\mathscr{O}}_i \cap \overline{\mathscr{O}}_j \subset \overline{\mathscr{O}}_k.$$

4. For all $0 \leq i \leq r$, one has $-\mathscr{O}_i = \mathscr{O}_{r-i}$.

Proof. Points (2), (3) and (4) follow from Equation (3.5.1). It remains to prove (1). It suffices to prove the first inclusion, the second one admitting an analogue proof. Recall the decomposition

$$\mathfrak{u}^- = \sum_{\beta \in \Sigma^+_{\{\alpha_r\}}} \mathfrak{g}_{-\beta},$$

We have already defined a linear form λ_r on $\mathfrak{u}^-$. But For all $1 \leq i \leq r$, we have $2\varepsilon_i \in \Sigma_{\{\alpha_r\}^+}$, and $\dim(\mathfrak{g}_{-2\varepsilon_i}) = 1$, so we can also define a linear form λ_i: any element $X \in \mathfrak{u}^-$ can be uniquely written $X = \lambda_i(X) v_i + X'$, with $\lambda_i(X) \in \mathbb{R}$ and $X'_i \in \sum_{\beta \in \Sigma^+_{\{\alpha_r\}} \smallsetminus \{2\varepsilon_i\}} \mathfrak{g}_{-\beta}$. The map λ_i is then a linear form on $\mathfrak{u}^-$. We define

$$n_\ell(X) := |\{1 \leq i \leq r \mid \lambda_i(\ell \cdot X) > 0\}| \quad \forall \ell \in L^0;$$
$$n(X) := \max_{\ell \in L^0} n_\ell(X).$$

For $1 \leq i \leq r$, the integer $n(X)$ is constant on the L^0-orbit $\mathscr{O}_i$. By Lemma 3.5.11, we know that $\lambda_r(\mathrm{Ad}(\ell) \cdot X_{r,0}) > 0$ for all $\ell \in L^0$.

Let W_{L_s} be the restricted Weyl group of L_s, in the sense of Seciotn 2.2.4. Then for all $1 \leq i \leq r$, we have $\varepsilon_i \in W_{L_s} \cdot \varepsilon_r$ (see e.g. [GK98]). Hence we have

$$\lambda_i(\mathrm{Ad}(\ell) \cdot X_{r,0}) > 0 \quad \forall \ell \in L^0. \tag{3.5.4}$$

Thus $n(X_{r,0}) = r$.

For $1 \leq j \leq r$, we have $X_{j,r-j} = w_j \cdot X_r$, where $w_j \in W$ is the unique element of the Weyl group sending ε_k to $-\varepsilon_k$ for all $k \geq j+1$, and fixing ε_k for all $k \leq j$. Thus we have $n(X) = j$.

To prove the first inclusion of Point (1), by L^0-invariance of c^0, it suffices to prove that $X_{j,r-j} + c^0 \subset \bigcup_{k \geq j} \overline{\mathscr{O}}_k$. Let $Z \in c^0$. Then for all $1 \leq i \leq j$, we have

$$\lambda_i(X_{j,r-j} + Z) = -\lambda_i(X_{j,r-j}) + \lambda_i(Z) > \lambda_i(X_{j,r-j}) > 0,$$

the second last inequality holding by Equation (3.5.4). Thus we have $n(X_{j,r-j} + Z) \geq j$. Equation (3.5.1) then gives $X_{j,r-j} + c^0 \subset \bigcup_{k \geq j} \overline{\mathscr{O}}_k$. $\square$

Remark 3.5.13. Let us reprove Lemma 3.5.12.(1) in the case where $\mathfrak{g} = \mathfrak{sp}(2r, \mathbb{R})$ for some $r \geq 2$.

According to the description of the $\mathscr{O}_i$'s given in Equation (3.5.2), it suffices to show that for any two symmetric matrices X, Y with Y positive-definite and X of signature $(i, r-i, 0)$, the signature of $X + Y$ is $(j, k, r-j-k)$ with $j \geq i$. This follows directly from the definition of the signature: if V is an i-dimensional real vector subspace of $\mathbb{R}^r$, then by the positivity of Y, one has ${}^t\mathsf{v}(X+Y)\mathsf{v} > 0$ for all $\mathsf{v} \in V \smallsetminus \{0\}$. Hence, $j \geq i$.

The same reasoning on the signature allows one to prove Lemma 3.5.12.(1) in the cases $\mathfrak{g} = \mathfrak{su}(r,r)$ and $\mathfrak{g} = \mathfrak{so}(4r)^*$, where $r \geq 2$. However, the cases $\mathfrak{g} = \mathfrak{so}(n,2)$ with $n \geq 2$ and $\mathfrak{g} = \mathfrak{e}_{7(-25)}$ must be handled separately (even though the case $\mathfrak{g} = \mathfrak{so}(n,2)$ with $n \geq 2$ is well known). In order to treat all cases simultaneously, the strategy of our proof of Lemma 3.5.12.(1) is to reconstruct this notion of signature within $\mathfrak{u}^-$, in order to apply the reasoning from the previous paragraph: when $\mathfrak{g} = \mathfrak{sp}(2r, \mathbb{R})$, the integer $n(X)$, for $X \in \mathfrak{u}^-$, actually coincides exactly with the positive component of the signature of X when viewed as a symmetric $(r \times r)$-matrix.

3.5.2.3 Causal convexity in an affine chart. In this section, we generalize the notion of causal convexity, defined in Section 3.5.2.1, to general causal flag manifolds.

We fix an affine chart $\mathbb{A}$ of $\mathbf{Sb}(\mathfrak{g})$. Based on the discussion of Section 3.5.2.1, we define:

Definition 3.5.14. We say that a subset $X \subset \mathbb{A}$ is *causally convex in* $\mathbb{A}$ if for all $x, y \in X$ with $y \in \mathbf{J}^+_{\mathbb{A}}(x)$, the closed diamond $\mathbf{D}^c_{\mathbb{A}}(x,y)$ is contained in X.

Note that if $\Omega \subset \mathbb{A}$ is open, then it is causally convex if and only if for all $x, y \in \Omega$ such that $y \in \mathbf{I}^+_{\mathbb{A}}(x)$, one has $\mathbf{D}_{\mathbb{A}}(x,y) \subset \Omega$.

The intersection of two causally convex sets is still causally convex. This leads to the following definition:

Definition 3.5.15. Let $X \subset \mathbb{A}$. The *causally convex hull* $\mathrm{Conv}_{\mathbb{A}}(X)$ *of* X *in* $\mathbb{A}$ is the smallest causally convex subset of $\mathbb{A}$ containing X. Equivalently, the set $\mathrm{Conv}_{\mathbb{A}}(X)$ is the intersection of all causally convex subsets of $\mathbb{A}$ containing X.

Lemma 3.5.16. *Let $X \subset \mathbb{A}$ be a subset. The causally convex hull of X in $\mathbb{A}$ is equal to the union of all diamonds $\mathbf{D}^c_{\mathbb{A}}(x,y)$, for $x, y \in X$ and $y \in \mathbf{J}^+_{\mathbb{A}}(x)$. In particular, it is connected whenever X is.*

If $\Omega \subset \mathbb{A}$ be an open subset, then the causally convex hull of Ω in $\mathbb{A}$ is equal to the union of all diamonds $\mathbf{D}_{\mathbb{A}}(x,y)$, for $x, y \in X$ and $y \in \mathbf{I}^+_{\mathbb{A}}(x)$.

Proof. We prove the first assertion, the second one admitting a similar proof. Let us define $X' := \bigcup_{x,y \in X,\ y \in \mathbf{J}^+_{\mathbb{A}}(x)} \mathbf{D}^c_{\mathbb{A}}(x,y)$. By definition of the causally convex hull, we have $X' \subset \mathrm{Conv}_{\mathbb{A}}(X)$. Since $X \subset X'$, to prove the converse inclusion, it suffices to prove that X' is causally convex. Let $x, y \in X'$ be such that $y \in \mathbf{J}^+_{\mathbb{A}}(x)$. By definition of X', there exist $x_1, x_2 \in X$ such that $x_2 \in \mathbf{J}^+_{\mathbb{A}}(x_1)$ and $x \in \mathbf{D}^c_{\mathbb{A}}(x_1, x_2)$, and $y_1, y_2 \in \mathbb{A}$ such that $y_2 \in \mathbf{J}^+_{\mathbb{A}}(y_1)$ and $y \in \mathbf{D}^c_{\mathbb{A}}(y_1, y_2)$. Then by transitivity, we get that $y_2 \in \mathbf{J}^+_{\mathbb{A}}(x_1)$. Since $x_1, y_2 \in X$, by definition of X' we have $\mathbf{D}^c_{\mathbb{A}}(x,y) \subset \mathbf{D}^c_{\mathbb{A}}(x_1, y_2) \subset X'$. Hence X' is causally convex. □

Remark 3.5.17. Lemma 3.5.16 states a property of convexity studied in this section — causal convexity — that significantly distinguishes it from classical convexity in the real projective setting. Indeed, in the latter case, the convex hull of a set F is in general not equal to the union of the projective segments connecting two points of F; instead, every element of F is a convex combination of at most n points, where $F \subset \mathbb{P}(\mathbb{R}^n)$. In the case of causal convexity in causal flag manifolds, Lemma 3.5.16 is a consequence of the intrinsic causality of our definition of convexity, and will be crucial in the proof of the implication (1) ⇒ (2) of Proposition 4.3.2.

3.5.2.4 Link with dual convexity.

The goal of this section is to relate causal convexity and dual convexity, see Proposition 3.5.19 and Remark 3.5.20 below. To this end, we need the following lemma, which answers in particular a question of Neeb [Nee25, Problem 9.8]:

Lemma 3.5.18. *Every connected component of $\mathbb{A}_{\mathsf{std}} \smallsetminus \mathrm{Z}_{\mathfrak{p}^+} = \mathbf{Sb}(\mathfrak{g}) \smallsetminus (\mathrm{Z}_{\mathfrak{p}^+} \cup \mathrm{Z}_{\mathfrak{p}^-})$ is causally convex in $\mathbb{A}_{\mathsf{std}}$.*

Proof. Le $\mathscr{U}$ be a connected component of $\mathbb{A}_{\mathsf{std}} \smallsetminus \mathrm{Z}_{\mathfrak{p}^+}$. In the notation of Section 3.5.2.2, we know that $\mathscr{U} = \varphi_{\mathsf{std}}(\mathscr{O}_i)$ for some $0 \leq i \leq r$. Let $x, y \in \mathscr{U}$ be such that $y \in \mathbf{I}^+(x)$, and let $X, Y \in \mathscr{O}_i$ be such that $x = \varphi_{\mathsf{std}}(X)$, $y = \varphi_{\mathsf{std}}(Y)$. Then, by Points (1) and (3) of Lemma 3.5.12, one has

$$\mathbf{D}(x,y) = \varphi_{\mathsf{std}}\big((X + c^0) \cap (Y - c^0)\big) \subset \varphi_{\mathsf{std}}\Big(\big(\bigcup_{j \geq i} \overline{\mathscr{O}}_i\big) \cap \big(\bigcup_{j \leq i} \overline{\mathscr{O}}_i\big)\Big) \subset \overline{\mathscr{U}}.$$

Since $\mathbf{D}(x,y)$ is open, by Lemma 3.5.12.(2), we have $\mathbf{D}(x,y) \subset \mathscr{U}$. □

We can now prove:

Proposition 3.5.19. *Let $\mathbb{A}$ be an affine chart. Any dually convex domain contained in $\mathbb{A}$ is causally convex in $\mathbb{A}$.*

Proof. Since G acts transitively on $\mathbf{Sb}(\mathfrak{g})$, we assume that $\mathbb{A} = \mathbb{A}_{\mathsf{std}}$. Let Ω be a dually convex domain of $\mathbf{Sb}(\mathfrak{g})$ which is contained in $\mathbb{A}_{\mathsf{std}}$.

Let $x, y \in \Omega$, with $y \in \mathbf{I}^+(x)$. Assume that there exists $a \in \mathbf{D}(x,y) \cap \partial\Omega$. By dual convexity, there exists $z \in \mathbf{Sb}(\mathfrak{g})$ such that $\Omega \cap \mathrm{Z}_z = \emptyset$ and $a \in \mathrm{Z}_z$. Since U^- acts transitively on $\mathbb{A}_{\mathsf{std}}$, we may assume that $z = \mathfrak{p}^+$. By connectedness, the domain Ω is then contained in one of the connected components of $\mathbb{A}_{\mathsf{std}} \smallsetminus \mathrm{Z}_{\mathfrak{p}^+}$, let us denote it by $\mathscr{O}$. In particular, we have $x, y \in \mathscr{O}$. Then, by Lemma 3.5.18, we have $a \in \mathscr{O}$, which contradicts the fact that $a \in \mathrm{Z}_{\mathfrak{p}^+}$. Thus we must have $\mathbf{D}(x,y) \subset \Omega$. $\square$

Remark 3.5.20. The implication of Proposition 3.5.19 is not an equivalence, as there exist causally convex domains which are not dually convex. For instance, take a diamond D bounded in an affine chart $\mathbb{A}$, and consider $\Omega := D \smallsetminus \mathbf{J}^+_{\mathbb{A}}(x)$ for some point $x \in D$. Then Ω is causally convex but not dually convex.

3.5.2.5 Causal convexity in $\mathbf{Sb}(\mathfrak{g})$. In this section, our goal is to prove that causal convexity is an intrinsic notion in $\mathbf{Sb}(\mathfrak{g})$, namely, whenever a subset $X \subset \mathbf{Sb}(\mathfrak{g})$ is connected, the property for X of being causally convex does not depend on the choice of an affine chart containing X. This statement is contained in Proposition 3.5.22 below. We will need the following lemma:

Lemma 3.5.21. *Let $\Omega \subset \mathbf{Sb}(\mathfrak{g})$ be a domain contained in an affine chart. For any subset $X \subset \Omega$, the causally convex hull of X in $\mathbb{A}$ does not depend on the affine chart $\mathbb{A}$ containing Ω. We denote it by* $\mathrm{Conv}^{\Omega}(X)$.

Proof. Let $\mathbb{A}, \mathbb{A}'$ be two affine charts containing Ω. There exists $g \in G^0$ such that $g \cdot \mathbb{A}' = \mathbb{A}$. For all $x \in \Omega$, one has $g \cdot x \in g \cdot \mathbb{A}' = \mathbb{A}$.

Note that, given some point $x \in \mathbb{A}'$, one has $g \cdot \mathbf{I}^{\pm}_{\mathbb{A}'}(x) = \mathbf{I}^{\pm}_{\mathbb{A}}(g \cdot x)$. Let $C_{\mathbb{A}}(X)$ be the convex hull of X in $\mathbb{A}$, and $C_{\mathbb{A}'}(X)$ its convex hull in $\mathbb{A}'$. Note that one has $\mathbf{D}_{\mathbb{A}}(g \cdot x, g \cdot y) = g \cdot \mathbf{D}_{\mathbb{A}'}(x,y)$. Since Ω_0^{**} is contained in both $\mathbb{A}$ and $\mathbb{A}'$, by Proposition 3.5.19, it contains both $C_{\mathbb{A}}(X)$ and $C_{\mathbb{A}'}(X)$.

Let us prove that $C_{\mathbb{A}}(X) \subset C_{\mathbb{A}'}(X)$, the converse inclusion then holding by symmetry.

Let $x, y \in X$ be such that $y \in \mathbf{J}^+_{\mathbb{A}}(x)$, and $D := \mathbf{D}^c_{\mathbb{A}}(x,y) \subset C_{\mathbb{A}}(X)$. Let $(x_k), (y_k) \in \Omega^{\mathbb{N}}$ such that $x_k \to x$, $y_k \to y$, and $y_k \in \mathbf{I}^+_{\mathbb{A}}(x_k)$ for all k. For all $k \in \mathbb{N}$, we have $\overline{\mathbf{D}_{\mathbb{A}}(x_k, y_k)} \subset \overline{\Omega_0^{**}} \subset \mathbb{A}'$. Then by Lemma 3.5.8, we have $x_k \in \mathbf{I}^{\pm}_{\mathbb{A}'}(y_k)$. Thus $\mathbf{D}_{\mathbb{A}}(x_k, y_k) = \mathbf{D}_{\mathbb{A}'}(x_k, y_k)$ or $\mathbf{D}_{\mathbb{A}'}(y_k, x_k)$. This is true for all $k \in \mathbb{N}$, so $\mathbf{D}^c_{\mathbb{A}}(x,y) = \mathbf{D}^c_{\mathbb{A}'}(x,y) \subset C_{\mathbb{A}'}(X)$. By Lemma 3.5.16, this gives $C_{\mathbb{A}}(X) \subset C_{\mathbb{A}'}(X)$. $\square$

Lemma 3.5.21 allows us to prove:

Proposition 3.5.22. *Let $X \subset \mathbf{Sb}(\mathfrak{g})$ be a connected subset contained in two affine charts $\mathbb{A}, \mathbb{A}'$. Then we have*

$$\mathrm{Conv}_{\mathbb{A}'}(X) = \mathrm{Conv}_{\mathbb{A}}(X).$$

Proof. Since X is connected and contained in the open set $\mathbb{A} \cap \mathbb{A}'$, there exists a connected open neighborhhod Ω of X contained in both $\mathbb{A}$ and $\mathbb{A}'$. By Lemma 3.5.21, the set $\mathrm{Conv}^{\Omega}(X)$ is then equal to both $\mathrm{Conv}_{\mathbb{A}}(X)$ and $\mathrm{Conv}_{\mathbb{A}'}(X)$. $\square$

Proposition 3.5.22 implies in particular that the convex hull of a connected subset $X \subset \mathbf{Sb}(\mathfrak{g})$ contained in an affine chart does not depend on the affine chart containing it:

Definition 3.5.23. Let $X \subset \mathbf{Sb}(\mathfrak{g})$ be a connected subset contained in an affine chart. The *causally convex hull* $\mathrm{Conv}(X)$ is by definition the causally convex hull of X in any affine chart containing it. We say that X is *causally convex* if it is equal to its causally convex hull.

We can now prove:

Proposition 3.5.24 (see Proposition 1.4.1)**.** *Any dually convex domain of* $\mathbf{Sb}(\mathfrak{g})$ *different from* $\mathbf{Sb}(\mathfrak{g})$ *is causally convex.*

Proof. Since $\Omega \neq \mathbf{Sb}(\mathfrak{g})$, there exists $a \in \partial\Omega$. By dual convexity, there exists $z \in \mathbf{Sb}(\mathfrak{g})$ such that $a \in \mathrm{Z}_z$ and $\mathrm{Z}_z \cap \Omega = \emptyset$. Thus Ω is contained in an affine chart, namely $\mathbb{A}_z$. By Proposition 3.5.19, the domain Ω is causally convex in $\mathbb{A}_z$ and thus causally convex. $\square$

Another straightforward but useful corollary of Proposition 3.5.22 is the following:

Corollary 3.5.25. *Let* $X \subset \mathbf{Sb}(\mathfrak{g})$ *be a connected subset, contained in an affine chart, and let* $G \in \mathscr{G}_{\{\alpha_1\}}(\mathfrak{g})$*. Then the causally convex hull of* X *is* $\mathsf{Aut}_G(X)$*-invariant.*

3.5.2.6 The Einstein case: globally hyperbolic spacetimes. The content of this section comes from a work in collaboration with Adam Chalumeau [CG24]. We give a complete characterization of proper dually convex domains in the Lorentzian Einstein universe, in terms of causality. A conformal spacetime $(M, [g])$ is called *globally hyperbolic* if it admits a Cauchy hypersurface Σ, that is Σ is an acausal hypersurface such that any inextensible causal curve contained in M meets Σ exactly once (see [MS08, Def. 3.74]). Given two globally hyperbolic spacetimes M and N, a *Cauchy embedding* of M into N is a one-to-one conformal map $f : M \to N$ that sends a Cauchy hypersurface of M to a Cauchy hypersurface of N. We say that a globally hyperbolic spacetime M is *maximal* is any Cauchy embedding of M is onto (see [Sal13, Sect. 3]).

Concrete examples of globally hyperbolic spacetimes are causally convex domains of $\mathbb{R}^{n-1,1}$. These domains have been studied in [Bar05] and [Sma]. For simplicity we will only describe *bounded* causally convex domains of $\mathbb{R}^{1,n-1}$, or equivalently proper causally convex domains of $\mathrm{Ein}^{n-1,1}$. Let $\Omega \subset \mathbb{R}^{n-1,1}$ be a bounded causally convex domain. We fix a spacelike hyperplane H in $\mathbb{R}^{n-1,1}$ and we define V to be the image of the orthogonal projection of Ω on H. If $x \in V$, then the normal line to H through x intersects Ω in a bounded segment $(f_-(x), f_+(x))$. This defines two 1-Lipschitz functions $f_-, f_+ : V \to H^{\perp}$ which coincide on ∂V such that

$$\Omega = \{(x,t) \in V \times H^{\perp} \mid f_-(x) < t < f_+(x)\}.$$

Conversely, any domain defined in this way, where $f_-, f_+ : V \to H^\perp$ are 1-Lipschitz functions defined on a bounded domain $V \subset H$ that coincide on ∂V, is causally convex. All these domains are seen to be globally hyperbolic. A Cauchy hypersuface of Ω is exactly the graph of a 1-Lipschitz function $h : V \to H^\perp$ such that $f_- < h < f_+$. Moreover we know precisely when these domains are maximal as globally hyperbolic spacetimes: with the notation above, the domain Ω is maximal exactly when f_+ and f_- are *eikonal* (see [Sma, Prop. 15]). We make the link between the "abstract" notion of dual convexity and the more concrete and understood notion of causally convex maximal domains of the Minkowski space:

Proposition 3.5.26 (with Chalumeau, see [CG24]). *Let Ω be a proper causally convex domain of $\mathrm{Ein}^{n-1,1}$. Then Ω is dually convex if and only if it is globally hyperbolic maximal.*

Proof. We fix a suitable stereographic projection and identify Ω with a bounded subset of $\mathbb{R}^{n-1,1}$. We write $\mathbb{R}^{n-1,1} = H^\perp \oplus H$ for some spacelike hyperplane $H \subset \mathbb{R}^{n-1,1}$ and $\Omega = \{(t,x) \in H^\perp \times V \mid f_-(x) < t < f_+(x)\}$ where $V \subset H$ is a bounded domain and $f_-, f_+ : V \to H^\perp$ are two 1-Lipschitz functions that coincide on ∂V. We denote by $f : \partial V \to H^\perp$ the common value of f_+ and f_- on ∂V and $\pi : \mathbb{R}^{n-1,1} \to H$ the projection map.

If Ω is globally hyperbolic maximal, then (see [Sma, Sect. 7.1]) Ω is a connected component of the complementary $\mathbb{R}^{n-1,1} \smallsetminus C(\Lambda)$, where Λ is the graph of f and $C(\Lambda) = \cup_{x\in\Lambda} C(x)$. Therefore Ω is dually convex.

Assume now that Ω is dually convex. Let $\Omega' = \mathbb{R}^{n-1,1} \smallsetminus C(\Lambda)$ where Λ is the graph of f. Since Ω is causally convex, one has $\Omega \subset \Omega'$. We want to show that $\partial\Omega \subset C(\Lambda)$. Let $y \in \partial\Omega$ and let $z \in \mathrm{Ein}^{n-1,1}$ such that $C(z)$ is a supporting lightcone at y. If $y \in \Lambda$ then in particular $y \in C(\Lambda)$. Assume $y \notin \Lambda$, so that $y = (f_+(x), x)$ for some $x \in V$ (the case $y = (f_-(x), x)$ is similar). If $y \neq z$, let Λ be the unique photon through y and z (if $y = z$, take Λ to be any photon through y). The intersection $\Lambda \cap \overline{\Omega}$ is a lightlike segment containing y, we write it $[a,b]$ with $a \in \mathbf{I}^+(b)$. The projection $\pi([y,b]) = [x, \pi(b)]$ is a segment contained in $\overline{V}$. Assume by contradiction that this segment is contained in V. Let $v \in H$ be a unit vector collinear to $\pi(b) - x$ and $v' \in H^\perp$ be a unit future directed vector. For ε sufficiently small one has $f_-(b + \varepsilon v) < f_+(\pi(b)) - \varepsilon v' \leq f_+(b + \varepsilon v)$ so $[x, b + \varepsilon(v + v')/\sqrt{2}] \subset \overline{\Omega}$, a contradiction with the definition of $[a,b]$.

Therefore $\pi([y,b])$ intersects ∂V. We can always shorten $[y,b]$ and assume that $\pi([y,b))$ is contained in V and $\pi(b) \in \partial V$. Therefore $y \in \mathrm{Z}_b \subset C(\Lambda)$. This implies that Ω is a connected component of $\mathbb{R}^{n-1,1} \smallsetminus C(\Lambda)$, so Ω is globally hyperbolic maximal (see [Sma, Sect. 7.1]). $\square$

Chapter 4

Transverse groups preserving proper domains in flag manifolds

This chapter, motivated by Question 1.3.1, is devoted to the study of groups preserving proper domains in self-opposite flag manifolds, particularly in comparison with the real projective case.

In convex projective geometry, the conditions for an irreducible subgroup of $\mathrm{PGL}(n,\mathbb{R})$ to preserve a properly convex open subset of $\mathbb{P}(\mathbb{R}^n)$ are fairly weak. By [Ben00], a necessary and sufficient condition is that, up to finite index, the group Γ contains elements which are proximal in $\mathbb{P}(\mathbb{R}^n)$, and that all such elements have a real positive highest (in modulus) eigenvalue. The first goal of this chapter, addressed in Section 4.1, is to determine a necessary condition for a subgroup of a Lie group G to preserve a proper domain in one of the flag manifolds of G. In Proposition 4.1.5, we will see that for self-opposite flag manifolds, this property turns out to be quite restrictive.

Once a subgroup of $\mathrm{PGL}(n,\mathbb{R})$ preserves a properly convex domain, we can sometimes read the dynamical properties of Γ in the geometry of an open set and the convex subsets it preserves: for instance, by [DGK24], the group Γ is P_1-Anosov (in the sense of Section 2.3.2) if and only if it is *strongly convex cocompact* in $\mathbb{P}(\mathbb{R}^n)$ in the following sense:

Definition 4.0.1. [DGK24] A discrete subgroup $\Gamma \leq \mathrm{PGL}(n,\mathbb{R})$ is *strongly convex cocompact* if it preserves a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^n)$ and acts cocompactly on a closed convex subset $\mathscr{C}$ of Ω whose ideal boundary $\partial_i\mathscr{C} := \overline{\mathscr{C}} \cap \partial\Omega$ contains the *full orbital limit set* of Γ in Ω, and does not contain any nontrivial projective segment.

Here the *full orbital limit set* of Γ in Ω is, by definition, the set

$$\Lambda^{\mathrm{orb}}_{\Omega}(\Gamma) := \bigcup_{x\in\Omega} \overline{\Gamma\cdot x} \smallsetminus (\Gamma\cdot x).$$

If G is a real reductive linear Lie group and Θ a non-empty subset of the simple restricted roots of G, then by [GGKW17, Lem. 3.2] there exists an irreducible real Θ-proximal representation $\delta : G \to \mathrm{GL}(V)$, as defined in Section 2.3.3.2. Moreover, by Fact 2.3.4, there is a δ-equivariant embedding $\iota_\delta : \mathscr{F}(\mathfrak{g},\Theta) \to \mathbb{P}(V)$. Now let Γ be a group and

$\rho : \Gamma \to G$ a representation. If $\rho(\Gamma)$ already preserved a proper domain $\Omega \subset \mathscr{F}(\mathfrak{g}, \Theta)$, then it is easy to see that $\delta \circ \rho(\Gamma)$ will preserve a proper domain in $\mathbb{P}(V)$ (typically the interior of the convex hull of $\iota_\delta(\Omega)$). Furthermore, the representation ρ is Θ-Anosov if and only if $\delta \circ \rho$ is P_1-Anosov [GGKW17]. This gives:

Fact 4.0.2 ([GGKW17, DGK24]). *Let $\rho : \Gamma \to G$ be a representation preserving a proper domain $\Omega \subset \mathscr{F}(\mathfrak{g}, \Theta)$. The representation ρ is Θ-Anosov if and only if the image $\delta \circ \rho(\Gamma)$ is strongly convex cocompact in $\mathbb{P}(V)$.*

This provides a geometric characterization of Θ-Anosov groups of G that preserve proper domains in $\mathscr{F}(\mathfrak{g}, \Theta)$. An intrinsic geometric characterization within the flag manifold $\mathscr{F}(\mathfrak{g}, \Theta)$ would be more interesting (see Section 1.3.2 of the introduction); this is what we study in Section 4.3. However, the restrictive condition of preserving a proper domain (Proposition 4.1.5) excludes a number of such representations, such as the well-known maximal representations in HTT Lie groups (see e.g. [BILW05] for more information on maximal representations).

As seen in Chapter 3, convexity in general flag manifolds loses many structural properties when we move from real projective space to the general case. However, a lot of properties of projective convex cocompact groups relies on the "convexity" part of convex cocompactness in Definition 4.0.1. A first approach to studying groups that might be considered "convex cocompact" in a flag manifold $\mathscr{F}(\mathfrak{g}, \Theta)$ is to weaken the convexity requirement and instead focus on the properties of groups that preserve a proper domain $\Omega \subset \mathscr{F}(\mathfrak{g}, \Theta)$ and act cocompactly on a closed subset of Ω which is *not necessarily convex*. This is the purpose of Section 4.2.

In Section 4.3, we focus on groups satisfying a convex cocompactness property with respect to *causal convexity* in causal flag manifolds (see point (2) of Proposition 4.3.2). In Proposition 4.3.2, we show that this naive generalization of Definition 4.0.1 in causal flag manifolds does not characterize Θ-Anosov groups, but in fact all Θ-transverse groups. We will see that, even more than Proposition 4.1.5, this proposition highlights the restrictions on the topological and dynamical behavior of Θ-transverse groups preserving proper domains in $\mathscr{F}(\mathfrak{g}, \Theta)$ (see Remark 4.3.4).

Then, in Section 4.4, we construct examples of Zariski-dense Θ-Anosov groups of G preserving proper domains in $\mathscr{F}(\mathfrak{g}, \Theta)$ (for certain values of G and Θ).

This chapter highlights the challenge of defining a notion of convex cocompactness in flag manifolds that both generalizes the one in real projective space and characterizes the Anosov property.

4.1 Topological restrictions

In this section, we investigate the topological restrictions on groups preserving proper domains in flag manifolds, see Proposition 4.1.5 and its Corollary 4.1.7. These restrictions are related to the *relative positions* of 3 pairwise-transverse points of their limit set in the flag manifold. We first give reminders on this notion and its properties in Section 4.1.1 below.

4.1.1 Stable connected components

Let $\mathfrak{g}$ be a semisimple Lie algebra and $\Theta \subset \Delta$ be a self-opposite subset of the simple restricted roots of $\mathfrak{g}$. We know from Section 2.2.6.2 that if we remove from $\mathscr{F}(\mathfrak{g}, \Theta)$ a maximal proper Schubert variety Z_ξ ($\xi \in \mathscr{F}(\mathfrak{g}, \Theta)$), we get an affine chart, which is contractible. Now if we remove two such Schubert varieties in generic position, we obtain an open set that is in general no longer connected (it however has only finitely many connected components). The stability of its connected components under a well chosen symmetry (s_Θ in what follows) is related to several topological questions concerning the groups acting on $\mathscr{F}(\mathfrak{g}, \Theta)$ (see Remark 4.1.1 and Proposition 4.1.5).

Following [DGR24], consider the involution

$$\mathsf{s}_\Theta : \mathbb{A}_{\mathsf{std}} \longrightarrow \mathbb{A}_{\mathsf{std}}; \quad \varphi_{\mathsf{std}}(X) \longmapsto \varphi_{\mathsf{std}}(-X). \tag{4.1.1}$$

The map s_Θ induces a homeomorphism of $\mathbb{A}_{\mathsf{std}} \smallsetminus \mathrm{Z}_{\mathfrak{p}_\Theta^+}$ [DGR24]. We denote by $\mathscr{E}_\Theta$ the set of connected components of $\mathscr{F}(\mathfrak{g}, \Theta) \smallsetminus (\mathrm{Z}_{\mathfrak{p}_\Theta^+} \cup \mathrm{Z}_{\mathfrak{p}_\Theta^-})$. The map s_Θ induces a permutation of $\mathscr{E}_\Theta$.

Remark 4.1.1. Dey–Greenberg–Riestenberg proved [DGR24] that if $\mathscr{E}_\Theta$ has no s_Θ-invariant element, then any hyperbolic group Γ admitting a Θ-Anosov representation $\Gamma \to G$ (where $G \in \mathscr{G}_\Theta(\mathfrak{g})$) is either virtually free or virtually a surface group. The question of what flag manifolds $\mathscr{F}(\mathfrak{g}, \Theta)$ satisfy that $\mathscr{E}_\Theta$ has s_Θ-invariant elements is thus deeply related to questions on Θ-Anosov representations. It has been investigated by several authors:

1. If $\mathfrak{g} = \mathfrak{sl}(2p, \mathbb{R})$ with p odd, then we have $\mathscr{F}(\mathfrak{g}, \alpha_p) = \mathrm{Gr}_p(\mathbb{R}^{2p})$, $\mathbb{A}_{\mathsf{std}} \simeq \mathrm{Mat}_p(\mathbb{R})$, and $\mathbb{A}_{\mathsf{std}} \smallsetminus \mathrm{Z}_{\mathfrak{p}_\Theta^+} \simeq \mathscr{O}_1 \sqcup \mathscr{O}_{-1}$, with
$$\mathscr{O}_\varepsilon := \{X \in \mathrm{Mat}_p(\mathbb{R}) \mid \varepsilon \det(X) > 0\}.$$
Since $\det(-X) = -\det(X)$ for all $X \in \mathrm{Mat}_p(\mathbb{R})$, the set $\mathscr{E}_\Theta$ has no s_Θ-element. Recall Section 2.4.2 for more details on Grassmannians.
2. If $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{R})$, taking the root system given in Section 2.4.4.1, if Θ contains a simple root α_i for an odd $1 \leq i \leq n$, then $\mathscr{E}_\Theta$ has no s_Θ-element [DGR24].
3. If $\mathfrak{g} = \mathfrak{sl}(d, \mathbb{R})$ with $d \neq 5$ and $d = 2, 3, 4, 5 \mod 8$ and $\Theta = \Delta$, then $\mathscr{E}_\Theta$ has no s_Θ-invariant element [Dey22].
4. Kineider–Troubat have classified all the elements of $\mathscr{E}_\Theta$, where $\mathfrak{g} = \mathfrak{so}(p, q)$ and Θ is any subset of the simple restricted roots of $\mathfrak{g}$, and investigated which ones are s_Θ-invariant, see [KT24]. □

From now on, we will use the following terminology:

Definition 4.1.2. Let $F \subset \mathscr{F}(\mathfrak{g}, \Theta)$ be a subset. We denote by $F^{3,*}$ the set of triples of pairwise transverse points in F.

Note that if F is any subset of $\mathscr{F}(\mathfrak{g}, \Theta)$, then the set $F^{3,*}$ might be empty.

Until the end of this section, we fix $G := \mathsf{Aut}_\Theta(\mathfrak{g})$. The subgroup L_Θ of G acts on $\mathscr{E}_\Theta$. If $(x, y, z) \in \mathscr{F}(\mathfrak{g}, \Theta)$ are 3 pairwise transverse points, then there exists a unique $[g] \in G/L_\Theta$ such that $g \cdot (x, y) = (\mathfrak{p}_\Theta^+, \mathfrak{p}_\Theta^-)$. We denote by $\mathsf{type}(x, y, z)$ the L_Θ-orbit of the connected component $\mathscr{O}$ of $\mathscr{F}(\mathfrak{g}, \Theta) \smallsetminus (\mathrm{Z}_{\mathfrak{p}_\Theta^+} \cup \mathrm{Z}_{\mathfrak{p}_\Theta^-})$ containing $g \cdot y$.

Definition 4.1.3. The *type* of a triple $(x, y, z) \in \mathscr{F}(\mathfrak{g}, \Theta)^{3,*}$ is by definition the orbit $\mathsf{type}(x, y, z) \in \mathscr{E}_\Theta / L_\Theta$.

The type is a generalization of the well-known *Maslov index* for HTT Lie groups, which we will recall in Section 4.1.3. It is G-invariant.

Since s_Θ commutes with the action of L_Θ on $\mathfrak{u}_\Theta^-$, it induces a bijection of $\mathscr{E}_\Theta / L_\Theta$, still denoted by s_Θ. We then have

$$\mathsf{type}(x, y, z) = \mathsf{s}_\Theta(\mathsf{type}(x, z, y)) \quad \forall (x, y, z) \in \mathscr{F}(\mathfrak{g}, \Theta)^{3,*}. \tag{4.1.2}$$

Note that s_Θ induces a bijection between $\pi_0(\mathscr{F}(\mathfrak{g}, \Theta)^{3,*}/G)$ and $\mathscr{E}_\Theta / L_\Theta$.

Remark 4.1.4. If there exists $\ell \in L_\Theta$ such that $\ell_{|\mathbb{A}_{\mathsf{std}}} = \mathsf{s}_\Theta$, then Equation (4.1.2) becomes:

$$\mathsf{type}(x, y, z) = \mathsf{type}(x, z, y) \quad \forall (x, y, z) \in \mathscr{F}(\mathfrak{g}, \Theta)^{3,*}.$$

This is the case if and only if the flag manifold $\mathscr{F}(\mathfrak{g}, \Theta)$ is a *Nagano space*, see Remark 5.4.4.

4.1.2 The general case

In this section, we first study the topological restrictions imposed on a group by the property of preserving a proper domain. By "topological restrictions", we mean that these restrictions are on the type of triples of the Θ-limit set, and are thus related to a question of invariant connected components. Recall the notions introduced in Section 2.3.

Proposition 4.1.5. *Let G be a noncompact real semisimple Lie group, and Θ be a self-opposite subset of the simple restricted roots of G. Let $H \leq G$. If one of the two following conditions is satisfied, then there exists an s_Θ-invariant element $\mathscr{O} \in \mathscr{E}_\Theta$, and for any triple $(a, b, c) \in \Lambda_\Theta(H)^{3,*}$, we have $\mathsf{type}(a, b, c) = [\mathscr{O}] \in \mathscr{E}_\Theta / L_\Theta$:*

1. *The Θ-limit set $\Lambda_\Theta(H)$ contains at least four pairwise transverse points and H preserves a (not necessarily proper) domain $\Omega \subset \mathscr{F}(\mathfrak{g}, \Theta)$ such that*

$$\mathrm{Z}_p \cap \Omega = \emptyset \quad \forall p \in \Lambda_\Theta(H). \tag{4.1.3}$$

2. *The Θ-limit set $\Lambda_\Theta(H)$ contains at least 3 pairwise transverse points and H preserves a proper domain in $\mathscr{F}(\mathfrak{g}, \Theta)$.*

In particular, in this case, the set $\mathscr{E}_\Theta$ admits an s_Θ-invariant connected component.

Proof. 1. *Let us assume that (1) is satisfied.* First note that $a \in \overline{\Omega}$ for all $a \in \Lambda_\Theta(H)$: indeed, let $a \in \Lambda_\Theta(H)$, and let $(h_n) \in H^{\mathbb{N}}$ and $b \in \mathscr{F}(\mathfrak{g}, \Theta)$ such that (h_n) is Θ-contracting with respect to (a, b). Since Ω is open, there exists $x \in \Omega \smallsetminus \mathrm{Z}_b$. Thus $h_n \cdot x \to a$. Since Ω is H-invariant, we have $a \in \overline{\Omega}$.

Since G acts transitively on pairs of transverse points of $\mathscr{F}(\mathfrak{g}, \Theta)^2$, we may assume that $\mathfrak{p}_\Theta^+, \mathfrak{p}_\Theta^- \in \Lambda_\Theta(H)$ and that there exist two transverse points $x, y \in \Lambda_\Theta(H) \smallsetminus (\mathrm{Z}_{\mathfrak{p}_\Theta^+} \cup \mathrm{Z}_{\mathfrak{p}_\Theta^-})$. By Equation (4.1.3) and connectedness of Ω, there exists $\mathscr{O} \in \mathscr{E}_\Theta$ such that $\Omega \subset \mathscr{O}$. Since $x, y \in \overline{\Omega}$, we have $x, y \in \overline{\mathscr{O}}$. By transversality, we then have $x, y \in \mathscr{O}$. Let $X, Y \in \mathfrak{u}_\Theta^-$ be such that $(x, y) = (\varphi_{\mathsf{std}}(X), \varphi_{\mathsf{std}}(Y))$.

Let $Z \in \{X, Y\}$. Since $\varphi_{\mathsf{std}}(Z) \in \Lambda_\Theta(H)$, by Equation (4.1.3) and connectedness of Ω, there exists an element $\mathscr{O}' \in \mathscr{E}_\Theta$ such that $\Omega \subset \exp(Z) \cdot \mathscr{O}'$. Then $\mathfrak{p}_\Theta^+ \in \overline{\Omega} \subset \overline{\mathscr{O}}'$, and since $\mathfrak{p}_\Theta^+$ is transverse to $\varphi_{\mathsf{std}}(Z)$, we have $\mathfrak{p}_\Theta^+ \in \exp(Z) \cdot \mathscr{O}'$. Thus $\mathsf{s}_\Theta(\varphi_{\mathsf{std}}(Z)) = \exp(-Z) \cdot \mathfrak{p}_\Theta^+ \in \mathscr{O}'$, so $\varphi(Z) \in \mathsf{s}_\Theta(\mathscr{O}')$. Since $\varphi_{\mathsf{std}}(Z) \in \mathscr{O}$ and $\mathscr{O}, \mathsf{s}_\Theta(\mathscr{O}')$ are two connected components of $\mathscr{F}(\mathfrak{g}, \Theta) \smallsetminus (\mathrm{Z}_{\mathfrak{p}_\Theta^+} \cup \mathrm{Z}_{\mathfrak{p}_\Theta^-})$, we must have $\mathscr{O}' = \mathsf{s}_\Theta(\mathscr{O})$.

We have proven that $\Omega \subset \exp(Z)\mathsf{s}_\Theta(\mathscr{O})$. Since $x \in \overline{\Omega}$ and x is transverse to y, for $Z = Y$ we have
$$x \in \exp(Y)\mathsf{s}_\Theta(\mathscr{O}). \tag{4.1.4}$$
Similarly, taking $Z = X$, one gets $y \in \exp(X)\mathsf{s}_\Theta(\mathscr{O})$. Thus we have
$$\exp(-Y)\exp(X) \cdot \mathfrak{p}_\Theta^+ \in \mathsf{s}_\Theta(\mathscr{O})$$
and
$$\exp(-X)\exp(Y) \cdot \mathfrak{p}_\Theta^+ = \mathsf{s}_\Theta\big(\exp(-Y)\exp(X) \cdot \mathfrak{p}_\Theta^+\big) \in \mathsf{s}_\Theta(\mathscr{O}).$$
Thus $\mathsf{s}_\Theta(\mathscr{O}) = \mathsf{s}_\Theta(\mathsf{s}_\Theta(\mathscr{O})) = \mathscr{O}$. Moreover (4.1.4) implies that $[\mathscr{O}] = \mathsf{type}(x, y, \mathfrak{p}_\Theta^+)$. Now it is readily checked that for all $z \in \Lambda_\Theta(\Gamma) \smallsetminus (\mathrm{Z}_x \cup \mathrm{Z}_y)$, we have $\mathscr{O} \in \mathsf{type}(x, y, z)$.

2. *Let us assume that (2) is satisfied.* Let us prove that Equation (4.1.3) is satisfied. Let $a \in \Lambda_\Theta(H)$. There exists a Θ-contracting sequence $(g_k) \in H^{\mathbb{N}}$ such that (g_k) has Θ-limit a. Since Ω^* has nonempty interior, there exists $z \in \Omega^*$ such that $g_k \cdot z \to a$. Since Ω^* is H-invariant and closed, we have $a \in \Omega^*$. Thus by definition of Ω^*, one has $\mathrm{Z}_a \cap \Omega = \emptyset$. Thus Equation (4.1.3) is satisfied.

   The rest of the proof is similar to that of point (1). Since G acts transitively on pairs of transverse points of $\mathscr{F}(\mathfrak{g}, \Theta)^2$, we may assume that $\overline{\Omega} \subset \mathbb{A}_{\mathsf{std}}$, that $\mathfrak{p}_\Theta^+ \in \Lambda_\Theta(H)$, and that there exist two transverse points $x, y \in \Lambda_\Theta(H) \smallsetminus \mathrm{Z}_{\mathfrak{p}_\Theta^+}$. By Equation (4.1.3), there is an element $\mathscr{O} \in \mathscr{E}_\Theta$ such that $\Omega \subset \mathscr{O}$. Then $x, y \in \overline{\Omega} \subset \mathbb{A}_{\mathsf{std}} \cap \overline{\mathscr{O}}$, and since x, y are transverse to $\mathfrak{p}_\Theta^+$, we have $x, y \in \mathscr{O}$. As in point (1), we prove that $\mathscr{O} = \mathsf{s}_\Theta(\mathscr{O})$, and $\mathscr{O} \in \mathsf{type}(x, y, \mathfrak{p}_\Theta^+)$. Now if $z \in \Lambda_\Theta(H) \smallsetminus \mathrm{Z}_{\mathfrak{p}_\Theta^+}$, then it is readily checked that $[\mathscr{O}] = \mathsf{type}(x, y, z)$. □

Remark 4.1.6. One must be cautious here. Proposition 4.1.5 states that there exists an s_Θ-invariant element in $\mathscr{E}_\Theta$, which is significantly stronger than requiring the existence of an s_Θ-invariant element in $\mathscr{E}_\Theta / L_\Theta$. Indeed, as mentioned in Remark 4.1.4, if $\mathscr{F}(\mathfrak{g}, \Theta)$ is a Nagano space in the sense of Chapter 5 that follows, then all elements of $\mathscr{E}_\Theta / L_\Theta$ are s_Θ-invariant. However, if we consider, for instance, the Nagano space $\mathscr{F}(\mathfrak{sl}(2p, \mathbb{R}), \alpha_p) = \mathrm{Gr}_p(\mathbb{R}^{2p})$, with p odd, Remark 4.1.1 tells us that there is no element of $\mathscr{E}_\Theta$ that is s_Θ-invariant. Proposition 4.1.5 then tells us that there is no subgroup $H \leq \mathrm{PGL}(2p, \mathbb{R})$ satisfying condition (1) or (2).

4.1.3 Maslov index

In the case where G is a HTT Lie group of rank $r \geq 1$ and $\Theta = \{\alpha_r\}$, the relative positions of three transverse points of $\mathscr{F}(\mathfrak{g}, \Theta) = \mathbf{Sb}(\mathfrak{g})$ are described by the classical

Maslov index. In Corollary 4.1.7 below, we reformulate Proposition 4.1.5 in this particular context.

We take Notation 2.4.2 and the one of Section 3.5.2.2; recall in particular the definition of the domains $\mathscr{O}_i$ for $1 \leq i \leq r$. By Lemma 3.5.12.(4), and since $\mathfrak{l}$ contains an element H_0 such that $\mathrm{ad}(H_0)v = -v$ for all $v \in \mathfrak{u}^-$ by Section 2.4.4.2, the possible values of the type of a triple of pairwise transverse points $(x, y, z) \in \mathbf{Sb}(\mathfrak{g})$ are:

$$\mathsf{type}(x, y, z) = \left\{\varphi_{\mathsf{std}}(\mathscr{O}_i), \varphi_{\mathsf{std}}(\mathscr{O}_{r-i})\right\} \text{ for some } 0 \leq i \leq \frac{r}{2}. \tag{4.1.5}$$

The *Maslov index* of (x, y, z), denoted by $\mathsf{idx}(a, b, c)$, is then the well-defined integer $|r-2i|$, where $\mathsf{type}(x, y, z) = \{\varphi_{\mathsf{std}}(\mathscr{O}_i), \varphi_{\mathsf{std}}(\mathscr{O}_{r-i})\}$. With these notations, Proposition 4.1.5 gives:

Corollary 4.1.7. *Let G be a* HTT *Lie group of rank $r \geq 1$ and let $H \leq G$ be subgroup preserving a domain $\Omega \subset \mathbf{Sb}(\mathfrak{g})$. If one of the following conditions in satisfied, then r is even and $\mathsf{idx}(a, b, c) = 0$ for any triple $(a, b, c) \in \Lambda_{\{\alpha_r\}}(H)^{3,*}$:*

1. *If $\Lambda_{\{\alpha_r\}}(H)$ contains at least 4 transverse points and $\mathrm{Z}_{\mathfrak{p}^+} \cap \Omega = \emptyset$ for all $p \in \Lambda_{\{\alpha_r\}}(H)$;*
2. *If $\Lambda_{\{\alpha_r\}}(H)$ contains at least 3 transverse points and Ω is proper.*

Proof. By Proposition 4.1.5, we know that Points (1) and (2) both imply that there exists an $\mathsf{s}_{\{\alpha_r\}}$-invariant connected component $\mathscr{O}$ of $\mathbb{A}_{\mathsf{std}} \smallsetminus \mathrm{Z}_{\mathfrak{p}^+}$ such that $\mathscr{O} \in \mathsf{type}(x, y, z)$ for any triple of pairwise transverse points $(x, y, z) \in \Lambda_{\{\alpha_r\}}(H)$. But by Lemma 3.5.12.(4), the only connected component of $\mathbb{A}_{\mathsf{std}} \smallsetminus \mathrm{Z}_{\mathfrak{p}^+}$ which is $\mathsf{s}_{\{\alpha_r\}}$-invariant is $\varphi_{\mathsf{std}}(\mathscr{O}_{\frac{r}{2}})$, so $\mathscr{O} = \varphi_{\mathsf{std}}(\mathscr{O}_{\frac{r}{2}})$. In particular r is even. Moreover, by Equation (4.1.5), the Maslov index of any triple of distinct points $(x, y, z) \in \Lambda_{\{\alpha_r\}}(H)$ is equal to 0. □

Remark 4.1.8. Let Γ be the fundamental group of a closed surface, and let $\rho : \Gamma \to G$ be an $\{\alpha_r\}$-Anosov representation. By Proposition 1.4.2, if the group $\rho(\Gamma)$ preserves a proper domain, then the Maslov index of ρ is equal to 0, and in particular r is even. This implies in particular that *maximal* representations of Γ into G, i.e. those of Maslov index r, never preserve a proper domain in $\mathbf{Sb}(\mathfrak{g})$. On the contrary, in the context of groups preserving proper domains, we are interested in $\{\alpha_r\}$-Anosov representations $\rho : \Gamma \to G$ which are "as far as possible" from being maximal.

4.2 Groups acting cocompactly on a closed subset

The definition of projective convex cocompactness involves convexity; but in general flag manifolds, convexity loses many of the nice properties it had in real projective space (see Chapter 3). Thus, one difficulty in defining a notion of convex cocompactness in a general flag manifold $\mathscr{F}(\mathfrak{g}, \Theta)$ is to determine which convexity assumptions to make on the subsets of $\mathscr{F}(\mathfrak{g}, \Theta)$ which will play the role of Ω and $\mathscr{C}$ as introduced in Definition 4.0.1. A first step in studying groups that could be "convex cocompact" in a flag manifold $\mathscr{F}(\mathfrak{g}, \Theta)$ is to relax the convexity assumption and merely examine the properties of groups that preserve a proper domain $\Omega \subset \mathscr{F}(\mathfrak{g}, \Theta)$ and act cocompactly on a closed (not necessarily convex) subset of Ω. This is what we do in this section.

4.2.1 Dynamics of groups preserving proper domains

In this section, we fix a real noncompact semisimple Lie group G and a subset of the simple restricted roots Θ of G. We investigate basic dynamical properties of subgroups $H \leq G$ preserving a proper domain $\Omega \subset \mathscr{F}(\mathfrak{g}, \Theta)$ and acting cocompactly on a closed subset $\mathscr{C}$ of Ω. The *full orbital limit set* of H in Ω is:

$$\Lambda_\Omega^{\mathrm{orb}}(H) = \bigcup_{x \in \Omega} \overline{H \cdot x} \smallsetminus (H \cdot x).$$

Since H acts properly on Ω (recall Fact 3.1.3), we have

$$\Lambda_\Omega^{\mathrm{orb}}(H) \subset \partial\Omega. \tag{4.2.1}$$

Remark 4.2.1. If G is HTT and $\Theta = \{\alpha_r\}$, or if $G = \mathrm{SO}(p,q)$ and $\Theta = \{\alpha_1\}$, then we will prove in Theorems 8.3.1 and 8.4.1 that, whenever (4.2.1) is an equality, the domain Ω is a diamond.

Given a subset $\mathscr{C} \subset \Omega$ which is closed in Ω, the *ideal boundary $\partial_i\mathscr{C}$ of $\mathscr{C}$* is the set $\overline{\mathscr{C}} \smallsetminus \mathscr{C} = \overline{\mathscr{C}} \cap \partial\Omega$.

Definition 4.2.2. A closed subset $\mathscr{C}$ of Ω is said to have *strictly convex boundary* if for any two distinct points $x, y \in \partial_i\mathscr{C}$, one has $x \notin \mathscr{F}_\Omega^d(y)$.

The next Lemma 4.2.3 investigates the dynamical properties of a group preserving a proper domain and acting cocompactly on a closed subset with strictly convex boundary containing all the information on its dynamics, i.e. whose ideal boundary contains its full orbital limit set. Even not assuming any convexity assumption, we recover several properties of strongly convex cocompact subgroups of real projective space.

Lemma 4.2.3. *Let $\Omega \subset \mathscr{F}(\mathfrak{g}, \Theta)$ be a proper domain, and let us assume that there exist a subgroup $H \leq \mathsf{Aut}_G(\Omega)$ and a closed subset $\mathscr{C}$ of Ω such that:*

* *the set $\mathscr{C}$ is H-invariant and has strictly convex boundary;*
* *the group H acts cocompactly on $\mathscr{C}$;*
* *One has $\Lambda_\Omega^{\mathrm{orb}}(H) \subset \partial_i\mathscr{C}$.*

Then the following hold:

1. *For any $a \in \partial_i\mathscr{C}$, there exist $x_0 \in \mathscr{C}$ and $(g_k) \in H^{\mathbb{N}}$ such that $g_k \cdot x_0 \to a$.*
2. *For any $a \in \partial_i\mathscr{C}$ and any sequence $(g_k) \in H^{\mathbb{N}}$ such that there exists $x_0 \in \mathscr{C}$ with $g_k \cdot x_0 \to a$, the sequence (g_k) is Θ-contracting, with Θ-limit a.*
3. *The group H is Θ-divergent, and $\Lambda_\Theta(H) = \partial_i\mathscr{C} = \Lambda_\Omega^{\mathrm{orb}}(H)$.*
4. *For any $a \in \partial_i\mathscr{C}$, there exists $b \in \Omega^*$ such that $\mathrm{id} \in \mathsf{pos}^{(\Theta, \mathrm{i}(\Theta))}(a, b)$. If Θ is self-opposite, then this is equivalent to saying that $\mathrm{Z}_a \cap \Omega = \emptyset$.*

Proof. (1) Let $\mathsf{K} \subset \mathscr{C}$ be a compact subset such that $\mathscr{C} = H \cdot \mathsf{K}$. Let $x_k \in \mathscr{C}^{\mathbb{N}}$ such that $x_k \to a$. For all $k \in \mathbb{N}$ there exist $g_k \in \Gamma$ and $z_k \in \mathscr{C}$ such that $g_k \cdot z_k = x_k$. Up to extracting we may assume that there exists $x_0 \in \mathsf{K}$ such that $z_k \to x_0$. By $\mathsf{Aut}_G(\Omega)$-invariance of the Caratheodory metrics, in the notation of Fact 3.1.13, we have

$$C^{\rho}_{\Omega}(g_k \cdot z_k, g_k \cdot x_0) = C^{\rho}_{\Omega}(z_k, x_0) \longrightarrow 0. \tag{4.2.2}$$

Thus by Fact 3.1.13.(1), we have $g_k \cdot x_0 \to z_\infty \in \Lambda^{\mathrm{orb}}_{\Omega}(H)$.

(2) Let $y \in \Omega$ and let a' be a limit point of $(g_k \cdot y)$, then $a' \in \Lambda^{\mathrm{orb}}_{\Omega}(H) \subset \partial_i \mathscr{C}$. For all $k \in \mathbb{N}$ one has $C^{\rho}_{\Omega}(g_k \cdot x, g_k \cdot y) = C^{\rho}_{\Omega}(x, y) < +\infty$. By Fact 3.1.13.(2), this implies that $a' \in \mathscr{F}^{d}_{\Omega}(a)$. But $\partial_i \mathscr{C}$ is strictly convex, so $a' = a$.

We have proven that $g_k \cdot y \to a$ for all $y \in \Omega$. Now let $\mathsf{K}' \subset \Omega$ be a compact subset with nonempty interior. Then Fact 3.1.13.(1) implies that $g_k \cdot \mathsf{K}' \to \{a\}$ for the Hausdorff topology. Then by Fact 2.3.1.(1), the sequence (g_k) is Θ-contracting with Θ-limit a. This proves (2).

(3) Let $(g_k) \in H^{\mathbb{N}}$ be a sequence of distinct elements of H and let (δ_k) be a subsequence of (g_k). Let a be a limit point of $(\delta_k \cdot x)$. Then $a \in \partial_i \mathscr{C}$ and there exists a subsequence (δ'_k) of δ_k such that $\delta'_k \cdot x \to a$. Thus by point (2), the sequence (δ'_k) is Θ-contracting. By Fact 2.3.1.(1), the sequence (g_k) is thus Θ-divergent. This is true for every infinite sequence of H. Thus H is Θ-divergent. One then has $\Lambda_{\Theta}(H) \subset \Lambda^{\mathrm{orb}}_{\Omega}(H) \subset \partial_i \mathscr{C}$, and the converse inclusion follows from Points (1) and (2).

(4) Let $a \in \partial_i \mathscr{C}$. By Points (1) and (2), there exists a Θ-contracting sequence $(g_k) \in H^{\mathbb{N}}$ with limit a. Thus there exists $b \in \mathscr{F}(\mathfrak{g}, \Theta)^-$ such that $g_k \cdot y \to a$ for all $y \notin \mathrm{Z}_b$. Let $y \in \Omega^*$ be such that $w_0 \in \mathsf{pos}^{(\mathsf{i}(\Theta), \mathsf{i}(\Theta))}(b, y)$; such an element y exists because the set

$$\left\{ y \in \mathscr{F}(\mathfrak{g}, \Theta)^- \mid w_0 \in \mathsf{pos}^{(\mathsf{i}(\Theta), \mathsf{i}(\Theta))}(b, y) \right\}$$

is dense in $\mathscr{F}(\mathfrak{g}, \Theta)^-$, and Ω^* is open. Then there exists $a' \in \mathscr{F}(\mathfrak{g}, \Theta) \smallsetminus \mathrm{Z}_b$ such that $\mathrm{id} \in \mathsf{pos}^{(\mathsf{i}(\Theta), \Theta)}(y, a')$. Hence we have $g_k \cdot a' \to a$. On the other hand, up to extracting we may assume that $(g_k \cdot y)$ converges to some $c \in \Omega^*$. For all $k \in \mathbb{N}$ we have $\mathrm{id} \in \mathsf{pos}^{(\mathsf{i}(\Theta), \Theta)}(g_k \cdot y, g_k \cdot a')$, so taking the limit, one has $\mathsf{pos}^{(\mathsf{i}(\Theta), \Theta)}(c, a)$ (see [KLP18, Lem. 3.15]). ☐

In the notation of Lemma 4.2.3, Point (3) implies that the ideal boundary of $\mathscr{C}$ contains no more than the information about the dynamics of the elements of H.

Now, recall Definition 3.1.10.

Lemma 4.2.4. *Let $H \leq G$. Assume that there exist a proper H-invariant domain Ω and a closed H-invariant subset $\mathscr{C}$ of Ω, such that H acts cocompactly on $\mathscr{C}$. Then $\partial \mathscr{C} \cap \partial \Omega_0^{**} = \partial \mathscr{C} \cap \partial \Omega$. Moreover, if $\Lambda^{\mathrm{orb}}_{\Omega}(H) \subset \partial \mathscr{C} \cap \partial \Omega$ and $\mathscr{C}$ has strictly convex boundary for Ω, then $\Lambda^{\mathrm{orb}}_{\Omega_0^{**}}(H) \subset \partial \mathscr{C} \cap \partial \Omega_0^{**}$ and $\mathscr{C}$ still has strictly convex boundary for Ω_0^{**}.*

Proof. Note that Ω_0^{**} is H-stable. This inclusion $\partial \mathscr{C} \cap \partial \Omega_0^{**} \subset \partial \mathscr{C} \cap \partial \Omega$ is due to the fact that $\partial \Omega_0^{**} \subset \partial \Omega$. Now let $a \in \partial \Omega \cap \partial \mathscr{C}$. Then there exists $(x_k) \in \Omega^{\mathbb{N}}$ converging in Ω (and thus in Ω_0^{**}), and $(g_k) \in H^{\mathbb{N}}$ such that $g_k \cdot x_k \to a$. Since $a \notin \Omega$, the sequence (g_k) diverges in G. Since H acts properly on Ω_0^{**} and (x_k) converges in Ω_0^{**}, the sequence $(g_k \cdot x_k)$ diverges in Ω_0^{**}. Thus $p \in \partial \Omega_0^{**}$. Hence $\partial \mathscr{C} \cap \partial \Omega = \partial \mathscr{C} \cap \partial \Omega_0^{**}$.

Now assume that $\mathscr{C}$ has strictly convex boundary for Ω. Note that for all $a \in \mathscr{C} \cap \partial\Omega_0^{**}$, we have $\mathscr{F}^d_{\Omega_0^{**}}(a) \subset \mathscr{F}^d_{\Omega}(a)$. Thus $\mathscr{C}$ still has strictly convex boundary for Ω_0^{**}.

Now assume moreover that $\Lambda^{\mathrm{orb}}_{\Omega}(H) \subset \partial\mathscr{C} \cap \partial\Omega$. Let $a \in \Lambda^{\mathrm{orb}}_{\Omega_0^{**}}(H)$. There exists $y \in \Omega_0^{**}$ and $(g_k) \in H^{\mathbb{N}}$ such that $g_k \cdot y \to a$. Now let $x \in \mathscr{C}$. Since H acts properly on Ω, up to extracting we may assume that there exists $b \in \partial_i\mathscr{C}$ such that $g_k \cdot x \to b$. Then by Lemma 4.2.3.(2), the sequence (g_k) is Θ-contracting with limit point b. Let $b' \in \mathscr{F}(\mathfrak{g}, \Theta)^-$ be such that $g_k \cdot z \to b$ for all $z \in \mathscr{F}(\mathfrak{g}, \Theta) \smallsetminus \mathrm{Z}_{b'}$. Then $b' \in \Omega^*$ by the same argument as in the proof of Lemma 4.2.3, so $\Omega_0^{**} \cap \mathrm{Z}_{b'} = \emptyset$, and in particular $y \notin \mathrm{Z}_{b'}$. Thus $g_k \cdot y \to b$, and $a = b$. We have proven that $\Lambda^{\mathrm{orb}}_{\Omega_0^{**}}(H) \subset \partial\mathscr{C} \cap \partial\Omega_0^{**}$. The converse is clear by the equality $\partial\mathscr{C} \cap \partial\Omega = \partial\mathscr{C} \cap \partial\Omega_0^{**}$. $\square$

4.2.2 Finite generation

In this section, we prove Proposition 4.2.5 below, whose main consequence will be the implication "(2) $\Rightarrow$ Γ is finitely generated" of Proposition 1.4.4 (see also Proposition 4.3.2):

Proposition 4.2.5. *Let G be a noncompact linear reductive Lie group and $\Theta \subset \Delta$ a subset of the simple restricted roots. Let $\Gamma \leq G$ be a discrete subgroup preserving a proper domain in $\mathscr{F}(\mathfrak{g}, \Theta)$. Assume that there exists a closed Γ-invariant connected subset F of Ω, and a compact subset K of F such that $F = \Gamma \cdot \mathsf{K}$. Then Γ is finitely generated.*

The proof of Proposition 4.2.5 is contained in Sections 4.2.2.2 and 4.2.2.1 below.

4.2.2.1 Proof of Proposition 4.2.5 in the projective case. In this section, we take the notation of Proposition 4.2.5 when $G = \mathrm{PGL}(n, \mathbb{R})$ for some $n \in \mathbb{N}_{\geq 2}$ and Θ is the first simple restricted root of G, that is, $\mathscr{F}(\mathfrak{g}, \Theta) = \mathbb{P}(\mathbb{R}^n)$.

Up to considering the convex hull of Ω instead of Ω, we may assume that Ω is properly convex in $\mathbb{P}(\mathbb{R}^n)$, and we denote by H_Ω the classical Hilbert metric on Ω; recall Section 2.1.1.3 for the terminology. Let us chose $D \subset \mathbb{P}(\mathbb{R}^n)$ a properly convex domain, and denote by $\mathscr{D}$ the set of G-translates of D. Note that $\mathscr{D}$ forms a basis of neighborhoods of $\mathbb{P}(\mathbb{R}^n)$.

Let $\mathsf{K} \subset F$ be a compact set such that $F = \Gamma \cdot \mathsf{K}$. Now let us cover K with a finite number of elements $D_1, \dots, D_N \in \mathscr{D}$ intersecting K whose closure is contained in Ω, and set $\mathsf{K}' := \bigcup_{1 \leq i \leq N} \overline{D}_i \subset \Omega$. Let $F' := \Gamma \cdot \mathsf{K}'$. Then, by construction, the group Γ acts cocompactly on F'. Moreover, since Γ acts properly discontinuously on Ω (Fact 3.1.3), the set F' satisfies the following property:

Lemma 4.2.6. *For all $x \in F'$ and for any neighborhood $\mathscr{V}$ of x in Ω with closure contained in Ω, there exist some integer $m > 0$ and elements $A_1, \dots, A_m \in \mathscr{D}$ such that $\mathscr{V} \cap F' \subset \overline{A}_1 \cup \dots \cup \overline{A}_m$.*

Proof. Since Γ acts properly discontinuously on Ω, the set $S' := \{g \in \Gamma \mid g^{-1} \cdot \mathscr{V} \cap \mathsf{K}' \neq \emptyset\}$ is finite. By definition of F', we have

$$\mathscr{V} \cap F' \subset \bigcup_{g \in S'} g \cdot \mathsf{K}',$$

so we may take $\{A_1, \dots, A_N\} = \{g \cdot D_i \mid 1 \leq i \leq N,\ g \in S'\}$. $\square$

Given two points $x, y \in F'$, we denote by $\mathscr{C}^{F'}_{x,y}(\Omega)$ the set of continuous paths $\beta : [0,1] \to F'$ from x to y which are piecewise projective, that is, such that there exists a subdivision $t_0 = 0 < t_1 < \cdots < t_m < t_{m+1} = 1$ of $[0,1]$ such that $\beta([t_i, t_{i+1}])$ is a projective segment contained in F'. The length of such a path for the Hilbert metric H_Ω on Ω is then:

$$\mathsf{len}_\Omega(\beta) = \sum_{1 \leq i \leq m} \mathsf{H}_\Omega\big(\beta(t_i), \beta(t_{i+1})\big). \tag{4.2.3}$$

We define the following map on F':

$$\delta(x, y) = \inf_{\beta \in \mathscr{C}^{F'}_{x,y}(\Omega)} \mathsf{len}_\Omega(\beta). \tag{4.2.4}$$

Lemma 4.2.7. *The map δ is a proper geodesic Γ-invariant metric on F', generating the standard topology.*

Proof. The map δ obviously satisfies the triangle inequality and the symmetry property. Moreover, by construction of F', for all $x, y \in F'$, the set $\mathscr{C}^{F'}_{x,y}(\Omega)$ is nonempty, so by Equation (4.2.3), this implies that $\delta(x, y) < +\infty$. The Γ-invariance is straightforward. By definition, we have $\delta(x, y) \geq \mathsf{H}_\Omega(x, y)$ for all $x, y \in F'$. Since F' is closed in Ω and H_Ω is a proper metric, this implies that δ is a proper metric.

Now let $\mathscr{C}'_{x,y}(F')$ be the set of all rectifiable curves joining x and y in F'. By the definition of the length of a curve, one has $\delta(x, y) \leq \inf\big\{\ell_\delta(\beta) \mid \beta \in \mathscr{C}'_{x,y}(F')\big\}$ (where ℓ_δ is the length for the metric δ). Since the elements of $\mathscr{C}^{F'}_{x,y}(\Omega)$ are rectifiable, this last inequality is an equality. Hence δ is a length metric.

It remains to prove that δ generates the standard topology on F'. By Equation (4.2.4) and since H_Ω generates the standard topology, it suffices to prove that δ is continuous with respect to the standard topology. By the inequality

$$|\delta(x_0, y_0) - \delta(x, y)| \leq \delta(x_0, x) + \delta(y_0, y) \quad \forall x_0, y_0, x, y \in F',$$

one only needs to prove that for any $x_0 \in \Omega$ the map $x \mapsto \delta(x_0, x)$ is continuous at x_0. But this is a consequence of the fact that, by Lemma 4.2.6, for any $x_0 \in F'$ and any sequence $(x_k) \in (F')^{\mathbb{N}}$ such that $x_k \to x_0$, up to extracting there exists a G-translate D' of D such that $x_0 \in \overline{D}'$ and $x_k \in \overline{D}'$ for all $k \in \mathbb{N}$, and thus $\delta(x, x_k) = \mathsf{H}_\Omega(x_k, x)$.

We have proven that the metric space (F', δ) is proper length metric space, it is thus geodesic. $\square$

We have endowed F' with a Γ-invariant, proper, geodesic metric δ, generating the standard topology (and thus locally compact). Hence by Svarc-Milnor's Lemma, the group Γ is finitely generated. This ends the proof of Proposition 4.2.5 in the case where $G = \mathrm{PGL}(n, \mathbb{R})$ for some $n \geq 1$ and $\Theta = \{\alpha_1\}$.

4.2.2.2 Proof of Proposition 4.2.5 in the general case. We take the notation of Proposition 4.2.5. We fix a representation (V, ρ) such that (G, ρ, V) is a linear or projective Θ-proximal triple of $\mathfrak{g}$. Let ι_ρ, ι_ρ^- be the associate embeddings by Fact 2.3.4.

One can consider the connected component $\mathscr{O}$ of $\iota^-(\Omega^*)^*$ containing $\iota_\rho(\Omega)$. It is open, convex, and proper by Fact 2.3.5, and $\rho(\Gamma)$-invariant. Moreover, it contains the subset $\iota_\rho(F)$. We have:

Lemma 4.2.8. *The set $\iota_\rho(F)$ is closed in $\mathscr{O}$.*

Proof. Let $(x_k) \in F^{\mathbb{N}}$ be such that $\iota_\rho(x_k) \to y \in \mathscr{O}$. Since Γ acts cocompactly on F, there exists a converging sequence $(z_k) \in \Omega^{\mathbb{N}}$ and $(g_k) \in \Gamma^{\mathbb{N}}$ such that $g_k \cdot z_k = x_k$ for all $k \in \mathbb{N}$.

Since F is closed in Ω, if (x_k) does not converge in F, up to extracting it converges to a point $x \in \partial\Omega$. Thus the sequence (g_k) diverges in G, so $(\rho(g_k))$ diverges in $\mathrm{PGL}(V)$. Since $\mathscr{O}$ contains Ω, the sequence $(\iota_\rho(z_k))$ converges in $\mathscr{O}$. Since $\mathsf{Aut}_{\mathrm{PGL}(V)}(\mathscr{O})$ acts properly on $\mathscr{O}$, the sequence $\rho(g_k) \cdot \iota_\rho(z_k)$ cannot converge in $\mathscr{O}$, which is a contradiction. Thus (x_k) converges in F, so $y \in \iota_\rho(F)$. This proves that $\iota_\rho(F)$ is closed in $\mathscr{O}$. $\square$

Thus the discrete subgroup $\rho(\Gamma)$ of $\mathrm{PGL}(V)$ preserves a proper domain $\mathscr{O}$ and acts cocompactly on the closed (in $\mathscr{O}$) connected subset $\iota_\rho(F)$ of $\mathscr{O}$. Then by Section 4.2.2.1, the group $\rho(\Gamma)$ is finitely generated. Now since G is simple, the representation ρ has finite kernel, so Γ is finitely generated. This ends the proof of Proposition 4.2.5.

4.3 Transverse groups preserving proper domains in causal flag manifolds

In this section, we fix a HTT Lie group G of rank $r \geq 1$, and take Notation 2.4.2. We investigate the properties of $\{\alpha_r\}$-transverse groups preserving a proper domain in $\mathbf{Sb}(\mathfrak{g})$. Proposition 4.3.2 may seem surprising at first, as it appears to state that all $\{\alpha_r\}$-transverse groups are strongly convex cocompact, for a natural notion of strong convex cocompactness analoguous to the one of Definition 4.0.1. Indeed, in the projective case, as mentioned in the introduction of this chapter, any discrete subgroup of $\mathrm{PGL}(n, \mathbb{R})$ acting strongly convex cocompactly on a proper domain of $\mathbb{P}(\mathbb{R}^n)$ is $P_{\{\alpha_1\}}$-Anosov, and not merely $P_{\{\alpha_1\}}$-transverse.

Proposition 4.3.2 actually highlights the orthogonality between the notion of causal convexity (defined in Section 3.5.2.5), which is timelike (or, equivalently, of “maximal Maslov index”) by definition, and the spacelike (or, equivalently, of “Maslov index 0”) dynamical behavior of a group preserving a proper domain in $\mathbf{Sb}(\mathfrak{g})$, already observed in Corollary 4.1.7. See Remark 4.3.4 below for more details.

In analogy with the well-known convex cores in real hyperbolic geometry and in real projective geometry, we define:

Definition 4.3.1. Let $\Gamma \leq G$ be a discrete subgroup and let $\Omega \subset \mathbf{Sb}(\mathfrak{g})$ be a proper Γ-invariant domain. A *convex core of* (Ω, Γ) is a closed connected Γ-invariant causally convex subset $\mathscr{C}$ of Ω such that $\partial_i \mathscr{C}$ contains $\Lambda^{\mathrm{orb}}_{\Omega}(\Gamma)$.

Contrary to the projective case, where one can consider the convex hull of $\Lambda^{\mathrm{orb}}_{\Omega}(\Gamma)$ in Ω [DGK24], there is in our case no preferred convex core. The aim of this section is to prove:

Proposition 4.3.2 (see Proposition 1.4.4)**.** *Let G be a* HTT *Lie group and $\Gamma \leq G$ a discrete subgroup. The following are equivalent:*

1. *The group Γ is finitely generated, $\{\alpha_r\}$-transverse, preserves a proper domain $\Omega \subset \mathbf{Sb}(\mathfrak{g})$, and $\Lambda_{\{\alpha_r\}}(\Gamma)$ contains at least 3 points;*
2. *There exists a proper Γ-invariant* causally convex *domain $\Omega \subset \mathbf{Sb}(\mathfrak{g})$ such that Γ acts cocompactly on convex core $\mathscr{C}$ of (Ω, Γ) whose ideal boundary is transverse and contains at least 3 points;*
3. *There exists a proper Γ-invariant* dually convex *domain $\Omega' \subset \mathbf{Sb}(\mathfrak{g})$ such that Γ acts cocompactly on a convex core $\mathscr{C}'$ of (Ω', Γ) whose ideal boundary is transverse and contains at least 3 points.*

If these statements hold, then we have the equality

$$\partial_i \mathscr{C} = \Lambda_{\{\alpha_r\}}(\Gamma) = \Lambda_{\Omega}^{\mathrm{orb}}(\Gamma) = \Lambda_{\Omega'}^{\mathrm{orb}}(\Gamma) = \partial_i \mathscr{C}'.$$

Example 4.3.3. 1. Let $\Gamma \leq G$ be an infinite finitely generated discrete subgroup with symmetric generating set S, and let $x_0 \in \mathbb{H}^{n-1}$. Let $\mathscr{V} \subset \mathbb{H}^{n-1}$ be a bounded connected neighborhood of $S \cdot x_0$, and let $\mathscr{C} := \Gamma \cdot \overline{\mathscr{V}}$. The conformal identification $\mathbf{D}_{\mathsf{std}} \simeq \mathbb{H}^{n-1} \times (-\mathbb{R})$ gives an equivariant embedding $\mathbb{H}^{n-1} \hookrightarrow \mathbf{D}_{\mathsf{std}}$. Via this embedding, the set $\mathscr{C}$ is a subset of $\mathrm{Ein}^{n-1,1}$, causally convex (and even *acausal*) subset in $\mathbf{D}_{\mathsf{std}}$, on which Γ acts cocompactly, closed in the proper domain $\mathbf{D}_{\mathsf{std}}$, and such that $\mathscr{C}$ has strictly convex boundary. Thus Γ satisfies condition (2) of Proposition 1.4.4. This example illustrates why this condition (2) is not a good candidate for defining a notion of convex cocompactness in $\mathrm{Ein}^{n-1,1}$ (and in $\mathbf{Sb}(\mathfrak{g})$ for G a general HTT Lie group): the notion of convexity involved does not imply enough constraints of the *spatial* shape of $\mathscr{C}$.

2. Note that the assumtpion that $|\Lambda_{\{\alpha_r\}}(\Gamma)| > 2$ in (1) $\Rightarrow$ (2) is necessary. For instance, take $g \in L$ be an element with attracting fixed point $\mathfrak{p}^+$ and repelling fixed point $\mathfrak{p}^-$. Then $\Gamma := \langle g \rangle$ preserves $\mathbf{D}_{\mathsf{std}}$, and $\Lambda^{\mathrm{orb}}_{\mathbf{D}_{\mathsf{std}}}(\Gamma) = \{\mathfrak{p}^+, \mathfrak{p}^-\}$. The only closed causally convex subset of $\mathbf{D}_{\mathsf{std}}$ whose ideal boundary contains $\Lambda^{\mathrm{orb}}_{\mathbf{D}_{\mathsf{std}}}(\Gamma)$ is $\mathbf{D}_{\mathsf{std}}$ itself, and Γ clearly does not acts cocompactly on it.

Remark 4.3.4. Corollary 4.1.7 gives an intuition of why implication (1) $\Rightarrow$ (2) in Proposition 4.3.2 is true. If $\Gamma \leq G$ is an $\{\alpha_r\}$-transverse group preserving a proper domain $\Omega \subset \mathbf{Sb}(\mathfrak{g})$, then Proposition 1.4.2 suggests that its dynamics should be *spacelike*; in other words, if D is a diamond containing Ω, and if an observer is positioned at one extremity of D, then they have full access to all information regarding the geometric behavior of Γ. If Γ preserves and acts cocompactly on a closed subset $\mathscr{C}$ of Ω (which is always the case if Γ is $\{\alpha_r\}$-transverse), then considering the causally convex hull of $\mathscr{C}$ should not fundamentally alter what the observer perceives. Hence, requiring $\mathscr{C}$ to be causally convex ought to be a vacuous condition.

To obtain, in analogy with Definition 4.0.1, a notion of convex cocompactness in $\mathbf{Sb}(\mathfrak{g})$ equivalent to the property of being $\{\alpha_r\}$-Anosov, it would be necessary to introduce a notion of "spatial" convexity. In the case where $G = \mathrm{SO}(n, 2)$, Smaï [Sma22] gives a geometric interpretation of $\{\alpha_1\}$-Anosov representations in G (using our notation, we have $P_1 = P_{\{\alpha_2\}}$, see Section 2.4.4.4.2) that goes in this direction. However, it relies on the existence of *Cauchy hypersurfaces* in $\mathrm{Ein}^{n-1,1}$, which is deeply connected to the product structure $\mathbb{S}^{n-1} \times \mathbb{S}^1$ of the double cover of $\mathrm{Ein}^{n-1,1}$, and, in particular, on the fact that $\mathbb{S}^{n-1}$ is the boundary of a rank-one symmetric space. If G is a HTT Lie group of real-rank $r > 2$, then $\mathbf{Sb}(\mathfrak{g})$ admits a double cover by a product $N \times \mathbb{S}^1$, where N is the boundary of the *higher-rank* symmetric space of L_s. This property complicates the understanding of transversality in the product $N \times \mathbb{S}^1$ and the definition of Cauchy hypersurfaces. We believe that the higher rank of L_s should prevent one from defining a good notion of spatial convexity in this case, for reasons essentially similar to those of [Qui05].

4.3.1 Proof of implication (2) $\Rightarrow$ (1) of Proposition 4.3.2

Note that Lemma 4.2.3 does not give the implication implication (2) $\Leftrightarrow$ (1), as the transversality assumption on $\partial_i \mathscr{C}$ is *a priori* not equivalent to its strict convexity. Thus we need to prove that Γ is $\{\alpha_r\}$-divergent. It is done in the next lemma, whose proof is very similar to that of Lemma 4.2.3:

Lemma 4.3.5. *In the setting of Proposition 4.3.2, assume that Point (2) is satisfied. Then:*

1. *For any $a \in \partial_i \mathscr{C}$, there exist $x_0 \in \mathscr{C}$ and $(g_k) \in \Gamma^{\mathbb{N}}$ such that $g_k \cdot x_0 \to a$.*
2. *For any $a \in \partial_i \mathscr{C}$ and any sequence $(g_k) \in \Gamma^{\mathbb{N}}$ such that there exists $x_0 \in \mathscr{C}$ with $g_k \cdot x_0 \to a$, the sequence (g_k) is $\{\alpha_r\}$-contracting, with $\{\alpha_r\}$-limit a.*
3. *The group Γ is $\{\alpha_r\}$-divergent, and $\Lambda_{\{\alpha_r\}}(\Gamma) = \partial_i \mathscr{C} = \Lambda_\Omega^{\mathrm{orb}}(\Gamma)$.*
4. *For any $a \in \partial_i \mathscr{C}$, one has $\mathrm{Z}_a \cap \Omega = \emptyset$.*

Proof. (1) Let $\mathsf{K} \subset \mathscr{C}$ be a compact subset such that $\mathscr{C} = \Gamma \cdot \mathsf{K}$. Let $x_k \in \mathscr{C}^{\mathbb{N}}$ such that $x_k \to a$. For all $k \in \mathbb{N}$ there exists $g_k \in \Gamma$ and $z_k \in \mathscr{C}$ such that $g_k \cdot z_k = x_k$. Up to extracting we may assume that there exists $x_0 \in \mathsf{K}$ such that $z_k \to x_0$. By $\mathsf{Aut}_G(\Omega)$-invariance of the Caratheodory metrics, in the notation of Fact 3.1.13, we have

$$C_\Omega^\rho(g_k \cdot z_k, g_k \cdot x_0) = C_\Omega^\rho(z_k, x_0) \longrightarrow 0. \tag{4.3.1}$$

Thus by Fact 3.1.13.(1), we have $g_k \cdot x_0 \to z_\infty \in \Lambda_\Omega^{\mathrm{orb}}(\Gamma)$.

(2) Let $y \in \Omega$ and let a' be a limit point of $(g_k \cdot y)$, then $a' \in \Lambda_\Omega^{\mathrm{orb}}(\Gamma) \subset \partial_i \mathscr{C}$. Let us now use the terminology of Chapter 6. By Observation 6.4.2, there exists a *chain of photons* (as defined in Section 6.4.1) between y and the point x_0 determined in Point (1), contained in Ω. By induction, we may thus assume that y is on a photon through x_0. Hence for all $k \in \mathbb{N}$, the points $g_k \cdot y$ and $g_k \cdot x_0$ are on a same photon. Thus a and a' are on a same photon, they are thus nontransverse by Lemma 6.3.9. Since $\partial_i \mathscr{C}$ is transverse, we have $a = a'$.

We have proven that $g_k \cdot y \to a$ for all $y \in \Omega$. Now let $\mathsf{K}' \subset \Omega$ be a compact subset with nonempty interior. Then Fact 3.1.13.(1) implies that $g_k \cdot \mathsf{K}' \to \{a\}$ for the Hausdorff topology. Then by Fact 2.3.1.(1), the sequence (g_k) is $\{\alpha_r\}$-contracting with $\{\alpha_r\}$-limit a. This proves (2).

(3) Let $(g_k) \in \Gamma^{\mathbb{N}}$ be a sequence of distinct elements of H and let (δ_k) be a subsequence of (g_k). Let a be a limit point of $(\delta_k \cdot x)$. Then $a \in \partial_i \mathscr{C}$ and there exists a subsequence (δ'_k) of δ_k such that $\delta'_k \cdot x \to a$. Thus by point (2), the sequence (δ'_k) is $\{\alpha_r\}$-contracting. By Fact 2.3.1.(1), the sequence (g_k) is thus $\{\alpha_r\}$-divergent. This is true for every infinite

sequence of H. Thus H is $\{\alpha_r\}$-divergent. One then has $\Lambda_{\{\alpha_r\}}(\Gamma) \subset \Lambda_{\Omega}^{\mathrm{orb}}(\Gamma) \subset \partial_i \mathscr{C}$, and the converse inclusion follows from Points (1) and (2).

(4) Let $a \in \partial_i \mathscr{C}$. By Points (1) and (2), there exists a $\{\alpha_r\}$-contracting sequence $(g_k) \in \Gamma^{\mathbb{N}}$ with limit a. Thus there exists $b \in \mathbf{Sb}(\mathfrak{g})$ such that $g_k \cdot y \to a$ for all $y \notin \mathrm{Z}_b$. Let $y \in \Omega^* \smallsetminus \mathrm{Z}_b$; such an element y exists because the set $\mathbf{Sb}(\mathfrak{g}) \smallsetminus \mathrm{Z}_b$ is dense in $\mathbf{Sb}(\mathfrak{g})$, and Ω^* is open. Then $(g_k \cdot y)$ converges to a. By Γ-invariance and closedness of Ω^*, we have $a \in \Omega^*$. □

Let $\Gamma \leq G$ be a discrete subgroup satisfying Condition (2) of Proposition 4.3.2. Lemma 4.3.5.(3) and the transversality of $\partial_i \mathscr{C}$ imply that Γ is $\{\alpha_r\}$-transverse and $\Lambda_{\{\alpha_r\}}(\Gamma) = \partial_i \mathscr{C}$. Moreover, since $\partial_i \mathscr{C}$ contains at least 3 points, so does $\Lambda_{\{\alpha_r\}}(\Gamma)$. Finally, Proposition 4.2.5 implies that Γ is finitely generated.

Remark 4.3.6. Note that the discreteness assumption on Γ can be removed in Lemma 4.3.5. Moreover, in the setting of Chapter 5, we see that it generalizes to any self-opposite Nagano space with the exact same proof (replacing photons with *Helgason spheres*, see Section 6.5).

4.3.2 Proof of equivalence (2) ⇔ (3) of Proposition 4.3.2

Let us take the notation of Proposition 4.3.2 above.

Assume (2). Then Ω_0^{**} is a proper Γ-invariant dually convex domain of $\mathbf{Sb}(\mathfrak{g})$ containing $\mathscr{C}$, and the same proof as that of Lemma 4.2.4 gives us that we may replace Ω by Ω_0^{**}, so we get (3). Conversely, (3) $\Rightarrow$ (2) is just a consequence of the fact that any proper dually convex domain of $\mathbf{Sb}(\mathfrak{g})$ is causally convex (Proposition 1.4.1).

4.3.3 Proof of implication (1) ⇒ (2) of Proposition 4.3.2

Let us take the notation of Proposition 4.3.2 above, and assume that (1) is satisfied. We may assume that $\overline{\Omega} \subset \mathbb{A}_{\mathsf{std}}$ and that Ω is causally convex, up to considering the causally convex hull of Ω (in the sense of Definition 3.5.25) instead of Ω — note that this causally convex hull is still Γ-invariant, by Corollary 3.5.25.

Let $S := \{g_1, \dots, g_N\}$ be a symmetric family of generators of Γ, containing the identity element. Let $x \in \Omega$ and let $\mathscr{V}$ be a connected neighborhood of $S \cdot x$ such that $\overline{\mathscr{V}}_1 \subset \Omega$. The open set $\mathscr{A} := \Gamma \cdot \overline{\mathscr{V}}$ is thus connected, Γ-invariant, and closed in Ω.

Let $\mathscr{C}$ be the causally convex hull of $\mathscr{A}$, in the sense of Definition 3.5.25.

Lemma 4.3.7. *One has $\partial_i \mathscr{C} = \Lambda_{\{\alpha_r\}}(\Gamma)$.*

Proof. By construction, the set $\partial_i \mathscr{C}$ contains $\Lambda_{\{\alpha_r\}}(\Gamma)$. Let us prove the converse inclusion. Let $a \in \partial_i \mathscr{C}$. There exists $(z_k) \in (\mathscr{C} \cap \Omega)^{\mathbb{N}}$ such that $z_k \to a$. For all $k \in \mathbb{N}$, by definition of the convex hull, there exist $x_k, y_k \in \mathscr{A}$ such that $y_k \in \mathbf{J}^+(x_k)$ and $z_k \in \mathbf{D}^c(x_k, y_k)$. Let b, b' be the limits (up to extracting) of (x_k) and (y_k). For all $k \in \mathbb{N}$, we have $y_k \in \mathbf{J}^-(z_k)$, so $b' \in \mathbf{J}^-(a)$. Similarly, one has $q \in \mathbf{J}^+(a)$. Hence, by transitivity, one has $b' \in \mathbf{J}^-(b)$. But $b, b' \in \partial_i \mathscr{A} = \Lambda_{\{\alpha_r\}}(\Gamma)$, thus b and b' are either transverse or equal. If they are transverse, then $b' \in \mathbf{I}^-(b)$. Then $\Omega \subset \mathbf{I}^-(b)$. Similarly, $\Omega \subset \mathbf{I}^+(b')$. Hence $\Omega \subset \mathbf{D}(q, q')$. By causal convexity we must have $\mathbf{D}(b, b') = \Omega$.

But $|\Lambda_{\{\alpha_r\}}(\Gamma)| > 2$, so there exists $\eta \in \Lambda_{\{\alpha_r\}}(\Gamma) \smallsetminus \{b, b'\}$. Since $\Lambda_{\{\alpha_r\}}(\Gamma)$ is transverse, we must have $\eta \notin \mathrm{Z}_b \cup \mathrm{Z}_{b'}$. But $\eta \in \partial\Omega = \partial\mathbf{D}(b, b') \subset \mathrm{Z}_b \cup \mathrm{Z}_{b'}$, contradiction. Hence we have $b = b'$, so $p \in \mathbf{D}^c(b, b') = \{b\}$ is equal to b, and thus is in $\Lambda_{\{\alpha_r\}}(\Gamma)$. We have proven that $\partial_i\mathscr{C} = \Lambda_{\{\alpha_r\}}(\Gamma)$. $\square$

Implication (1) $\Rightarrow$ (2) of Proposition 4.3.2 is then a direct consequence of the following lemma:

Lemma 4.3.8. *The group Γ acts cocompactly on $\mathscr{C}$.*

Proof. Let $S' := \{g \in \Gamma \mid \overline{\mathscr{V}} \cap \mathbf{J}^+(\overline{\mathscr{V}}) \neq \emptyset \text{ or } \overline{\mathscr{V}} \cap \mathbf{J}^-(\overline{\mathscr{V}}) \neq \emptyset\}$. Assume that S' is infinite. Then there exists a sequence (g_k) of distinct elements of Γ, and $x_k, y_k \in \overline{\mathscr{V}}$ such that $x_k \in \mathbf{J}^+(g_k \cdot y_k)$ (for instance) for all $k \in \mathbb{N}$. Up to extracting one has $x_k \to x \in \overline{\mathscr{V}}$, and since Γ acts properly on Ω by Fact 3.1.3, we have $g_k \cdot y_k \to a \in \Lambda_{\{\alpha_r\}}(\Gamma)$. Then $x \in \mathbf{J}^+(a)$. Since $\mathbf{C}^+(a) \cap \Omega = \emptyset$, we have $x \in \mathbf{I}^+(a)$. Then by connectedness of Ω, one has $\Omega \subset \mathbf{I}^+(a)$.

Now let $b \in \Lambda_{\{\alpha_r\}}(\Gamma) \smallsetminus \{a\}$. By transversality, one has $b \notin \mathrm{Z}_a$. Since $b \in \overline{\Omega}$, by the previous paragraph we have $b \in \mathbf{I}^+(a)$. Since we have $\Omega \cap \mathrm{Z}_b = \emptyset$ and $a \in \mathbf{I}^-(b)$, one has $\Omega \subset \mathbf{I}^-(b)$. Thus $\Omega \subset \mathbf{D}(a, b)$. In particular, we have $\mathscr{C} \subset \mathbf{D}(a, b)$. Since $a, b \in \Lambda_{\alpha_r}(\Gamma) \subset \partial_i\mathscr{C}$, by causal convexity of $\mathscr{C}$ one has $\mathscr{C} = \mathbf{D}(a, b) = \Omega$. But then $\partial_i\mathscr{C} = \Lambda_{\{\alpha_r\}}(\Gamma)$ is not transverse, which is a contradiction.

Thus S' is finite. Let $\mathscr{B} := \bigcup_{g \in S'} g \cdot \overline{\mathscr{V}}$, and let K be the causal convex hull of $\mathscr{B}$ in $\mathbb{A}$. Then K is compact because $\mathscr{B}$ is. It is contained in Ω, as Ω is causally convex.

Thus it remains to prove that $\mathscr{C} \subset \Gamma \cdot \mathsf{K}$. Let $x \in \mathscr{C}$. There exist $a, b \in \overline{\mathscr{V}}$ and $g_1, g_2 \in \Gamma$ such that $x \in \mathbf{D}^c(g_1 \cdot a, g_2 \cdot b)$. Set $x' := g_1^{-1} \cdot x$ and $g := g_1^{-1} g_2$. Then $x' \in \mathbf{D}^c(a, g \cdot b)$ with $a, b \in \overline{\mathscr{V}}$ and $g \in S$, so $x' \in \mathsf{K}$ and $x \in g_1 \cdot \mathsf{K}$. Hence $\mathscr{C} \subset \Gamma \cdot \mathsf{K} \subset \Omega$. Since $\mathscr{C}$ is closed in Ω, the group Γ acts cocompactly on $\mathscr{C}$. $\square$

4.4 Zariski-dense Anosov subgroups preserving proper domains

In this section, taking Notation 2.4.2 we construct examples of Zariski-dense $\{\alpha_r\}$-Anosov subgroups of HTT Lie groups G of real rank $r \geq 1$ preserving a proper domain in $\mathbf{Sb}(\mathfrak{g})$. We will use the structural stability of $\{\alpha_r\}$-Anosov subgroups of G and deform well-chosen Anosov representations from Γ to L. However, the property for an $\{\alpha_r\}$-Anosov representation into G to be the deformation of a representation of Γ into L turns out to be restrictive, in particular when the real rank of G is greater than 2, as we will see in Section 4.4.3.

4.4.1 Openness property

Lemma 2.3.6 implies a stability property for Θ-Anosov representations preserving a proper domain in $\mathscr{F}(\mathfrak{g}, \Theta)$:

Corollary 4.4.1. *Let G be a real noncompact semisimple Lie group and Θ a subset of the simple restricted roots of G. Let Γ be a word hyperbolic group and let $\rho : \Gamma \to G$ be a Θ-Anosov representation. Assume that there exists a proper $\rho(\Gamma)$-invariant* *domain $\Omega \subset \mathscr{F}(\mathfrak{g}, \Theta)$. Then there exists a neighborhood $\mathscr{U}$ of ρ in* $\mathrm{Hom}_{\Theta-\mathsf{An}}(\Gamma, G)$ *such that for every representation $\rho' \in \mathscr{U}$, there exists a proper $\rho'(\Gamma)$-invariant domain $\Omega' \subset \mathscr{F}(\mathfrak{g}, \Theta)$.*

Proof. It suffices to prove that for any sequence $(\rho_k) \in \mathrm{Hom}_{\Theta-\mathsf{An}}(\Gamma, G)^{\mathbb{N}}$ such that $\rho_k \to \rho$, there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ there exists a proper $\rho_k(\Gamma)$-invariant domain $\Omega_k \subset \mathscr{F}(\mathfrak{g}, \Theta)$.

Let $z_0 \in \Omega^*$ and $x_0 \in \Omega$. Let us fix some symmetric generating set S of Γ containing the identity. By Lemma 2.3.6, the sets $k_k := \overline{\rho_k(\Gamma) \cdot z_0}$ converge to $X := \overline{\rho(\Gamma) \cdot z_0}$ for the Hausdorff topology as $k \to +\infty$.

Since $X \subset \overline{\Omega^*}$, we have $\mathrm{Z}_x \cap \Omega = \emptyset$ for all $x \in X$. Thus for k large enough, there exists a connected neighborhood $\mathscr{V}_k$ of $\{\rho_k(g) \cdot x_0 \mid g \in S\}$ such that $\mathrm{Z}_x \cap \mathscr{V}_k = \emptyset$ for all $x \in x_k$; Then $\Omega_k := \rho_k(\Gamma) \cdot \mathscr{V}_k$ is a connected, $\rho_k(\Gamma)$-invariant domain, such that

$$\overline{\Omega}_k = \bigcup_{g \in \Gamma} \overline{\rho_k(g) \cdot \mathscr{V}_k} \cup \xi_{\rho_k}(\partial_\infty \Gamma) \subset \mathbb{A}_{z_0}.$$

Thus Ω_k is proper in $\mathbb{A}_{z_0}$. □

4.4.2 Deformations of Anosov representations into L_s

In this section, we take Notation 2.4.2. Using Corollary 4.4.1, we construct Zariski-dense $\{\alpha_r\}$-Anosov subgroups of HTT Lie groups G preserving a proper domain in $\mathbf{Sb}(\mathfrak{g})$. More precisely, we prove:

Proposition 4.4.2 (see Proposition 1.4.5)**.** *Let $r = 2p$, with $p \in \mathbb{N}_{>0}$. If G is a* HTT *Lie group of real rank r, then there exist Zariski-dense $\{\alpha_r\}$-Anosov surface groups in G preserving a proper domain in $\mathbf{Sb}(\mathfrak{g})$.*

Recall from Corollary 4.1.7 that the assumption that r is even is necessary in Proposition 4.4.2. Note that Example 4.4.6.(2) below allows, in the case where p is even, to produce examples that are neither virtually free nor surface groups.

For the rest of this section, we fix once and for all a HTT Lie group G of real rank r. Recall the strongly orthogonal roots $2\varepsilon_1, \ldots, 2\varepsilon_r$ introduced in Section 2.4.4.1. We fix $(v_0^+, v_0^-) \in \mathfrak{g}_{\varepsilon_1+\varepsilon_r} \times \mathfrak{g}_{-\varepsilon_1-\varepsilon_r} \subset \mathfrak{u}^+ \times \mathfrak{u}^-$ such that $(v_0^+, h_{\varepsilon_1+\varepsilon_r}, v_0^-)$ is an $\mathfrak{sl}_2$-triple, where $h_{\varepsilon_1+\varepsilon_r}$ is defined in Section 2.2.3. We will need the following lemma:

From now on, we assume that $r = 2p$ with $p \in \mathbb{N}$. A fundamental system of simple restricted roots of $\mathfrak{l}_s$ is

$$\{\alpha_1, \ldots, \alpha_{r-1}\}, \tag{4.4.1}$$

of type A_{r-1}. We denote by $\beta_{i,j}$ the positive root $\varepsilon_i - \varepsilon_j$, for $i > j$. Let $(h_{i,j} := h_{\beta_{i,j}})$ be the associated co-roots (see Section 2.2.3). For all i, j $(i < j)$ there is a unique pair $(e_{i,j}, f_{i,j}) \in (\mathfrak{l}_s)_{\beta_{i,j}} \times (\mathfrak{l}_s)_{-\beta_{i,j}}$ such that $(e_{i,j}, h_{i,j}, f_{i,j})$ is an $\mathfrak{sl}_2$-triple. Now let

$$h := \sum_{k=1}^{p} h_{k,p+k}; \quad e := \sum_{k=1}^{p} e_{k,k+p}; \quad f := \sum_{k=1}^{p} f_{k,k+p}.$$

Then (e, h, f) is an $\mathfrak{sl}_2$-triple. Note that v_0^+ and v_0^- both commute with h.

The $\mathfrak{sl}_2$-triple (e,h,f) induces a Lie algebras embedding $\mathfrak{sl}_2(\mathbb{R}) \hookrightarrow \mathfrak{l}_s$, which itself induces a group homomorphism $\tau_p : \mathrm{SL}(2,\mathbb{R}) \to L_s^0$ with kernel $\{\pm I_2\}$. Let Γ be the fundamental group of a closed surface $\mathscr{S}$ of genus $g \geq 2$. The natural inclusion $\Gamma \hookrightarrow \mathrm{SL}(2,\mathbb{R})$ induces a representation $\rho : \Gamma \hookrightarrow \mathrm{SL}(2,\mathbb{R}) \xrightarrow{\rho} L_s^0$, which is $\{\alpha_p\}$-Anosov.

Now the representation $\tau_0 : \Gamma \hookrightarrow \mathrm{SL}(2,\mathbb{R}) \xrightarrow{\tau_p} L_s^0 \hookrightarrow G$, which is $\{\alpha_r\}$-Anosov by [GW12, Prop. 4.4]. Note that τ_0 preserves the two diamonds of $\mathbf{Sb}(\mathfrak{g})$ with endpoints $\mathfrak{p}^+$ and $\mathfrak{p}^-$ (see Fact 3.5.5). Thus, by Corollary 4.4.1, any small deformation of τ_0 in $\mathrm{Hom}_{\{\alpha_r\}-\mathsf{An}}(\Gamma, G)$ still preserves a proper domain in $\mathbf{Sb}(\mathfrak{g})$. Since $\{\alpha_r\}$-Anosov representations are discrete and faithful, Corollary 4.4.1 and the following proposition imply in particular Proposition 4.4.2:

Proposition 4.4.3. *There exists $g_0 \geq 2$ such that for all $g \geq g_0$ and any neighborhood $\mathscr{U}$ of τ_0 in $\mathrm{Hom}_{\{\alpha_r\}-\mathsf{An}}(\Gamma, G)$, there exists a Zariski-dense representation $\rho \in \mathscr{U}$.*

Proposition 4.4.3 is just a consequence of a theorem of Kim–Pansu, saying that surface groups of sufficiently large genus in classical real Lie groups G' admit small Zariski-dense deformations, unless G' is Hermitian *not* of tube type [KP14, KP15]. Indeed, here the condition that the real rank r is even implies that the HTT Lie group G we consider is *not* locally isomorphic to $E_{7(-25)}$ (which is of real rank 3), so G a classical Lie group, see Table 2.1.

Remark 4.4.4. It is possible, with the same method, to produce Θ-Anosov subgroups of G, where $\mathscr{F}(\mathfrak{g},\Theta)$ is a *Nagano space*, preserving a proper domain in $\mathscr{F}(\mathfrak{g},\Theta)$ (see Chapter 5 for the definition of Nagano spaces, and Section 5.1.3 for the description of the embedding of a symmetric domain in a Nagano space).

4.4.3 Restrictions on Anosov representations induced from L

In Section 4.4.2, taking Notation 2.4.2, we have constructed Zariski-dense $\{\alpha_r\}$-Anosov subgroups of HTT Lie groups G preserving a proper domain of $\mathbf{Sb}(\mathfrak{g})$, by deformations of $P_{\{\alpha_{r/2}\}}$-Anosov representations of surface groups into L_s into G, where r is even. In the present section, we investigate the topological restrictions on discrete subgroups of G built from deformations of representations $\rho_0 : \Gamma \to L_s \hookrightarrow G$. Since our considerations are topological, it goes back to investigating the topological restrictions on representations $\Gamma \to L_s$ such that $\Gamma \to L_s \hookrightarrow G$ is $\{\alpha_r\}$-Anosov.

We know from Equation (4.4.1) that the Weyl chamber of L_s associated with its root system $\{\alpha_1, \dots, \alpha_{r-1}\}$ is:

$$\overline{\mathfrak{a}}^+_{L_s} := \{X \in \mathfrak{a} \mid \varepsilon_i(X) \geq \varepsilon_{i+1}(X) \quad \forall 1 \leq i \leq r-1\}.$$

We denote by $\mu_{L_s} : L_s \to \overline{\mathfrak{a}}^+_{L_s}$ the Cartan projection of L_s. The following proposition is a consequence of work of Kassel [Kas08]:

Proposition 4.4.5. *Let Γ be a non-virtually cyclic Gromov-hyperbolic group, and let $\rho : \Gamma \to L_s^0$ be a representation. We denote by $\iota : L_s^0 \hookrightarrow G$ the natural inclusion.*

If r is odd, then $\iota \circ \rho$ cannot be $\{\alpha_r\}$-Anosov.

If r is even, then $\iota \circ \rho$ is $\{\alpha_r\}$-Anosov if and only if all but finitely many elements of $\mu_{L_s}(\rho(\Gamma))$ are contained in

$$\{X \in \overline{\mathfrak{a}}^+_{L_s} \mid \varepsilon_{\frac{r}{2}}(X) > 0 > \varepsilon_{\frac{r}{2}+1}(X)\},$$

and in this case, the representation ρ is $\{\alpha_{r/2}\}$-Anosov.

Proof. Let S be a finite symmetric generating set of Γ containing the identity, and $|\cdot|_S$ be the associated word length. By [GGKW17], the condition that ρ is $\{\alpha_r\}$-Anosov implies that there exist $C, C' \in \mathbb{R}$ such that

$$\langle \alpha_r, \mu(\iota \circ \rho(g))\rangle \geq C|g| - C' \quad \forall g \in \Gamma. \tag{4.4.2}$$

Let us fix $g \in \Gamma$. According to the description of the restricted root system of L_s in Equation (4.4.1), and since the Weyl group of G acts by *signed* permutations on the (ε_i), there exists a permutation $\sigma \in \mathfrak{S}_r$ such that

$$|\langle \varepsilon_i, \mu_{L_s}(\rho(g))\rangle| = \langle \varepsilon_{\sigma(i)}, \mu(\iota \circ \rho(g))\rangle \geq \langle \varepsilon_r, \mu(\iota \circ \rho(g))\rangle = \frac{1}{2}\langle \alpha_r, \mu(\iota \circ \rho(g))\rangle \tag{4.4.3}$$

for all $1 \leq i \leq r$.

Equation (4.4.2) then gives that $|\langle \varepsilon_i, \mu_{L_s}(\rho(g))\rangle| \to +\infty$ as $|g|_S \to \infty$. Then by [Kas08], there exists a connected component C of $\overline{\mathfrak{a}}^+_{L_s} \smallsetminus \left(\bigcup_{i=1}^{r-1} \ker(\varepsilon_i)\right)$ such that for all but finitely many $g \in \Gamma$, we have $\mu_{L_s}(\rho(g)) \in C$, and moreover, since Γ is not virtually cyclic, this connected component is invariant under the opposition involution of L_s. Since L_s is of type A_{r-1}, we must have that $r-2$ is even and

$$C = \{X \in \overline{\mathfrak{a}}^+_{L_s} \mid \varepsilon_{\frac{r}{2}}(X) > 0 > \varepsilon_{\frac{r}{2}+1}(X)\}.$$

Hence we have

$$|\langle \varepsilon_{r/2} - \varepsilon_{r/2+1}, \mu_{L_s}(\rho(g))\rangle| = |\langle \varepsilon_{r/2}, \mu_{L_s}(\rho(g))\rangle| + |\langle \varepsilon_{r/2+1}, \mu_{L_s}(\rho(g))\rangle|$$

for all but finitely many $g \in \Gamma$. By Equations (4.4.3) and (4.4.2), this implies

$$|\langle \varepsilon_{r/2} - \varepsilon_{r/2+1}, \mu_{L_s}(\rho(g))\rangle| \geq \langle \alpha_r, \mu(\iota \circ \rho(g))\rangle \geq C|g| - C' \quad \forall g \in \Gamma.$$

Hence, by [GGKW17], the representation ρ is $P_{\{\alpha_{r/2}\}}$-Anosov. □

Example 4.4.6. Assume that $r = 2p$, where $p \in \mathbb{N}_{>0}$.

1. If $G = \mathrm{Sp}(2r, \mathbb{R})$ and p is odd, then Tsouvalas proved that if Γ is vitually either a free group or a surface group [Tso20].
2. If p is even, then there exist P_p-Anosov subgroups Γ of $\mathrm{SL}(r, \mathbb{R})$ (and even in $\mathrm{SL}(r, \mathbb{K})$ for a general $\mathbb{K}$) which are not virtually free or surface groups, see e.g. [Tso20, Ex. 4.1]: an explicit example is the group $\Gamma = \Gamma_1 * F_2$, where Γ_1 is the fundamental group of a closed surface of genus $g \geq 2$ and F_2 is the free group on two generators. For g large enough, one can reproduce the proof of Theorem 4.4.3 verbatim, to deform Γ_1 into a Zariski-dense $\{\alpha_r\}$-Anosov subgroup Γ_2 of G. For small enough deformations, the group $\Gamma_2 * F_2$ still preserves a proper domain of $\mathbf{Sb}(\mathfrak{g})$, and is still $\{\alpha_r\}$-Anosov.

4.5 Anosov subgroups preserving proper domains in the Einstein universe

In this section, we take $G = \mathrm{PO}(n,2)$, and taking Notation 2.4.2, we investigate examples of $\{\alpha_2\}$-Anosov subgroups of G preserving proper domains in $\mathbf{Sb}(\mathfrak{g}) = \mathrm{Ein}^{n-1,1}$, beyond those of Section 4.4.2. Here we use the convention of Example 2.2.1 on the roots of $\mathfrak{so}(n,2)$, meaning that our "$\{\alpha_2\}$-Anosov representations" coincide with the "$\{\alpha_1\}$

A subset $F \subset \mathrm{Ein}^{n-1,1}$ is said to be *negative* 3 *by* 3 (in the sense of [DGK24]) if for every triple of pairwise distinct points $(a,b,c) \in F^3$, one has $\mathsf{idx}(a,b,c) = 0$; recall from Section 4.1.3 that idx is the Maslov index. In the notation of Section 2.4.1, this is equivalent to saying that there exists a lift $\widetilde{F}$ of F in $\mathbb{P}(\mathbb{R}^{n,2})$ such that $\mathbf{b}(u,v) < 0$ for every pair of distinct points $u, v \in \widetilde{F}$.

Let $\Gamma \leq G$ be an $\{\alpha_r\}$-transverse subgroup, and let

$$\Omega_\Gamma := \left\{x \in \mathrm{Ein}^{n-1,1} \mid \mathsf{idx}(\xi_1, x, \xi_2) = 0 \;\; \forall \xi_1, \xi_2 \in \Lambda_{\{\alpha_1\}}(\Gamma),\; \xi_1 \neq \xi_2\right\}.$$

The set Ω_Γ is open, but not necessarily proper or connected. Since Γ is Θ-divergent, it acts properly discontinuously on Ω_Γ. Moreover:

Lemma 4.5.1. *Assume that $\Lambda_{\{\alpha_2\}}(\Gamma)$ is negative* 3 *by* 3*. Then the set Ω_Γ is* photon-convex*, that is, for every photon $\Lambda \subset \mathrm{Ein}^{n-1,1}$, the intersection $\Lambda \cap \Omega_\Gamma$ is connected. Moreover, we have $\overline{\Lambda \cap \Omega} = \Lambda \cap \overline{\Omega}$.*

Proof. Let us take the notation of Section 2.4.1. By [DGK18], the set $\Lambda_{\{\alpha_1\}}(\Gamma)$ lifts to a cone F of $\mathbb{R}^{n,2}$ on which $\mathbf{b}$ is negative. The set

$$C := \mathbb{P}(\{u \in \mathbb{R}^{n,2} \mid \mathbf{b}(u,v) < 0 \quad \forall v \in F\})$$

is convex in an affine chart of $\mathbb{P}(\mathbb{R}^{n,2})$, and one has $\Omega_\Gamma = \mathrm{Ein}^{n-1,1} \cap C$ (see [Sma22]). Since every photon Λ of $\mathrm{Ein}^{n-1,1}$ is a projective line which is contained in $\mathrm{Ein}^{n-1,1}$, we conclude that $\Omega_\Gamma \cap \Lambda = C \cap \Lambda$ is connected, and that $\overline{\Lambda \cap \Omega} = \overline{\Lambda \cap C} = \Lambda \cap \overline{C} = \Lambda \cap \overline{\Omega}$. ☐

Lemma 4.5.2. *If Ω_Γ is nonempty and not connected, then every connected component of Ω_Γ is proper.*

Proof. Write $\Omega_\Gamma = \bigsqcup_{i \in I} \Omega_i$, where the Ω_i are pairwise disjoint connected components of Ω_Γ. Let $i \in I$, and let us assume for a contradiction that there exist $j \in I \smallsetminus \{i\}$ and $(x,y) \in \Omega_i \times \Omega_j$ such that x and y are not transverse. Note that this implies that x and y are on a common photon Λ. By Lemma 4.5.1, the intersection $\Lambda \cap \Omega_\Gamma$ must be connected, and gives a continuous path from x to y in Ω_Γ. This contradicts the fact that x and y belong to distinct connected components of Ω_Γ.

We have proven that $\Omega_j \subset \Omega_i^*$ for all $j \neq i$. Since Ω_j is open, this implies that Ω_i is proper in $\mathrm{Ein}^{n-1,1}$. ☐

Lemma 4.5.3. *If Γ is an $\{\alpha_2\}$-Anosov subgroup of G and if Ω_Γ is nonempty and connected, then there exists a Γ-invariant proper domain $\Omega' \subset \Omega_\Gamma$.*

Proof. Let S be a finite symmetric generating set of Γ. Let $\mathscr{S} \subset \Omega_\Gamma$ be a Γ-invariant *acausal Cauchy hypersurface*, that is, a closed subset of Ω_Γ such that $\mathsf{idx}(x,y,z) = 0$ for every triple of pairwise distinct points $(x,y,z) \in \mathscr{S}^3$ and such that every photon meeting Ω_Γ meets $\mathscr{S}$ in exactly one point. Note that, in particular, two distinct points of $\mathscr{S}$ are always transverse. Then one has $\partial \mathscr{S} \smallsetminus \Omega_\Gamma = \Lambda_{\{\alpha_1\}}(\Gamma)$. By [Sma22], such a closed subset exists. Let $x_0 \in \mathscr{S}$. By [Sma22], the action of Γ on $\mathscr{S}$ is properly discontinuous (and cocompact). Thus there exists a neighborhood $\mathscr{V}$ of x such that $\mathscr{V} \cap \Gamma \cdot x = \{x\}$. Let $z \in (\mathscr{V} \smallsetminus \{x\}) \cap \mathscr{S}$. Since $\mathscr{S}$ is acausal, we have $\mathscr{S} \cap \bigcup_{g\in\Gamma} \mathrm{Z}_{g\cdot x} = \Gamma \cdot x$, so $\mathscr{S} \smallsetminus (\Gamma \cdot x)$ is connected. Thus there exists a connected neighborhood $\mathscr{V}'$ of $S \cdot z$ in Ω_Γ such that $\overline{\mathscr{V}}' \cap \bigcup_{g\in\Gamma} \mathrm{Z}_{g\cdot x} = \emptyset$. Let $\Omega' := \Gamma \cdot \mathscr{V}'$. Then Ω' is connected, open, and

$$\overline{\Omega}' = \Gamma \cdot \overline{\mathscr{V}}' \cup \Lambda_{\{\alpha_1\}}(\Gamma) \subset \mathbb{A}_x.$$

The domain Ω' is thus Γ-invariant and proper. $\square$

In the notation of Section 2.4.1 we write the decomposition $\mathbb{R}^{n,2} = V_1 \oplus V_2$, where $V_1 = \mathrm{Span}(e_1, \dots, e_n)$ and $V_2 := \mathrm{Span}(e_{n+1}, e_{n+2})$. Let $\mathscr{S} := \mathbb{P}(V_1) \cap \mathrm{Ein}^{n-1,1}$ and $\mathbb{S} := \mathbb{P}(V_2) \cap \mathrm{Ein}^{n-1,1}$. Then $\mathscr{S}$ is a conformal $(n-1)$-sphere, and $\mathbb{S}$ is a circle, with respective Riemannian distance functions denoted by $d_{\mathscr{S}}$ and $d_{\mathbb{S}}$. Moreover, one has a double covering $\pi : \mathscr{S} \times \mathbb{S}^1 \to \mathrm{Ein}^{n-1,1}$.

Let Γ be a word-hyperbolic group and let $\rho : \Gamma \to \mathrm{PO}(n,2)$ be an $\{\alpha_1\}$-Anosov representation. Let $\widetilde{\Lambda}$ be a lift of $\xi_\rho(\partial_\infty \Gamma)$ in $\mathscr{S} \times \mathbb{S}^1$. By [Sma22], one has

$$\Omega_{\rho(\Gamma)} = \pi\Big(\{(s,t) \in \mathscr{S} \times \mathbb{S} \mid d_{\mathscr{S}}(s,s') > d_{\mathbb{S}}(t,t') \quad \forall (s',t') \in \widetilde{\Lambda}\}\Big).$$

By [DGK24, Lem. 11.9], the boundary map $\xi_\rho : \partial_\infty\Gamma \to \mathrm{Ein}^{n-1,1}$ is homotopic to an embedding $f : \partial_\infty\Gamma \to \mathrm{Ein}^{n-1,1}$ with image contained in $\pi(\mathscr{S}) \simeq \mathscr{S}$. Thus we may assume that $\xi_\rho = f$. We then have:

$$\Omega_{\rho(\Gamma)} = \pi\Big(\{(s,t) \in \mathscr{S} \times \mathbb{S} \mid d_{\mathscr{S}}(s,s') > d_{\mathbb{S}}(t,0) \quad \forall (s',0) \in \widetilde{\Lambda}\}\Big). \tag{4.5.1}$$

If Γ has cohomological dimension $\leq n-1$, then $\partial_\infty\Gamma$ has covering dimension $\leq n-2$, so $\pi(\mathscr{S}) \smallsetminus \xi_\rho(\partial_\infty\Gamma)$ is nonempty. Then (4.5.1) implies that $\Omega_{\rho(\Gamma)}$ is nonempty. Now there are two possibilities:

1. If Γ has cohomological dimension $\leq n-2$, then $\partial_\infty\Gamma$ has covering dimension $\leq n-3$, so $\pi(\mathscr{S}) \smallsetminus \xi_\rho(\partial_\infty\Gamma)$ is connected. Then Equation (4.5.1) implies that $\Omega_{\rho(\Gamma)}$ is connected.
2. If $\partial_\infty\Gamma$ is an $(n-2)$-sphere, then by [Sma22], the domain $\Omega_{\rho(\Gamma)}$ is the union of a diamond and its dual; thus an index-two subgroup of Γ preserves a diamond.

This analysis, together with Lemma 4.5.3, gives:

Corollary 4.5.4. *Let $n \geq 2$. Let $\Gamma \leq \mathrm{SO}(n,2)$ be a word-hyperbolic group and let $\rho : \Gamma \to \mathrm{SO}(n,2)$ be an $\{\alpha_2\}$-Anosov representation.*

If Γ has cohomological dimension $\leq n-2$, then $\rho(\Gamma)$ preserves a proper domain $\Omega \subset \mathrm{Ein}^{n-1,1}$.

If $\partial_\infty\Gamma$ is an $(n-2)$-sphere, then there is a subgroup of Γ, with index at most two, that preserves a diamond.

Example 4.5.5. Let $n \geq 3$, and let $1 \leq k \leq n-3$. Let M be a closed negatively curved Riemannian k-manifold, with $k \leq n-3$. Consider the natural embedding $\mathrm{PO}(n,1) \hookrightarrow \mathrm{PO}(n,2)$. Then the induced representation $\pi_1(M) \to \mathrm{PO}(n,2)$ is $\{\alpha_2\}$-Anosov and negative 3 by 3 (see [DGK24, Ex. 11.12]). By Corollary 4.5.4, it preserves a proper domain in $\mathrm{Ein}^{n-1,1}$.

Chapter 5

Preliminaries on Nagano spaces

In this chapter, we introduce a family of flag manifolds known as *Nagano spaces*. This family includes the families of key examples of Sections 2.4.1, 2.4.2, 2.4.3, and 2.4.4, along with many others; see Table 8.1.

A *Nagano space*, also referred to as an *extrinsic symmetric space* or an *R-symmetric space* (R for "root"), is a flag manifold $\mathscr{F}(\mathfrak{g}, \Theta)$ that is also a symmetric space with isometry group contained in a Lie group $G \in \mathscr{G}_\Theta(\mathfrak{g})$. Nagano has classified such spaces in [Nag65]. As mentioned in the introduction of this thesis, Nagano spaces have been the subject of extensive research, starting with [KN64, KN65, Nag65, Tak65, TK68, Mak73, Nag88, Tak88, Kan98, Kan06, Kan11]. We provide an overview of the main deep results on these spaces in Section 5.1.

In a second Section 5.2, we establish a useful lemma concerning linear and projective Θ-proximal triples for Nagano spaces (Lemma 5.2.1). This lemma will be useful in the study of *photons* and the comparison of their Hilbert metric with the projective cross ratios in Section 6.3.3 (see in particular Lemma 6.3.13). This, in turn, will allow us to compare the Kobayashi metric (defined in Section 6.4) with the Caratheodory metrics in Section 6.4.5, and to obtain Proposition 6.4.10.

In Section 5.3, we observe that the specific structure of Nagano spaces enables us to provide a sufficient condition for the dual of a proper domain to be connected (and even contractible), a property that is not guaranteed in general (see Remark 3.1.8).

Finally, in Section 5.4, we establish elementary characterizations of Nagano spaces among flag manifolds.

5.1 Reminders on Nagano spaces

In this section, we define Nagano spaces and recall their well-known properties.

5.1.1 Graded Lie algebra structure

Let $\mathfrak{g}$ be a real semisimple Lie algebra with no compact factors and Θ be a subset of the simmple restricted roots of $\mathfrak{g}$. We say that the pair $(\mathfrak{g}, \Theta)$ is a *Nagano pair*, and the flag manifold $\mathscr{F}(\mathfrak{g}, \Theta)$ is a *Nagano space*, if $\mathfrak{u}_\Theta^-$ is abelian. When Θ is a singleton $\{\alpha\}$, in order to simplify the notation, we will sometimes denote the Nagano pair $(\mathfrak{g}, \Theta)$ by $(\mathfrak{g}, \alpha)$ instead of $(\mathfrak{g}, \{\alpha\})$.

Since $\mathfrak{g}$ is semisimple, there exist $N > 0$, simple Lie subalgebras $\mathfrak{g}_1, \dots, \mathfrak{g}_N$ of $\mathfrak{g}$ and subsets Θ_i of the simple restricted roots of $\mathfrak{g}_i$ for $1 \leq i \leq N$ such that $\Theta = \Theta_1 \cup \cdots \cup \Theta_N$ and $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_N$. We then have a natural identification

$$\mathscr{F}(\mathfrak{g}, \Theta) \simeq \mathscr{F}(\mathfrak{g}_1, \Theta_1) \times \cdots \times \mathscr{F}(\mathfrak{g}_N, \Theta_N). \tag{5.1.1}$$

The pair $(\mathfrak{g}, \Theta)$ is a Nagano pair if and only if all of the $(\mathfrak{g}_i, \Theta_i)$ are, for $1 \leq i \leq N$. It motivates the definition of an *irreducible Nagano pair*:

If $\mathfrak{g}$ is simple and $(\mathfrak{g}, \Theta)$ is a Nagano pair, then Θ is a singleton $\Theta = \{\alpha\}$, and the longest root α_Δ of Σ can be written

$$\alpha_\Delta = \alpha + \sum_{\beta \in \Delta \smallsetminus \{\alpha\}} n_\beta \beta, \tag{5.1.2}$$

where $n_\beta \in \mathbb{N}$ (see for instance [Tak88]). We say in this case that $(\mathfrak{g}, \alpha)$ is an *irreducible Nagano pair* and that $\mathscr{F}(\mathfrak{g}, \alpha)$ is an *irreducible Nagano space*.

Any Nagano space is then a product of irreducible Nagano spaces.

The list of irreducible Nagano pairs, established by Nagano [Nag65], is given in Table 8.1. Note that, by [Nag65], we will always be able to assume that a Nagano pair is irreducible.

Remark 5.1.1. In the notation of Section 2.2.6, it is important to keep in mind that a Nagano space is a flag manifold, and not just a coset space. For instance, the real projective space $\mathbb{P}(\mathbb{R}^{2n})$ (i.e. the flag manifold $\mathscr{F}(\mathfrak{sl}(2n, \mathbb{R}), \alpha_1)$ by Section 2.5) is a Nagano space, while the flag manifold $\mathscr{F}(\mathfrak{sp}(2n, \mathbb{R}), \alpha_1)$ is not ; they are however $\mathrm{Sp}(2n, \mathbb{R})$-equivariantly diffeomorphic.

Let us now describe the structure of Nagano spaces. We will admit the following results, which are well-known. First, let us set some notations, that we will use for the rest of this memoir.

Notation 5.1.2. Given an irreducible Nagano pair $(\mathfrak{g}, \alpha)$ and $G \in \mathscr{G}_{\{\alpha\}}(\mathfrak{g})$, we will use the following simplified notations:

$$\mathfrak{u}^\pm = \mathfrak{u}^\pm_{\{\alpha\}},\ \mathfrak{l} = \mathfrak{l}_{\{\alpha\}},\ L = L_{\{\alpha\}},\ U^\pm = U^\pm_{\{\alpha\}},$$
$$\mathfrak{p}^+ = \mathfrak{p}^+_{\{\alpha\}},\ P = P^+_{\{\alpha\}},\ \mathfrak{p}^- = \mathfrak{p}^-_{\{\alpha\}}, P^- = P^-_{\{\alpha\}}.$$

It will be convenient to fix $v^+ \in \mathfrak{g}_\alpha,\ v^- \in \mathfrak{g}_{-\alpha}$ such that

$$\mathsf{t}_{\mathsf{std}} := (v^+, h_\alpha, v^-) := (v^+_\alpha, h_\alpha, v^-_\alpha) \tag{5.1.3}$$

is an $\mathfrak{sl}_2$-triple (where h_α is defined in Section 2.2.1).

The main consequence of the fact that $\mathfrak{u}^-$ is abelian is the following consequence of a more general result of Kostant:

Fact 5.1.3. *[Kos10] Let $G \in \mathscr{G}_{\{\alpha\}}(\mathfrak{g})$. The identity component L^0 of L acts irreducibly on $\mathfrak{u}^-$ (resp. on $\mathfrak{u}^+$).*

If we set $\mathfrak{g}_1 = \mathfrak{u}^+$ and $\mathfrak{g}_{-1} = \mathfrak{u}^-$, then $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ satisfies $[\mathfrak{g}_k, \mathfrak{g}_l] \subset \mathfrak{g}_{k+1}$ (setting $\mathfrak{g}_k = \{0\}$ for $k \notin \{-1, 0, 1\}$). This decomposition endows $\mathfrak{g}$ with the structure of a *graded Lie algebra*. Thus there exists a Cartan involution σ_0 of $\mathfrak{g}$ (as defined in Section 2.2.2 for some maximal compact subalgebra of $\mathfrak{g}$) such that $\sigma_0 \mathfrak{g}_k = \mathfrak{g}_{-k}$ for $k \in \{-1, 0, 1\}$, and an element $H_0 \in \mathrm{Centr}(\mathfrak{g}_0)$ such that $\mathfrak{g}_k$ is the eigenspace of $\mathrm{ad}(H_0)$ for the eigenvalue k [KN64].

Note that, as in Section 2.4.4.2, the existence of H_0 allows to define dilations in affine charts. For all $t \in \mathbb{R}_{>0}$ we define $\ell_0(t) = \exp\left(-\log(t)\, H_0\right) \in L$. The element $\mathrm{Ad}(\ell_0(t))$ acts on $\mathfrak{u}^\pm$ by

$$\mathrm{Ad}(\ell_0(t))X = \begin{cases} tX & \forall X \in \mathfrak{u}^-; \\ \frac{1}{t}X & \forall X \in \mathfrak{u}^+. \end{cases} \tag{5.1.4}$$

Hence any positive dilation of $\mathbb{A}_{\mathsf{std}}$ at $\mathfrak{p}^+ = \varphi_{\mathsf{std}}(0)$ can be realized as the restriction to $\mathbb{A}_{\mathsf{std}}$ of a map of the form $x \mapsto \ell_0(t) \cdot x$ of $\mathscr{F}(\mathfrak{g}, \alpha)$ for some $t \in \mathbb{R}_{>0}$.

Finally, before concluding this section, let us prove the following auxiliary lemma, which follows from the multiplicity of α in the highest root α_Δ (Equation (5.1.2)):

Lemma 5.1.4. *Let $(\mathfrak{g}, \alpha)$ be an irreducible Nagano pair. Let $\beta \in \Sigma^+_{\{\alpha\}} \smallsetminus \{\alpha\}$, and let $Y_\beta \in \mathfrak{g}_\beta$. Then there exists $Z \in \mathrm{Lie}(P)$ such that $[v^-, Z] = 0$ and $\mathrm{Ad}(\exp(Y_\beta))v^- = v^- + Z$.*

Proof. One has

$$\mathrm{Ad}(\exp(Y_\beta))v^- = \exp(\mathrm{ad}(Y_\beta))v^- = \sum_{k=0}^{\infty} \frac{\mathrm{ad}(Y_\beta)^k v^-}{k!}.$$

We compute the terms $\mathrm{ad}(Y_\beta)^k v^-$ for $k \in \mathbb{N}$. For all $k \in \mathbb{N}$, one has $\mathrm{ad}(Y_\beta)^k v^- \in \mathfrak{g}_{k\beta - \alpha}$. Since the multiplicity of α in the longest root is 1 (Equation (5.1.2)), one has $\mathfrak{g}_{k\beta-\alpha} \subset \mathfrak{l}$. On the other hand, one has

$$[v^-, \mathrm{ad}(X)^k v^-] \in \mathfrak{g}_{k\beta - 2\alpha} = \{0\}$$

(because $k\beta - 2\alpha$ is not a restricted root, again by Equation (5.1.2)). Thus one has $\mathrm{Ad}(\exp(X))v^- = v^- + Y$, where $Y \in \mathfrak{l}$ commutes with v^-. Thus $Z := \sum_{k=1}^{\infty} \frac{\mathrm{ad}(Y_\beta)^k v^-}{k!}$ works. ☐

5.1.2 Symmetric structure

If $\mathfrak{g}$ is a real semisimple Lie algebra of noncompact type and Θ is a subset of the simple restricted roots of $\mathfrak{g}$, then for all $G \in \mathscr{G}_\Theta(\mathfrak{g})$, any maximal compact subgroup K of G acts transitively on $\mathscr{F}(\mathfrak{g}, \Theta)$. By compactness of K, one can easily construct a Riemannian metric on $\mathscr{F}(\mathfrak{g}, \Theta)$, whose isometry group contains K. However, the Riemannian space built this way is not necessarily symmetric. This last property actually characterizes Nagano spaces among flag manifolds. As compact irreducible symmetric spaces, they admit a noncompact dual and turn out to contain it as a proper symmetric divisible domain. In this section, we describe the symmetric structure on Nagano spaces, and the embedding of their noncompact dual.

5.1.2.1 Construction of the symmetric structure. In this section, we describe the compact symmetric space structure on an irreducible Nagano space. By definition, the compact symmetric structure on (not necessarily irreducible) Nagano spaces follows immediately. We fix $(\mathfrak{g}, \alpha)$ an irreducible Nagano pair. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{h}$ be the Cartan decomposition associated with σ_0. If we fix some $G \in \mathscr{G}_{\{\alpha\}}(\mathfrak{g})$, by definition, there exists a maximal compact subgroup K of G with Lie algebra $\mathfrak{k}$. The group K acts transitively on $\mathscr{F}(\mathfrak{g}, \alpha)$, and the stabilizer of $\mathfrak{p}^+$ in K has Lie algebra $\mathfrak{k}_0$. Thus, we have the natural identification $\mathscr{F}(\mathfrak{g}, \alpha) \simeq K_{\sigma_0}/K_0$. Thus there exists a natural K-invariant metric $g_{\mathfrak{g},\alpha}$ on $\mathscr{F}(\mathfrak{g}, \alpha)$. The element $\exp(i\pi H_0) \in K_0$ then acts as a symmetry on $T_{\mathfrak{p}^+}\mathscr{F}(\mathfrak{g}, \alpha)$, making $(\mathscr{F}(\mathfrak{g}, \alpha), g_{\mathfrak{g},\alpha})$ a compact Riemannian symmetric space.

Definition 5.1.5. The *rank* of an irreducible Nagano pair (resp. an irreducible Nagano space) $(\mathfrak{g}, \alpha)$ (resp. $\mathscr{F}(\mathfrak{g}, \alpha)$) will be, by definition, the rank of the compact symmetric space $(\mathscr{F}(\mathfrak{g}, \alpha), g_{\mathfrak{g},\alpha})$. This integer does not depend on the choices above, and will be denoted by $\mathrm{rk}(\mathfrak{g}, \alpha)$. The irreducible Nagano pair $(\mathfrak{g}, \alpha)$ (resp. the Nagano space $\mathscr{F}(\mathfrak{g}, \alpha)$) will be said to be *of higher rank* if its rank is ≥ 2.

Remark 5.1.6. The *rank* of a (not necessarily irreducible) Nagano space is then simply the sum of the ranks of its irreducible factors.

5.1.2.2 Nagano's characterization. We just detailed in Section 5.1.2.1 above the construction that equips an irreducible Nagano space with the structure of a compact Riemannian symmetric space. The Riamannian symmetric structure on (not necessarily irreducible) Nagano spaces follows by Equation (5.1.1).

Conversely, Nagano has characterized irreducible compact symmetric spaces that are Nagano spaces. Let (M, g) be a compact symmetric space. A *transformation group* G of (M, g) is a noncompact semisimple Lie group acting effectively on M, i.e. transitively with finite kernel, containing the isometry group $\mathrm{Isom}(M, g)$ of (M, g) as a Lie subgroup.

Fact 5.1.7 ([Nag65])**.** *Let (M, g) be a compact symmetric space admitting a noncompact transformation group. Then M is a Nagano space. In particular, the stabilizer of a basepoint $x \in M$ is a parabolic subgroup P of G, so that G/P identifies G-equivariantly with M.*

Historically, this theorem is the starting point of the study of Nagano spaces.

5.1.3 Embedding the noncompact dual

Let $(\mathfrak{g}, \alpha)$ be an irreducible Nagano pair and s be the real rank of the symmetric space $(\mathscr{F}(\mathfrak{g}, \alpha), g_{\mathfrak{g},\alpha})$ constructed in Section 5.1.2.1. One can choose a maximal system of strongly orthogonal roots of the same length $\beta_1, \ldots, \beta_s \in \Sigma_\Theta^+$ such that $\beta_1 = \alpha_\Delta$ [Tak88] (recall the longest root α_Δ defined in (5.1.2)). By "same length," we refer to the length defined by the Killing form.

We set $s_0 = \mathrm{id}$ and $s_\ell := s_{\beta_1} \cdots s_{\beta_\ell} \in W$ for $1 \leq \ell \leq s$. The following is proven in [Tak88]:

Fact 5.1.8 ([Tak88])**.** *The set of symmetries $\{s_1, \dots, s_s\}$ is a complete set of representatives of $W_{\Delta \smallsetminus \{\alpha\}} \backslash W / W_{\Delta \smallsetminus \{\alpha\}}$, and for all $1 \leq \ell < \ell' \leq s$, one has $[s_\ell] \leq [s_{\ell'}]$.*

We take the notations of Section 5.1.2.1. The map $\sigma := \mathrm{Ad}(\exp(\pi i H_0))$ is an involutive automorphism of $\mathfrak{g}$, which commutes with σ_0 and is equal to id on $\mathfrak{l}$ and $-\mathrm{id}$ on $\mathfrak{m} := \mathfrak{u}^+ \oplus \mathfrak{u}^-$. Since σ_0 and σ commute, the Lie algebra $\mathfrak{g}$ can be decomposed into four spaces:

$$\mathfrak{g} = \mathfrak{k}_0 \oplus \mathfrak{m}_{\mathfrak{k}} \oplus \mathfrak{h}_0 \oplus \mathfrak{m}_{\mathfrak{h}},$$

where $\mathfrak{k}_0 = \mathfrak{l} \cap \mathfrak{k}$, $\mathfrak{h}_0 = \mathfrak{l} \cap \mathfrak{h}$, $\mathfrak{m}_{\mathfrak{k}} = \mathfrak{m} \cap \mathfrak{k}$ and $\mathfrak{m}_{\mathfrak{h}} = \mathfrak{m} \cap \mathfrak{h}$.

For all $1 \leq i \leq s$, let $E_i \in \mathfrak{g}_{\beta_i}$ be such that $(\sigma_0(E_i), \beta_i', E_i)$ is an $\mathfrak{sl}_2$-triple, where $\beta_i' = \frac{2}{B(\beta_i, \beta_i)} \beta_i$. Now let $m_i := E_i + \sigma_0(E_i) \in \mathfrak{m}_{\mathfrak{h}}$ and

$$\mathfrak{c} := \sum_{i=1}^{s} \mathbb{R} m_i.$$

The following is well-known:

Fact 5.1.9. *1. [Kan87] The space $\mathfrak{c}$ is a maximal abelian subspace in $\mathfrak{m}_{\mathfrak{h}}$;*

2. [Nag65, Tak65] Let $\mathfrak{g}^* := \mathfrak{k}_0 \oplus \mathfrak{m}_{\mathfrak{h}} \subset \mathfrak{g}$. Then $\mathfrak{g}^*$ is a subalgebra of $\mathfrak{g}$, and the triple $(\mathfrak{g}^*, \mathfrak{k}_0, \sigma_0)$ is the noncompact dual of the symmetric triple $(\mathfrak{k}, \mathfrak{k}_0, \sigma_{\mathfrak{g}})$.

By Fact 5.1.9.(2), the symmetric space defined by the symmetric triple $(\mathfrak{g}^*, \mathfrak{k}_0, \sigma_0)$ is uniquely defined by the pair $(\mathfrak{g}, \alpha)$. We will denote it by $\mathbb{X}(\mathfrak{g}, \alpha)$. Facts 5.1.8 and 5.1.9 lead to the following key observation:

Observation 5.1.10. We have

$$\mathrm{rk}(\mathfrak{g}, \alpha) = \mathrm{rk}_{\mathbb{R}}\big(\mathbb{X}(\mathfrak{g}, \alpha)\big) = |W_{\Delta \smallsetminus \Theta} \backslash W / W_{\Delta \smallsetminus \Theta}| - 1. \tag{5.1.5}$$

The ranks of all irreducible Nagano pairs are given in Table 8.2.

In the setting of Fact 5.1.9, there exists $G \in \mathscr{G}_{\{\alpha\}}(\mathfrak{g})$ such that the connected subgroup G^* of G with Lie algebra $\mathfrak{g}^*$ identifies with the identity component of the isometry group of $\mathbb{X}(\mathfrak{g}, \alpha)$. Nagano proved that $\mathbb{X}(\mathfrak{g}, \alpha)$ embeds into $\mathscr{F}(\mathfrak{g}, \alpha)$ as a symmetric domain, proper in $\mathscr{F}(\mathfrak{g}, \alpha)$ [Nag65]. Let us make this embedding explicit. Let $\Omega = G^* \cdot \mathfrak{p}^+$, and let $x_0 \in \mathbb{X}(\mathfrak{g}, \alpha) \simeq G^*/K_0$ be the class of K_0. There is a G^*-equivariant diffeomorphism

$$\mathsf{F}_{(\mathfrak{g}, \alpha)} : \mathbb{X}(\mathfrak{g}, \alpha) \longrightarrow \Omega;\ g \cdot x_0 \longrightarrow g \cdot \mathfrak{p}. \tag{5.1.6}$$

Note that this embedding depends on the basepoint $\mathfrak{p}^+ \in \mathscr{F}(\mathfrak{g}, \alpha)$.

Definition 5.1.11. We will say that a domain $\Omega \subset \mathscr{F}(\mathfrak{g}, \alpha)$ is a *realization of* $\mathbb{X}(\mathfrak{g}, \alpha)$ if there exists $g \in G$ such that $\Omega = g \cdot \mathsf{F}_{(\mathfrak{g}, \alpha)}(\mathbb{X}(\mathfrak{g}, \alpha))$.

The following fact follows by definition, and generalizes, to any Nagano space, Proposition 3.3.2 and Facts 3.4.4 and 3.5.5:

Fact 5.1.12. *The automorphism group* $\mathsf{Aut}_G(\Omega)$ *of a realization* Ω *of* $\mathbb{X}(\mathfrak{g},\alpha)$ *is conjugate to* G' *in* G*. Thus* Ω *is symmetric, divisible and homogeneous.*

The divisibility comes from the fact that G^* is reductive, so it admits cocompact lattices by Borel's Theorem, and it acts transitively on Ω.

The groups G^* (up to finite index) associated with all irreducible Nagano pairs $(\mathfrak{g},\alpha)$ are given in Table 8.2.

Example 5.1.13. The main examples we are interested in are:

1. Let $\mathfrak{g}$ be a HTT Lie algebra and $\alpha=\alpha_r$. Then $\mathfrak{g}^*$ is isomorphic to $\mathfrak{l}$. The realizations of $\mathbb{X}(\mathfrak{g},\alpha)\simeq\mathbb{X}_{L_s}\times\mathbb{R}$ are the diamonds defined in Section 3.5.1.
2. If $\mathfrak{g}=\mathfrak{so}(p+1,q+1)$ and $\alpha=\alpha_1$, then $\mathfrak{g}^*=\mathfrak{so}(p,1)\oplus\mathfrak{so}(1,q)$, and the realizations of $\mathbb{X}(\mathfrak{g},\alpha)\simeq\mathbb{H}^p\times\mathbb{H}^q$ in $\mathrm{Ein}^{p,q}$ are exactly the diamonds defined in Section 3.4.2.
3. If $\mathfrak{g}=\mathfrak{sl}(p+q,\mathbb{R})$ and $\alpha=\alpha_p$, then $\mathfrak{g}^*=\mathfrak{so}(p,q)$ and the realizations of $\mathbb{X}(\mathfrak{g},\alpha)\simeq\mathrm{PO}(p,q)/\,\mathrm{P}(\mathrm{O}(p)\times\mathrm{O}(q))$ in $\mathrm{Gr}_p(\mathbb{R}^{p+q})$ are exactly the domains $\mathbb{B}(\mathbf{b})$, for $\mathbf{b}$ a bilinear form of signature (p,q) on $\mathbb{R}^{p+q}$ (see Section 3.3.1).

Remark 5.1.14. The embedding of the noncompact dual generalizes the well-known Harish-Chandra embedding of Hermitian symmetric space of noncompact type as bounded domains of complex vector spaces.

5.2 Proximal representations of Nagano spaces

In this section, we prove the following lemma, concerning proximal representations of some Lie groups inducing embeddings of Nagano spaces:

Lemma 5.2.1. *Let* $(\mathfrak{g},\alpha)$ *be an irreducible Nagano pair and assume that* $\dim(\mathfrak{g}_\alpha)=1$*. Let* (G,ρ,V) *be a linear or projective* $\{\alpha\}$*-proximal triple of* $\mathfrak{g}$*, with highest weight* $\chi:=N\omega_\alpha$ *for some* $N\in\mathbb{N}$*. Let* $\mathsf{v}_0\in V^\chi\smallsetminus\{0\}$*. Then* $\rho_*(v^-)^k\cdot\mathsf{v}_0\neq 0$ *for all* $k\leq N$*, and* $\rho_*(v^-)^k\cdot\mathsf{v}_0=0$ *for all* $k\geq N+1$*.*

Proof. By the definition of ω_α, one has $0\neq\rho_*(v^-)^k\cdot\mathsf{v}_0\in V^{N\omega_r-k\alpha_r}$ for all $0\leq k\leq N$ (see e.g. [GW09, Lem. 3.2.9]). Let $k\geq N+1$. One has $\rho_*(v^-)^k\cdot\mathsf{v}_0\in V^{N\omega_\alpha-k\alpha}$, so it suffices to prove that $V^{N\omega_\alpha-k\alpha}=\{0\}$. Since $\dim(\mathfrak{g}_\alpha)=1$, this is satisfied whenever $N\omega_\alpha-k\alpha\notin\mathrm{Conv}(W\cdot(N\omega_\alpha))$ (see e.g. [GW09, Prop. 3.2.10])), where W is the restricted Weyl group of $\mathfrak{g}$ defined in Section 2.2.4. Let us check this property. In particular, it suffices to prove that $\omega_\alpha-\lambda\alpha\notin\mathrm{Conv}(W\cdot\omega_\alpha)$ for all $\lambda>1$.

Recall that we denote by B the Killing form on $\mathfrak{g}$. Its restriction to $\mathfrak{a}$ induces a W-invariant inner product.

For any root $\beta\in\Sigma$, let $h'_\beta\in\mathfrak{a}$ be such that $\beta=B(\cdot,h'_\beta)$. Then the element h_β defined in Section 2.2.3 is just $h_\beta=\frac{2h'_\beta}{B(h'_\beta,h'_\beta)}$. By W-invariance of B, we have

$$w\cdot h_\beta=h_{w^{-1}\cdot\beta}\tag{5.2.1}$$

for all $w \in W$. By Equation (5.1.2), for all $\beta \in \Sigma$ we have $\beta = \sum_{\beta' \in \Delta \smallsetminus \{\alpha\}} n_{\beta'} \beta' + \delta\alpha$, with $n_{\beta'} \in \mathbb{N}$ and $\delta \in \{-1, 0, 1\}$. Thus a direct computation gives

$$h_\beta = X + \delta \frac{B(h'_\alpha, h'_\alpha)}{B(h'_\beta, h'_\beta)} h_\alpha,$$

where $X \in \sum_{\beta' \in \Delta \smallsetminus \{\alpha\}} \mathbb{R} h_{\beta'}$. Thus by definition of ω_α (see Section 2.2.3), we have $\omega_\alpha(h_\beta) = \delta \frac{B(h'_\alpha, h'_\alpha)}{B(h'_\beta, h'_\beta)}$. If moreover we have $\beta = w^{-1} \cdot \alpha$ for some $w \in W$, by Equation (5.2.1) and W-equivariance of B, we have $\omega_\alpha(h_{w^{-1}\cdot\alpha}) = \delta$. Thus

$$|\omega_\alpha(h_{w^{-1}\cdot\alpha})| \leq 1. \tag{5.2.2}$$

Let $\lambda \in \mathbb{R}_{>0}$ be such that $\omega_\alpha - \lambda\alpha \in \mathrm{Conv}(W \cdot \omega_\alpha)$. Then there exist $(\lambda_w) \in \mathbb{R}_{>0}^{|W|}$ such that $\sum_{w \in W} \lambda_w = 1$ and $\omega_\alpha - \lambda\alpha = \sum_{w \in W} \lambda_w w \cdot \omega_\alpha$. Evaluating in h_α and taking the absolute value, we get

$$\begin{aligned} 2\lambda - 1 \leq |1 - 2\lambda| = |\omega_\alpha(h_\alpha) - \lambda\alpha(h_\alpha)| &= \Big| \sum_{w \in W} \lambda_w w \cdot \omega_\alpha(h_\alpha) \Big| \\ &\overset{\text{by (5.2.1)}}{=} \Big| \sum_{w \in W} \lambda_w \omega_\alpha(h_{w^{-1}\cdot\alpha}) \Big| \leq \sum_{w \in W} \lambda_w |\omega_\alpha(h_{w^{-1}\cdot\alpha})| \overset{\text{by (5.2.2)}}{\leq} \sum_{w \in W} \lambda_w = 1 \end{aligned} \tag{5.2.3}$$

Thus $\lambda \leq 1$. This proves the lemma. $\square$

Remark 5.2.2. In Lemma 5.2.1, the assumption that $\dim(\mathfrak{g}_\alpha) = 1$ is necessary. Indeed, if $(\mathfrak{g}, \alpha) = (\mathfrak{so}(n, 1), \alpha_1)$, where α_1 is the unique simple restricted root of $\mathrm{PO}(n, 1)$, then the triple $(\mathrm{PO}(n, 1), \rho_1, \mathbb{R}^{n+1})$ defined in Equation (2.4.11) is a projective $\{\alpha_1\}$-proximal triple of $\mathfrak{so}(n, 1)$ with highest weight ω_{α_1}. However, one has $V^{\omega_\alpha - k\alpha} \neq 0$ for all $1 \leq k \leq 2$.

5.3 Convexity in Nagano spaces

In projective space, the dual of any properly convex domain Ω is also a properly convex domain. Moreover, one has $\Omega = \Omega^{**}$. In general, in a flag manifold $\mathscr{F}(\mathfrak{g}, \Theta)$, we only know that if Ω is dually convex, then it is equal to a connected component of its bidual (see the proof of Proposition 3.1.11). As seen in Sections 3.3.2 and 3.4, dual convexity neither guarantees the simple connectedness of Ω nor the connectedness of Ω^*.

In the case where $\mathscr{F}(\mathfrak{g}, \Theta)$ is a Nagano space, we have a sufficient condition for a proper dually convex domain to be equal to its bidual:

Proposition 5.3.1. *Let $(\mathfrak{g}, \alpha)$ be an irreducible Nagano pair and let $S \subset \mathscr{F}(\mathfrak{g}, \alpha)$ be a subset. For all $(a, b) \in F \times F^*$, if S is starshaped in $\mathbb{A}_b$ at a, then $S^* \subset \mathbb{A}_a$ is starshaped ath b.*

In particular, if S is starshaped in an affine chart of $\mathscr{F}(\mathfrak{g}, \alpha)$, then the set S^* is connected (and even contractible).

Proof. We take Notation 5.1.2. Assume that there exists $(a,b) \in F \times F^*$ such that S is starshaped in $\mathbb{A}_b$ at a. We may assume that $b = \mathfrak{p}^-$ and $a = \mathfrak{p}^+$. Then $S \subset \mathbb{A}_{\mathsf{std}}$ and that S is starshaped in $\mathbb{A}_{\mathsf{std}}$ at $\mathfrak{p}^+$. This means

$$\forall X \in \varphi_{\mathsf{std}}^{-1}(S),\ \forall t \in [0,1], \quad \varphi_{\mathsf{std}}(tX) \in S.$$

Recall the element $\ell_0(t) \in L$ defined in Equation (5.1.4). The fact that S is starshaped as P translates as:

$$\forall x \in S,\ \forall t \in]0,1], \quad \ell_0(t) \cdot x \in S.$$

Since $P \in S$, we know that $S^* \subset \mathbb{A}_{\mathfrak{p}^+} = \exp(\mathfrak{u}^+) \cdot \mathfrak{p}^-$. Since $S \subset \mathbb{A}_{\mathsf{std}}$, one has $\mathfrak{p}^- \in S^*$. Let $y = \exp(Y) \cdot \mathfrak{p}^- \in S^*$, with $Y \in \mathfrak{u}^+$. Then for all $t \in]0,1]$ and for all $x \in S$:

$$\ell_0(t) \cdot x \in S \Longrightarrow (\ell_0(t) \cdot x) \,\overline{\pitchfork}\, y \Longrightarrow x \,\overline{\pitchfork}\, (\ell_0(t)^{-1} \cdot y).$$

This is true for all $x \in S$, so for all $t \in]0,1]$, the set S^* contains

$$\ell_0(\frac{1}{t}) \cdot y = \ell_0(t)^{-1} \exp(Y) \ell_0(t) \cdot \mathfrak{p}^- = \exp(\mathrm{Ad}(\ell_0(t))^{-1} Y) \cdot \mathfrak{p}^- = \exp(tY) \cdot \mathfrak{p}^-.$$

Since S^* also contains $\mathfrak{p}^-$, the element $\exp(tY) \cdot \mathfrak{p}^-$ belongs to S^* for all $t \in [0,1]$. This is true for all $y \in S^*$, so S^* is starshaped in $\mathbb{A}_{\mathfrak{p}^+}$ at $\mathfrak{p}^-$. ☐

Corollary 5.3.2. *Let $(\mathfrak{g}, \alpha)$ be an irreducible Nagano pair and $\Omega \subset \mathscr{F}(\mathfrak{g}, \alpha)$ be a proper dually convex domain, which is starshaped in an affine chart. Then $\Omega^{**} = \Omega$.*

Proof. By Proposition 5.3.1, the set Ω^* is starshaped in an affine chart. Thus, again by Proposition 5.3.1, the set Ω^{**} is starshaped in an affine chart. It is thus connected. By Lemma 8.5.15, we thus have $\Omega = \Omega_0^{**} = \Omega^{**}$. ☐

Remark 5.3.3. Although we do not have a proof that the condition of being starshaped in an affine chart is necessary for $\Omega^{**} = \Omega$ (in fact, we believe it is not), we do know that several examples of proper dually convex open sets that are not starshaped in an affine chart differ from their bidual. Take for instance the domain Ω of Example 3.5.9. Then $\Omega^{**} = D \smallsetminus \mathrm{Z}_z \neq \Omega$. In particular Ω^{**} is not connected.

5.4 Characterizations of Nagano spaces

In this section, we give additional elementary characterizations of Nagano spaces among flag manifolds, and of real projective space among Nagano spaces, coming from the classification of Nagano spaces [Nag65] and of their symmetric domains [Mak73]. We will not use these characterizations in the rest of this memoir, except in Section 8.9.

Nagano has proven that the noncompact dual $\mathbb{X}(\mathfrak{g}, \alpha)$ of an irreducible Nagano space embeds as a proper symmetric domain of $\mathscr{F}(\mathfrak{g}, \alpha)$. However, the realizations of $\mathbb{X}(\mathfrak{g}, \alpha)$ into $\mathscr{F}(\mathfrak{g}, \alpha)$ are not necessarily the only proper symmetric domains in $\mathscr{F}(\mathfrak{g}, \alpha)$. Makarevic [Mak73] has listed all possible symmetric domains of Nagano spaces $\mathscr{F}(\mathfrak{g}, \alpha)$ that have a reductive transitive automorphism group. Given an irreducible Nagano pair $(\mathfrak{g}, \alpha)$, any realization of $\mathbb{X}(\mathfrak{g}, \alpha)$ is part of this list. In general, there is at least one other domain in the list (up to translation) strictly contained in $\mathscr{F}(\mathfrak{g}, \alpha)$. However, they are not necessarily proper, and if one asks for properness, one actually has:

Lemma 5.4.1. *Let $(\mathfrak{g},\alpha)$ be an irreducible Nagano pair and assume that there exists a proper symmetric domain $\Omega \subset \mathscr{F}(\mathfrak{g},\alpha)$ with transitive and reductive automorphism group, such that Ω is not a realization of $\mathbb{X}(\mathfrak{g},\alpha)$. Then there exists $n \geq 3$ such that $(\mathfrak{g},\alpha) = (\mathfrak{sl}(n,\mathbb{R}),\alpha_1)$ or $(\mathfrak{sl}(n,\mathbb{R}),\alpha_{n-1})$, i.e. $\mathscr{F}(\mathfrak{g},\alpha)$ is either the real projective space of dimension $n-1$ or its dual.*

Proof. Let $G \in \mathscr{G}_{\{\alpha\}}(\mathfrak{g})$. By Fact 3.1.3, the stabilizer of a point $x \in \Omega$ is a compact Lie subgroup of $\mathsf{Aut}_G(\Omega)$. But in the list of [Mak73], whenever

$$(\mathfrak{g},\alpha) \notin \big\{(\mathfrak{sl}(n,\mathbb{R}),\alpha_1),(\mathfrak{sl}(n,\mathbb{R}),\alpha_{n-1}) \mid n \in \mathbb{N}_{\geq 3}\big\},$$

the only cases where the stabilizer of a point is compact is when Ω is a realization of $\mathbb{X}(\mathfrak{g},\alpha)$. ☐

Nagano's theorem and Makarevic's list tell us that Nagano spaces contain many symmetric domains. Reciprocally:

Lemma 5.4.2. *Let $\mathfrak{g}$ be a real semisimple Lie algebra of noncompact type and Θ be a subset of the simple restricted roots of $\mathfrak{g}$. If $\mathscr{F}(\mathfrak{g},\Theta)$ contains a (not necessarily proper) symmetric domain, then it is a Nagano space.*

Proof. Let $G := \mathsf{Aut}_\Theta(\mathfrak{g})$ and K be a maximal compact subgroup of G. Then K acts transitively on $\mathscr{F}(\mathfrak{g},\Theta)$ and there exists a K-invariant metric $g_{\mathscr{F}(\mathfrak{g},\Theta)}$ on $\mathscr{F}(\mathfrak{g},\Theta)$. By Fact 5.1.7, we just need to show that $\mathscr{F}(\mathfrak{g},\Theta)$ is a Riemannian symmetric space for this metric. Since K acts transitively on $\mathscr{F}(\mathfrak{g},\Theta)$, it suffices to show that there exists $x \in \mathscr{F}(\mathfrak{g},\Theta)$ and $k \in K$ such that k stabilizes x and $d_xk = -\operatorname{id}_{T_x\mathscr{F}(\mathfrak{g},\Theta)}$. But this is just a consequence of the existence of a symmetric domain Ω in $\mathscr{F}(\mathfrak{g},\Theta)$. Indeed, let $x \in \Omega$ and let $s_x \in \mathsf{Aut}_G(\Omega)$ be a symmetry. Since s_x has finite order, up to translating Ω by an element of G, we may assume that $s_x \in K$. Since x is the only fixed point of s_x in Ω, we have $d_xs_x = -\operatorname{id}_{T_x\Omega}$, and since Ω is open in $\mathscr{F}(\mathfrak{g},\Theta)$, one has $\operatorname{id}_{T_x\Omega} = \operatorname{id}_{T_x\mathscr{F}(\mathfrak{g},\Theta)}$, so $d_xs_x = -\operatorname{id}_{T_x\mathscr{F}(\mathfrak{g},\Theta)}$, and the lemma is proven. ☐

The proof of Lemma 5.4.2 is analytic and uses the characterization of Nagano spaces of Fact 5.1.7. The following lemma is more general, and admits a more algebraic proof:

Lemma 5.4.3. *Let $\mathfrak{g}$ be a real semisimple Lie algebra of noncompact type and Θ be a subset of the simple restricted roots of $\mathfrak{g}$. Let $G \in \mathscr{G}_\Theta(\mathfrak{g})$, and assume that there exist $\lambda \in \mathbb{R} \smallsetminus \{0,1\}$, $g \in L_\Theta$ and $x \in \mathscr{F}(\mathfrak{g},\Theta)$ such that $d_xg = \lambda \operatorname{id}$. Then $(\mathfrak{g},\Theta)$ is a Nagano pair.*

Proof. We may assume that $x = \mathfrak{p}_\Theta^+$. The map φ_{std} induces an identification $d_0\varphi_{\mathsf{std}} : \mathfrak{u}_\Theta^- \simeq T_x\mathbb{A}_{\mathsf{std}} = T_x\mathscr{F}(\mathfrak{g},\Theta)$. Since $g \in L_\Theta$ preserves $\mathfrak{u}_\Theta^-$, this identification gives $d_xg \cdot \circ(d_0\varphi_{\mathsf{std}}) = (d_0\varphi_{\mathsf{std}}) \circ \operatorname{Ad}(g)$. This gives, for all $X, Y \in \mathfrak{u}_\Theta^-$:

$$\lambda[X,Y] = \operatorname{Ad}(g)\cdot[X,Y] = [\operatorname{Ad}(g)\cdot, \operatorname{Ad}(g)\cdot Y] = [\lambda X, \lambda Y] = \lambda^2[X,Y].$$

Since $\lambda \notin \{0,1\}$, this implies that $[X,Y] = 0$. Thus $\mathfrak{u}_\Theta^-$ is abelian, which implies by definition (see Section 5.1) that $(\mathfrak{g},\Theta)$ a Nagano pair. ☐

Remark 5.4.4. A direct consequence of Lemma 5.4.2 is that the Nagano spaces are exactly the flag manifolds $\mathscr{F}(\mathfrak{g}, \Theta)$, where $\mathfrak{g}$ is a real semisimple Lie algebra of noncompact type and Θ a subset of the simple restricted roots of $\mathfrak{g}$, such that there exists $\ell \in L_\Theta$ (where L_Θ is associated with $G := \mathsf{Aut}_\Theta(\mathfrak{g})$) whose restriction to $\mathfrak{u}_\Theta^-$ is equal to s_Θ.

Chapter 6

Photons and the Kobayashi metric in Nagano spaces

The ultimate goal of this section is to develop a theory analogous to projective geometry in Nagano spaces. We have already introduced tools to study proper domains in general flag manifolds in Chapter 3. Since Nagano spaces have a richer structure, we can expect new tools of study to emerge, which is the case whenever the Nagano space is *of real type* (see Definition 6.1.1). For instance, we define the *photons* of Nagano spaces $\mathscr{F}(\mathfrak{g}, \alpha)$ of real type in Section 6.3, which serve as analogs of projective lines. These embeddings of the real projective line into $\mathscr{F}(\mathfrak{g}, \alpha)$ allow us to define a *Kobayashi metric* on any proper domain $\Omega \subset \mathscr{F}(\mathfrak{g}, \alpha)$ in Section 6.4. This metric, generalizing the Hilbert metric in convex projective geometry, enables the study of the boundary of Ω (for instance, an analogue of Fact 1.1.1 will be proven in Section 7.2, more precisely in Lemma 7.2.10).

The general properties of the Kobayashi metric will be particularly useful in Chapter 8, where we investigate proper almost-homogeneous domains in flag manifolds. Therefore, in the present chapter, we conduct a detailed study of its properties (see e.g. Propositions 6.4.5 and 6.4.8). Comparing it with the Caratheodory metrics in Section 6.4.5 (see Proposition 6.4.10) provides a proof that it is a proper geodesic metric whenever Ω is proper and dually convex (see Corollary 6.4.12). This property, which generalizes the projective case involving the Hilbert metric and classical convexity, will be crucial in Chapter 7 for studying the boundary of proper domains in Nagano spaces of real type (Section 7.2.2). Moreover, the fact that this metric is geodesic will allow us in Chapter 8 to apply the Švarc–Milnor Lemma to discrete groups acting cocompactly on Ω, a key feature in the classical theory of divisible convex sets in real projective space.

As an example, in Section 6.4.7, we conclude the chapter with a study of the geodesics of realizations of the noncompact dual for the Kobayashi metric.

Notation 6.0.1. For all this chapter, whenever we consider an irreducible Nagano pair $(\mathfrak{g}, \alpha)$, we will use Notation 5.1.2.

6.1 Nagano spaces of real type

The Nagano spaces we consider in this chapter are those of *real type*, see Definition 6.1.1 below. It is only in this context that the *photons* (which we will define in Section 6.3) truly behave similarly to projective lines of real projective space; see Remark 6.3.4. In Section 6.5, we will investigate a possible generalization of our construction to any Nagano space.

Definition 6.1.1. Let $(\mathfrak{g}, \Theta)$ be a Nagano pair. We say that it is *of real type* if $\dim(\mathfrak{g}_\alpha) = 1$ for all $\alpha \in \Theta$. In this case, we say that $\mathscr{F}(\mathfrak{g}, \Theta)$ is a *Nagano space of real type*.

If $(\mathfrak{g}, \alpha)$ is an irreducible Nagano pair, then it is of real type if and only if $\dim(\mathfrak{g}_\alpha) = 1$. Thus a Nagano space is of real type if and only if all of its irreducible factors are of real type.

In Table 8.1, we give the dimensions of $\mathfrak{g}_\alpha$, for all irreducible Nagano pairs $(\mathfrak{g}, \alpha)$. We can thus see that all our key examples (i.e. the Grassmannians, causal flag manifolds and the Einstein universes of signature (p, q) with $p, q \geq 1$) are of real type. There is also another classical example, which is the Nagano space defined by $(\mathfrak{so}(n, n), \alpha_n)$. It is the space of totally isotropic subspaces of $\mathbb{R}^{n,n}$. There are two exceptional examples.

Remark 6.1.2. 1. By Table 8.1, the Einstein universe $\mathrm{Ein}^{p,q}$ is always an irreducible Nagano space, but it is of real type if, and only if, $p, q \geq 1$.

2. Note that Nagano spaces of real type all satisfy the assumptions of Proposition 3.2.1 (see also Remark 3.2.2).

3. Since in our considerations we can always assume that a Nagano space is irreducible, we will only consider *irreducible* Nagano spaces in this section, but everything can be generalized to Nagano spaces.

Tables 8.1 and 8.2 tell us that the only irreducible Nagano pairs of real type which have rank one (in the sense of Observation 5.1.10) are the pairs $(\mathfrak{sl}(n, \mathbb{R}), \alpha_1)$, $n \in \mathbb{N}_{>0}$, and their duals $\mathscr{F}(\mathfrak{sl}(n, \mathbb{R}), \alpha_{n-1})$. They correspond to the real projective spaces of all dimensions and their duals. All the other Nagano spaces of real type are of higher rank. We will see in the rest of this memoir that, depending on if an irreducible Nagano space of real type $(\mathfrak{g}, \alpha)$ is of higher rank or not, its geometry is slightly different (see e.g. Lemma 6.3.9 or Remark 6.4.16.(2)).

6.2 Embedding the projective line into $\mathscr{F}(\mathfrak{g}, \alpha)$

For this section, we fix an irreducible Nagano pair $(\mathfrak{g}, \alpha)$ and $G \in \mathscr{G}_{\{\alpha\}}(\mathfrak{g})$. We construct embeddings of the projective line into $\mathscr{F}(\mathfrak{g}, \alpha)$. The images of these embeddings are what we will call *photons* in Section 6.3. In Lemmas 6.2.2 and 6.2.2, we investigate the action of the unipotent radical U^+ on these images.

Let us consider the $\mathfrak{sl}_2$-triple $\mathsf{t}_{\mathsf{std}}$ defined in Notation 5.1.2. The map $\mathrm{j}_{\mathsf{t}_{\mathsf{std}}}$ associated with $\mathsf{t}_{\mathsf{std}}$ (see Section 2.2.1) induces a group homomorphism $\tau : \mathrm{SL}(2, \mathbb{R}) \hookrightarrow G$ with kernel contained in $\{\pm \mathrm{id}\}$ and with differential $\tau_* = \mathrm{j}_{\mathsf{t}_{\mathsf{std}}}$ at id.

Lemma 6.2.1. *The stabilizer of P in* $\mathrm{SL}(2,\mathbb{R})$ *is the standard Borel subgroup P_1 of* $\mathrm{SL}(2,\mathbb{R})$.

Proof. Let us denote by S this stabilizer. Note that $e^{\mathrm{E}}, e^{\mathrm{H}} \in S$, so the identity component P_1^0 of P_1 is contained in S. Since the orbit of $P \in \mathscr{F}(\mathfrak{g},\alpha)$ is nontrivial, we have $P_1^0 \subset S \subset P_1$. It remains to show that $g := \tau(-\mathrm{id})$ is in P. Noticing that $g = k^2$, where $k = \exp(\frac{\pi}{2}(v^+ - v^-))$ is a representative of s_α in K, we get that $\mathrm{Ad}(g)$ acts as $s_\alpha^2 = \mathrm{id}$ on Σ. Then, by Equation (2.2.1), the element g normalizes $\mathfrak{u}^+$. Hence $g \in P$. This proves that $S = P_1$. $\square$

By Lemma 6.2.1 above, the group homomorphism τ induces a τ-equivariant embedding $\zeta : \mathbb{P}(\mathbb{R}^2) \hookrightarrow \mathscr{F}(\mathfrak{g},\alpha)$. It will be convenient to write this map explicitly:

$$\zeta([1:t]) = \exp(tv^-) \cdot \mathfrak{p}^+ \quad \forall t \in \mathbb{R}. \tag{6.2.1}$$

Lemma 6.2.2. *Let $(\mathfrak{g},\alpha)$ be an irreducible Nagano pair of real type. For any $Y \in \mathfrak{u}^+$, write $Y = \lambda v^+ + \sum_{\beta \in \Sigma^+_{\{\alpha\}} \smallsetminus \{\alpha\}} X_\beta$, with $X_\beta \in \mathfrak{g}_\beta$ for all $\beta \in \Sigma^+_{\{\alpha\}} \smallsetminus \{\alpha\}$, and $\lambda \in \mathbb{R}$. Then*

$$\exp(Y) \cdot \zeta = \zeta \circ e^{\lambda \mathrm{F}}.$$

In particular, for all $t \in \mathbb{R} \smallsetminus \{-\lambda^{-1}\}$ (with $-\lambda^{-1} = \infty$ if $\lambda = 0$), one has:

$$\exp(Y) \cdot \zeta([1:t]) = \zeta\Big(\Big[1 : \frac{t}{1+\lambda t}\Big]\Big).$$

Proof. Since $\dim(\mathfrak{g}_\alpha) = 1$, we can write $Y = \lambda v^+ + \sum_{\beta \in \Sigma^+_\Theta \smallsetminus \{\alpha\}} Y_\beta$ with $Y_\beta \in \mathfrak{g}_\beta$ for all $\beta \in \Sigma^+_\Theta \smallsetminus \{\alpha\}$ and with $\lambda \in \mathbb{R}$. Since $\mathfrak{u}^+$ is abelian, one has:

$$\exp(Y) \cdot \zeta([1:t]) = \exp(\lambda v^+) \Bigg(\prod_{\beta \in \Sigma^+_\alpha \smallsetminus \{\alpha\}} \exp(Y_\beta) \Bigg) \cdot \zeta([1:t]). \tag{6.2.2}$$

Let $\beta \in \Sigma^+_\alpha \smallsetminus \{\alpha\}$. Since $Y_\beta \in \mathfrak{p}^+$, one has

$$\begin{aligned}\exp(Y_\beta) \cdot \zeta([1:t]) = \exp(Y_\beta)\exp(tv^-) \cdot \mathfrak{p}^+ &= \exp(Y_\beta)\exp(tv^-)\exp(-Y_\beta) \cdot \mathfrak{p}^+ \\ &= \exp\big(t \,\mathrm{Ad}\,(\exp(Y_\beta))\, v^-\big) \cdot \mathfrak{p}.\end{aligned}$$

By Lemma 5.1.4, there exists $Z \in \mathfrak{p}^+$ such that $[v^-, Z] = 0$ and

$$\begin{aligned}\exp\big(t\,\mathrm{Ad}\,(\exp(Y_\beta))\, v^-\big) \cdot \mathfrak{p}^+ &= \exp(t(v^- + Z)) \cdot \mathfrak{p}^+ \\ &= \exp(tv^-)\exp(tZ) \cdot \mathfrak{p}^+ = \exp(tv^-) \cdot \mathfrak{p}^+,\end{aligned}$$

the second last equality holding because v^- and Z commute, and the last one holding because $Z \in \mathfrak{p}^+$. Hence by induction, Equation (6.2.2) becomes:

$$\begin{aligned}\exp(Y) \cdot \zeta([1:t]) = \exp(\lambda v^+) \cdot \zeta([1:t]) &= \tau(e^{\lambda \mathrm{E}}) \cdot \zeta([1:t]) \\ &= \zeta(e^{\lambda \mathrm{E}} \cdot [1:t]) = \zeta\Big(\Big[1 : \frac{t}{1+\lambda t}\Big]\Big),\end{aligned}$$

the last equality holding by an elementary computation and the second last by τ-equivariance of ζ. $\square$

6.3 Photons and cross ratio

In this section, we define *photons* in Nagano spaces and investigate their properties. We do not need the Nagano space to be of real type to define photons. However, we will need the Nagano space to be of real type to get suitable invariance properties of the photons; see e.g. Remark 6.3.4 and Example 6.3.8.

6.3.1 Photons in Nagano spaces

Let $(\mathfrak{g}, \alpha)$ be an irreducible Nagano pair. Recall the map ζ of Equation (6.2.1). We define the topological circle

$$\Lambda_{\mathsf{std}} := \zeta(\mathbb{P}(\mathbb{R}^2)),$$

called the *standard photon* of $\mathscr{F}(\mathfrak{g}, \alpha)$. The map ζ is a parametrization of Λ_{std}.

Definition 6.3.1. A *photon* of $\mathscr{F}(\mathfrak{g}, \alpha)$ is an $\mathsf{Aut}_\Theta(\mathfrak{g})$-translate of Λ_{std} in $\mathscr{F}(\mathfrak{g}, \alpha)$. We denote by $\mathscr{L}$ the set of all photons of $\mathscr{F}(\mathfrak{g}, \alpha)$.

See Section 6.3.4 for a description of photons in concrete examples. In particular, for $(\mathfrak{g}, \alpha) = (\mathfrak{sl}(n, \mathbb{R}), \alpha_1)$, photons are simply the projective lines of $\mathbb{P}(\mathbb{R}^n)$.

Remark 6.3.2. Photons have been defined and investigated for general flag manifolds in [BGL$^+$24], to prove that the property of being Θ-positive (in the sense of [GLW21]) for representations of a surface groups is closed.

Note that for all $G \in \mathscr{G}_{\{\alpha\}}(\mathfrak{g})$, one has

$$\mathscr{L} = \{g \cdot \Lambda_{\mathsf{std}} \mid g \in G\}.$$

Lemma 6.3.3. *Let $(\mathfrak{g}, \alpha)$ be an irreducible Nagano pair of real type. Then*

1. $U^+ \cdot \Lambda_{\mathsf{std}} = \Lambda_{\mathsf{std}}$*;*
2. *L acts transitively on the set of photons through $\mathfrak{p}^+$.*

Proof. 1. Let $Y \in \mathfrak{u}^+$. By Lemma 6.2.2, there exists $\lambda \in \mathbb{R}$ such that for all $t \in \mathbb{R} \smallsetminus \{-\lambda^{-1}\}$, one has $\exp(Y) \cdot \zeta([1:t]) \in \Lambda_{\mathsf{std}}$. Thus

$$\exp(Y) \cdot \Lambda_{\mathsf{std}} = \overline{\{\exp(Y) \cdot \zeta([1:t]) \mid t \neq -\lambda^{-1}\}} \subset \Lambda_{\mathsf{std}}.$$

The same argument applied to $-Y$ gives that $\exp(-Y) \cdot \Lambda_{\mathsf{std}} \subset \Lambda_{\mathsf{std}}$. Thus $\exp(Y) \cdot \Lambda_{\mathsf{std}} = \Lambda_{\mathsf{std}}$.

2. Let Λ be a photon through $\mathfrak{p}^+$. There exists $g \in G$ such that $\Lambda = g \cdot \Lambda_{\mathsf{std}}$.

If $g \cdot \mathfrak{p}^+ \in \mathbb{A}_{\mathsf{std}}$, then by Equation (2.4.6), one can write $g = \exp(X)\ell\exp(Y)$ for some $X \in \mathfrak{u}^-$, $\ell \in L$ and $Y \in \mathfrak{u}^+$. Since $\mathfrak{p}^+ \in \Lambda$, one has

$$\mathfrak{p}^+ = \ell^{-1} \cdot \mathfrak{p}^+ \in \ell^{-1} g \cdot \Lambda_{\mathsf{std}} \overset{\text{point (1)}}{=} \exp(\mathrm{Ad}(\ell^{-1}) \cdot X) \cdot \Lambda_{\mathsf{std}}.$$

Thus there exists $(t_k) \in \mathbb{R}^{\mathbb{N}}$ such that $\mathfrak{p}^+ = \lim_{k \to +\infty} \varphi_{\mathsf{std}}(\mathrm{Ad}(\ell^{-1}) \cdot X + t_k v^-)$. In particular $X \in \mathfrak{g}_{-\alpha}$: there exists $t \in \mathbb{R}$ such that $\mathrm{Ad}(\ell^{-1}) \cdot X = tv^-$. Thus

$$\Lambda = \exp(t\,\mathrm{Ad}(\ell) \cdot v^-)\ell\exp(Y) \cdot \Lambda_{\mathsf{std}} = \ell^{-1}\exp(tv^-)\exp(Y) \cdot \Lambda_{\mathsf{std}} = \ell^{-1} \cdot \Lambda_{\mathsf{std}}.$$

Now if $g \cdot \mathfrak{p}^+ \notin \mathbb{A}_{\mathsf{std}}$, there exists $Y \in \mathfrak{u}^+$ such that $\exp(Y)g \cdot \mathfrak{p}^+ \in \mathbb{A}_{\mathsf{std}}$. Since $\exp(Y) \cdot \mathfrak{p}^+ \in \exp(Y) \cdot \Lambda$, according to the previous case, there exists $\ell \in L$ such that $\exp(Y) \cdot \Lambda = \ell \cdot \Lambda_{\mathsf{std}}$. Then

$$\Lambda = \exp(-Y)\ell \cdot \Lambda_{\mathsf{std}} = \ell \exp(-\operatorname{Ad}(Y)) \cdot \Lambda_{\mathsf{std}} \overset{\text{point (1)}}{=} \ell \cdot \Lambda_{\mathsf{std}}. \square$$

Remark 6.3.4. Lemma 6.3.3 is not necessarily true when $\dim(\mathfrak{g}_\alpha) \geq 2$. For instance, let $n \geq 3$ and $\mathfrak{g} = \mathfrak{so}(n, 1)$. Then $\dim(\mathfrak{g}_\alpha) = \dim(\mathfrak{u}^+) = n - 1$. The standard photon Λ_{std} is a geodesic of the Riemannian symmetric space $\mathbb{S}^{n-1} \simeq \mathrm{SO}(n)/\mathrm{SO}(n-1)$ through $\mathfrak{p}^+$ and $\mathfrak{p}^-$ (which can be chosen to be the south pole and the north pole). Since U^+ acts transitively on the affine chart $\mathbb{S}^{n-1} \smallsetminus \{\mathfrak{p}^+\}$, there exists $u \in U^+$ such that $u \cdot \mathfrak{p}^- \notin \Lambda_{\mathsf{std}}$. Then $u \cdot \Lambda_{\mathsf{std}} \neq \Lambda_{\mathsf{std}}$.

Lemma 6.3.5. *Let $(\mathfrak{g}, \alpha)$ be an irreducible Nagano pair of real type. For all $x, y \in \mathscr{F}(\mathfrak{g}, \alpha)$ there exists at most one photon through x and y.*

Proof. We may assume that $x = \mathfrak{p}^+$ and $y \in \Lambda_{\mathsf{std}} \cap \mathbb{A}_{\mathsf{std}}$. Let us fix $G \in \mathscr{G}_{\{\alpha\}}(\mathfrak{g})$. Up to dilating (see Section 5.1.1), we may also assume that $y = \exp(v^-) \cdot \mathfrak{p}^+$. Assume that there exists another photon Λ through $\mathfrak{p}^+$ and $\exp(v^-) \cdot \mathfrak{p}^+$. By Lemma 6.3.3.(2), there exists $\ell \in L$ such that $\Lambda = \ell \cdot \Lambda_{\mathsf{std}}$. Since $\exp(v^-) \cdot \mathfrak{p}^+ \in \ell \cdot \Lambda_{\mathsf{std}}$, we have:

$$\ell^{-1}\varphi_{\mathsf{std}}(v^-) = \varphi_{\mathsf{std}}(\operatorname{Ad}(\ell)^{-1} \cdot v^-) = \varphi_{\mathsf{std}}(tv^-) \tag{6.3.1}$$

for some $t \in \mathbb{R}$. By injectivity of φ_{std}, we have $\operatorname{Ad}(\ell)^{-1} \cdot v^- = tv^-$. Now by injectivity of $\operatorname{Ad}(\ell)$, we necessarily have $t \neq 0$. Since $\operatorname{Ad}(\ell)^{-1}$ is linear on $\mathfrak{u}^-$, this implies that $\operatorname{Ad}(\ell)^{-1} \cdot \mathfrak{g}_{-\alpha} = \mathfrak{g}_{-\alpha}$; in other words, we have $\ell \cdot (\Lambda_{\mathsf{std}} \cap \mathbb{A}_{\mathsf{std}}) = \Lambda_{\mathsf{std}} \cap \mathbb{A}_{\mathsf{std}}$. Taking the closure, we get $\ell \cdot \Lambda_{\mathsf{std}} = \Lambda_{\mathsf{std}}$. Thus $\Lambda = \Lambda_{\mathsf{std}}$. $\square$

Given a photon $\Lambda \in \mathscr{L}$, there exists $g \in G$ such that $\Lambda = g \cdot \Lambda_{\mathsf{std}}$. The map

$$\begin{array}{rccc} \zeta_g : & \mathbb{P}(\mathbb{R}^2) & \longrightarrow & \mathscr{F}(\mathfrak{g}, \alpha) \\ & x & \longmapsto & g \cdot \zeta(x) \end{array} \tag{6.3.2}$$

is then a parametrization of Λ. A priori, this parametrization depends on the choice of $g \in G$ such that $\Lambda = g \cdot \Lambda_{\mathsf{std}}$ (although its image does not). The next lemma shows that two parametrizations given by different choices of $g \in G$ such that $\Lambda = g \cdot \Lambda_{\mathsf{std}}$ only differ by a projective reparametrization of $\mathbb{P}(\mathbb{R}^2)$.

Lemma 6.3.6. *Let $(\mathfrak{g}, \alpha)$ be an irreducible Nagano pair of real type, and $G \in \mathscr{G}_{\{\alpha\}}(\mathfrak{g})$. One has*

$$\operatorname{Stab}_G(\Lambda_{\mathsf{std}}) = \tau(\mathrm{SL}(2, \mathbb{R})) \times \operatorname{Cent}_G(\tau(\mathrm{SL}(2, \mathbb{R}))),$$

where $\operatorname{Stab}_G(\Lambda_{\mathsf{std}})$ *is the stabilizer of* Λ_{std} *in* G *and* $\operatorname{Cent}_G(\tau(\mathrm{SL}(2, \mathbb{R})))$ *is the centralizer in* G *of the group* $\tau(\mathrm{SL}(2, \mathbb{R}))$.

By the definition of Λ_{std}, the centralizer of $\tau(\mathrm{SL}(2, \mathbb{R}))$ acts trivially on Λ_{std}. Therefore Lemma 6.3.6 implies that $\operatorname{Stab}_G(\Lambda_{\mathsf{std}})$ acts on Λ_{std} by projective transformations. By equivariance, for any photon Λ, the group $\operatorname{Stab}_G(\Lambda)$ acts on Λ by projective transformations.

Proof of Lemma 6.3.6. The inclusion $\tau(\mathrm{SL}(2,\mathbb{R})) \times \mathrm{Cent}_G(\tau(\mathrm{SL}(2,\mathbb{R}))) \subset \mathrm{Stab}_G(\Lambda_{\mathsf{std}})$ follows from the definition of Λ_{std}. Let us prove the converse inclusion.

Let $g \in \mathrm{Stab}_G(\Lambda_{\mathsf{std}})$. First assume that $g \cdot \mathfrak{p}^+ \in \mathbb{A}_{\mathsf{std}}$. Then, by Equation (2.2.7), we can write $g = \exp(X)\ell\exp(Y)$ with $Y \in \mathfrak{u}^+$, $X \in \mathfrak{u}^-$ and $\ell \in L$. By Lemma 6.2.2, we have $\exp(X)\ell \cdot \Lambda_{\mathsf{std}} = \Lambda_{\mathsf{std}}$. For all $t \in \mathbb{R}$, we have

$$\exp(X)\ell \cdot \zeta([1:t]) = \varphi_{\mathsf{std}}(X + t\,\mathrm{Ad}(\ell)(v^-)) \in \varphi_{\mathsf{std}}(\mathbb{R}v^-).$$

This implies that there exists $\delta \in \mathbb{R}$ such that $X = \delta v^-$. It also implies that $\mathrm{Ad}(\ell)$ preserves the line $\mathfrak{g}_{-\alpha} = \mathbb{R}v^-$. Since $\mathrm{Ad}(\ell)$ induces an endomorphism of $\mathfrak{u}^-$, it induces an endomorphism of $\mathfrak{g}_{-\alpha}$. Thus there exists $\mu \in \mathbb{R}$ such that $\mathrm{Ad}(\ell)_{|\mathfrak{g}_{-\alpha}} = \mu\,\mathrm{id}$.

On the other hand, by Lemma 6.2.2, there exists $\lambda \in \mathbb{R}$ such that $\exp(Y) \cdot \zeta = \zeta \circ e^{\lambda \mathrm{F}}$. Thus we have, for all $t \in \mathbb{R} \smallsetminus \{-\lambda^{-1}\}$:

$$\begin{aligned}\ell\exp(Y) \cdot \zeta([1:t]) &= \ell \cdot \zeta\Big(\Big[1 : \frac{t}{1+\lambda t}\Big]\Big) = \exp\Big(\frac{t}{1+\lambda t}\,\mathrm{Ad}(\ell)(v^-)\Big) \cdot \mathfrak{p}^+ \\ &= \exp\Big(\frac{\mu t}{1+\lambda t}v^-\Big) \cdot \mathfrak{p}^+ = \zeta\Big(\Big[1 : \frac{\mu t}{1+\lambda t}\Big]\Big) = \zeta(e^{\mu \mathrm{H}}e^{\lambda \mathrm{E}} \cdot [1:t]).\end{aligned}$$

Moreover, one has $\exp(X) = \tau\left(e^{\delta \mathrm{E}}\right)$. Then, by τ-equivariance of ζ, one has $g \cdot \zeta([1:t]) = \zeta(A \cdot [1:t])$ with $A = e^{\delta \mathrm{F}}e^{\mu \mathrm{H}}e^{\lambda \mathrm{E}} \in \mathrm{SL}(2,\mathbb{R})$. By continuity, this equality holds for all $x \in \mathbb{P}(\mathbb{R}^2)$. In particular, the element $\tau(A)^{-1}g$ fixes every point of Λ_{std}. By the definition of Λ_{std}, this implies that $\tau(A)^{-1}g \in \mathrm{Cent}_G(\tau(\mathrm{SL}(2,\mathbb{R})))$.

Now if $g \cdot \mathfrak{p}^+ \notin \mathbb{A}_{\mathsf{std}}$, since g preserves Λ_{std} one must have $g \cdot \mathfrak{p}^+ = \zeta([0:1])$. Since $\mathrm{SL}(2,\mathbb{R})$ acts transitively on $\mathbb{P}(\mathbb{R}^2)$, there exists $B \in \mathrm{SL}(2,\mathbb{R})$ such that

$$g \cdot \mathfrak{p}^+ = \zeta([0:1]) = \zeta(B \cdot [1:0]) = \tau(B) \cdot \zeta([1:0]),$$

so that $\tau(B)^{-1}g \in \mathrm{Stab}_G(\Lambda_{\mathsf{std}})$ satisfies $\tau(B)^{-1}g \cdot \mathfrak{p}^+ = \mathfrak{p}^+ \in \mathbb{A}_{\mathsf{std}}$. Then by the previous case, one has $\tau(B)^{-1}g \in \tau(\mathrm{SL}(2,\mathbb{R})) \times \mathrm{Cent}_G(\tau(\mathrm{SL}(2,\mathbb{R})))$. Hence

$$g = \tau(B)\tau(B)^{-1}g \in \tau(\mathrm{SL}(2,\mathbb{R})) \times \mathrm{Cent}_G(\tau(\mathrm{SL}(2,\mathbb{R}))) \quad \square$$

The following lemma states that if $\dim(\mathfrak{g}_\alpha) = 1$, then photons intersecting $\mathbb{A}_{\mathsf{std}}$ are compactifications of certain affine lines of $\mathbb{A}_{\mathsf{std}}$:

Lemma 6.3.7. *Let $(\mathfrak{g}, \alpha)$ be an irreducible Nagano pair of real type. Let $\Lambda \in \mathscr{L}$. If $\Lambda \cap \mathbb{A}_{\mathsf{std}}$ is nonempty, then it is an affine line in $\mathbb{A}_{\mathsf{std}}$, and $\Lambda \cap \mathrm{Z}_{\mathfrak{p}^-}$ is a singleton.*

Lemma 6.3.7 implies that for all $\xi \in \mathscr{F}(\mathfrak{g}, \alpha)^-$ such that $\Lambda_{\mathsf{std}} \not\subset \mathrm{Z}_\xi$, the set $\Lambda_{\mathsf{std}} \cap \mathrm{Z}_\xi$ is a singleton.

Proof of Lemma 6.3.7. Assume that $\Lambda \cap \mathbb{A}_{\mathsf{std}} \neq \emptyset$. Let $g \in G$ be such that $\Lambda = g \cdot \Lambda_{\mathsf{std}}$. Since Λ is not contained in $\mathrm{Z}_{\mathfrak{p}^-}$, there exists $t \in \mathbb{R}$ such that $g\exp(tv^-) \cdot \mathfrak{p}^+ \in \mathbb{A}_{\mathsf{std}}$. Since $\exp(tv^-)$ stabilizes Λ_{std}, we have $g\exp(tv^-) \cdot \Lambda_{\mathsf{std}} = \Lambda$. Hence, up to replacing g with $g\exp(tv^-)$, we may assume that $g \cdot \mathfrak{p}^+ \in \mathbb{A}_{\mathsf{std}}$.

By Equation (2.2.7), one can thus write $g = \exp(X)\ell\exp(Y)$ with $X \in \mathfrak{u}^-$, $Y \in \mathfrak{u}^+$ and $\ell \in L$. By Lemma 6.3.3, one then has $g \cdot \Lambda_{\mathsf{std}} = \exp(X)\ell \cdot \Lambda_{\mathsf{std}}$. Hence

$$\varphi_{\mathsf{std}}^{-1}(\Lambda \cap \mathbb{A}_{\mathsf{std}}) = \varphi_{\mathsf{std}}^{-1}\big((g \cdot \Lambda_{\mathsf{std}}) \cap \mathbb{A}_{\mathsf{std}}\big) = X + \mathrm{Ad}(\ell) \cdot \mathfrak{g}_{-\alpha} \subset \mathfrak{u}^- \tag{6.3.3}$$

is an affine line of $\mathfrak{u}^-$. Hence $\Lambda \cap \mathbb{A}_{\mathsf{std}}$ is an affine line of $\mathbb{A}_{\mathsf{std}}$ for the canonical affine structure.

Moreover, the map $\zeta_{g'}$, with $g' = \exp(X)\ell$, is a parametrization of Λ. By Equation (6.3.3), one has $\zeta_{g'}([1:t]) = \exp(X + t\,\mathrm{Ad}(\ell)v^-) \cdot \mathfrak{p}^+ \in \mathbb{A}_{\mathsf{std}}$ for all $t \in \mathbb{R}$. Hence the only point of $\Lambda \cap \mathrm{Z}_{\mathfrak{p}^-}$ is $\zeta_{g'}([0:1])$, and $\Lambda \cap \mathrm{Z}_{\mathfrak{p}^-}$ is a singleton. □

Example 6.3.8. Let us go back to the example of Remark 6.3.4. For $\mathfrak{g} = \mathfrak{so}(n,1)$ (and necessarily $\alpha = \alpha_1$ the unique restricted root of $\mathfrak{g}$), take for instance $G = \mathrm{PO}(n,1)$. Consider the canonical embedding $\rho_1 : G \hookrightarrow \mathrm{PGL}(n+1,\mathbb{R})$ defined in Equation (2.4.11). Then a subset $F \subset \mathbb{S}^{n-1}$ is a photon if and only if its image under ι_{ρ_1} is the *nontrivial* intersection of a 2-plane in $\mathbb{P}(\mathbb{R}^{n+1})$ with $\mathbb{S}^{n-1}$. Unlike the cases studied in the previous sections, the nonempty intersection of a photon Λ with an affine chart is not always an affine line. For instance, it can be a circle, in which case Λ is bounded in the affine chart.

To prove the following lemma, we rely only on the results established in this section. In Section 6.5, deep results on Nagano spaces proven by Takeuchi [Tak88] will also imply it (see Theorem 6.5.8).

Lemma 6.3.9. *Let $(\mathfrak{g},\alpha)$ be an irreducible Nagano pair of real type and of higher rank (in the sense of Definition 5.1.5). Let $\xi \in \mathscr{F}(\mathfrak{g},\alpha)^-$ and $x \in \mathscr{F}(\mathfrak{g},\alpha)$ be such that* $\mathrm{id} \in \mathsf{pos}^{(\{\alpha\},\{\mathrm{i}(\alpha)\})}(x,\xi)$. *Let Λ be a photon through x. Then $\Lambda \subset \mathrm{Z}_\xi$.*

Proof. By Lemma 2.2.6, it suffices to prove that

$$\Lambda \subset (\mathrm{Z}_x)^* := \{x' \in \mathscr{F}(\mathfrak{g},\Theta) \mid w_0 \notin \mathsf{pos}^{(\{\alpha\},\{\alpha\})}(x,x')\}.$$

We may assume that $x = \mathfrak{p}^+$ and $\Lambda = \Lambda_{\mathsf{std}}$. To simplify the notation, we write $\mathsf{pos} := \mathsf{pos}^{(\{\alpha\},\{\alpha\})}$. Let $k := \exp(\frac{\pi}{2}(v^+ - v^-))$ be a representative for s_α in W. One has:

$$k \cdot \mathfrak{p}^+ = \tau(e^{\frac{\pi}{2}(\mathrm{E}-\mathrm{F})}) \cdot \mathfrak{p}^+ = \zeta\big(e^{\frac{\pi}{2}(\mathrm{E}-\mathrm{F})} \cdot [1:0]\big) \in \Lambda_{\mathsf{std}} \smallsetminus \{\mathfrak{p}^+\}. \tag{6.3.4}$$

Since $k \cdot \mathfrak{p}^+ \in \Lambda_{\mathsf{std}} \smallsetminus \{\mathfrak{p}^+\}$, one has $\Lambda_{\mathsf{std}} \smallsetminus \{\mathfrak{p}^+\} = \overline{\{\ell_0(t)k \cdot \mathfrak{p}^+ \mid t \in \mathbb{R}\}} \smallsetminus \{\mathfrak{p}^+\}$. But for all $t \in \mathbb{R}$, one has

$$\begin{aligned} s_\alpha \in \mathsf{pos}(\mathfrak{p}^+, k \cdot \mathfrak{p}^+) &= \mathsf{pos}\big(\ell_0(t) \cdot \mathfrak{p}^+, \ell_0(t)k \cdot \mathfrak{p}^+\big) \\ &= \mathsf{pos}\big(\mathfrak{p}^+, \ell_0(t)k \cdot \mathfrak{p}^+\big). \end{aligned}$$

Thus we have

$$s_\alpha \in \mathsf{pos}(x'', \mathfrak{p}^+) \quad \forall x'' \in \Lambda_{\mathsf{std}} \smallsetminus \{\mathfrak{p}^+\}.$$

We thus have $\big(\bigcup_{\ell \in L} \ell \cdot \Lambda_{\mathsf{std}}\big) \smallsetminus \{\mathfrak{p}^+\} \subset \mathbf{C}_{\overline{s_\alpha}}(\mathfrak{p}^+)$.

Reciprocally, let $x'' \in \mathscr{F}(\mathfrak{g},\alpha)$ and assume that $s_\alpha \in \mathsf{pos}(x'', \mathfrak{p}^+)$. Using the Bruhat decomposition, write $x'' = pk_w \cdot \mathfrak{p}^+$, with $p \in P+$, $w \in W$ and $k_w \in K$ a representative of w. Then $s_\alpha \in \mathsf{pos}(x'', \mathfrak{p}^+) = \mathsf{pos}(k_w \cdot \mathfrak{p}^+, \mathfrak{p}^+)$, so $\overline{w} = \overline{s_\alpha}$ in $|W_{\Delta \smallsetminus \{\alpha\}} \backslash W / W_{\Delta \smallsetminus \{\alpha\}}|$. Thus there exist $a, b \in W_{\Delta \smallsetminus \{\alpha\}}$ such that $w = a s_\alpha b$. Since $W_{\Delta \smallsetminus \{\alpha\}}$ stabilizes $\mathfrak{p}^+$, we have $x'' = pk_a k \cdot \mathfrak{p}^+$, with $p = u\ell$, $u \in U^+$ and $\ell \in L$, and $k_a \in K$ a representative of a. By

Equation (6.3.4), one thus has $x'' \in pk_a \cdot (\Lambda_{\mathsf{std}} \smallsetminus \{\mathfrak{p}^+\}) = (p \cdot \Lambda_{\mathsf{std}}) \smallsetminus \{\mathfrak{p}^+\} = (\ell \cdot \Lambda_{\mathsf{std}}) \smallsetminus \{\mathfrak{p}^+\}$ (the last equality holding by Lemma 6.3.3). We have just proved:

$$\Big(\bigcup_{\ell \in L} \ell \cdot \Lambda_{\mathsf{std}}\Big) \smallsetminus \{\mathfrak{p}^+\} = \mathbf{C}_{\overline{s_\alpha}}(\mathfrak{p}^+).$$

Thus $\overline{s_\alpha}$ is minimal in $(W_{\Delta \smallsetminus \{\alpha\}} \backslash W / W_{\Delta \smallsetminus \{\alpha\}}) \smallsetminus \{\mathrm{id}\}$: if $\overline{w} \leq \overline{s_\alpha}$ is different from $\overline{s_\alpha}$, then by Fact 2.2.4, one has $\mathbf{C}_{\overline{w}}(\mathfrak{p}^+) \subset \overline{\mathbf{C}_{\overline{s_\alpha}}(\mathfrak{p}^+)} \smallsetminus \mathbf{C}_{\overline{s_\alpha}}(\mathfrak{p}^+) = \{\mathfrak{p}^+\}$, so $\overline{w} = \overline{\mathrm{id}}$.

Assume for a contradiction that $\overline{w_0} = \overline{s_\alpha}$ in $W_{\Delta \smallsetminus \{\alpha\}} \backslash W / W_{\Delta \smallsetminus \{\alpha\}}$. Then since $\overline{s_\alpha}$ is minimal and $\overline{w_0}$ is a maximum, for all $w \in W$ one has $\overline{w} = \overline{s_\alpha}$ or $\overline{\mathrm{id}}$, so $W_{\Delta \smallsetminus \{\alpha\}} \backslash W / W_{\Delta \smallsetminus \{\alpha\}} = \{\overline{\mathrm{id}}, \overline{w_0}\}$ has cardinal 2 and, by Observation 5.1.10, the irreducible Nagano pair $(\mathfrak{g}, \alpha)$ has rank 1, contradiction. ☐

To end this section, note that Fact 5.1.3 directly gives the following fundamental fact:

Fact 6.3.10. *The set* $\mathrm{Ad}(L^0) \cdot v^- = \{\mathrm{Ad}(\ell) \cdot v^- \mid \ell \in L^0\}$ *generates* $\mathfrak{u}^-$ *as a vector space.*

Lemma 6.3.3.(2) and Fact 6.3.10 imply that for all $x \in \mathscr{F}(\mathfrak{g}, \alpha)$, there exist photons through x *in general position*, i.e. if $\mathbb{A}$ is an affine chart containing x, considering $\mathbb{A}$ as a vector space with basepoint x, then there exist $N := \dim(\mathscr{F}(\mathfrak{g}, \alpha))$ photons $\Lambda_1, \dots, \Lambda_N$ through x (whose intersections with $\mathbb{A}$ are vector lines of $\mathbb{A}$ by Lemma 6.3.7), such that the vector lines $\Lambda_1 \cap \mathbb{A}, \dots, \Lambda_N \cap \mathbb{A}$ generate $\mathbb{A}$ as a vector space.

6.3.2 Intersection polynomials

In this section, we define the *intersection polynomials*, which algebraically describe the intersection between the standard photon Λ_{std} and the nontransverse set Z_ξ of a point $\xi \in \mathscr{F}(\mathfrak{g}, \alpha)^-$ such that $\Lambda_{\mathsf{std}} \not\subset \mathrm{Z}_\xi$.

We fix an irreducible Nagano pair $(\mathfrak{g}, \alpha)$ of real type and an $\{\alpha\}$-proximal triple (G, ρ, V) of $\mathfrak{g}$ with highest weight $\chi = N\omega_r$ for some $N \in \mathbb{N}_{>0}$. We let $\iota_\rho : \mathscr{F}(\mathfrak{g}, \alpha) \hookrightarrow \mathbb{P}(V)$ and $\iota_\rho^- : \mathscr{F}(\mathfrak{g}, \alpha)^- \hookrightarrow \mathbb{P}(V^*)$ be the two embeddings associated with ρ by Fact 2.3.4, and we fix a vector $\mathsf{v}_0 \in V^\chi \smallsetminus \{0\}$. By Lemma 5.2.1 and Equation (6.2.1), we have, for $t \in \mathbb{R}$,

$$\iota_\rho \circ \zeta([1:t]) = [e^{t\rho_*(v^-)} \cdot \mathsf{v}_0] = \big[\mathsf{v}_0 + t\rho_*(v^-) \cdot \mathsf{v}_0 + \cdots + \frac{t^N}{N!}\rho_*(v^-)^N \cdot \mathsf{v}_0\big] \qquad (6.3.5)$$

Let us define the dense open subset

$$\mathscr{X} := \big\{\xi \in \mathscr{F}(\mathfrak{g}, \alpha)^- \mid \Lambda_{\mathsf{std}} \not\subset \mathrm{Z}_\xi\big\} = \bigcup_{x \in \Lambda_{\mathsf{std}}} \big(\mathscr{F}(\mathfrak{g}, \alpha)^- \smallsetminus \mathrm{Z}_x\big)$$

of $\mathscr{F}(\mathfrak{g}, \alpha)^-$. Given $\xi \in \mathscr{X}$, we choose any lift $f \in V^* \smallsetminus \{0\}$ of $\iota_\rho^-(\xi) \in \mathbb{P}(V^*)$. Since $\Lambda_{\mathsf{std}} \not\subset \mathrm{Z}_\xi$, by Fact 2.3.4, the polynomial defined by

$$f\big(e^{t\rho_*(v^-)} \cdot \mathsf{v}_0\big) = f(\mathsf{v}_0) + tf(\rho_*(v^-) \cdot \mathsf{v}_0) + \cdots + \frac{t^N}{N!}f(\rho_*(v^-)^N \cdot \mathsf{v}_0) \quad \forall t \in \mathbb{R}$$

is nonzero. Then there exists a maximal $0 \leq n(\xi) \leq N$ such that $f(\rho_*(v^-)^{n(\xi)} \cdot \mathsf{v}_0) \neq 0$, and $n(\xi)$ does not depend on the choice of the lift f of $\iota_\rho^-(\xi)$. Hence we may choose f

such that $\frac{1}{n(\xi)!} f(\rho_*(v^-)^{n(\xi)} \cdot \mathsf{v}_0) = 1$. This defines a polynomial Q^ρ_ξ with coefficients in $\mathbb{R}$, depending only on ξ and (V, ρ):

$$\mathsf{Q}^\rho_\xi(t) = f\big(e^{t\rho_*(v^-)} \cdot \mathsf{v}_0\big) = f(\mathsf{v}_0) + t f(\rho_*(v^-) \cdot \mathsf{v}_0) + \cdots + t^{n(\xi)} \quad \forall t \in \mathbb{R}. \tag{6.3.6}$$

Definition 6.3.11. Given a point $\xi \in \mathscr{X}$, the polynomial Q^ρ_ξ defined in Equation (6.3.6) is called the *intersection polynomial of ξ associated with the representation* (V, ρ).

Now let

$$\mathscr{A}_\rho := \big\{\xi \in \mathscr{F}(\mathfrak{g}, \alpha)^- \mid [\rho_*(v^-)^N \cdot \mathsf{v}_0] \notin \iota^-_\rho(\xi)\big\} \subset \mathscr{X} \tag{6.3.7}$$

be the set of all elements $\xi \in \mathscr{X}$ such that $n(\xi)$ is maximal, i.e. such that $n(\xi) = N$.

Lemma 6.3.12. *The set $\mathscr{A}_\rho$ is open and dense in $\mathscr{F}(\mathfrak{g}, \alpha)^-$.*

Proof. By irreducibility of ρ, for any open set $\mathscr{O} \subset \mathscr{F}(\mathfrak{g}, \alpha)^-$, there exist $\xi_1, ..., \xi_D \in \mathscr{O}$ such that $\iota^-_\rho(\xi_1) \oplus ... \oplus \iota^-_\rho(\xi_D) = V^*$ (see e.g. [Zim18a, Lem. 4.7]). If the set

$$\mathscr{A}_\rho = \{\xi \in \mathscr{F}(\mathfrak{g}, \alpha)^- \mid [\rho_*(v^-)^N \cdot \mathsf{v}_0] \in \iota^-_\rho(\xi)\}$$

had nonempty interior, then we would have $f(\rho_*(v^-)^N \cdot \mathsf{v}_0) = 0$ for all $f \in V^*$. This is absurd because $\rho_*(v^-)^N \cdot \mathsf{v}_0 \neq 0$ by Lemma 5.2.1, so $\mathscr{A}^c_\rho$ has empty interior and $\mathscr{A}_\rho$ is a dense open subset of $\mathscr{F}(\mathfrak{g}, \alpha)^-$. $\square$

Let $\xi \in \mathscr{X}$ and $t \in \mathbb{R}$. One has

$$\begin{aligned} \mathsf{Q}^\rho_\xi(t) = 0 \iff & f\big(e^{t\rho_*(v^-)} \cdot \mathsf{v}_0\big) \iff \iota_\rho(\exp(tv^-) \cdot \mathfrak{p}^+) \in \iota^-_\rho(\xi) \\ \iff & \exp(tv^-) \cdot \mathfrak{p}^+ \in \mathrm{Z}_\xi \iff \zeta([1:t]) = \exp(tv^-) \cdot \mathfrak{p}^+ \in \Lambda_{\mathsf{std}} \cap \mathrm{Z}_\xi, \end{aligned}$$

the last equivalence holding by Equation (6.2.1) and Fact 2.3.4. Hence the real roots of Q^ρ_ξ describe the intersection points of $\Lambda_{\mathsf{std}} \cap \mathbb{A}_{\mathsf{std}}$ with Z_ξ. By Lemma 6.3.7, the set $\Lambda_{\mathsf{std}} \cap \mathrm{Z}_\xi$ is a singleton, so the polynomial Q^ρ_ξ has at most one real root t, satisfying $\zeta([1:t]) \in \mathrm{Z}_\xi$. We will see in Section 6.4.5 that the complex roots of Q^ρ_ξ also describe the intersection of two sets, corresponding to complexifications of $\Lambda_{\mathsf{std}} \cap \mathbb{A}_{\mathsf{std}}$ and Z_ξ. This is why we call the polynomial Q^ρ_ξ the intersection polynomial of ξ.

6.3.3 Comparison of two crossratios

The aim of this section is to prove Lemma 6.3.13 below. We keep the notation of Section 6.3.2.

Let us fix once and for all some notation for the rest of this section. Let $(\mathfrak{g}, \alpha)$ be an irreducible Nagano pair of real type, and let (G, ρ, V) be a linear $\{\alpha\}$-proximal triple of $\mathfrak{g}$ with highest weight $\chi = N\omega_r$ for some $N \in \mathbb{N}_{>0}$. We let $\iota_\rho : \mathscr{F}(\mathfrak{g}, \alpha) \hookrightarrow \mathbb{P}(V)$ and $\iota^-_\rho : \mathscr{F}(\mathfrak{g}, \alpha)^- \hookrightarrow \mathbb{P}(V^*)$ be the two embeddings associated with ρ by Fact 2.3.4, and fix a vector $\mathsf{v}_0 \in V^\chi \smallsetminus \{0\}$.

Recall that, by Lemma 6.3.7, given a point $\xi \in \mathscr{X}$, the set $\mathrm{Z}_\xi \cap \Lambda_{\mathsf{std}}$ is a singleton. We can then define a projection $\mathrm{pr}_{\mathsf{std}} : \mathscr{X} \to \Lambda_{\mathsf{std}}$ by setting $\mathrm{pr}_{\mathsf{std}}(\xi) := a$ where $\{a\} = \mathrm{Z}_\xi \cap \Lambda_{\mathsf{std}}$ if $\xi \in \mathscr{X}$. Using Lemma 6.3.15 and Corollary 6.3.16, we now establish a comparison between two cross ratios, involving the projection $\mathrm{pr}_{\mathsf{std}}$:

Lemma 6.3.13. *Let $\xi_1, \xi_2 \in \mathscr{X}$, and for $i \in \{1,2\}$, let $b_i \in \mathbb{P}(\mathbb{R}^2)$ be such that $\mathrm{pr}_{\mathsf{std}}(\xi_i) = \zeta(b_i)$. Then for all $a_1, a_2 \in \mathbb{P}(\mathbb{R}^2)$, one has*

$$\log \left|[\xi_1 : \zeta(a_1) : \zeta(a_2) : \xi_2]_\rho\right| = N |\log(b_1 : a_1 : a_2 : b_2)|.$$

Remark 6.3.14.

1. Lemma 6.3.13 is established in general flag manifolds defined by roots of multiplicity 1 (not just Nagano spaces) in [BGL+24], in [Gal24] for causal flag manifolds, in [Zim15, Lem. 10.4] for the Grassmannians $\mathrm{Gr}_p(\mathbb{R}^n)$ when $N = 1$ and in [LZ17, Lem. 2.9] for the full flag manifold of $\mathrm{SL}(n, \mathbb{R})$.
2. Given two points $x, y \in \Lambda_{\mathsf{std}}$, Lemma 6.3.13 expresses that the cross ratios $[\xi_1, x, y, \xi_2]_\rho$, where $\xi_1, \xi_2 \in \mathscr{X}$, depends only on the projections of ξ_1 and ξ_2 to Λ_{std}.

The argument to prove Lemma 6.3.13 uses the intersection polynomials introduced in Section 6.3.2. We have seen that the real roots of an intersection polynomial Q_ξ^ρ for $\xi \in \mathscr{X}$ are geometrically described by the intersection points of $\iota_\rho(\Lambda_{\mathsf{std}})$ with $\iota_\rho^-(\xi)$, so that there is at most one, by Lemma 6.3.7. But our argument will rely on the fact that the intersection polynomials have only one *complex* root, so that they are split (see Corollary 6.3.16). The complex roots of Q_ξ^ρ will be geometrically described as the intersection points of the complexification of $\iota_\rho^-(\xi)$ and a set, denoted $\mathscr{P}_N$, that plays the role of the complexification of $\iota_\rho(\Lambda_{\mathsf{std}})$. We describe this intersection in Lemma 6.3.15 below and actually prove that it is still a singleton. To this end, we work in a complexification of the representation (V, ρ), and use the notation of Section 2.1.

Let $V^{\mathbb{C}} := V \otimes \mathbb{C}$ be the complexification of V. For any $\xi \in \mathscr{F}(\mathfrak{g}, \alpha)^-$ and any lift $f \in V^* \smallsetminus \{0\}$ of $\iota_\rho^-(\xi)$, the map f extends uniquely to a linear form $f^{\mathbb{C}} : V^{\mathbb{C}} \to \mathbb{C}$. We denote by $\iota_\rho^-(\xi)^{\mathbb{C}}$ the class $[f^{\mathbb{C}}]_c$ of $f^{\mathbb{C}}$ in $\mathbb{P}_c\left((V^{\mathbb{C}})^*\right)$. This definition does not depend on the choice of the lift f of $\iota_\rho^-(\xi)$ in $V^* \smallsetminus \{0\}$. As for the real case, we identify $\mathbb{P}_c\left((V^{\mathbb{C}})^*\right)$ with the set of projective hyperplanes of $\mathbb{P}_c(V^{\mathbb{C}})$.

Similarly, for any $g \in G$, the operator $\rho(g)$ uniquely extends to an automorphism of $V^{\mathbb{C}}$. We will still denote by $\rho(g)$ this extension.

The map $\tau : \mathrm{SL}(2, \mathbb{R}) \to G$ induces a group homomorphism $\mathrm{SL}(2, \mathbb{R}) \to G \to \mathrm{GL}(V)$ with kernel included in $\{\pm \mathrm{id}\}$, which extends uniquely to a homomorphism $\tau_{\mathbb{C}} : \mathrm{SL}(2, \mathbb{C}) \to \mathrm{GL}(V^{\mathbb{C}})$ (with kernel $\{\pm \mathrm{id}\}$), such that

$$\tau_{\mathbb{C}}\left(e^{\mathrm{E}}\right) = e^{\rho_*(v^+)}, \quad \tau_{\mathbb{C}}\left(e^{\mathrm{F}}\right) = e^{\rho_*(v^-)}, \quad \tau_{\mathbb{C}}\left(e^{\mathrm{H}}\right) = e^{\rho_*(h_r)}.$$

The stabilizer of v_0 in $\mathrm{SL}(2, \mathbb{C})$ is the standard Borel subgroup $P_1^{\mathbb{C}}$ of $\mathrm{SL}(2, \mathbb{C})$ (the proof of this fact is the same as the one of Lemma 6.2.1 — it is even easier since $P_1^{\mathbb{C}}$ is connected). Hence the map $\tau_{\mathbb{C}}$ induces a $\tau_{\mathbb{C}}$-equivariant embedding $\zeta^{\mathbb{C}} : \mathbb{P}_c(\mathbb{C}^2) \hookrightarrow \mathbb{P}_c(V^{\mathbb{C}})$. The image of $\zeta^{\mathbb{C}}$ is denoted by $\mathscr{P}_N$. Explicitly, the set $\mathscr{P}_N$ is the closure in $\mathbb{P}_c(V^{\mathbb{C}})$ of

$$\left\{\zeta^{\mathbb{C}}\left([1 : z]_c\right) = e^{z\rho_*(v^-)} \cdot [\mathsf{v}_0]_c \;\middle|\; z \in \mathbb{C}\right\}.$$

Lemma 6.3.15. *1. One has $\rho(U^+) \cdot \mathscr{P}_N = \mathscr{P}_N$.*

2. Let $\xi \in \mathscr{X}$. Then the set $\mathscr{P}_N \cap \iota_\rho^-(\xi)^{\mathbb{C}}$ is a singleton.

Proof. Let us first prove (1). As in the proof of Lemma 6.2.2, for all $Y \in \mathfrak{u}^+$, using Lemma 5.1.4, we can find $\lambda \in \mathbb{R}$ such that for all $z \in \mathbb{C} \smallsetminus \{-\lambda^{-1}\}$ (with $-\lambda^{-1} = \infty$ if $\lambda = 0$), one has

$$\rho\left(\exp(Y)\right) \cdot \zeta^{\mathbb{C}}\left([1:z]_c\right) = \zeta^{\mathbb{C}}\Big(\left[1 : \frac{z}{1+z\lambda}\right]_c\Big) \in \mathscr{P}_N.$$

Hence $\rho\left(\exp(Y)\right) \cdot \zeta^{\mathbb{C}}\left([1:z]_c\right) \in \mathscr{P}_N$ for all $z \in \mathbb{C} \smallsetminus \{-\lambda^{-1}\}$. Taking the closure, we get $\rho\left(\exp(Y)\right) \cdot \mathscr{P}_N \subset \mathscr{P}_N$. The converse inclusion also holds by the same argument applied to $-Y$ instead of Y. Therefore $\rho\left(\exp(Y)\right) \cdot \mathscr{P}_N = \mathscr{P}_N$.

Now let us prove (2).

Step 1. Let us first prove (2) for $\xi = \mathfrak{p}^- \in \mathscr{X}$. Let $f_0 \in V^* \smallsetminus \{0\}$ be any lift of $\iota_\rho^-(\mathfrak{p}^-)$ and let $x \in \iota_\rho^-(\mathfrak{p}^-)^{\mathbb{C}} \cap \mathscr{P}_N$.

There exists $(z_k) \in \mathbb{C}^{\mathbb{N}}$ be such that $\zeta^{\mathbb{C}}\left(e^{z_k \mathrm{F}} \cdot [1:0]_c\right) \to x$ as $k \to +\infty$. The choice of ξ gives

$$f_0^{\mathbb{C}}\big(e^{z\rho_*(v^-)} \cdot \mathsf{v}_0\big) = f_0(\mathsf{v}_0) + z f_0\left(\rho_*(v^-) \cdot \mathsf{v}_0\right) + \cdots + \frac{z^N}{N!} f_0\left(\rho_*(v^-)^N \cdot \mathsf{v}_0\right) = f_0(\mathsf{v}_0) \neq 0$$

for all $z \in \mathbb{C}$. Thus we must have $|z_k| \to +\infty$. Then $e^{z_k \mathrm{F}} \cdot [1:0]_c \to [0:1]_c$. Hence x has to be equal to $\zeta^{\mathbb{C}}\left([0:1]_c\right)$.

Step 2. Now let $\xi \in \mathscr{X}$ be any point, and let $g \in G$ be such that $\xi = g^{-1} \cdot \mathfrak{p}^-$. Since $\Lambda_{\mathsf{std}} \not\subset \mathrm{Z}_\xi$, there exists $t \in \mathbb{R}$ such that $g^{-1}\exp(tv^-) \cdot \mathfrak{p}^+ \in \mathbb{A}_{\mathsf{std}}$. Since $\exp(tv^-)$ preserves $\mathfrak{p}^-$, one has $g^{-1}\exp(tv^-) \cdot \mathfrak{p}^- = \xi$. Hence, up to replacing g with $\exp(tv^-)g$, we may assume that $g \cdot \mathfrak{p}^+ \in \mathbb{A}_{\mathsf{std}}$. Then, by Equation (2.2.7), we can write $g = h\exp(Y)$ with $Y \in \mathfrak{u}^+$ and $h \in \mathfrak{p}^-$. Since $\rho(h^{-1})$ preserves $\iota_\rho^-(\mathfrak{p}^-)$, its $\mathbb{C}$-extension preserves $\iota_\rho^-(\mathfrak{p}^-)^{\mathbb{C}}$. Thus $\rho(h^{-1}) \cdot \iota_\rho^-(\mathfrak{p}^-)^{\mathbb{C}} = \iota_\rho^-(\mathfrak{p}^-)^{\mathbb{C}}$. This gives

$$\begin{aligned}
\mathscr{P}_N \cap \iota_\rho^-(\xi)^{\mathbb{C}} &= \mathscr{P}_N \cap \Big(\rho\left(\exp(-Y)\right)\rho\left(h^{-1}\right) \cdot \iota_\rho^-(\mathfrak{p}^-)^{\mathbb{C}}\Big) \\
&= \rho\left(\exp(-Y)\right) \cdot \Big(\rho(\exp(Y))\mathscr{P}_N \cap \iota_\rho^-(\mathfrak{p}^-)^{\mathbb{C}}\Big) \\
&= \rho\left(\exp(-Y)\right) \cdot \Big(\mathscr{P}_N \cap \iota_\rho^-(\mathfrak{p}^-)^{\mathbb{C}}\Big),
\end{aligned}$$

the last equality holding by point (1). Then, by Step 2, the set $\mathscr{P}_N \cap \iota_\rho^-(\xi)^{\mathbb{C}}$ is a singleton. $\square$

Lemma 6.3.15.(2) above admits the following corollary:

Corollary 6.3.16. *Let $\xi \in \mathscr{X}$. Then the intersection polynomial Q_ξ^ρ of ξ has only one complex root. If moreover $\Lambda_{\mathsf{std}} \cap \mathrm{Z}_\xi \subset \mathbb{A}_{\mathsf{std}}$, then the unique complex root of Q_ξ^ρ is equal to the unique $t \in \mathbb{R}$ satisfying $\zeta([1:t]) \in \mathrm{Z}_\xi$, and Q_ξ^ρ is split over $\mathbb{R}$.*

Proof. With the notation of Section 6.3.2, let $f \in V^*$ be the unique lift of $\iota_\rho^-(\xi)$ such that $f(\rho_*(v^-)^{n(\xi)} \cdot \mathsf{v}_0) = n(\xi)!$. For all $z \in \mathbb{C}$, one has:

$$\mathsf{Q}_\xi^\rho(z) = f(\mathsf{v}_0) + z f(\rho_*(v^-) \cdot \mathsf{v}_0) + \cdots + z^{n(\xi)} = f^{\mathbb{C}}\big(e^{z\rho_*(v^-)} \cdot \mathsf{v}_0\big).$$

Hence one has:

$$\mathsf{Q}_\xi^\rho(z) = 0 \iff f^{\mathbb{C}}\big(e^{z\rho_*(v^-)} \cdot \mathsf{v}_0\big) = 0 \iff \zeta^{\mathbb{C}}\left([1:z]_c\right) \in \mathscr{P}_N \cap \iota_\rho^-(\xi)^{\mathbb{C}}. \tag{6.3.8}$$

By Lemma 6.3.15.(2), the intersection $\mathscr{P}_N \cap \iota_\rho^-(\xi)^{\mathbb{C}}$ is a singleton, so the injectivity of $\zeta^{\mathbb{C}}$ and the equivalence of Equation (6.3.8) above give that Q_ξ^ρ has only one complex root.

If moreover $\Lambda_{\mathsf{std}} \cap \mathrm{Z}_\xi \subset \mathbb{A}_{\mathsf{std}}$, then there exists $t \in \mathbb{R}$ such that $\zeta([1:t]) \in \mathrm{Z}_\xi$. Then $\mathsf{Q}_\xi^\rho(t) = 0$, so $t \in \mathbb{R}$ is the unique complex root of Q_ξ^ρ. □

Proof of Lemma 6.3.13. Let us set $x := \zeta(a_1)$, $y := \zeta(a_2)$, $p_1 := \zeta(b_1)$ and $p_2 := \zeta(b_2)$. We may assume that $x = \mathfrak{p}^+$, $a_1 = [1:0]$ and $a_2 = [1:1]$, and that there exist $t_1, t_2 \in \mathbb{R}$ such that $b_i = [1:t_i]$ for $i \in \{1,2\}$, and that $t_1 < 0 < 1 < t_2$, the case where there are equalities then following by a continuity argument. Then the four distinct points p_1, x, y, p_2 are aligned on $\Lambda_{\mathsf{std}} \cap \mathbb{A}_{\mathsf{std}}$ in this order. Note that $\iota_\rho(x) = [\mathsf{v}_0]$. Let $\mathsf{v}_1 := \mathsf{v}_0 + \rho_*(v^-)\cdot\mathsf{v}_0 + \cdots + (1/N!)\rho_*(v^-)^N\mathsf{v}_0$. Then, by Equation (6.3.5), one has $\iota_\rho(y) = [\mathsf{v}_1]$.

Recall the open set $\mathscr{A}_\rho$ of Equation (6.3.7). By Lemma 6.3.12, the set $\mathscr{A}_\rho$ is a dense open subset of $\mathscr{F}(\mathfrak{g},\alpha)^-$. Hence for $i \in \{1,2\}$, we can find a sequence $(\xi_{i,k}) \in \mathscr{A}_\rho^{\mathbb{N}}$ such that $\xi_{i,k} \to \xi_i$. For all $k \in \mathbb{N}$, let $p_{i,k} := \mathrm{pr}_{\mathsf{std}}(\xi_{i,k})$. Then, by continuity of $\mathrm{pr}_{\mathsf{std}}$, one has $p_{i,k} \to p_i \in \mathbb{A}_{\mathsf{std}}$, so up to extracting we may assume that $p_{i,k} \in \mathbb{A}_{\mathsf{std}}$ for all $k \in \mathbb{N}$.

Let $f_i \in V^* \smallsetminus \{0\}$ (resp. $f_{i,k} \in V^* \smallsetminus \{0\}$) be a lift of $\iota_\rho^-(\xi_i)$ (resp. $\iota_\rho^-(\xi_{i,k})$). For every $k \in \mathbb{N}$ we choose $f_{i,k}$ such that $f_{i,k}(\rho_*(v^-)^N \cdot \mathsf{v}_0) = N!$. For any $k \in \mathbb{N}$ and $i \in \{1,2\}$, the intersection polynomial

$$\mathsf{Q}_{i,k}(z) := \mathsf{Q}^\rho_{\xi_{i,k}}(z) = (f_{i,k})^{\mathbb{C}}(e^{z\rho_*(v^-)}\cdot\mathsf{v}_0) = f_{i,k}(\mathsf{v}_0) + zf_{i,k}(\rho_*(v^-)\cdot\mathsf{v}_0) + \cdots + z^N \tag{6.3.9}$$

is nonzero, so there exists $t \in \mathbb{R}$ such that $\mathsf{Q}_{i,k}(t) = f_{i,k}(e^{t\rho_* v^-}\cdot\mathsf{v}_0) \neq 0$. This implies in particular that $\Lambda_{\mathsf{std}} \not\subset \mathrm{Z}_{\xi_{i,k}}$, i.e. $\xi \in \mathscr{X}$. By Corollary 6.3.16, the polynomial $\mathsf{Q}_{i,k}$ is thus split. But we also know that $\Lambda_{\mathsf{std}} \cap \mathrm{Z}_{\xi_{i,k}} = \{p_{i,k}\}$ is contained in $\mathbb{A}_{\mathsf{std}}$. Hence by the "moreover" part of Corollary 6.3.16, and since $n(\xi_{i,k}) = N$, the polynomial $\mathsf{Q}_{i,k}$ can be written $\mathsf{Q}_{i,k}(z) = (z - t_{i,k})^N$, with $t_{i,k} \in \mathbb{R}$ satisfying $\zeta([1:t_{i,k}]) = p_{i,k}$. Since $(p_{i,k})$ converges to p_i, the sequence $(t_{i,k})$ converges to t_i. One then has:

$$\begin{aligned}
\log(t_1 : [1:0] : [1:1] : t_2) &= \log\left|\frac{t_1\cdot(t_2-1)}{t_2\cdot(t_1-1)}\right| \\
&= \lim_{k\to+\infty}\log\left|\frac{t_{1,k}\cdot(t_{2,k}-1)}{t_{2,k}\cdot(t_{1,k}-1)}\right| \\
&= \frac{1}{N}\lim_{k\to+\infty}\log\left|\frac{\mathsf{Q}_{1,k}(0)\mathsf{Q}_{2,k}(1)}{\mathsf{Q}_{2,k}(0)\mathsf{Q}_{1,k}(1)}\right| \\
&= \frac{1}{N}\lim_{k\to+\infty}\log\left|\frac{f_{1,k}(\mathsf{v}_0)f_{2,k}(\mathsf{v}_1)}{f_{2,k}(\mathsf{v}_0)f_{2,k}(\mathsf{v}_1)}\right| \\
&= \frac{1}{N}\log\left|\frac{f_1(\mathsf{v}_0)f_2(\mathsf{v}_1)}{f_2(\mathsf{v}_0)f_2(\mathsf{v}_1)}\right| = \frac{1}{N}\log\left|\,[\xi_1 : x : y : \xi_2]_\rho\,\right|. \quad \square
\end{aligned}$$

6.3.4 Examples

Let us describe the photons in the explicit examples of Sections 2.4.1 up to 2.4.4.

6.3.4.1 Photons in Grassmannians. Limbeek–Zimmer have introduced a notion of "rank-one lines" in Grassmannians.

Recall the representation $\rho_0 : \mathrm{PGL}(p+q,\mathbb{R}) \to \mathrm{PGL}(\bigwedge^p \mathbb{R}^{p+q})$ and the embedding $\iota_{\rho_0} : \mathrm{Gr}_p(\mathbb{R}^{p+q}) \to \mathbb{P}(\bigwedge^p \mathbb{R}^{p+q})$ defined in Equation (2.4.2.2) and Equation (2.4.5). More generally, recall the notations of Section 2.4.2.

A projective line $\Lambda \subset \mathbb{P}(\bigwedge^p \mathbb{R}^{p+q})$ is said to be *of rank one* if $\Lambda \subset \iota_{\rho_0}(\mathrm{Gr}_p(\mathbb{R}^{p+q}))$. We then identify Λ with its pre-image by ι_{ρ_0}. More intrinsically, a photon is uniquely defined by an element $y \in \mathrm{Gr}_{p+1}(\mathbb{R}^{p+q})$: it is the set of p-planes contained in y. This gives a natural one-to-one correspondance, for all $x \in \mathrm{Gr}_p(\mathbb{R}^{p+q})$:

$$\{\text{photons through } x\} \longleftrightarrow \mathbb{P}(\mathbb{R}^{p+q}/x).$$

There exists a characterization of the intersection of rank-one lines with affine charts (recall the map φ_p of Equation (2.4.7)):

Lemma 6.3.17. *[LZ19]*

1. *If Λ is a rank-one line satisfying $\Lambda \cap \mathbb{A}_{\mathsf{std}} \neq \emptyset$, then there exist $X, S \in \mathrm{Mat}_{q,p}(\mathbb{R})$ with $\mathrm{rk}(S) = 1$, such that*
$$\Lambda \cap \mathbb{A}_{\mathsf{std}} = \varphi_p(\{X + tS \mid t \in \mathbb{R}\}).$$
2. *Conversely, if $X, S \in \mathbb{A}_{\mathsf{std}}$ with $\mathrm{rk}(S) = 1$, then the closure of $\iota_{\rho_0} \circ \varphi_p(\{X + tS \mid t \in \mathbb{R}\})$ in $\mathbb{P}(\bigwedge^p \mathbb{R}^{p+q})$ is a rank-one line.*

Lemma 6.3.18. *Rank-one lines are exactly the images in $\mathbb{P}(\bigwedge^p \mathbb{R}^{p+q})$ of the photons we have defined in Section 6.3, in the case where $(\mathfrak{g}, \alpha) = (\mathfrak{sl}(p+q,\mathbb{R}), \alpha_p)$.*

Proof. This Lemma will be a consequence of the more general Proposition 7.1.4, see Remark 7.1.5. It would also follow from an explicit computation of $\mathfrak{g}_{-\alpha_p}$ and Lemma 6.3.17. □

Remark 6.3.19. In particular, if $(\mathfrak{g}, \alpha)$ is the pair $(\mathfrak{sl}(n,\mathbb{R}), \alpha_1)$, then the photons of $\mathscr{F}(\mathfrak{g}, \alpha) = \mathbb{P}(\mathbb{R}^n)$ are exactly the projective lines of $\mathbb{P}(\mathbb{R}^n)$.

6.3.4.2 Photons in Shilov boundaries. We use the notation from Section 2.4.4. Let G be an HTT Lie group of rank $r \geq 2$. Let $\mathbb{A}$ be an affine chart of $\mathbf{Sb}(\mathfrak{g})$. By the definition of c^0, any photon Λ through $x \in \mathbb{A}$ satisfies $\Lambda \cap \mathbb{A} \subset \mathbf{C}_{\mathbb{A}}(x)$.

Now, let us look at what photons look like with specific examples. We take the notation of Sections 2.4.4.4.1 and 3.5.2.2. One has $v^- = v_r = \begin{pmatrix} 0_r & 0_r \\ E_{r,r} & 0_r \end{pmatrix}$. By Equation (2.4.18), one has

$$\mathrm{Ad}(L) \cdot v^- = \left\{ \begin{pmatrix} 0_r & 0_r \\ X & 0_r \end{pmatrix} \mid X \in \mathrm{Mat}_r(\mathbb{K}),\ {}^t\overline{X} = X,\ \mathrm{rk}(X) = 1 \right\}.$$

By Lemma 6.3.3.(2), a photon intersecting $\mathbb{A}_{\mathsf{std}}$ is thus a subset of $\mathrm{Lag}_r(\mathbb{K}^{2r})$ of the form

$$\overline{\left\{ \begin{bmatrix} I_r \\ tX \end{bmatrix} \mid t \in \mathbb{R} \right\}},$$

where $X \in \mathrm{Mat}_r(\mathbb{K})$ is a rank-one matrix such that ${}^t\overline{X} = X$. Note that, for the natural embedding $\mathrm{Lag}_r(\mathbb{R}^{2r}) \subset \mathrm{Gr}_r(\mathbb{R}^{2r})$, by Lemmas 6.3.17 and 6.3.18, the photons of $\mathrm{Lag}_r(\mathbb{R}^{2r})$ are exactly the photons of $\mathrm{Gr}_r(\mathbb{R}^{2r})$ that are contained in $\mathrm{Lag}_r(\mathbb{R}^{2r})$.

6.3.4.3 Photons in the Einstein universes. Let $p, q \geq 1$. In the case where $(\mathfrak{g}, \alpha) = (\mathfrak{so}(p+1, q+1), \alpha_1)$, recall from Section 2.4.3 that we have $\mathscr{F}(\mathfrak{g}, \alpha) = \mathrm{Ein}^{p,q}$. Photons as defined in this Chapter coincide with the classical photons already mentioned in Section 2.4.3.1. To see this, a direct computation gives a parametrization of photons in affine charts. Another way to see it is to use Proposition 7.1.4 and the fact that the photons defined in Section 2.4.3.1 are exactly the projective lines of $\mathbb{P}(\mathbb{R}^{p+q+2})$ that are contained in $\mathrm{Ein}^{p,q}$, for the natural embedding ι_{ρ_1} defined in Equation (2.4.12); now one just needs to apply Proposition 7.1.4.

As mentioned in Section 2.4.3.1.2, for all $x \in \mathrm{Ein}^{p,q}$ we have

$$\mathrm{Z}_x = \bigcup_{\Lambda \text{ photon through } x} \Lambda.$$

This property is specific to the Einstein case. Algebraically, it follows from Lemma 6.3.9 and the fact that $|W_{\Delta \smallsetminus \{\alpha_1\}} \backslash W / W_{\Delta \smallsetminus \{\alpha_1\}}| = 3$, see Section 2.4.3.1.

6.4 The Kobayashi metric

In this section, we define the *Kobayashi metric* on a proper domain Ω contained in an irreducible Nagano space $\mathscr{F}(\mathfrak{g}, \alpha)$ of real type. Constructions of Kobayashi metrics are classical and were initiated by S. Kobayashi [Kob67, Sho84]. The properties of the Kobayashi metric (in particular, its properness, see Corollary 6.4.12) in the dually convex case will allow us to relate the geometry of the boundary of a proper almost-homogeneous domain to the dynamics of its automorphism group in Section 7.2.2.

Notation 6.4.1. We fix in this section an irreducible Nagano pair $(\mathfrak{g}, \alpha)$ of real type, an arbitrary group $G \in \mathscr{G}_{\{\alpha\}}(\mathfrak{g})$, and adopt Notation 5.1.2.

6.4.1 Chains

Let $\Omega \subset \mathscr{F}(\mathfrak{g}, \alpha)$ be a domain, not necessarily proper. We say that two points $x_1, x_2 \in \Omega$ are *photon-related* (in Ω), denoted by $x_1 \leftrightsquigarrow x_2$, if they belong to the same photon Λ and are in the same connected component of $\Lambda \cap \Omega$. Note that this relation is not an equivalence relation, as it is not transitive (in general).

Now let $x, y \in \Omega$ be any two points. An *N-chain from x to y* ($N \in \mathbb{N}$) is a sequence of $(N+1)$ elements $(x_0 = x, \dots, x_N = y)$ of Ω such that $x_i \leftrightsquigarrow x_{i+1}$ for all $0 \leq i \leq N-1$. We denote by $\mathscr{C}_{x,y}(\Omega)$ (resp. $\mathscr{C}^N_{x,y}(\Omega)$) the set of all chains (resp. N-chains) from x to y in Ω.

By Fact 6.3.10, there exist $\ell_1, \dots, \ell_n \in L$ (where $n = \dim(\mathscr{F}(\mathfrak{g}, \alpha))$) such that $\mathsf{e} := (\mathrm{Ad}(\ell_1) \cdot v^-, \dots, \mathrm{Ad}(\ell_n) \cdot v^-)$ is a basis of $\mathfrak{u}^-$. Let $\langle \cdot, \cdot \rangle$ be a scalar product on $\mathfrak{u}^-$ for which e is an orthonormal basis. Let $\mathbb{B}$ be the associate euclidean ball.

Let $Y \in \mathbb{B}$ and $y := \exp(Y)$. There exist $n \in \mathbb{N}_{>0}$ and $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ such that $Y = \sum_{k=1}^n \lambda_k \mathrm{Ad}(\ell_k) \cdot v^-$. For $1 \leq i \leq n$, we set

$$x_i := \varphi_{\mathsf{std}}\Big(\sum_{k=1}^{i} \lambda_k \mathrm{Ad}(\ell_k) \cdot v^-\Big).$$

Then $(x_0 = \mathfrak{p}^+, x_1, \dots, x_n = y) \in \mathscr{C}^n_{\mathfrak{p}^+,y}(\varphi_{\mathsf{std}}(\mathbb{B}))$. Now by concatenation, for all $y, z \in \varphi_{\mathsf{std}}(\mathbb{B})$, one has $\mathscr{C}^{\mathsf{n}(G)}_{y,z}(\varphi_{\mathsf{std}}(\mathbb{B})) \neq \emptyset$, with $\mathsf{n}(G) := 2n$.

Note that $\{g \cdot \varphi_{\mathsf{std}}(\mathbb{B}) \mid g \in G\}$ forms a basis of neighborhoods of $\mathscr{F}(\mathfrak{g}, \Theta)$. This implies:

Observation 6.4.2. There exists an integer $\mathsf{n}(G)$ only depending on G and a basis $\mathscr{V}$ of the topology of $\mathscr{F}(\mathfrak{g}, \Theta)$ such that for all $x \in \mathscr{F}(\mathfrak{g}, \alpha)$ and for any $V \in \mathscr{V}$ and $x, y \in \mathscr{V}$, one has $\mathscr{C}^{\mathsf{n}(G)}_{x,y}(\mathscr{V}) \neq \emptyset$.

Remark 6.4.3. It would actually follow from the definitions of Section 5.1 that for all $x, y \in \mathbb{X}(\mathfrak{g}, \alpha)$, there exists an s-chain from x to y in $\mathbb{X}(\mathfrak{g}, \alpha)$, where s is the rank of $(\mathfrak{g}, \alpha)$ in the sense of Definition 5.1.5. Thus we may replace our family of neighborhoods $\{g \cdot \varphi_{\mathsf{std}}(\mathbb{B}) \mid g \in G\}$ by $\{g \cdot \varphi_{\mathsf{std}}(\mathbb{X}(\mathfrak{g}, \alpha)) \mid g \in G\}$ and get $\mathsf{n}(G) = s$.

6.4.2 The pseudo-metric

In this section, we define the Kobayashi pseudo-metric on domains of $\mathscr{F}(\mathfrak{g}, \alpha)$.

Let $\Omega \subset \mathscr{F}(\mathfrak{g}, \alpha)$ be a (not necessarily proper) domain. We assume moreover that there exists no photon $\Lambda \subset \Omega$. Let $x, y \in \Omega$ be two photon-related points. Let $\Lambda_{x,y} \in \mathscr{L}$ be a photon containing x and y (which is unique if $x \neq y$, see Lemma 6.3.5), and $g \in G$ be such that $\Lambda_{x,y} = g \cdot \Lambda_{\mathsf{std}}$. We denote by $I_{x,y}$ the connected component of $\Omega \cap \Lambda_{x,y}$ containing x and y. Recall the parametrization $\zeta_g : \mathbb{P}(\mathbb{R}^2) \to \Lambda_{x,y}$ of Equation (6.3.2). Let

$$\mathsf{k}_\Omega(x, y) := H_{\zeta_g^{-1}(I_{x,y})}\left(\zeta_g^{-1}(x), \zeta_g^{-1}(y)\right).$$

Recall that we denote by H_I the Hilbert pseudo-metric of an interval I of $\mathbb{P}(\mathbb{R}^2)$ (see Section 2.1). Due to the $\mathrm{SL}(2, \mathbb{R})$-invariance of the cross ratio on $\mathbb{P}(\mathbb{R}^2)$ and Lemma 6.3.6, the quantity $\mathsf{k}_\Omega(x, y)$ does not depend on the choice of $g \in G$ such that $\Lambda_{x,y} = g \cdot \Lambda_{\mathsf{std}}$.

Definition 6.4.4. Given a domain $\Omega \subset \mathscr{F}(\mathfrak{g}, \alpha)$, we define $K_\Omega : \Omega \times \Omega \to \mathbb{R}_+ \cup \{+\infty\}$ by:

$$\forall x, y \in \Omega, \quad K_\Omega(x, y) = \inf \left\{ \sum_{i=0}^{N} \mathsf{k}_\Omega(x_i, x_{i+1}) \mid N \in \mathbb{N}^*, \quad (x_0, \dots, x_N) \in \mathscr{C}_{x,y}(\Omega) \right\}.$$

For x, y sufficiently close to each other, there exists $g \in G$ such that $x, y \in g \cdot \mathbb{B} \subset \Omega$; see Section 6.4.1. In that case $\mathscr{C}_{x,y}(\Omega)$ (and even $\mathscr{C}^{\mathsf{n}(G)}_{x,y}(\Omega)$) is nonempty by Observation 6.4.2. Hence the relation "x and y can be joined by a chain" is locally trivial. Since it is an equivalence relation and Ω is connected, it is the trivial relation. Hence the set $\mathscr{C}_{x,y}(\Omega)$ is never empty for two points $x, y \in \Omega$. Since $K_\Omega(a, b)$ is always finite whenever a, b are photon-related, we know by Observation 6.4.2 that the quantity $K_\Omega(x, y)$ is thus always finite as well. Thus K_Ω is actually a map $\Omega \times \Omega \to \mathbb{R}_+$. We moreover have:

Proposition 6.4.5. *Let Ω_1 and Ω_2 be two domains of $\mathscr{F}(\mathfrak{g}, \alpha)$, and $G \in \mathscr{G}_{\{\alpha\}}(\mathfrak{g})$. Then:*

1. *If $\Omega_1 \subset \Omega_2$, then for any $x, y \in \Omega_1$ one has $K_{\Omega_2}(x, y) \leq K_{\Omega_1}(x, y)$.*
2. *For any $g \in G$, for any $x, y \in \Omega_1$, one has $K_{g \cdot \Omega_1}(g \cdot x, g \cdot y) = K_{\Omega_1}(x, y)$. In particular, the metric K_{Ω_1} is $\mathsf{Aut}_G(\Omega_1)$-invariant.*

Proof. This is a consequence of the definition of K_Ω and of the fact that an element g of G induces a natural bijection between $\mathscr{C}^N_{x,y}(\Omega)$ and$\mathscr{C}^N_{g\cdot x, g\cdot y}(g\cdot\Omega)$, for all $N \in \mathbb{N}$. $\square$

Note that K_{Ω_2} and K_{Ω_1} do not need to be metrics in Proposition 6.4.5.

As one can concatenate and reverse the orientation of a chain, the map K_Ω is symmetric and satisfies the triangle inequality. It is thus a pseudo-metric, and we call it the *Kobayashi pseudo-metric*. In the next section, we investigate when K_Ω is a metric.

Remark 6.4.6. In the case where $\mathscr{F}(\mathfrak{g}, \alpha) = \mathrm{Ein}^{n-1,1}$, our Kobayashi pseudo-metric coincides with the *Markowitz pseudo-metric* introduced in [Mar81]. In the case where $\mathscr{F}(\mathfrak{g}, \alpha) = \mathrm{Gr}_p(\mathbb{R}^{p+q})$, it is the Kobayashi pseudo-metric defined in [LZ19]. In the case where $\mathscr{F}(\mathfrak{g}, \alpha) = \mathbb{P}(\mathbb{R}^n)$, then it is the classical Kobayashi pseudo-metric on Ω [Sho84], and if $\Omega \subset \mathbb{P}(\mathbb{R}^n)$ is properly convex, then it is the Hilbert metric.

6.4.3 Kobayashi-hyperbolicity

In this section, we define the *Kobayashi-hyperbolicity* of a domain and prove that any proper domain is Kobayashi-hyperbolic.

Definition 6.4.7. We say that a domain $\Omega \subset \mathscr{F}(\mathfrak{g}, \alpha)$ is *Kobayashi-hyperbolic* if K_Ω is a metric, that is, if K_Ω separates points. The map K_Ω defined in Definition 6.4.4 is then called the *Kobayashi metric* of Ω.

Proposition 6.4.8. *Proper domains of $\mathscr{F}(\mathfrak{g}, \alpha)$ are Kobayashi-hyperbolic. Moreover, for any proper domain $\Omega \subset \mathscr{F}(\mathfrak{g}, \alpha)$, the metric K_Ω generates the standard topology.*

Proof. Let us first show that any properly convex domain C of $\mathbb{A}_{\mathsf{std}}$ is Kobayashi-hyperbolic, where $\mathbb{A}_{\mathsf{std}}$ is endowed with its standard affine structure (given in Equation (2.2.7)). Since C is a properly convex domain of the affine space $\mathbb{A}_{\mathsf{std}}$, it has a classical Hilbert metric H_C (see Remark 2.1.1). By the definition of H_C and the fact that the trace of a photon in $\mathbb{A}_{\mathsf{std}}$ is either empty or an affine line of $\mathbb{A}_{\mathsf{std}}$ (Lemma 6.3.7), if $a, b \in C$ are photon-related then one has

$$\mathsf{H}_C(a,b) = \mathsf{k}_C(a,b). \tag{6.4.1}$$

Now let $x, y \in C$ and $\gamma = (x_0, \dots, x_N) \in \mathscr{C}_{x,y}(\Omega)$ be a chain from x to y. Since H_C satisfies the triangle inequality, one has:

$$\sum_{i=0}^{N-1} \mathsf{k}_C(x_i, x_{i+1}) = \sum_{i=0}^{N-1} \mathsf{H}_C(x_i, x_{i+1}) \geq \mathsf{H}_C(x,y).$$

This is true for all $\gamma \in \mathscr{C}_{x,y}(\Omega)$, so by taking the infimum we get

$$K_C(x,y) \geq \mathsf{H}_C(x,y) > 0. \tag{6.4.2}$$

Therefore K_C separates the points and C is Kobayashi-hyperbolic.

Now let Ω be any domain. We may assume that Ω is proper in $\mathbb{A}_{\mathsf{std}}$ (Remark 3.1.2). Let C be any domain, properly convex in $\mathbb{A}_{\mathsf{std}}$, containing Ω (for instance the convex hull of Ω in $\mathbb{A}_{\mathsf{std}}$). Then, by Proposition 6.4.5:

$$K_C(x,y) \leq K_\Omega(x,y) \quad \forall x,y \in \Omega. \tag{6.4.3}$$

Since K_C separates the points, so does K_Ω.

Let $\mathscr{T}_{\mathsf{std}}$ be the standard topology on Ω, and let $\mathscr{T}$ be the one induced by K_Ω. We will prove that $\mathscr{T}_{\mathsf{std}} = \mathscr{T}$. By Equations (6.4.2) and (6.4.3) and the fact that the Hilbert metric generates the standard topology on properly convex domains, one has $\mathscr{T}_{\mathsf{std}} \subset \mathscr{T}$. To prove the reverse inclusion, one need to show that K_Ω is continuous with respect to the standard topology. By the inequality

$$|K_\Omega(x_0,y_0) - K_\Omega(x,y)| \leq K_\Omega(x_0,x) + K_\Omega(y_0,y) \quad \forall x_0,y_0,x,y \in \Omega,$$

one only needs to show that for any $x_0 \in \Omega$ the map $x \mapsto K_\Omega(x_0,x)$ is continuous at x_0. For this we see $\mathbb{A}_{\mathsf{std}}$ we see $\mathbb{A}_{\mathsf{std}}$ as a Euclidean space and fix any euclidean norm $||.||$ on $\mathbb{A}_{\mathsf{std}}$. Up to dilating at x_0, we may assume that the Euclidean ball B of center x_0 and of radius 1 is contained in Ω. For any $0 < \delta < 1$, let B_δ be the Euclidean ball of center x_0 and of radius δ. Let $N := \mathsf{n}(G)$ given by Observation 6.4.2. For any $0 < \delta < 1$, tere exists an N-chain $(x_0, x_1, \ldots, \ x_N = x)$ contained in B_δ. Then, by Proposition 6.4.5 and Equation (6.4.1), one has

$$\begin{aligned}
K_\Omega(x,x_0) \leq K_B(x_0,x) &\leq \sum_{k=0}^{N-1} K_B(x_i,x_{i+1}) = \sum_{k=0}^{N-1} \mathsf{H}_B(x_i,x_{i+1}) \\
&\leq \sum_{k=0}^{N-1} \mathsf{H}_B(x_0,x_i) + \mathsf{H}_B(x_0,x_{i+1}) \\
&= \sum_{k=1}^{N-1} \log\frac{1+||x_i-x_0||}{1-||x_i-x_0||} + \log\frac{1+||x_{i+1}-x_0||}{1-||x_{i+1}-x_0||} \\
&\leq \sum_{k=1}^{N-1} 2\log\frac{1+\delta}{1-\delta} \underset{\delta\to 0}{\longrightarrow} 0
\end{aligned}$$

This proves that $K_\Omega(x_0,x) \to 0$ as $x \to x_0$. □

Remark 6.4.9.
1. The proof of Proposition 6.4.8 gives that for any two sequences $(x_k),(y_k) \in \Omega^{\mathbb{N}}$ such that $x_k \to a \in \partial\Omega$ and $y_k \to b \in \overline{\Omega}$, if $K_\Omega(x_k,y_k) \to 0$ then one has $a = b$.
2. Proposition 6.4.8 is a generalization, to any irreducible Nagano space of real type, of the results we obtained in [Gal24, CG24] for causal flag manifolds and Einstein universes respectively (the latter with A. Chalumeau).

6.4.4 Kobayashi length

In this section, we recall some definitions and fix some notation. Let Ω be a proper domain. For a continuous path $\gamma : [0,1] \to \Omega$, we define the *Kobayashi length* or K_Ω*-length* of γ in the the usual way, as

$$\mathsf{len}_\Omega(\gamma) = \sup \sum_{i=0}^{N} K_\Omega(\gamma(t_i), \gamma(t_{i+1})),$$

where the supremum is taken over all finite subdivisions of $[0, 1]$.

Let $x, y \in \Omega$ be two distinct photon-related points, and let $\Lambda_{x,y}$ be the unique photon containing x and y. We denote by $[x, y]$ the closure of the only connected component of $\Lambda_{x,y} \smallsetminus \{x, y\}$ that is contained in Ω (both components cannot be contained in Ω by properness of Ω). By 6.2.1, it can be parametrized by

$$[t_1, t_2] \longrightarrow [x, y];\ t \longmapsto \zeta_g([1 : t]),$$

where $g \in G$ is such that $\Lambda_{x,y} = g \cdot \Lambda_{\mathsf{std}}$ and $x = \zeta_g([1 : t_1])$ and $y = \zeta_g([1 : t_2])$. This parametrization depends on the choice of $g \in G$.

Now, let $x, y \in \Omega$ be any two points. Any element of $u = (x_0, \dots, x_N) \in \mathscr{C}_{x,y}(\Omega)$ gives rise to a continuous path γ from x to y, defined as the concatenation of all segments $[x_0, x_1], \dots, [x_{N-1}, x_N]$ in this order, endowed with a parametrization as described above. This path is uniquely defined by u up to reparametrization. The K_Ω-length $\mathsf{len}_\Omega(\gamma)$ of γ does not depend on the choice of parametrization of the $[x_i, x_{i+1}]$ for $0 \leq i \leq N$. This defines a unique K_Ω-length for the chain $(x_0, \dots, x_N)$.

In the rest of the memoir, we will identify a chain with the unique (up to parametrization) path it defines by the process described above. In particular, this will allow us to consider the K_Ω-length of a chain.

6.4.5 Comparison with the Caratheodory metrics

The goal of this section is to prove Proposition 6.4.10 below, where we compare the Kobayashi metric with the Caratheodory metrics defined in Section 3.1.2. Together with Corollary 6.4.12 in the next section, it will imply Proposition 1.4.7.

Proposition 6.4.10. *Let $(\mathfrak{g}, \alpha)$ be an irreducible Nagano pair of real type, $G \in \mathscr{G}_{\{\alpha\}}(\mathfrak{g})$ and let (G, ρ, V) a linear $\{\alpha\}$-proximal triple of $\mathfrak{g}$ with highest weight $\chi := N\omega_\alpha$ for some $N \in \mathbb{N}_{>0}$. Let $\Omega \subset \mathscr{F}(\mathfrak{g}, \Theta)$ be a proper dually convex domain, and let C^ρ_Ω be the Caratheodory metric on Ω induced by (V, ρ). Then for any two photon-related points $x, y \in \Omega$, one has*

$$\mathsf{k}_\Omega(x, y) = \frac{1}{N} C^\rho_\Omega(x, y). \tag{6.4.4}$$

In particular,

1. *one has $K_\Omega \geq N^{-1} C^\rho_\Omega$;*
2. *the metric K_Ω is a length metric, and given two photon-related points $x, y \in \Omega$, the K_Ω-length of the 1-chain (x, y) is equal to $\mathsf{k}_\Omega(x, y) = K_\Omega(x, y)$.*

Remark 6.4.11. We will see in Propositions 6.4.13 and 6.4.15 that the inequality of Proposition 6.4.10.(1) is an equality for the realizations of the noncompact duals of the Naganon spaces $\mathrm{Ein}^{p,q}$ and $\mathrm{Gr}_p(\mathbb{R}^{p+q})$, with $p, q \geq 1$. The proof of these propositions outline a method to compute the Kobayashi mertic on a realization of the noncompact dual of an irreducible Nagano space of real type, which we generalize to any Nagano space of real type in [Gal25]. We think that, in higher rank, the equality case characterizes realizations of the noncompact dual, see Conjecture 1.4.8.

Proof of Proposition 6.4.10. Up to translating Ω by an element of G, one may assume that $x, y \in \Lambda_{\mathsf{std}}$. Then there exist $a_1, a_2 \in \mathbb{P}(\mathbb{R}^2)$ such that $x = \zeta(a_1)$ and $y = \zeta(a_2)$.

Recall that we denote by $I_{x,y}$ the connected component of $\Lambda_{\mathsf{std}} \cap \Omega$ containing x and y. Let $p_1, p_2 \in \partial\Omega$ be the endpoints of $I_{x,y}$, such that p_1, x, y, p_2 are aligned on Λ_{std} in this order. Then there exist $b_1, b_2 \in \mathbb{P}(\mathbb{R}^2)$ such that b_1, a_1, a_2, b_2 are aligned in this order and $p_1 = \zeta(b_1)$, and $p_2 = \zeta(b_2)$.

By dual convexity, for $i \in \{1, 2\}$ there exists $\xi_i \in \Omega^*$ such that $p_i \in \mathrm{Z}_{\xi_i}$. Then, by Lemma 6.3.13, one has

$$\mathsf{k}_\Omega(x, y) = \log |(b_1 : a_1 : a_2 : b_2)| = \frac{1}{N} \log \big| [\xi_1 : x : y : \xi_2]_\rho \big|.$$

By the definition of C^ρ_Ω, this implies that $\mathsf{k}_\Omega(x, y) \leq N^{-1} C^\rho_\Omega(x, y)$.

For the converse inequality, let $\eta_1, \eta_2 \in \Omega^*$ be such that

$$C^\rho_\Omega(x, y) = \log |[\eta_1 : x : y : \eta_2]_\rho| .$$

For $i \in \{1, 2\}$, let $b'_i \in \mathbb{P}(\mathbb{R}^2)$ be such that $\zeta(b'_i) = \mathrm{pr}_{\mathsf{std}}(\eta_i)$. Then, again by Lemma 6.3.13:

$$\big| \log \big|(b'_1 : a_1 : a_2 : b'_2)\big| \big| = \frac{1}{N} \log |[\eta_1 : x : y : \eta_2]_\rho| .$$

Since $\eta_1, \eta_2 \in \Omega^*$, the two points $\zeta(b'_1), \zeta(b'_2)$ are not contained in $I_{x,y}$. Thus one has

$$\big| \log \big|(b'_1 : a_1 : a_2 : b'_2)\big| \big| \leq \log |(b_1 : a_1 : a_2 : b_2)| = \mathsf{k}_\Omega(x, y).$$

Hence one has $N^{-1} C^\rho_\Omega(x, y) \leq \mathsf{k}_\Omega(x, y)$. We have proven that $N^{-1} C^\rho_\Omega(x, y) = \mathsf{k}_\Omega(x, y)$.

Now let us prove that $K_\Omega \geq N^{-1} C^\rho_\Omega$. Let $x, y \in \Omega$ be any two points, and let $(x_0, \ldots, x_M) \in \mathscr{C}_{x,y}(\Omega)$. Then one has

$$\begin{aligned}
\sum_i \mathsf{k}_\Omega(x_i, x_{i+1}) = \frac{1}{N} \sum_i C^\rho_\Omega(x_i, x_{i+1}) &\geq \sum_i \frac{1}{N} \sup_{\xi^i_1, \xi^i_2 \in \Omega^*} \log[\xi^i_1 : x_i : x_{i+1} : \xi^i_2]_\rho \\
&\geq \frac{1}{N} \sup_{\xi_1, \xi_2 \in \Omega^*} \sum_i \log[\xi_1 : x_i : x_{i+1} : \xi_2]_\rho \\
&\geq \frac{1}{N} \sup_{\xi_1, \xi_2 \in \Omega^*} \log[\xi_1 : x : y : \xi_2]_\rho \\
&= \frac{1}{N} C^\rho_\Omega(x, y).
\end{aligned} \tag{6.4.5}$$

Since this is true for all $(x_0, \ldots, x_M) \in \mathscr{C}_{x,y}(\Omega)$, by taking the infimum we get the inequality $K_\Omega(x, y) \geq N^{-1} C^\rho_\Omega(x, y)$.

Now let us show that K_Ω is a length metric. In Equation (6.4.5), take x and y

to be two photon-related points. The fact that $C^{\rho}_{\Omega}(x,y) = N\mathsf{k}_{\Omega}(x,y)$ and Equation (6.4.5) imply that the segment $[x,y]$ has K_{Ω}-length $\mathsf{k}_{\Omega}(x,y)$. Hence the K_{Ω}-length of $\gamma = (x_0, \ldots, x_M) \in \mathscr{C}_{x,y}(\Omega)$ is

$$\mathsf{len}_{\Omega}(\gamma) = \sum_i K_{\Omega}(x_i, x_{i+1}). \tag{6.4.6}$$

Then one has $K_{\Omega}(x,y) = \inf\{\mathsf{len}_{\Omega}(\gamma) \mid \gamma \in \mathscr{C}_{x,y}(\Omega)\}$.

Now let $\mathscr{C}'_{x,y}(\Omega)$ the set of all rectifiable curves joining x and y in Ω. By the definition of the K_{Ω}-length of a curve, one has $K_{\Omega}(x,y) \leq \inf\left\{\mathsf{len}_{\Omega}(\gamma) \mid \gamma \in \mathscr{C}'_{x,y}(\Omega)\right\}$. Since chains are rectifiable (for the identification with continuous paths, see Section 6.4.4), this last inequality is an equality. Hence K_{Ω} is a length metric. □

6.4.6 Properness

In this section, we state a corollary of Proposition 6.4.10, which is the properness of the Kobayashi metric on a proper dually convex domain of $\mathscr{F}(\mathfrak{g}, \alpha)$.

Let us fix (G, ρ, V) a linear $\{\alpha\}$-proximal triple of $\mathfrak{g}$ with highest weight $\chi := N\omega_{\alpha}$. Let Ω be a proper domain of $\mathscr{F}(\mathfrak{g}, \alpha)$, and let C^{ρ}_{Ω} be the Caratheodory metric on Ω induced by (V, ρ). In [Zim18a, Thm 9.1], A. Zimmer proves that the following three assertions are equivalent:

1. The domain Ω is dually convex.
2. The metric C^{ρ}_{Ω} is a proper metric.
3. The metric C^{ρ}_{Ω} is a complete metric.

The equivalence (1) $\Leftrightarrow$ (3) is stated in [Zim18a, Thm 9.1], and the equivalence (1) $\Leftrightarrow$ (2) is a consequence of its proof.

If Ω is dually convex, then C^{ρ}_{Ω} is proper, and on the other hand, by Proposition 6.4.10, one has $K_{\Omega} \geq N^{-1} C^{\rho}_{\Omega}$. Thus the Kobayashi metric K_{Ω} is also proper:

Corollary 6.4.12. *Let $(\mathfrak{g}, \alpha)$ be an irreducible Nagano pair of real type. If $\Omega \subset \mathscr{F}(\mathfrak{g}, \alpha)$ is a proper dually convex domain, then K_{Ω} is a proper metric. In particular, if $\Omega \subset \mathscr{F}(\mathfrak{g}, \alpha)$ is a proper almost-homogeneous domain, then K_{Ω} is a proper metric.*

Proposition 6.4.10.(2) and Corollary 6.4.12 imply in particular that K_{Ω} is geodesic as soon as Ω is a proper almost-homogeneous (and even just dually convex) domain.

6.4.7 Example: the Kobayashi metric on symmetric domains

In this section, we compute some geodesics for the Kobayashi metric on realizations Ω of the noncompact dual of two families of Nagano spaces of real type: the Einstein universes and the Grassmannians. We also exhibit some geodesics that are also chains. In particular, the Kobayashi metric is not uniquely geodesic, whenever the considered Nagano space is of higher rank.

As we will see in Propositions 6.4.13 and 6.4.15 below, in this case, the inequality 6.4.4 is an equality. Our proof relies on the existence of chains contained in a flat of Ω (for its symmetric space structure); these sequences give two points ξ, ξ' of the boundary in which the Caratheodory metrics and the Kobayashi metric are achieved simultaneously, and this is why their K_Ω-length is equal to both the Kobayashi metric and the Caratheodory metrics.

6.4.7.1 Geodesics of the diamonds. In this Section, we consider the irreducible Nagano pair of real type $(\mathfrak{g}, \alpha) = (\mathfrak{so}(p+1, q+1), \alpha_1)$ for $p, q \geq 1$. The aim is to prove:

Proposition 6.4.13. *Let $p, q \geq 1$. Let $D \subset \mathrm{Ein}^{p,q}$ be a diamond. Then for all $x, y \in D$ there exists a 2-chain between x and y in D, which is geodesic for K_D. Moreover, for any linear or projective $\{\alpha_1\}$-proximal triple (G, ρ, V) of $\mathfrak{so}(p+1, q+1)$ with highest weight $N\omega_{\alpha_1}$, one has $K_D = \frac{1}{N} C^\rho_D$.*

Moreover, let $\varphi : \mathbb{H}^p \times \mathbb{H}^q \simeq D$ be a conformal identification, as given in Section 3.4.2. Then for all $x = (\mathbf{x}_p, \mathbf{x}_q), y = (\mathbf{y}_p, \mathbf{y}_q) \in \mathbb{H}^p \times \mathbb{H}^q$, one has

$$K_\Omega(\varphi(x), \varphi(y)) = \max\big\{d_p(\mathbf{y}_p, \mathbf{x}_p), d_q(\mathbf{y}_q, \mathbf{x}_q)\big\}, \tag{6.4.7}$$

where d_p and d_q are the hyperbolic metrics on $\mathbb{H}^p$ and $\mathbb{H}^q$ respectively.

Proof. We take the notation of Construction 3.4.5. In particular, recall that we write $\mathbb{A}_{\mathsf{std}} = H_p \oplus H_q$, where H_p is a positive-definite p-plane in $\mathbb{A}_{\mathsf{std}}$, and H_q is a negative-definite q-plane orthogonal to H_p. We write an element $y \in \mathbb{A}_{\mathsf{std}}$ by $y = y_p + y_q$ in this decomposition. Recall the diamond D defined in Construction 3.4.5. Recall that it is defined by $D = \{x \in \mathbb{A}_{\mathsf{std}} \mid |x|_{p,q} < 1\}$, where the norm $|\cdot|_{p,q}$ is defined in (3.4.1).

Note that it suffices to prove Proposition 6.4.13 on a dense subset of $D \times D$, since the property stated is a closed property.

Let $x, y \in D$. If x, y are on the same photon, we know from Proposition 6.4.10 that $K_\Omega = \frac{1}{N} C^\rho_D$. Thus we may assume that x and y are transverse. Thus we may assume that $x = 0$, and that $\mathbf{b}_{p,q}(y, y) \neq 0$. Up to exchanging p and q, we may also assume that $\mathbf{b}_{p,q}(y, y) > 0$, i.e.

$$\mathbf{b}_{p,q}(y_p, y_p) > -\mathbf{b}_{p,q}(y_q, y_q). \tag{6.4.8}$$

Since $\{y \in D \mid \mathbf{b}_{p,q}(y_p, y_p), \mathbf{b}_{p,q}(y_q, y_q) \neq 0\}$ is dense in D, we may also assume that $\mathbf{b}_{p,q}(y_p, y_p), \mathbf{b}_{p,q}(y_q, y_q) \neq 0$.

We are searching for a point $z = z_p + z_q \in H_p \oplus H_q$, such that (x, z, y) is a 2-chain. We moreover search such a point z in $\mathbb{R}y_p \oplus \mathbb{R}y_q$, i.e. $z_p = \lambda y_p$ with $\lambda \in \mathbb{R}$ and $z_q = \mu y_q$ with $\mu \in \mathbb{R}$. The condition that (x, y, z) is a 2-chain is then equivalent to

$$\mathbf{b}_{p,q}(z, z) = \mathbf{b}_{p,q}(z - y, z - y) = 0 \text{ and } |z|_{p,q} < 1. \tag{6.4.9}$$

If we set $t_p := \sqrt{\mathbf{b}(y_p, y_p)}$, $t_q := \sqrt{-\mathbf{b}(y_q, y_q)}$, then the two solutions of this system are the two pairs (λ_+, μ_+) and (λ_-, μ_-) such that

$$\lambda_\pm = \frac{1}{2t_p} \frac{t_p^2 + t_q^2}{t_p \pm t_q}; \quad \mu_\pm = \frac{1}{2t_q} \frac{t_p^2 + t_q^2}{\pm t_p + t_q},$$

which indeed satisfy $|\lambda| t_p + |\mu| t_q < 1$, i.e. $z \in D$. Note that there are thus two possible points z working, and we choose the one defined by (λ_+, μ_+). Note that Equation (6.4.8) implies that $\lambda_+, \mu_+ > 0$.

Now let $a := \frac{-y_p}{\sqrt{\mathbf{b}(y_p,y_p)}}$, $b := \frac{y_p}{\sqrt{\mathbf{b}(y_p,y_p)}} \in S_{p-1}$ (see Figure 6.2). Then by Construction 3.4.5, the intersection $(\mathbb{R}y_p \oplus \mathbb{R}y_q) \cap \mathrm{Z}_a$ is a union of two photons through a, and the same holds replacing a with b.

Let Λ_1 be the photon through x and z, and Λ_2 be the photon through z and y. Let a_1, b_1, resp. a_2, b_2 be the endpoints of $\Lambda_1 \cap D$, resp. $\Lambda_2 \cap D$, such that a_1, x, z, b_1, resp. a_2, z, y, b_2, are aligned in this order. According to the previous paragraph and by choice of z, we have $a_1, a_2 \in \mathrm{Z}_a$ and $b_1, b_2 \in \mathrm{Z}_b$. Thus we have

$$a_1 = \mathsf{pr}_{\Lambda_{x,z}}(a),\ a_2 = \mathsf{pr}_{\Lambda_{z,y}}(a),\ b_1 = \mathsf{pr}_{\Lambda_{x,z}}(b),\ b_2 = \mathsf{pr}_{\Lambda_{z,y}}(b).$$

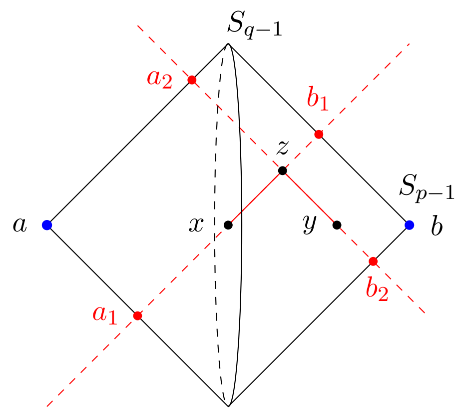

Figure 6.1 – Proof of Prop. 6.4.13 when $(p,q) = (1,2)$

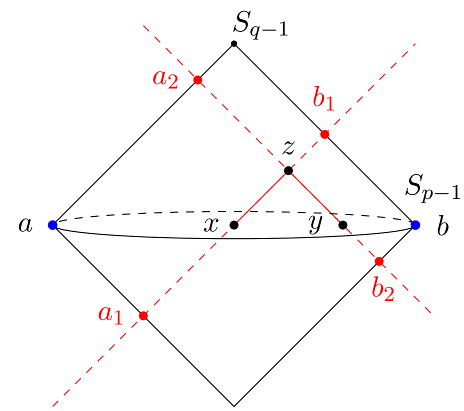

Figure 6.2 – Proof of Prop. 6.4.13 when $(p,q) = (2,1)$

We have:

$$\begin{aligned}
C_D^\rho(x,y) &\geq |\log[a:x:y:b]_\rho| = |\log[a:x:z:b]_\rho + |\log[a:z:y:b]_\rho| \\
&\overset{\text{Lem. 6.3.13}}{=} N|\log(\mathsf{pr}_{\Lambda_{x,z}}(a):x:z:\mathsf{pr}_{\Lambda_{x,z}}(b)) + \log(\mathsf{pr}_{\Lambda_{z,y}}(a):z:y:\mathsf{pr}_{\Lambda_{z,y}}(b))| \\
&\overset{(*)}{=} N|\log(\mathsf{pr}_{\Lambda_{x,z}}(a):x:z:\mathsf{pr}_{\Lambda_{x,z}}(b))| + N|\log(\mathsf{pr}_{\Lambda_{z,y}}(a):z:y:\mathsf{pr}_{\Lambda_{z,y}}(b))| \\
&= NK_D(x,z) + NK_D(z,y) \\
&= N\mathsf{len}_D(x,z,y) \geq NK_D(x,y).
\end{aligned} \tag{6.4.10}$$

To simplify the notation, we have denoted by $(c:x:y:d)$ the cross ratio of four points on the same photon, which does not depend on the projective parametrization given in (6.3.2).

The equality $(*)$ holds because all the considered cross ratios are positive, due to the choices of a and b.

Since we also have $NK_D \geq C_D^{\rho_1}$ by Proposition 6.4.10, all inequalities in (6.4.10) are equalities. In particular, we have $K_D(x,y) = \mathsf{len}_\Omega(x,z,y)$, so (x,z,y) is a geodesic for K_D. It follows that

$$K_D(x,y) = \frac{1}{N} C_D^{\rho_1}(x,y).$$

It remains now to prove (6.4.7). The first line of (6.4.10) gives:

$$K_D(x,y) = \Big| \log \frac{\mathbf{b}(\widetilde{a},\widetilde{y})\mathbf{b}(\widetilde{b},\widetilde{x})}{\mathbf{b}(\widetilde{b},\widetilde{y})\mathbf{b}(\widetilde{a},\widetilde{x})} \Big|, \tag{6.4.11}$$

where $\widetilde{z}$ is any lift of $z \in \mathrm{Ein}^{p,q}$ in $\mathbb{R}^{p+q+2}$ and $\mathbf{b}$ is the chosen bilinear form on $\mathbb{R}^{p+q+2}$ of signature $(p+1, q+1)$ defining $\mathrm{Ein}^{p,q}$.

Note that Equality (6.4.7) does not depend on the conformal identification φ, by Fact 2.4.1 and the $(\mathrm{PO}(p,1) \times \mathrm{PO}(1,q))$-invariance of the two terms of the equality. Let us consider the natural identifications

$$\mathbb{H}^p \simeq \{z_p \in H_p \mid |z_p|_{p,q} < 1\}; \quad \mathbb{H}^q \simeq \{z_q \in H_q \mid -|z_q|_{p,q} < 1\}.$$

They naturally induce a conformal identification $\mathbb{H}^p \times (-\mathbb{H}^q) \simeq D$. write $y = (\mathbf{y}_p, \mathbf{y}_q)$ for this identification. In this identification, Equation (6.4.11) gives $K_D(x,y) = d_p(\mathbf{y}_p, \mathbf{x}_p)$, where d_p is the hyperbolic metric on $\mathbb{H}^p$.

Note that we have made the choice that $\mathbf{b}_{p,q}(y,y) > 0$ (i.e. $d_p(\mathbf{y}_p, \mathbf{x}_p) > d_q(\mathbf{y}_q, \mathbf{x}_q)$). If we had chosen $\mathbf{b}_{p,q}(y,y) < 0$, then we would have $\mathbf{b}_{p,q}(y_p,y_p) < -\mathbf{b}_{p,q}(y_q,y_q)$ (i.e. $d_p(\mathbf{y}_p, \mathbf{x}_p) < d_q(\mathbf{y}_q, \mathbf{x}_q)$) instead, then we would have found $K_D(x,y) = d_q(\mathbf{y}_q, \mathbf{x}_q)$. Thus (6.4.7) holds. $\square$

Remark 6.4.14.

1. In the proof of Proposition 6.4.13, the construction of a 2-chain between x and y is simlilar to that of the proof of Lemma 3.4.7, but it uses Construction 3.4.5 instead of the conformal identification $D \simeq \mathbb{H}^p \times (-\mathbb{H}^q)$. The benefit of this point of view is that it is extrinsic: we see $D \simeq \mathbb{H}^p \times (-\mathbb{H}^q)$ *inside* $\mathrm{Ein}^{p,q}$, and thus are able to argue with points of ∂D (in particular, we can find points $a, b \in \mathrm{Ein}^{p,q}$ in which the suprema defining the Caratheodory metrics are achieved).
2. Equality (6.4.7) is also proven in [CG24, Ex. 4.2] and [Cha24] with other techniques.

Note that Equality 6.4.7 is also proven in [CG24, Ex. 4.2] and [Cha24] with other techniques.

Note that, in the resolution of Equation (6.4.9), we have made the choice of a point z between two solutions. Choosing the other one would have given another 2-chain between x and y being a geodesic. We think that this gives the only two geodesic between x and y that are 2-chains. We see that the metric K_D is *not uniquely geodesic* on a diamond D.

6.4.7.2 Geodesics of the symmetric domains of $\mathrm{Gr}_p(\mathbb{R}^{p+q})$. In this section, we take the notations in the Grassmannians defined in Sections 2.4.2 and 3.3.1, assuming that $q \geq p$. Let us add some notation : given $s_1, \ldots, s_p \in \mathbb{R}$, the matrix $\mathrm{atdiag}(s_1, \ldots, s_p)$ is the matrix (d_{ij}), where $d_{ij} = s_i$ if $p - j + 1 = i$ and 0 otherwise. We consider the bilinear form $\mathbf{b}$ on $\mathbb{R}^{p+q}$ whose matrix in the canonical basis is

$$S = \begin{pmatrix} 0 & 0 & J_p \\ 0 & I_{q-p} & 0 \\ J_p & 0 & 0 \end{pmatrix},$$

where $J_p = \mathrm{atdiag}(1,\dots,1)$ (as in Example 2.2.1, but replacing $p+1$ with p and $q+1$ with q). If $K \subset \mathrm{PO}(\mathbf{b})$ is the stabilizer of $\varphi_p\Big(\begin{pmatrix} 0_{q,p} \\ J_p \end{pmatrix}\Big) \in \mathbb{B}(\mathbf{b})$, then by Proposition 3.3.2, the group K is a maximal compact subgroup of $\mathrm{PO}(\mathbf{b})$, and we have a $\mathrm{PO}(\mathbf{b})$-equivariant diffeomorphism

$$\psi_{\mathbf{b}} : \mathbb{X}(\mathfrak{sl}(p+q,\mathbb{R}),\alpha_p) \simeq \mathrm{PO}(\mathbf{b})/K \longrightarrow \mathbb{B}(\mathbf{b}),$$

such that $\psi_{\mathbf{b}}([\mathrm{id}]) = \varphi_p\Big(\begin{pmatrix} 0_{q,p} \\ J_p \end{pmatrix}\Big)$. By Example 2.2.1, the space

$$F := \big\{ \big[\mathrm{diag}(s_1,\dots,s_p, I_{q-p}, s_p^{-1},\dots,s_1^{-1})\big]\cdot K \mid s_i \in \mathbb{R}_{>0}\ \forall 1 \le i \le p \big\}$$

is a flat of $\mathrm{PO}(\mathbf{b})/K$. The image of F by $\psi_{\mathbf{b}}$ is

$$\Big\{ \varphi_p\Big(\begin{pmatrix} 0_{q,p} \\ \mathrm{atdiag}(t_1,\dots,t_p) \end{pmatrix}\Big) \mid t_1,\dots,t_p \in \mathbb{R}_{>0} \Big\}. \tag{6.4.12}$$

This setting allows us to prove:

Proposition 6.4.15. *Let $p,q \in \mathbb{N}_{>0}$ and let Ω be a realization of $\mathbb{X}(\mathfrak{sl}(p+q,\mathbb{R}),\alpha_p)$ in $\mathrm{Gr}_p(\mathbb{R}^{p+q})$. Then for all $x,y \in \Omega$, there exists an s-chain between x and y which is a geodesic for K_Ω, where $s \le \min(p,q) = \mathrm{rk}(\mathfrak{sl}(p+q,\mathbb{R}),\alpha_p)$. Moreover, if $N \in \mathbb{N}_{>0}$ and (G,ρ,V) is an $\{\alpha_p\}$-proximal triple of $\mathfrak{sl}(p+q,\mathbb{R})$ with highest weight $N\omega_{\alpha_p}$, then one has $K_\Omega = (1/N)C_\Omega^\rho$.*

Moreover, the metric K_Ω is Finsler, and the restriction of its infinitesimal form to a flat of Ω (for its identification with $\mathbb{X}(\mathfrak{sl}(p+q,\mathbb{R}),\alpha_p)$) is $2||\cdot||_1$, where $||\cdot||_1$ is the L^1-norm.

Proof. Assume $p \le q$ for instance. By Proposition 3.3.2, we may assume that $\Omega = \mathbb{B}(\mathbf{b})$. In this proof, we give an explicit formula for a chain between two points $x,y \in \mathbb{B}(\mathbf{b})$ which is a geodesic for the metric $K_{\mathbb{B}(\mathbf{b})}$.

Now let $x,y \in \mathbb{B}(\mathbf{b})$. Then $\psi_{\mathbf{b}}^{-1}(x), \psi_{\mathbf{b}}^{-1}(y)$ are in a common maximal flat of $\mathbb{X}(\mathfrak{sl}(p+q,\mathbb{R}),\alpha_p)$. Since K acts transitively on the set of maximal flats of $\mathbb{X}(\mathfrak{sl}(p+q,\mathbb{R}),\alpha_p)$ and $K_{\mathbb{B}(\mathbf{b})}$ is K-invariant, we may assume that they are both in F, and even that

$$x = \varphi_p\Big(\begin{pmatrix} 0_{q,p} \\ I_p \end{pmatrix}\Big) = \psi_{\mathbf{b}}([\mathrm{id}]) \ \text{ and } \ y = \varphi_p\Big(\begin{pmatrix} 0_{q,p} \\ \mathrm{atdiag}(t_1,\dots,t_p) \end{pmatrix}\Big),$$

where $t_1,\dots,t_p \in \mathbb{R}_{>0}$. By density, we may assume that $t_i \neq 1$ for all $0 \le i \le p$. For all $1 \le i \le p$, we set

$$x_i := \varphi_p\Big(\begin{pmatrix} 0_{q,p} \\ \mathrm{atdiag}(t_1,\dots,t_i,1,\dots,1) \end{pmatrix}\Big)$$

(in particular, we have $x_0 = x$ and $x_p = y$). For all $0 \le i \le p-1$, let Λ_i be the unique photon through x_i, x_{i+1}, and let a_i, b_i be the endpoints of $\Lambda_i \cap \mathbb{B}(\mathbf{b})$ such that a_i, x_i, x_{i+1}, b_i are aligned in this order.

Let us write a_i, b_i explicitly. To this end, let $\mathscr{J} := \{1 \leq i \leq p \mid t_i > 1\}$. For all $0 \leq i \leq p-1$, let

$$v_i = \left[\begin{array}{c} I_p \\ \hline 0_{q-p,p} \\ \hline \operatorname{atdiag}(t_1, \ldots, t_{i-1}, 0, 1, \ldots, 1) \end{array}\right];$$

$$w_i = \left[\begin{array}{c} \begin{array}{c|c} I_{i-1} & 0_{i-1,p-i+1} \end{array} \\ \hline 0_{1,p} \\ \hline \begin{array}{c|c} 0_{p-i,i} & I_{p-i} \end{array} \\ \hline 0_{q-p,p} \\ \hline \operatorname{atdiag}(t_1, \ldots, t_{i-1}, 1, 1, \ldots, 1) \end{array}\right].$$

We then have

$$a_i = \begin{cases} v_i & \text{if } i \in \mathscr{J}, \\ w_i & \text{if } i \notin \mathscr{J}. \end{cases} \quad \text{and} \quad b_i = \begin{cases} w_i & \text{if } i \in \mathscr{J}, \\ v_i & \text{if } i \notin \mathscr{J}. \end{cases}$$

Recall that we denote by $(e_1, \ldots, e_{p+q})$ the canonical basis of $\mathbb{R}^{p+q}$. Given a set $\mathscr{I} \subset \{1, \ldots, p\}$, we define $M_{\mathscr{I}} \in \operatorname{Mat}_{p+q,q}(\mathbb{R})$ as the matrix whose i-th column is e_i if $i \in \mathscr{I}$, and e_{q+i} otherwise.

Let us moreover set $\xi_1 := [M_{\mathscr{J}}]$ and $\xi_2 := [M_{^c\mathscr{J}}]$. Note that $\xi_1, \xi_2 \in \overline{\mathbb{B}(\mathbf{b})^-}$, and for all $0 \leq i \leq p-1$, we have $a_i \in \mathrm{Z}_{\xi_1}$ and $b_i \in \mathrm{Z}_{\xi_2}$. In particular, we have $a_i = \mathsf{pr}_{\Lambda_i}(\xi_1)$ and $b_i = \mathsf{pr}_{\Lambda_i}(\xi_2)$.

Now let $N \in \mathbb{N}_{>0}$ and (G, ρ, V) be an $\{\alpha_p\}$-proximal triple of $\mathfrak{sl}(p+q, \mathbb{R})$ with highest weight $N\omega_{\alpha_p}$. We have:

$$\begin{aligned} C^{\rho}_{\mathbb{B}(\mathbf{b})}(x,y) \geq |\log[\xi_1 : x : y : \xi_2]_\rho| &= \left|\sum_{i=0}^{p-1} \log[\xi_1 : x_i : x_{i+1} : \xi_2]_\rho\right| \\ &\overset{\text{Lem. 6.3.13}}{=} N \sum_{i=0}^{p-1} |\log(\mathsf{pr}_{\Lambda_i}(\xi_1) : x_i : x_{i+1} : \mathsf{pr}_{\Lambda_i}(\xi_2))| \\ &\overset{(*)}{=} N \sum_{i=0}^{p-1} |\log(a_i : x_i : x_{i+1} : b_i))| \\ &= N \sum_{i=0}^{p-1} K_{\mathbb{B}(\mathbf{b})}(x_i, x_{i+1}) \\ &= N \operatorname{len}_{\mathbb{B}(\mathbf{b})}(x_0, \ldots, x_p) \geq N K_{\mathbb{B}(\mathbf{b})}(x,y), \end{aligned} \tag{6.4.13}$$

where Equality $(*)$ follows from the fact that all the considered cross ratios are positive, due to the choices of ξ_1 and ξ_2. Since we also have $NK_{\mathbb{B}(\mathbf{b})}(x,y) \geq C^{\rho}_{\mathbb{B}(\mathbf{b})}(x,y)$ by Proposition 6.4.10, all inequalities in Equation (6.4.13) are equalities. We thus have $NK_{\mathbb{B}(\mathbf{b})}(x,y) = C^{\rho}_{\mathbb{B}(\mathbf{b})}(x,y)$. Moreover, by density and continuity of the Kobayashi and Caratheodory metrics, this equality extends to the case where there exists $1 \leq i \leq p$

such that $t_i = 1$. Finally, if we had assumed that $q \leq p$ instead of $p \leq q$, we would have found the same equality, but with a q-chain instead of a p-chain.

Let us now compute $K_{\mathbb{B}(\mathbf{b})}$. Recall the Cartan subspace $\mathfrak{a}$ of $\mathfrak{so}(p,q)$, defined in Example 2.2.1.(2) (replacing $p+1$ by p and $q+1$ by q). Let $||\cdot||_1$ be the L^1-norm on $\mathfrak{a}$:

$$\big\| \operatorname{diag}(\lambda_1, \dots, \lambda_p, 0_{q-p}, -\lambda_p, \dots, -\lambda_1) \big\|_1 = \sum_{i=1}^{p} |\lambda_i|.$$

Let $x, y \in \mathbb{B}(\mathbf{b})$, and first assume that $x = \varphi_p([\mathrm{id}])$. Let

$$\mathbf{p}_{\mathfrak{a}}(y) = \operatorname{diag}(\lambda_1, \dots, \lambda_p, 0_{q-p}, -\lambda_p, \dots, -\lambda_1)$$

be the projection of $\psi_{\mathbf{b}}^{-1}(y)$ on $\mathfrak{a}$, i.e. the unique element of $\mathfrak{a}$ such that there exists $k \in K$ such that $\psi_{\mathbf{b}}^{-1}(y) = k \exp(\mathbf{p}_{\mathfrak{a}}(y)) \cdot K$. By K-equivariance of $K_{\mathbb{B}(\mathbf{b})}$ and Equation (6.4.13), we have

$$K_{\mathbb{B}(\mathbf{b})}(x,y) = K_{\mathbb{B}(\mathbf{b})}\big(x, \psi_{\mathbf{b}}([\mathbf{p}_{\mathfrak{a}}(y)] \cdot K)\big) = \sum_{i=0}^{p} 2|\lambda_i| = 2\|\mathbf{p}_{\mathfrak{a}}(y)\|_1.$$

Now if x is not necessarily equal to $\varphi_p([\mathrm{id}])$, let $\ell \in \mathrm{PO}(\mathbf{b})$ be such that $x = \varphi_p([\ell])$. Then by $\mathrm{PO}(\mathbf{b})$-equivariance of $K_{\mathbb{B}(\mathbf{b})}$, we have:

$$K_{\mathbb{B}(\mathbf{b})}(x,y) = 2\big\|\mathbf{p}_{\mathfrak{a}}(\ell^{-1} \cdot y)\big\|_1. \quad \square$$

Remark 6.4.16.

1. In the proof of Proposition 6.4.15, we have made the choice of a chain $(x_0, \dots, x_p)$ between x and y (in the notation of the proof). Permuting the t_i, we could have chosen 2^{p-1} such chains. We think that these 2^{p-1} choices for the t_i describe all possible chains between x and y being geodesics.
2. The Kobayashi metrics computed in this section and in Section 6.4.7.1 on realizations of the noncompact dual are not induced by Riemannian metrics on $\mathbb{X}(\mathfrak{g}, \alpha)$, except in the case where $\min(p,q) = 1$. In this latter case, we recover $\mathbb{X}(\mathfrak{g}, \alpha) \simeq \mathbb{H}^{\max(p,q)}$, and classically, the metric $K_\Omega = \mathsf{H}_\Omega$ of a realization Ω of $\mathbb{X}(\mathfrak{g}, \alpha)$ is a multiple of the pushforward of the hyperbolic metric on $\mathbb{H}^{\max(p,q)}$, and Ω is an ellipsoïd.

   Thus, except in the real projective case, the Kobayashi metric on a realization of $\mathbb{X}(\mathfrak{g}, \alpha)$ is not a multiple of the Riemannian metric of $\mathbb{X}(\mathfrak{g}, \alpha)$ (however, we see that it is Finsler). This property distinguishes the higher-rank case from the rank-one (i.e. the real-projective) case.

6.5 An introduction to Helgason spheres

Let us end this chapter with a discussion of possible generalizations of photons and of the Kobayashi pseudo-metric to Nagano spaces which are *not* of real type.

In the construction of the Kobayashi pseudo-metric in Section 6.4, we required that $\dim(\mathfrak{g}_\alpha) = 1$. On the contrary, if we consider the irreducible Nagano pair $(\mathfrak{so}(3,1), \alpha_1)$, the associated Nagano space is $\mathbb{S}^2$, and we have $\dim(\mathfrak{g}_{\alpha_1}) = 2$. Let us consider $\Omega := \mathbb{H}^2 \subset \mathbb{S}^2$. If we naively define the Kobayashi pseudo-metric as in Section 6.4, then it is not a metric on Ω. The reason is that photons are no longer the appropriate objects for defining a metric. Based on the work of Peterson, Nagano, and Takeuchi [Pet87, Nag88, Tak88], it appears that the correct notion should be that of *Helgason spheres*.

In this section, we recall fundamental results on Helgason spheres, discuss their connections with our photons, and propose a construction of a Kobayashi pseudo-metric using Helgason spheres instead of photons.

6.5.1 Definition

In this section, we consider a compact symmetric space M.

Definition 6.5.1. [Tak88] A submanifold $\mathbb{S}$ of M is said to be a *Helgason sphere* if it is a totally geodesic sphere of M with minimal radius, and if it has maximal dimension among all submanifolds of M satisfying this property.

Let $\mathfrak{g}'_\alpha := \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_\alpha \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \simeq \mathfrak{o}(n,1)$ (where $n = \dim(\mathfrak{g}'_\alpha) - 1$). The inclusion $\mathfrak{g}'_\alpha \subset \mathfrak{g}$ gives a Lie group homomorphism $\tau_H : \mathrm{SO}(n,1) \to G$ with kernel $\{\pm 1\}$. The stabilizer of P in $\mathrm{SO}(n,1)$ is then the standard minimal parabolic subgroup P_1 of $\mathrm{O}(n,1)$ [Tak88]. This defines a τ_H-equivariant embedding $\zeta_H : \mathbb{S}^{n-1} = \mathrm{SO}(n,1)/P_1 \to G/P = \mathscr{F}(\mathfrak{g}, \alpha)$.

An explicit construction of such Helgason spheres for Nagano spaces is given in [Tak88].

Theorem 6.5.2. *[Tak88, Lem. 5.7, Thm 5.10] Every Helgason sphere of $\mathscr{F}(\mathfrak{g}, \alpha)$ is a G-translate of the manifold*

$$\mathbb{S}_\alpha := \exp(\mathfrak{g}'_\alpha) \cdot \mathfrak{p}^+ = \zeta_H(\mathbb{S}^{n-1}) = \overline{\exp(\mathfrak{g}_{-\alpha}) \cdot \mathfrak{p}^+}.$$

We can prove a lemma similar to Lemma 6.3.3:

Lemma 6.5.3. *One has*

1. $U^+ \cdot \mathbb{S}_\alpha = \mathbb{S}_\alpha$.
2. *For all $x, y \in \mathscr{F}(\mathfrak{g}, \alpha)$ such that $x \neq y$, there is at most one Helgason sphere through x and y.*

Proof. 1. Since $\mathfrak{u}^+$ is abelian and generated by the $\mathfrak{g}_\beta$ for $\beta \in \Sigma^+_\Theta$, it suffices to prove that $\exp(\mathfrak{g}_\beta) \cdot \mathbb{S}_\alpha = \mathbb{S}_\alpha$ for all $\beta \in \Sigma^+_\Theta$.

Let us first consider $\beta \in \Sigma^+_\Theta \smallsetminus \{\alpha\}$ and $X \in \mathfrak{g}_\beta$. Then for all $X_\alpha \in \mathfrak{g}_{-\alpha}$, one has

$$\exp(X)\exp(X_\alpha) \cdot \mathfrak{p}^+ = \exp(\mathrm{Ad}(\exp(X))X_\alpha) \cdot \mathfrak{p}^+. \tag{6.5.1}$$

Thus one should compute $\mathrm{Ad}(\exp(X))X_\alpha = \sum_{k=0}^\infty \frac{\mathrm{ad}(X)^k}{k!} X_\alpha$. For all $k \in \mathbb{N}$, one has $\mathrm{ad}(X)^k X_\alpha \in \mathfrak{g}_{k\beta-\alpha}$. Since the multiplicity of α in the longest root is 1 (see Equation (5.1.2)), one has $\mathfrak{g}_{k\beta-\alpha} \subset \mathfrak{l}$. On the other hand, one has $[X_\alpha, \mathrm{ad}(X)^k X_\alpha] \in \mathfrak{g}_{k\beta-2\alpha} = \{0\}$ (because $k\beta - 2\alpha$ is not a restricted root, again because of the multiplicity of α). Thus one has $\mathrm{Ad}(\exp(X))X_\alpha = X_\alpha + Y$, where $Y \in \mathfrak{l}$ commutes with X_α. Thus Equation (6.5.1) becomes:

$$\exp(X)\exp(X_\alpha) \cdot \mathfrak{p}^+ = \exp(X_\alpha + Y) \cdot \mathfrak{p}^+ = \exp(X_\alpha)\exp(Y) \cdot \mathfrak{p}^+ = \exp(X_\alpha) \cdot \mathfrak{p}^+$$

because $L \subset P^+$. Thus $\exp(X)$ fixes every point of $\mathbb{S}_\alpha$.

Now it remains to prove that $\exp(\mathfrak{g}_\alpha) \cdot \mathbb{S}_\alpha = \mathbb{S}_\alpha$. But this is by definition:

$$\exp(\mathfrak{g}_\alpha) \cdot \mathbb{S}_\alpha \in \tau_H\big(\exp(\mathfrak{o}(n,1))\big) \cdot \zeta_H(\mathbb{S}^{n-1}) = \zeta_H(\mathbb{S}^{n-1}) = \mathbb{S}_\alpha. \square$$

2. The proof is similar to the one of Lemma 6.3.3.(2). □

The following lemma admits the exact same proof as Lemma 6.3.6:

Lemma 6.5.4. *One has*

$$\mathrm{Stab}_G(\mathbb{S}_\alpha) = \tau_H(\mathrm{SO}(n,1) \times \mathrm{Cent}_G\left(\tau_H(\mathrm{SO}(n,1))\right),$$

where $\mathrm{Stab}_G(\mathbb{S}_\alpha)$ *is the stabilizer of* $\mathbb{S}_\alpha$ *in* G *and* $\mathrm{Cent}_G\left(\tau_H(\mathrm{SO}(n,1))\right)$ *is the centralizer in* G *of the group* $\tau_H(\mathrm{SO}(n,1))$.

The following lemma admits the same proof as Lemma 6.3.7:

Lemma 6.5.5. *Let* $\mathbb{S}$ *be a Helgason sphere. If* $\mathbb{S} \cap \mathbb{A}_{\mathsf{std}}$ *is nonempty, then it is an affine vector subspace of* $\mathbb{A}_{\mathsf{std}}$ *of dimension* $\dim(\mathfrak{g}_\alpha)$*, and* $\mathbb{S} \cap \mathrm{Z}_{\mathfrak{p}^-}$ *is a singleton.*

In particular, for any $\xi \in \mathscr{F}(\mathfrak{g},\alpha)^-$ and any Helgason sphere $\mathbb{S}$, if $\mathbb{S} \not\subset \mathrm{Z}_\xi$, then $\mathbb{S} \cap \mathrm{Z}_\xi$ is a singleton.

Theorem 6.5.2 directly gives:

Lemma 6.5.6. *Let* $x \in \mathscr{F}(\mathfrak{g},\alpha)$ *and* Λ *be a photon through* x*. Let* $\mathbb{S}_x$ *be a Helgason sphere through* x*. Then* $\Lambda \subset \mathbb{S}_x$*. In particular, if* $\dim(\mathfrak{g}_\alpha) = 1$*, then the photons of* $\mathscr{F}(\mathfrak{g},\alpha)$ *as defined in Section 6.3.1 are exactly the Helgason spheres of* $\mathscr{F}(\mathfrak{g},\alpha)$*.*

6.5.2 Arithmetic distance

The *arithmetic distance* between two distinct points $x, y \in \mathscr{F}(\mathfrak{g},\alpha)$ is the integer

$$\begin{aligned} d_H(x,y) = \min\Big\{ & k \in \mathbb{N} \mid \exists x_0 := x, x_1, \ldots, x_k := y \in \mathscr{F}(\mathfrak{g},\alpha), \\ & x_i \text{ and } x_{i+1} \text{ are on a common Helgason sphere}\Big\}. \end{aligned} \tag{6.5.2}$$

If $x = y$, we set $d_H(x,y) = 0$.

Example 6.5.7. In the following examples, we have $\dim(\mathfrak{g}_\alpha) = 1$, so Helgason spheres are exactly the photons as defined in Section 6.3.

1. Let $(\mathfrak{g},\alpha) = (\mathfrak{sl}(p+q,\mathbb{R}),\alpha_p)$. According to the description of photons in Section 6.3.4.1, the arithmetic distance between two points $x, y \in \mathrm{Gr}_p(\mathbb{R}^{p+q})$ is given by $d_H(x,y) = \dim(x+y) - p \in \{0,\ldots,p\}$.
2. Let $(\mathfrak{g},\alpha) = (\mathfrak{so}(p+1,q+1),\alpha_1)$, $p,q \geq 1$. Then $d_H(x,y) \in \{0,1,2\}$ for all $x, y \in \mathrm{Ein}^{p,q}$. It is 0 if $x = y$, it is 1 if x, y are on the same photon but not equal, and it is 2 if x, y are transverse; see also Equation (2.4.10).

3. Let $\mathfrak{g}$ be a HTT Lie algebra of real rank r and $\alpha = \alpha_r$, with $r \geq 4$. In the notation of Section 2.4.4.4.1, we then have $\mathfrak{g} = \mathfrak{g}_{\mathbb{K}}$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, and $\mathbf{Sb}(\mathfrak{g}) = \mathrm{Lag}_r(\mathbb{K}^{2r})$. According to Section 6.3.4.2, given two points $x, y \in \mathbf{Sb}(\mathfrak{g})$, one has $d_H(x,y) = \dim_{\mathbb{K}}(x+y) - r \in \{0, \dots, r\}$.

Now if we consider the Nagano pair $(\mathfrak{g}, \alpha) = (\mathfrak{sl}(p+q, \mathbb{C}), \alpha_p)$, which is not of real type, then the Helgason spheres are 2-dimensional (see Table 8.1). As for the real case, we have an identification $\mathscr{F}(\mathfrak{g}, \alpha_p) \simeq \mathrm{Gr}_p(\mathbb{C}^{p+q})$, for which we have $d_H(x,y) = \dim_{\mathbb{C}}(x+y) - p \in \{0, \dots, p\}$.

Finally, if $(\mathfrak{g}, \alpha) = (\mathfrak{so}(n,1), \alpha)$, where α is the unique simple restricted root of $\mathfrak{so}(n,1)$, then we have $d_H(x,y) \in \{0,1\}$; see also Equation (2.4.9).

A direct consequence of [Tak88, Thm 6.4] is the following:

Theorem 6.5.8. *Let $x_0 \in \mathscr{F}(\mathfrak{g}, \alpha)$. One has*

$$\{x \in \mathscr{F}(\mathfrak{g}, \alpha) \mid d(x_0, x) \leq |W_{\Delta \smallsetminus \{\alpha\}} \backslash W / W_{\Delta \smallsetminus \{\alpha\}}| - 2\} \subset (\mathrm{Z}_{x_0})^* := \mathscr{F}(\mathfrak{g}, \alpha) \smallsetminus \mathsf{C}_{\overline{w_0}}(x_0).$$

In particular, if $(\mathfrak{g}, \alpha)$ has higher rank — in the sense of Definition 5.1.5 — then any Helgason sphere through x_0 is contained in $(\mathrm{Z}_{x_0})^*$.

Theorem 6.5.8 and Lemma 6.5.6 imply in particular that if $(\mathfrak{g}, \alpha)$ has higher rank, then any photon through $x_0 \in \mathscr{F}(\mathfrak{g}, \alpha)$ is contained in $(\mathrm{Z}_{x_0})^*$. It implies in particular Lemma 6.3.9.

To finish this section, let us give a result of Takeuchi that states that the arithmetic metric allows to characterize the group G among the diffeomorphisms of $\mathscr{F}(\mathfrak{g}, \alpha)$:

Theorem 6.5.9. *[Tak88] If $(\mathfrak{g}, \alpha)$ is an irreducible Nagano pair of rank ≥ 2, then the transformation group G of the Nagano space $\mathscr{F}(\mathfrak{g}, \alpha)$ is exactly the group of diffeomorphisms of $\mathscr{F}(\mathfrak{g}, \alpha)$ which preserves the arithmetic distance on M.*

6.5.3 The pseudo-metric

In this section, we discuss a possible generalization of the Kobayashi pseudo-metric defined for Nagano spaces of real type, to the non-real-type case. The idea is to replace chains of photons with chains of Helgason spheres (in the real-type case, we recover the Kobayashi pseudo-metric defined in Section 6.4) and by replacing the Hilbert pseudo-metric on intervals of photons with a pseudo-metric on domains of the sphere. For instance, one could consider the following construction:

1. We start with the definition of a geodesic metric on proper domains of the Nagano space $\mathbb{S}^{n-1} = \mathscr{F}(\mathfrak{so}(n,1), \alpha)$, where α is the unique simple restricted root of $\mathfrak{so}(n,1)$. We will use the well-known metric introduced by Kulkarni–Pinkall [KP94] and Apanasov [Apa91]. To this end, fix a basepoint $o \in \mathbb{H}^{n-1}$. We endow $\mathbb{H}^{n-1}$ with its natural Riemannian metric $g_{\mathbb{H}^{n-1}}$. Recall that the conformal sphere $\mathbb{S}^{n-1}$ admits a Riemannian conformal structure, in the sense of Section 2.4.3.3. Let g be any metric in the conformal class defining this conformal structure. If $\Omega \subset \mathbb{S}^{n-1}$ is a domain, then the restriction of g to Ω is a Riemannian metric. For $x \in \Omega$, we denote by $\mathrm{Conf}_{o,x}(\mathbb{H}^{n-1}, \Omega)$ the space of all conformal maps f from $\mathbb{H}^{n-1}$ to Ω

such that $f(o) = x$. For all $f \in \mathrm{Conf}_{o,x}(\mathbb{H}^{n-1}, \Omega)$, there exists a unique $\lambda(f) \in \mathbb{R}_{>0}$ such that $\frac{T_o f}{\lambda(f)}$ is an isometry between $(T_o\mathbb{H}^{n-1}, (g_{\mathbb{H}^{n-1}})_o)$ and $(T_x\mathbb{S}^{n-1}, g_x)$. The *conformal distortion* at x is then:

$$\nu_\Omega(x) := \sup\{\lambda(f) \mid f \in \mathrm{Conf}_{o,x}(\mathbb{H}^{n-1}, \Omega)\}.$$

A theorem of Kulkarni–Pinkall [KP94] and Apanasov [Apa91] implies that the pseudo-metric $\frac{1}{\nu_\Omega^2} g$ is a complete $\mathsf{Aut}_G(\Omega)$-invariant Riemannian metric (for all $G \in \mathscr{G}_{\{\alpha_1\}}(\mathfrak{so}(n,1))$) whenever Ω is proper in $\mathbb{S}^{n-1}$. By integration, it defines a complete Riemannian metric on Ω, which we denote by d_Ω^{KP}. This metric is in particular geodesic.

2. Now let $(\mathfrak{g}, \alpha)$ be an irreducible Nagano space, and $\Omega \subset \mathscr{F}(\mathfrak{g}, \alpha)$ be a domain. Two points $x_1, x_2 \in \Omega$ are *H-related*, denoted by $x_1 \sim_H x_2$, if they belong to a common Helgason sphere $\mathbb{S}$ and are in the same connected component of $\mathbb{S} \cap \Omega$. For $N \in \mathbb{N}_{>0}$, an *N-Helgason chain from x to y* is a sequence of $(N+1)$ elements $(x_0 = x, \dots, x_N = y)$ of Ω such that $x_i \sim_H x_{i+1}$ for all $0 \leq i \leq N-1$. We denote by $\mathscr{C}_{x,y}(\Omega)_H$ (resp. $\mathscr{C}_{x,y}^N(\Omega)_H$) the set of all Helgason chains (resp. N-Helgason chains) from x to y in Ω. Since photons are contained in Helgason spheres (Lemma 6.5.6), Observation 6.4.2 gives that there exists an integer $\mathsf{n}(G)$ only depending on G and a basis of neighborhoods $\mathscr{V}$ of $\mathscr{F}(\mathfrak{g}, \Theta)$ such that for any $V \in \mathscr{V}$ and any $x, y \in V$, one has $\mathscr{C}_{x,y}^{\mathsf{n}(G)}(V)_H \neq \emptyset$.

3. Given two H-related points $x, y \in \Omega$, let $\mathbb{S}_{x,y} \in \mathscr{L}$ be the unique Helgason sphere containing x and y, and let $g \in G$ be such that $\mathbb{S}_{x,y} = g \cdot \mathbb{S}_\alpha$. We denote by $\mathscr{O}_{x,y}$ the connected component of $\Omega \cap \mathbb{S}_{x,y}$ containing x and y. The map $g \cdot \zeta_H$ is a parametrization of $\mathbb{S}_{x,y}$. Let

$$\mathsf{k}_\Omega^{\mathbb{S}}(x,y) := d^{KP}_{\zeta_H^{-1}(g^{-1} \cdot \mathscr{O}_{x,y})} \left(\zeta_H^{-1}(g^{-1} \cdot x), \zeta_H^{-1}(g^{-1} \cdot y) \right).$$

By Lemma 6.5.4, the quantity $\mathsf{k}_\Omega^{\mathbb{S}}(x,y)$ does not depend on the choice of $g \in G$ such that $\mathbb{S}_{x,y} = g \cdot \mathbb{S}_\alpha$.

4. The definition of the Kobayashi pseudo-metric is then the same as in Section 6.4.2, replacing chains by Helgason chains. Given a domain $\Omega \subset \mathscr{F}(\mathfrak{g}, \alpha)$ and $x, y \in \Omega$, we define:

$$K_\Omega^{\mathbb{S}}(x,y) = \inf \Big\{ \sum_{i=0}^{N} \mathsf{k}_\Omega(x_i, x_{i+1}) \mid N \in \mathbb{N}^*, \quad (x_0, \dots, x_N) \in \mathscr{C}_{x,y}(\Omega)_H \Big\}.$$

As in Section 6.4.2, it is easy to check that this defines a pseudo-metric, and we call it the *Kobayashi–Helgason pseudo-metric*. A domain $\Omega \subset \mathscr{F}(\mathfrak{g}, \alpha)$ is said to be *Kobayashi–Helgason hyperbolic* if $K_\Omega^{\mathbb{S}}$ is a metric, that is, if $K_\Omega^{\mathbb{S}}$ separates points, in which case it is called *the Kobayashi–Helgason metric of* Ω.

Further investigation on the Gromov-hyperbolicity or properness of this Kobayashi–Helgason pseudo-metric appears to be more challenging. It is work in progress.

In the case where $\dim(\mathfrak{g}_\alpha) = 2$, the Nagano space $\mathscr{F}(\mathfrak{g}, \alpha)$ carries a complex structure. In this setting, there already exists an $\mathrm{Aut}_G(\Omega)$-invariant (and even biholomorphism-invariant) Kobayashi pseudo-metric on proper domains of $\mathscr{F}(\mathfrak{g}, \alpha)$ (and even of $\mathbb{C}^n$), sharing similarities with our Kobayashi–Helgason pseudo-metric. Conditions for the Gromov-hyperbolicity of this metric have been studied (see, e.g., [Zim16, NTT16, Zim17, GS18]).

Remark 6.5.10. There are several metrics we could have chosen on proper domains of $\mathbb{S}^{n-1}$ in the first step of the construction of the Kobayashi–Helgason metric. For instance, we could have chosen the Caratheodory metrics of proper domains of the sphere, instead of d_Ω^{KP}. This construction would also lead to a Kobayashi–Helgason pseudo-metric generalizing the Kobayashi pseudo-metric introduced in Section 6.4.2. However, there exist proper domains of the sphere for which the Caratheodory metric is not geodesic. Since we want, in the end, a Kobayashi–Helgason metric which will be geodesic, it seems more reasonable to start the construction with a metric which is geodesic.

Chapter 7

Geometry of proper domains in Nagano spaces of real type

Nagano spaces of real type admit photons in generic position, as seen in Chapter 6 (in particular in Fact 6.3.10 and Observation 6.4.2). This additional structure provides new tools to deepen the study of proper dually convex domains in Nagano spaces of real type, initiated in Chapter 3 for general flag manifolds.

Let $(\mathfrak{g}, \alpha)$ be an irreducible Nagano pair of real type. In this chapter, building on work of Limbeek–Zimmer [LZ19] in Grassmannian case, we define subsets of the boundary of a proper domain $\Omega \subset \mathscr{F}(\mathfrak{g}, \alpha)$, referred to as *$\mathscr{R}$-faces*, generalizing faces in the real projective setting. The Kobayashi metric and $\mathscr{R}$-faces are intrinsically related, as both are defined from of photons. Just as in the projective case, where the Hilbert metric and faces interact (see Section 1.1.1.2), the interplay between these the Kobayashi metric and $\mathscr{R}$-faces allows us to study the dynamics of the automorphism group of a proper almost-homogeneous domain in an irreducible Nagano space of real type; see Sections 7.2.2 and 7.2.3.

Points whose $\mathscr{R}$-face is trivial are called *$\mathscr{R}$-extremal points*. We establish a generalization of Fact 1.1.1 to any Nagano space of real type, see Lemma 7.2.10. The geometric consequences of this Lemma for $\mathscr{R}$-extremal points (Theorem 7.2.6) are related to the structure of maximal proper Schubert subvarieties Z_z (see Equation (2.2.5)) of $\mathscr{F}(\mathfrak{g}, \alpha)$. In particular, these consequences are empty in the projective case but become increasingly significant as $|W_{\Delta \smallsetminus \{\alpha\}} \backslash W / W_{\{\Delta \smallsetminus \mathrm{i}(\alpha)\}}|$ grows; see Remark 7.2.7.(1). They reach their maximal strength when $\mathscr{F}(\mathfrak{g}, \alpha)$ is self-opposite, impacting the rigidity of proper almost-homogeneous domains, as we will see in Chapter 8.

The existence of $\mathscr{R}$-extremal points in generic position is proven in Lemma 7.2.5. Several properties of $\mathscr{R}$-faces (see e.g. Proposition 7.2.3 and Lemma 7.2.5) come from the existence of *Plücker triples* for Nagano spaces of real type, defined in the following Section 7.1.

Notation 7.0.1. For all this chapter, whenever we consider an irreducible Nagano pair $(\mathfrak{g}, \alpha)$, we will use Notation 5.1.2.

7.1 Plücker triples of Nagano spaces of real type

If $(\mathfrak{g}, \alpha)$ is an irreducible Nagano pair of real type, then there exists a projective $\{\alpha\}$-proximal triple (G, ρ, V) of $\mathfrak{g}$ with highest weight ω_α. Such a triple is called a *Plücker triple* of $(\mathfrak{g}, \alpha)$. Recall that the associate embeddings by Fact 2.3.4 are denoted by ι_ρ, ι_ρ^-. In this section, we investigate the consequences on the structure of Nagano spaces of real type of the existence of Plücker triples.

The following examples are fundamental, as we will use them directly in the proofs of Theorems 8.4.1 and 8.5.1 in Sections 8.4 and 8.5.

Example 7.1.1. 1. If $(\mathfrak{g}, \alpha) = (\mathfrak{sl}(p+q, \mathbb{R}), \alpha_p)$ defines the Grassmannian $\mathrm{Gr}_p(\mathbb{R}^{p+q})$, then the triple $(\mathrm{PGL}(p+q, \mathbb{R}), \rho_0, \bigwedge^p \mathbb{R}^{p+q})$ defined in Equation (2.4.2.2) is a Plücker triple of $(\mathfrak{g}, \alpha)$.

2. If $(\mathfrak{g}, \alpha) = (\mathfrak{so}(p+1, q+1), \alpha_1)$ with $p, q \geq 1$, defines the Einstein universe $\mathrm{Ein}^{p,q}$, then the triple $(\mathrm{PO}(p+1, q+1), \rho_1, \mathbb{R}^{p+q+2})$ defined in Equation (2.4.11) is a Plücker triple of $(\mathfrak{g}, \alpha)$.

3. If $(\mathfrak{g}, \alpha) = (\mathfrak{so}(n, n), \alpha_n)$ for some even $n \geq 2$, then

7.1.1 Images of photons

Plücker triples will be of fundamental importance in our study of proper domains in Nagano spaces of real type. The structural result from which all their main properties will follow is:

Lemma 7.1.2. *Let $(\mathfrak{g}, \alpha)$ be an irreducible Nagano pair of real type and (G, ρ, V) a Plücker triple of $(\mathfrak{g}, \alpha)$. Let $\Lambda \subset \mathscr{F}(\mathfrak{g}, \alpha)$ be a photon. Then $\iota_\rho(\Lambda)$ is a projective line in $\mathbb{P}(V)$.*

Proof. Let $\mathsf{v}_0 \in V^{\omega_\alpha} \smallsetminus \{0\}$. By the ρ-equivariance of ι_ρ, we may assume that $\Lambda = \Lambda_{\mathsf{std}}$. For all $t \in \mathbb{R}$, we have:

$$\rho(\exp(tv^-)) \cdot \mathsf{v}_0 = \exp(t\rho_*(v^-)) \cdot \mathsf{v}_0 = \sum_{k=0}^{\infty} \frac{t^k \rho_*(v^-)^k}{k!} \cdot \mathsf{v}_0 = \mathsf{v}_0 + t\rho_*(v^-) \cdot \mathsf{v}_0, \tag{7.1.1}$$

the last equality holding by Lemma 5.2.1. Thus $\iota_\rho(\Lambda_{\mathsf{std}}) = \overline{\{[\mathsf{v}_0 + t\rho_*(v^-) \cdot \mathsf{v}_0] \mid t \in \mathbb{R}\}}$ is a projective line. ☐

Remark 7.1.3. 1. When $(\mathfrak{g}, \alpha) = (\mathfrak{so}(n, 1), \alpha_1)$ for $n \geq 3$, the natural inclusion of $\mathrm{PO}(n, 1)$ into $\mathrm{PGL}(n+1, \mathbb{R})$ is irreducible, proximal, and of highest weight $\omega_{\alpha_1} = \alpha_1$, but Lemma 5.2.1 does not hold. The reason is that $(\mathfrak{g}, \alpha)$ is not a Nagano pair *of real type*, see Remark 6.1.2. More particularly, Lemma 5.2.1, which we use in the proof of Lemma 7.1.2, does not hold for the irreducible Nagano pair $(\mathfrak{so}(n, 1), \alpha_1)$; see Remark 5.2.2.

2. Let $(\mathfrak{g}, \alpha)$ be an irreducible Nagano pair of real type. If, instead of considering a Plücker triple of $(\mathfrak{g}, \alpha)$, we take a projective $\{\alpha\}$-proximal triple (G, ρ', V') of $\mathfrak{g}$

with highest weight $2\omega_\alpha$, then Lemma 7.1.2 fails. Indeed, by Lemma 5.2.1, Equation (7.1.1) becomes

$$\rho(\exp(tv^-)) \cdot \mathsf{v}_0 = \mathsf{v}_0 + t\rho_*(v^-) \cdot \mathsf{v}_0 + \frac{t^2}{2}\rho_*(v^-)^2 \cdot \mathsf{v}_0, \tag{7.1.2}$$

which implies that the images of photons are conics.

Such representations ρ are particular cases of what are called *spherical representations*, which are characterized by the fact that the maximal compact subgroup K of G admits a fixed point $\mathbb{P}(V)$. For such representations, the image $\iota_\rho(\mathscr{F}(\mathfrak{g}, \alpha))$ is then a proper sphere for some Riemannian metric on $\mathbb{P}(V)$. This explains why the photons, which are contained in this sphere, are of the form of Equation (7.1.2); in particular, they are contained in affine charts of $\mathbb{P}(V)$.

The property of photons stated in Lemma 7.1.2 actually characterizes them among projective lines:

Proposition 7.1.4. *In the notation of Lemma 7.1.2, the images of photons in $\mathbb{P}(V)$ under ι_ρ are exactly the projective lines of $\mathbb{P}(V)$ that are contained in $\iota_\rho(\mathscr{F}(\mathfrak{g}, \alpha))$.*

Proof. The fact that images of photons are projective lines is a consequence of Lemma 7.1.2. For the converse implication, let $\Lambda \subset \mathbb{P}(V)$ be a projective line contained in $\iota_\rho(\mathscr{F}(\mathfrak{g}, \alpha))$. Up to translating Λ by an element of G, we may assume that $\iota_\rho(\mathfrak{p}^+) \in \Lambda$.

Let $\{\alpha_1, \dots, \alpha_{\dim(V)-1}\}$ be a system of simple restricted roots of $\mathfrak{sl}(V)$ such that $V^{\omega_\alpha} = V^{\alpha_1}$ and $V^{<\omega_\alpha} = V^{<\alpha_1}$ (for the natural action of $\mathrm{SL}(V)$ on V). Recall that we denote by $\mathfrak{u}^-_{\{\alpha_1\}}$ the unipotent radical for this system of simple restricted roots defined by α_1. This radical parametrizes the affine chart $\mathbb{A}^{\mathbb{P}(V)}_{\mathsf{std}} := \mathbb{P}(V) \smallsetminus \iota^-_\rho(\mathfrak{p}^-)$, as in Equation (2.2.6): the map

$$\begin{array}{cccc} \varphi^{\mathbb{P}(V)}_{\mathsf{std}} : & \mathfrak{u}^-_{\{\alpha_1\}} & \longrightarrow & \mathbb{A}^{\mathbb{P}(V)}_{\mathsf{std}} \\ & X & \longmapsto & [\exp(X) \cdot \mathsf{v}_0] \end{array}$$

is a diffeomorphism. By definition of a projective line through $\iota(\mathfrak{p}^+) = [\mathsf{v}_0]$, there exists $X \in \mathfrak{u}_{\alpha_1})^-$ such that

$$\Lambda = \overline{\{[\exp(tX) \cdot \mathsf{v}_0] \mid t \in \mathbb{R}\}}.$$

Since $\Lambda \subset \iota(\mathscr{F}(\mathfrak{g}, \alpha))$, we have in particular

$$d_0\varphi^{\mathbb{P}(V)}_{\mathsf{std}}(X) = \frac{d}{dt}|_{t=0}[\exp(tX) \cdot \mathsf{v}_0] \in T_{[\mathsf{v}_0]}\iota_\rho(\mathscr{F}(\mathfrak{g}, \alpha)) = (\iota_\rho)_*(T_{\mathfrak{p}^+}\mathscr{F}(\mathfrak{g}, \alpha)).$$

We thus have

$$(\iota_\rho)_* \circ d_0\varphi_{\mathsf{std}}(Y) = d_0\varphi^{\mathbb{P}(V)}_{\mathsf{std}} \circ \rho_*(Y)$$

for some $Y \in \mathfrak{u}^-$. Thus $X = \rho_*(Y)$.

By Fact 6.3.10, there exist $N \in \mathbb{N}$ and $\ell_1, \dots, \ell_N \in L$ such that the family $(\mathrm{Ad}(\ell_i) \cdot v^-)_{1 \le i \le N}$ is free and such that $Y = \sum_{i=1}^N \mathrm{Ad}(\ell_i) \cdot v^-$ for some $N \in \mathbb{N}$. Then for all $t \in \mathbb{R}$, one has

$$[\exp(tX) \cdot \mathsf{v}_0] = \Big[\exp\Big(t\rho_*(\sum_{i=1}^{N} \mathrm{Ad}(\ell_i) \cdot v^-)\Big) \cdot \mathsf{v}_0\Big] = \rho\Big(\exp\Big(t(\sum_{i=1}^{N} \mathrm{Ad}(\ell_i) \cdot v^-)\Big) \cdot [\mathsf{v}_0]$$
$$= \rho\Big(\prod_{i=1}^{N} \exp\big(t\,\mathrm{Ad}(\ell_i) \cdot v^-\big)\Big) \cdot [\mathsf{v}_0] = \prod_{i=1}^{N} \exp(t\rho_*(\mathrm{Ad}(\ell_i) \cdot v^-)) \cdot [\mathsf{v}_0],$$

the third equality holding because $\mathfrak{u}^-$ is abelian. This describes a projective line if, and only if, $N = 1$. Thus there exists $\ell \in L$ such that $Y = \mathrm{Ad}(\ell) \cdot v^-$. Then Λ is the image in $\mathbb{P}(V)$ of the photon $\ell \cdot \Lambda_{\mathsf{std}}$. $\square$

Remark 7.1.5. By Example 7.1.1.(1) and the definition of rank-one lines in $\mathrm{Gr}_p(\mathbb{R}^{p+q})$ (see Section 6.3.4.1), Proposition 7.1.4 implies in particular Lemma 6.3.18.

7.1.2 The convex hull of Ω

Let $\mathscr{F}(\mathfrak{g}, \alpha)$ be an irreducible Nagano space of real type, and let $\Omega \subset \mathscr{F}(\mathfrak{g}, \alpha)$ be a proper domain. In general, given a projective $\{\alpha\}$-proximal triple (G, ρ, V) of $\mathfrak{g}$, one can define the convex hull of $\iota_\rho(\Omega)$ in $\mathbb{P}(V)$. However, this convex hull is not necessarily open in $\mathbb{P}(V)$. Nevertheless, as we will see in this section, if (G, ρ, V) is a Plücker triple of $(\mathfrak{g}, \alpha)$, then thanks to Lemma 7.1.2, this property holds. This will allow for a deeper study of the properties of Ω.

Let $(\mathfrak{g}, \alpha)$ be an irreducible Nagano pair of real type, and let (G, ρ, V) be a Plücker triple of $\mathscr{F}(\mathfrak{g}, \alpha)$. Let $\Omega \subset \mathscr{F}(\mathfrak{g}, \alpha)$ be a proper domain. There exists $\xi_0 \in \mathscr{F}(\mathfrak{g}, \alpha)^-$ such that $\mathrm{Z}_{\xi_0} \cap \overline{\Omega} = \emptyset$. Then $\iota_\rho(\overline{\Omega}) \cap \iota_\rho^-(\xi_0) = \emptyset$, which means that $\iota_\rho(\Omega)$ is proper in $\mathbb{P}(V)$. Since it is connected, one can lift it to a proper connected cone $\widetilde{\iota_\rho(\Omega)} \subset V \smallsetminus \{0\}$. Then we define $\widetilde{\mathscr{O}}^\rho_\Omega := \mathrm{Conv}(\widetilde{\iota_\rho(\Omega)})$ its convex hull in V, which is a properly convex cone of V, a priori not necessarily open. We define

$$\mathscr{O}^\rho_\Omega := \mathbb{P}(\widetilde{\mathscr{O}}^\rho_\Omega).$$

It is a properly convex subset of $\mathbb{P}(V)$, and it does not depend on the choice of affine chart containing $\overline{\Omega}$. In particular, the domain $\mathscr{O}^\rho_\Omega$ is $\rho(\mathsf{Aut}_G(\Omega))$-invariant.

Definition 7.1.6. Let $(\mathfrak{g}, \alpha)$ be an irreducible Nagano pair of real type. Given a proper domain $\Omega \subset \mathscr{F}(\mathfrak{g}, \alpha)$ and a Plücker triple (G, ρ, V) of $(\mathfrak{g}, \alpha)$, the properly convex domain $\mathscr{O}^\rho_\Omega$ is unique and called the *convex hull of* Ω *in* $\mathbb{P}(V)$.

For the rest of this section, we fix an irreducible Nagano pair $(\mathfrak{g}, \alpha)$ of real type, a Plücker triple (G, ρ, V) of $\mathscr{F}(\mathfrak{g}, \alpha)$, and $\mathsf{v}_0 \in V^{\omega_\alpha} \smallsetminus \{0\}$.

Proposition 7.1.7. *Let $\Omega \subset \mathscr{F}(\mathfrak{g}, \alpha)$ be a proper domain. The set $\mathscr{O}^\rho_\Omega$ is open. Moreover, if Ω is dually convex, then $\iota_\rho(\partial\Omega) \subset \partial\mathscr{O}^\rho_\Omega$ and $\iota_\rho(\Omega)$ is a connected component of $\mathscr{O}^\rho_\Omega \cap \iota_\rho(\mathscr{F}(\mathfrak{g}, \alpha))$.*

Proof. For the openness, by definition of $\mathscr{O}_\Omega^\rho$, it suffices to prove that for all $x \in \iota_\rho(\Omega)$, there exists a neighborhood $\mathscr{V}$ of x contained in $\mathscr{O}_\Omega^\rho$.

Now let $x, y \in \Omega$, and $(x_0 = x, \dots, x_N = y) \in \mathscr{C}_{x,y}(\Omega)$. For all i, by Lemma 7.1.2, the points x_i and x_{i+1} lie on the interior of a common projective segment contained in $\iota_\rho(\Omega) \subset \mathscr{O}_\Omega^\rho$. Thus x, y are on the same open face of $\mathscr{O}_\Omega^\rho$. This proves that $\iota_\rho(\Omega)$ is contained in a face of $\mathscr{O}_\Omega$. By Fact 2.3.5, this face must be the interior $\mathrm{int}(\mathscr{O}_\Omega^\rho)$ of $\mathscr{O}_\Omega^\rho$ (because otherwise we would have that $\iota_\rho(\Omega)$ is contained in a the proper projective subspace of $\mathbb{P}(V)$ generated by the face containing it). But then, the set $\mathrm{int}(\mathscr{O}_\Omega^\rho)$ is a convex set containing $\iota_\rho(\Omega)$ and contained in the convex hull $\mathscr{O}_\Omega^\rho$ of $\iota_\rho(\Omega)$. By definition of the convex hull, we must have $\mathscr{O}_\Omega^\rho = \mathrm{int}(\mathscr{O}_\Omega^\rho)$. Thus $\mathscr{O}_\Omega^\rho$ is open.

Now let us prove the second assertion. Assume that Ω is dually convex. Then for all $a \in \partial\iota_\rho(\Omega) = \iota_\rho(\partial\Omega)$, there exists $\xi \in \Omega^*$ such that $a \in \iota_\rho^-(\xi)$. Let $f \in V^* \smallsetminus \{0\}$ be a lift of $\iota_\rho^-(\xi)$; Since $\iota_\rho(\Omega) \cap \iota_\rho^-(\xi) = \emptyset$, one has $f(x) \neq 0$ for all $x \in \widetilde{\iota_\rho(\Omega)} \smallsetminus \{0\}$. By connectedness of Ω, we may assume that $f(x) > 0$ for all $x \in \widetilde{\iota_\rho(\Omega)} \smallsetminus \{0\}$. Then taking the convex envelope one has $f(x) > 0$ for all $x \in \widetilde{\mathscr{O}_\Omega^\rho}$. Thus $\mathscr{O}_\Omega^\rho \cap \iota_\rho^-(\xi) = \emptyset$. In particular, since $a \in \iota_\rho^-(\xi)$, one has $a \notin \mathscr{O}_\Omega^\rho$. On the other hand, one has $a \in \overline{\iota_\rho(\Omega)} \subset \overline{\mathscr{O}_\Omega^\rho}$. Thus $a \in \partial\mathscr{O}_\Omega^\rho$.

We have just proven that $\partial\iota_\rho(\Omega) \subset \partial\mathscr{O}_\Omega^\rho \cap \iota_\rho(\mathscr{F}(\mathfrak{g}, \alpha))$. Thus $\iota_\rho(\Omega)$ is closed in $\mathscr{O}_\Omega^\rho \cap \iota_\rho(\mathscr{F}(\mathfrak{g}, \alpha))$. It is also open, so it is a connected component of $\mathscr{O}_\Omega^\rho \cap \iota_\rho(\mathscr{F}(\mathfrak{g}, \alpha))$. $\square$

Another consequence of Proposition 7.1.7 is the following:

Corollary 7.1.8. *Let $\Omega \subset \mathscr{F}(\mathfrak{g}, \alpha)$ be a proper dually convex domain. Let Λ be a photon such that $\Lambda \cap \Omega \neq \emptyset$. Then $\Lambda \cap \Omega$ is connected and $\overline{\Lambda \cap \Omega} = \Lambda \cap \overline{\Omega}$.*

Proof. Since $\mathscr{O}_\Omega^\rho$ is convex and $\iota_\rho(\Lambda)$ is a projective line (Lemma 7.1.2), the intersection $\iota_\rho(\Lambda) \cap \mathscr{O}_\Omega^\rho$ is connected. Since $\Lambda \subset \mathscr{F}(\mathfrak{g}, \alpha)$, it is equal to the intersection $\iota_\rho(\Lambda) \cap \mathscr{O}_\Omega^\rho \cap \iota_\rho(\mathscr{F}(\mathfrak{g}, \alpha))$.

Now since $\iota_\rho(\Omega)$ is both open and closed in $\mathscr{O}_\Omega^\rho \cap \iota_\rho(\mathscr{F}(\mathfrak{g}, \alpha))$ (Proposition 7.1.7), the set $\iota_\rho(\Lambda) \cap \iota_\rho(\Omega)$ is both open closed in $\iota_\rho(\Lambda) \cap \mathscr{O}_\Omega^\rho \cap \iota_\rho(\mathscr{F}(\mathfrak{g}, \alpha)) = \iota_\rho(\Lambda) \cap \mathscr{O}_\Omega^\rho$. It is thus a union of connected components of $\iota_\rho(\Lambda) \cap \mathscr{O}_\Omega^\rho$. But by convexity of $\mathscr{O}_\Omega^\rho$ and Lemma 7.1.2, the set $\iota_\rho(\Lambda) \cap \mathscr{O}_\Omega^\rho$ is connected. Hence we have

$$\iota_\rho(\Lambda \cap \Omega) = \iota_\rho(\Lambda) \cap \iota_\rho(\Omega) = \iota_\rho(\Lambda) \cap \mathscr{O}_\Omega^\rho, \tag{7.1.3}$$

and in particular, the set $\iota_\rho(\Lambda \cap \Omega)$ is connected. Thus so is $\Lambda \cap \Omega$.

To prove that $\overline{\Lambda \cap \Omega} = \Lambda \cap \overline{\Omega}$, we just need to prove the inclusion $\Lambda \cap \overline{\Omega} \subset \overline{\Lambda \cap \Omega}$, the other one being straightforward. By (7.1.3), one has

$$\iota_\rho(\Lambda) \cap \overline{\mathscr{O}_\Omega^\rho} = \overline{\iota_\rho(\Lambda) \cap \iota_\rho(\Omega)} = \iota_\rho(\overline{\Lambda \cap \Omega}),$$

the first equality holding again by convexity of $\mathscr{O}_\Omega^\rho$. On the other hand, we have

$$\iota_\rho(\Lambda \cap \overline{\Omega}) = \iota_\rho(\Lambda) \cap \overline{\iota_\rho(\Omega)} \subset \iota_\rho(\Lambda) \cap \overline{\mathscr{O}_\Omega^\rho}.$$

Hence, by injectivity of ι_ρ, we have $\Lambda \cap \overline{\Omega} \subset \overline{\Lambda \cap \Omega}$. This proves the second assertion. $\square$

Let us finish this section by stating a natural property of proper dually convex domains in Nagano spaces of real type, which they share with domains that are properly convex in an affine chart, as announced in Remark 3.3.3:

Proposition 7.1.9. *Let $\Omega \subset \mathscr{F}(\mathfrak{g}, \alpha)$ be a proper dually convex domain. We have $\Omega = \mathrm{int}(\overline{\Omega})$.*

Proof. Since Ω is an open set contained in $\overline{\Omega}$, it is contained in $\mathrm{int}(\overline{\Omega})$. Let us prove the converse inclusion. We may assume that Ω is a proper domain in $\mathbb{A}_{\mathsf{std}}$.

Let us assume for a contradiction that there exists $a \in \mathrm{int}(\overline{\Omega}) \cap \partial\Omega$. By Proposition 7.1.7, we have $\iota_\rho(a) \in \partial\mathscr{O}^\rho_\Omega$.

On the other hand, since $\iota_\rho(\Omega) \subset \mathscr{O}^\rho_\Omega$, we have

$$\mathrm{int}(\overline{\Omega}) \subset \mathrm{int}(\iota_\rho^{-1}(\overline{\mathscr{O}^\rho_\Omega})).$$

But since ι is a homeomorphism onto its image, we have $\mathrm{int}(\iota_\rho^{-1}(\overline{\mathscr{O}^\rho_\Omega})) = \iota_\rho^{-1}(\overline{\mathscr{O}^\rho_\Omega}))$. Hence:

$$\mathrm{int}(\overline{\Omega}) \subset \iota_\rho^{-1}(\mathrm{int}(\overline{\mathscr{O}^\rho_\Omega})).$$

Now one has $\mathrm{int}(\overline{\mathscr{O}^\rho_\Omega}) = \mathscr{O}^\rho_\Omega$ by openness (again by Proposition 7.1.7) and convexity of $\mathscr{O}^\rho_\Omega$. Hence we have:

$$\mathrm{int}(\overline{\Omega}) \subset \iota_\rho^{-1}(\mathscr{O}^\rho_\Omega),$$

in other words $\iota(\mathrm{int}(\overline{\Omega})) \subset \mathscr{O}^\rho_\Omega$. Thus we have $\iota(a) \in \mathscr{O}^\rho_\Omega \cap \partial\mathscr{O}^\rho_\Omega$, contradiction. Thus $\in \mathrm{int}(\overline{\Omega}) \subset \Omega$. ☐

Remark 7.1.10. Proposition 7.1.9 does not hold anymore in the conformal sphere $\mathscr{F}(\mathfrak{so}(n,1), \alpha_1) = \mathbb{S}^{n-1}$. Indeed, take for instance $x \in \mathsf{F}_{(\mathfrak{g},\alpha)}(\mathbb{X}(\mathfrak{so}(n,1), \alpha_1))$ (recall the map $\mathsf{F}_{(\mathfrak{g},\alpha)}$ from Equation (5.1.6)) and let $\Omega := \mathsf{F}_{(\mathfrak{g},\alpha)}(\mathbb{X}(\mathfrak{so}(n,1), \alpha_1)) \smallsetminus \{a\}$. Then by Example 3.1.12, the domain Ω is proper and dually convex in $\mathbb{S}^{n-1}$, but $\mathrm{int}(\overline{\Omega}) = \mathsf{F}_{(\mathfrak{g},\alpha)}\big(\mathbb{X}(\mathfrak{so}(n,1), \alpha_1)\big) \neq \Omega$.

7.2 $\mathscr{R}$-extremality

In this section, generalizing notions from projective geometry, we define the *$\mathscr{R}$-faces* and the *$\mathscr{R}$-extremal* points of a proper domain Ω in an irreducible Nagano space of real type; see Section 7.2.1. In Section 7.2.2, we investigate the connection between $\mathscr{R}$-faces and the Kobayashi metric. This allows us to relate the structure of the boundary of a proper domain to the dynamics of its automorphism group and to establish a generalization of Fact 1.1.1 in Lemma 7.2.10. In particular, Theorem 7.2.6 will play a key role in the rigidity proofs of Chapter 8. This theorem distinguishes the geometric properties of proper almost-homogeneous domains in higher-rank Nagano spaces from those of real projective space; see Remark 7.2.7.

7.2.1 $\mathscr{R}$-faces

In [LZ19], a notion of $\mathscr{R}$-face and $\mathscr{R}$-extremal point is defined using rank-one lines (i.e. photons, in the setting of this memoir) in Grassmannians. In [Gal24, ?], inspired by the definition of Limbeek–Zimmer, we introduce analogous notions, using the classical photons of Einstein universe in the first case, and the photons in Shilov boundaries associated with HTT Lie groups in the second (see Section 6.3.4.2). Here we generalize these three notions of faces to the context of Nagano spaces of real type.

We fix an irreducible Nagano pair of real type $(\mathfrak{g}, \alpha)$. The following definition extends to all of $\overline{\Omega}$ the relation $\leftrightsquigarrow$, introduced on a proper domain $\Omega \subset \mathscr{F}(\mathfrak{g}, \alpha)$ in Section 6.4.1:

Definition 7.2.1. Let $\Omega \subset \mathscr{F}(\mathfrak{g}, \alpha)$ be a proper domain, and let $a, b \in \overline{\Omega}$. We say that $a \leftrightsquigarrow_{\mathscr{R}} b$ (or simply $a \leftrightsquigarrow b$ if the context is clear) if there exists a photon Λ through a and b, such that a and b belong to the same connected component of the relative interior of $\Lambda \cap \overline{\Omega}$ in Λ.

The $\mathscr{R}$-face of a, denoted by $\mathscr{F}_{\Omega}^{\mathscr{R}}(a)$, is the set of points $c \in \partial\Omega$ for which there exist $N \in \mathbb{N}$ and a sequence $a_0 = a, a_1, \dots, a_N = c \in \partial\Omega$ such that for all $0 \leq i < N$, we have $a_i \leftrightsquigarrow a_{i+1}$.

A point $a \in \partial\Omega$ is said to be *$\mathscr{R}$-extremal* if $\mathscr{F}_{\Omega}^{\mathscr{R}}(a) = \{a\}$.

Example 7.2.2. In all the following examples (except Point(3)), the closures of $\mathscr{R}$-faces of the domain coincide with its dual faces defined in Section 3.1.1.3:

1. Let $x, y \in \mathrm{Ein}^{n-1,1}$ be two transverse points contained in $\mathbb{A}_{\mathsf{std}}$ such that $y \in \mathbf{I}^+(x)$, and let $D := \mathbf{D}(x, y)$. Then
$$\mathrm{Extr}_{\mathscr{R}}(D) = \{x, y\} \cup (\mathbf{C}^+(x) \cap \mathbf{C}^-(y)).$$
The nontrivial $\mathscr{R}$-faces of D are the intervals of photons having either x or y as an extremity. See Figure 7.1.

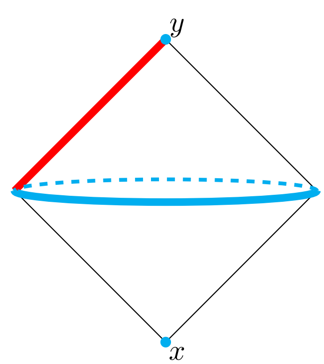

Figure 7.1 – $\mathscr{R}$-faces of the diamond : in blue, the $\mathscr{R}$-extremal points; in red, a nontrivial $\mathscr{R}$-face.

2. In the notation of Section 3.3.1, let $\Omega := \mathbb{B} \subset \mathrm{Gr}_p(\mathbb{R}^{p+q})$. In the notation of Proposition 3.3.1, one has
$$\mathbb{B} = \left\{ \begin{bmatrix} I_p \\ X \end{bmatrix} \mid ||X||_2 < 1 \right\}.$$

One has $\partial\mathbb{B} = \bigsqcup_{i=0}^{p-1} \mathscr{B}_i$, where

$$\begin{aligned}\mathscr{B}_i &:= \{V \in \mathrm{Gr}_p(\mathbb{R}^{p+q}) \mid \mathrm{sgn}(\mathbf{b}_{|V\times V}) = (i,0,p-i)\}\\ &= \mathrm{PO}(p,q)\cdot \mathrm{Span}(e_1+e_{p+1},\dots,e_i+e_{p+i},e_{i+1},\dots,e_p)\\ &= \mathrm{PO}(\mathbf{b}_{\mathsf{std}})\cdot \begin{bmatrix} I_p & \\ I_i & 0\\ 0 & 0\end{bmatrix}.\end{aligned}$$

Since $\mathrm{PO}(\mathbf{b}_{\mathsf{std}}) = \mathsf{Aut}_{\mathrm{PGL}(p+q,\mathbb{R})}(\Omega)$, it suffices to determine the $\mathscr{R}$-faces of the points $x_i := \begin{bmatrix} I_p & \\ I_i & 0\\ 0 & 0\end{bmatrix}$, with $0 \leq i \leq p-1$. An explicit computation gives

$$\mathscr{F}_\Omega^{\mathscr{R}}(x_i) = \Big\{ \begin{bmatrix} I_p & \\ I_i & 0\\ 0 & M\end{bmatrix} \mid M \in \mathrm{Mat}_{q-i,p-i}(\mathbb{R}),\ ||M||_2 < 1\Big\}.$$

Thus the $\mathscr{R}$-extremal points of $\mathbb{B}$ are the elements of the orbit $\mathrm{PO}(p,q)\cdot x_0$, i.e. the totally isotropic p-planes of $(\mathbb{R}^{p,q}, \mathbf{b}_{\mathsf{std}})$.

3. If $\Omega \subset \mathscr{F}(\mathfrak{g},\alpha)$ is a proper dually convex domain, then Ω is an $\mathscr{R}$-face of Ω.
4. If $(\mathfrak{g},\alpha) = (\mathfrak{sl}(n,\mathbb{R}),\alpha_1)$, then by Remark 6.3.19 the $\mathscr{R}$-faces of a properly convex domain of $\mathscr{F}(\mathfrak{g},\alpha) = \mathbb{P}(\mathbb{R}^n)$ coincide with its classical projective faces.

The following proposition generalizes a well-known fact of convex projective geometry:

Proposition 7.2.3. *Let $\Omega \subset \mathscr{F}(\mathfrak{g},\alpha)$ be a proper domain. Let $a \in \partial\Omega$ and let $b \in \Omega^*$ be such that $a \in \mathrm{Z}_b$. One has $\mathscr{F}_\Omega^{\mathscr{R}}(a) \subset \mathrm{Z}_b$.*

Proof. Let Λ be a photon through x such that $x \in \mathrm{intrel}_\Lambda(\partial\Omega \cap \Lambda)$. We may assume that $\overline{\Omega} \subset \mathbb{A}_{\mathsf{std}}$, $x = \mathfrak{p}^+$ and $\Lambda = \Lambda_{\mathsf{std}}$.

Let (G,ρ,V) be a Plücker triple of $\mathscr{F}(\mathfrak{g},\alpha)$. Let $\mathsf{v}_0 \in V^{\omega_\alpha} \smallsetminus \{0\}$, and let $f \in V^* \smallsetminus \{0\}$ be a lift of $\iota_\rho^-(b)$. By Lemma 5.2.1, we have, for all $t \in \mathbb{R}$:

$$f(\exp(tv^-)\cdot\mathfrak{p}^+) = f(\mathsf{v}_0 + t\rho_*(v^-)\cdot\mathsf{v}_0) = tf(\rho_*(v^-)\cdot\mathsf{v}_0). \tag{7.2.1}$$

Note that v_0 is a lift of $\iota_\rho(x) \smallsetminus \{0\}$. Since Ω is connected and $b \in \Omega^*$, we may lift $\iota_\rho(\Omega)$ to a proper connected cone C of V containing v_0 such that $f(\mathsf{v}) \neq 0$ for all $\mathsf{v} \in C$. By connectedness of C, we may assume that $f(\mathsf{v}) > 0$ for all $\mathsf{v} \in C$.

Note that there exists $\epsilon > 0$ such that $[\mathsf{v}_0 + tv^-] \in \iota_\rho(\partial\Omega)$ for all $t \in]-\epsilon,\epsilon[$. Thus we have $tf(\rho_*(v^-)\cdot\mathsf{v}_0) = f(v^-) \geq 0$ for all $t \in]-\epsilon,\epsilon[$. This implies that $f(v^-) = 0$. Hence by Equation (7.2.1), we have $f(\exp(tv^-)\cdot\mathsf{v}_0) = 0$ for all $t \in \mathbb{R}$. This implies that $\Lambda \subset \mathrm{Z}_b$.

By definition of $\mathscr{F}_\Omega^{\mathscr{R}}(x)$, we have $\mathscr{F}_\Omega^{\mathscr{R}}(x) \subset \mathrm{Z}_b$. $\square$

Remark 7.2.4. Proposition 7.2.3 implies in particular that the $\mathscr{R}$-face of a boundary point of a proper dually convex domain of $\mathscr{F}(\mathfrak{g},\alpha)$ is included in its dual face.

In projective space $\mathbb{P}(\mathbb{R}^n)$, the extremal points of a properly convex open subset generate $\mathbb{R}^n$, by Krein–Milman's Theorem. In Nagano spaces of real type, the existence of Plücker triples allows to recover this property for $\mathscr{R}$-extremal points of proper domains:

Lemma 7.2.5. *Let $(\mathfrak{g}, \alpha)$ be n irreducible Nagano pair of real type and (G, ρ, V) a Plücker triple of $(\mathfrak{g}, \alpha)$. Let $\Omega \subset \mathscr{F}(\mathfrak{g}, \alpha)$ be a proper domain. Then (projective) extremal points of $\mathscr{O}_\Omega$ are images of $\mathscr{R}$-extremal points of Ω. In particular, the set $\mathrm{Extr}_{\mathscr{R}}(\Omega)$ is nonempty and generates V, in the sense that there exist $x_1, \dots, x_D \in \mathrm{Extr}_{\mathscr{R}}(\Omega)$ such that*

$$V = \iota_\rho(x_1) \oplus \cdots \oplus \iota_\rho(x_D).$$

Proof. Let $\mathscr{O} := \mathscr{O}^\rho_\Omega$ be the convex hull of $\iota_\rho(\Omega)$ in $\mathbb{P}(V)$ defined in Section 7.1.2. Given two distinct points $x, y \in \overline{\mathscr{O}}$, we denote by (x, y) the unique connected component of $\ell \smallsetminus \{x, y\}$ intersecting $\mathscr{O}$, where ℓ is the unique projective line containing x and y.

Let x be an extremal point of $\mathscr{O}$. Then by definition of $\mathscr{O}$, there exists $z \in \overline{\Omega}$ such that $x = \iota_\rho(z)$. Moreover, $z \notin \Omega$ because $\iota_\rho(\Omega) \subset \mathscr{O}$. Then $z \in \partial\Omega$. If z is not $\mathscr{R}$-extremal, then there exists a rank-one line ℓ and $a, b \in \ell \cap \partial\Omega$ such that a, z, b are distinct and aligned in this order. Then z is included in the nontrivial projective interval $(\iota_\rho(a), \iota_\rho(b))$ of $\overline{\mathscr{O}}$, which contradicts the fact that x is $\mathscr{R}$-extremal.

Hence every $\mathscr{R}$-extremal point of $\mathscr{O}$ is the image of an $\mathscr{R}$-extremal point of $\partial\Omega$. Thus by Krein–Milman's theorem, the convex set $\mathscr{O}$ is the open convex hull (in a suitable affine chart) of $\iota_\rho(\mathrm{Extr}_{\mathscr{R}}(\Omega))$. Hence the result by openness of $\mathscr{O}$. □

7.2.2 Geometric and dynamical properties of extremal points

In [LZ19], Limbeek–Zimmer investigate the geometric properties of $\mathscr{R}$-extremal points of proper divisible domains of $\mathrm{Gr}_p(\mathbb{R}^{2p})$. In this section, following their strategy, we investigate the relation between the structure of the boundary of a proper almost-homogeneous domain in a Nagano spacce of real type and the dynamics of its automorphism group. Whenever Ω is almost-homogeneous, $\mathscr{R}$-extremal points satisfy the following geometric property:

Theorem 7.2.6. *Let $(\mathfrak{g}, \alpha)$ be an irreducible Nagano pair of real type. Assume that Ω is a proper almost-homogeneous domain of $\mathscr{F}(\mathfrak{g}, \alpha)$. Let $a \in \partial\Omega$ be an $\mathscr{R}$-extremal point. Then there exists $b \in \Omega^*$ such that* $\mathrm{id} \in \mathsf{pos}^{(\alpha, \mathsf{i}(\alpha))}(p, q)$. *In particular, if $\mathscr{F}(\mathfrak{g}, \alpha)$ is self-opposite, then $a \in \Omega^*$.*

Remark 7.2.7. 1. If $(\mathfrak{g}, \alpha) = (\mathfrak{sl}(n, \mathbb{R}), \alpha_1)$, then Theorem 7.2.6 simply states that every extremal point of a proper almost-homogeneous domain Ω of $\mathscr{F}(\mathfrak{g}, \alpha) = \mathbb{P}(\mathbb{R}^n)$ is contained in a projective hyperplane tangent to Ω. But this directly follows from the convexity of Ω (since almost-homogeneity implies convexity by Proposition 3.1.11) and does not distinguish extremal points from other points of $\partial\Omega$. This phenomenon arises because, in this case, we have $|W_{\Delta \smallsetminus \{\alpha\}} \backslash W / W_{\Delta \smallsetminus \{\mathsf{i}(\alpha)\}}| = 2$, meaning that for any pair $(x, H) \in \mathbb{P}(\mathbb{R}^n) \times \mathbb{P}((\mathbb{R}^n)^*)$, the only possible transversality degrees between x and H are: either $x \in H$ or $x \notin H$. This example illustrates that the smaller the cardinality of the set $W_{\{\alpha\}} \backslash W / W_{\{i(\alpha)\}}$ is, the weaker the geometric property of $\mathscr{R}$-extremal points stated in Theorem 7.2.6 becomes.

2. Theorem 7.2.6 tells us that proper almost-homogeneous domains in higher-rank Nagano spaces of real type $\mathscr{F}(\mathfrak{g}, \alpha)$ cannot be strictly convex, in the sense that not all points of $\partial\Omega$ can be $\mathscr{R}$-extremal. Indeed, let $x \in \Omega$, and let Λ be a photon through x. Then by properness there exists $a \in \Lambda \cap \partial\Omega$. If $a \in \mathrm{Extr}_{\mathscr{R}}(\Omega)$, then by Theorem 7.2.6, there exists $b \in \Omega^*$ such that $\mathrm{id} \in \mathsf{pos}^{(\{\alpha\},\{i(\alpha)\})}(a, b)$. Then, by Lemma 6.3.9 and the higher-rank assumption, we must have $x \in \mathrm{Z}_b$, contradicting the assumption that $b \in \Omega^*$. This shows that there exist points in $\partial\Omega$ that are not $\mathscr{R}$-extremal.

   This observation thus distinguishes proper almost-homogeneous domains in real projective space (which can be strictly convex, as we will discuss in Section 8.2) from those in higher-rank Nagano spaces of real type. We will further explore this distinction in Section 8.2.

For the proof of Theorem 7.2.6, we fix an irreducible Nagano pair $(\mathfrak{g}, \alpha)$ of real type and $G \in \mathscr{G}_{\{\alpha\}}(\mathfrak{g})$, and follow the strategy of [LZ19, Thm 7.4]. We will need the following definition:

Definition 7.2.8. Let Ω be a proper domain of $\mathscr{F}(\mathfrak{g}, \Theta)$ and $x, y \in \Omega$, and $N \in \mathbb{N}^*$. Let us define

$$K_\Omega^N(x, y) := \inf \left\{ \mathsf{len}_\Omega(\gamma) \middle| \, \gamma \in \mathscr{C}_{x,y}^N(\Omega) \right\}.$$

Recall that $\mathsf{len}_\Omega(\gamma)$ is the K_Ω-length of the path γ (see Section 6.4.4). The quantity $K_\Omega^N(x, y)$ is finite if and only if the set $\mathscr{C}_{x,y}^N(\Omega)$ is nonempty.

The map $K_\Omega^N : \Omega \times \Omega \to \mathbb{R} \cup \{\infty\}$ is $\mathsf{Aut}_G(\Omega)$-invariant. The sequence $(K_\Omega^N(x, y))_{N \in \mathbb{N}}$ is nonincreasing, eventually finite, and one has

$$K_\Omega(x, y) = \lim_{N \to +\infty} K_\Omega^N(x, y).$$

In classical convex projective geometry, two sequences of points of a properly convex domain Ω of real projective space remaining at bounded Hilbert distance converge to points lying on a common projective face of Ω. The following lemma is a generalization of this fact.

Lemma 7.2.9. *Let $\Omega \subset \mathscr{F}(\mathfrak{g}, \Theta)$ be a proper dually convex domain. Let $a, b \in \partial\Omega$. Assume that there exist $(x_k), (y_k) \in \Omega^{\mathbb{N}}$ such that $x_k \to a$ and $y_k \to b$, and such that there exist $N \in \mathbb{N}$ and $M > 0$ such that $K_\Omega^N(x_k, y_k) \le M$ for all $k \in \mathbb{N}$. Then $\mathscr{F}_\Omega^{\mathscr{R}}(a) = \mathscr{F}_\Omega^{\mathscr{R}}(b)$. In particular, if $a \in \mathrm{Extr}_{\mathscr{R}}(\Omega)$, then $a = b$.*

Proof. Note that we just need to prove that $b \in \mathscr{F}_\Omega^{\mathscr{R}}(a)$. For any $k \in \mathbb{N}$, let $\gamma_k = (x_k^0 := x_k, \dots, x_k^N := y_k) \in \mathscr{C}_{x_k,y_k}^N(\Omega)$ be such that

$$\sum_{i=0}^{N-1} K_\Omega(x_i^k, x_{i+1}^k) = \mathsf{len}_\Omega(\gamma_k) \le K_\Omega^N(x_k, y_k) + 1 \le M + 1,$$

the first equality holding because of Equation (6.4.6). Then, one has $K_\Omega(x_k^i, x_k^{i+1}) \le M+1$ for all $0 \le i \le N-1$. Hence one can assume that $N = 1$, and the lemma follows by induction.

Let us then assume that $N = 1$. For all k, the two points x_k and y_k lie in the same connected component of the intersection $I_k := \Lambda_k \cap \Omega$ of a photon Λ_k with Ω. Let c_k, d_k be the endpoints of I_k such that c_k, x_k, y_k, d_k are aligned in this order. If $g_k \in G$ is such that $\Lambda_k = g_k \cdot \Lambda_{\mathsf{std}}$ and if we define $\zeta^k := \zeta_{g_k^{-1}}$ (recall Equation (6.3.2)), then there exist $r_k, s_k, t_k, u_k \in \mathbb{P}(\mathbb{R}^2)$, aligned in this order, such that

$$\zeta^k(r_k) = c_k; \quad \zeta^k(s_k) = x_k; \quad \zeta^k(t_k) = y_k; \quad \zeta^k(u_k) = d_k.$$

Then Proposition 6.4.10 implies that

$$\log(r_k : s_k : t_k : u_k) = \mathsf{k}_\Omega(x_k, y_k) = K_\Omega(x_k, y_k) \leq M + 1.$$

Up to extracting, we may assume that there exist $c, d \in \partial\Omega$ such that $c_k \to c$ and $d_k \to d$ as $k \to +\infty$, and also that there exist $r, s, t, u \in \mathbb{P}(\mathbb{R}^2)$ such that $(r_k, s_k, t_k, u_k) \to (r, s, t, u)$. For all $k \in \mathbb{N}$, the points c_k, x_k, y_k, d_k lie on the same photon in this order, so c, a, b, d lie on the same photon, in this order. Moreover, since for all $k \in \mathbb{N}$ we have $(c_k, d_k) \subset \overline{\Omega}$, we have $(c, d) \subset \overline{\Omega}$. We have

$$\log(r : s : t : u) \leq M + 1.$$

Thus either $s, t \in (r, u)$ or $s = t$. Thus either $a, b \in (c, d)$ or $a = b$. We have just proven that $b \in \mathscr{F}_\Omega^{\mathscr{R}}(a)$. □

We can now prove a generalization of Fact 1.1.1:

Lemma 7.2.10. *Let Ω be a proper dually convex domain of $\mathscr{F}(\mathfrak{g}, \alpha)$, and let $a \in \mathrm{Extr}_{\mathscr{R}}(\Omega)$. If there exist $(g_k) \in \mathsf{Aut}_G(\Omega)^{\mathbb{N}}$ and $x \in \Omega$ such that $g_k \cdot x \to a$, then for every compact subset $\mathsf{K} \subset \Omega$, one has $g_k \cdot \mathsf{K} \to \{a\}$ for the Hausdorff topology. In particular, this sequence (g_k) is $\{\alpha\}$-contracting.*

Proof. Let $y \in \Omega$ and $N \in \mathbb{N}$ such that $K_\Omega^N(x, y) < +\infty$. Then, by $\mathsf{Aut}_G(\Omega)$-invariance of K_Ω^N, one has

$$K_\Omega^N(g_k \cdot x, g_k \cdot y) = K_\Omega^N(x, y) \quad \forall k \in \mathbb{N}.$$

Thus by Lemma 7.2.9, we have $g_k \cdot y \to a$. This holds for all $y \in \Omega$.

Let $\mathsf{K} \subset \Omega$ be a compact subset. If the sequence $g_k \cdot \mathsf{K}$ does not converge to $\{a\}$ for the Hausdorff topology, then there is a neighborhood $\mathscr{V}$ of a in $\mathscr{F}(\mathfrak{g}, \Theta)$ and a sequence $(y_k) \in \mathsf{K}^{\mathbb{N}}$ such that $g_k \cdot y_k \notin \mathscr{V}$ for all $k \in \mathbb{N}$. Since K is a compact subset of Ω by Corollary 6.4.12, up to extracting we may assume that there exists $y \in \mathsf{K}$ such that $y_k \to y$. Then $(g_k \cdot y)$ converges to a. But $K_\Omega(g_k \cdot y_k, g_k \cdot y) \to 0$, so $g_k \cdot y_k \to a$. But this is impossible, since we assumed that $y_k \notin \mathscr{V}$ for all k.

Hence $g_k \cdot \mathsf{K} \to \{a\}$ for the Hausdorff topology. □

The proof of Theorem 7.2.6 will now follow from the next lemma:

Lemma 7.2.11. *Let Ω is a proper domain of $\mathscr{F}(\mathfrak{g}, \alpha)$. Let $a \in \Lambda_{\{\alpha\}}(\mathsf{Aut}_G(\Omega))$. Then there exists $b \in \Omega^*$ such that $\mathsf{pos}^{(\{\alpha\}, \{\mathsf{i}(\alpha)\})}(a, b) = \overline{\mathrm{id}}$ In particular, if $\mathscr{F}(\mathfrak{g}, \alpha)$ is self-opposite, then $a \in \Omega^*$.*

Proof of Lemma 7.2.11. By definition of $\Lambda_{\{\alpha\}}(\mathsf{Aut}_G(\Omega))$, there exist a sequence $(g_k) \in \mathsf{Aut}_G(\Omega)^{\mathbb{N}}$ and a point $b \in \mathscr{F}(\mathfrak{g},\alpha)^-$ such that $g_k \cdot x \to a$ uniformly on compact subsets of $\mathscr{F}(\mathfrak{g},\alpha) \smallsetminus \mathrm{Z}_b$. Since Ω has nonempty interior, there exists $x \in \Omega \smallsetminus \mathrm{Z}_b$. Hence $g_k \cdot x \to a$. Since the set

$$\left\{y \in \mathscr{F}(\mathfrak{g},\Theta)^- \mid w_0 \in \mathsf{pos}^{(\{\mathsf{i}(\alpha)\},\{\mathsf{i}(\alpha)\})}(b,y)\right\}$$

is dense in $\mathscr{F}(\mathfrak{g},\Theta)^-$ and Ω^* has nonempty interior (see Section 3.1.1.2), we can fix $y \in \Omega^*$ such that $w_0 \in \mathsf{pos}^{(\{\mathsf{i}(\alpha)\},\{\mathsf{i}(\alpha)\})}(b,y)$. Now let $a' \in \mathscr{F}(\mathfrak{g},\Theta)$ such that $\mathsf{pos}^{(\{\alpha\},\{\mathsf{i}(\alpha)\})}(a',y) = \overline{\mathrm{id}}$. Then necessarily by Lemma 2.2.6 one has $\mathsf{pos}^{(\{\alpha\},\{\mathsf{i}(\alpha)\})}(a',b) = \overline{w_0}$. Hence we have $g_k \cdot a' \to a$. On the other hand, up to extracing we may assume that $(g_k \cdot y)$ converges to some $c \in \Omega^*$. For all $k \in \mathbb{N}$, we have $\mathsf{pos}^{(\{\alpha\},\{\mathsf{i}(\alpha)\})}(g_k \cdot a', g_k \cdot y) = \overline{\mathrm{id}}$, so by [KLP18, Lem. 3.15], we can take the limit and get $\mathsf{pos}^{(\{\alpha\},\{\mathsf{i}(\alpha)\})}(a,c) = \overline{\mathrm{id}}$. ☐

We can now prove Theorem 7.2.6:

Proof of Theorem 7.2.6. Since Ω is almost-homogeneous, we can find $x \in \Omega$ and $(g_k) \in \mathsf{Aut}_G(\Omega)^{\mathbb{N}}$ such that $g_k \cdot x \to a$. Since Ω is dually convex By Proposition 3.1.11, we know that Ω is dually convex. We can thus apply Lemma 7.2.10, and there exists $b \in \mathscr{F}(\mathfrak{g},\alpha)^-$ such that (g_k) is $\{\alpha\}$-contracting with respect to (a,b), in the sense of Section 2.3.1. Thus $a \in \Lambda_{\{\alpha\}}(\mathsf{Aut}_G(\Omega))$. The theorem then follows by Lemma 7.2.11. ☐

7.2.3 The proximal limit set

In this section, using the results from Section 7.2.2, we prove Proposition 7.2.12 below, which generalizes a well-known fact in convex projective geometry.

Let G be a noncompact semisimple Lie group and Θ be a subset of the simple restricted roots of G. An element $g \in G$ is Θ-proximal if it has two transverse fixed points $x \in \mathscr{F}(\mathfrak{g},\Theta)$ and $y \in \mathscr{F}(\mathfrak{g},\Theta)^-$ such that $g^n \cdot z \to x$ for all $z \in \mathscr{F}(\mathfrak{g},\Theta) \smallsetminus \mathrm{Z}_y$. The points x and y are then uniquely defined by g. A subgroup $H \leq G$ is Θ-proximal if it contains at least one proximal element. In this case, we define

$$\Lambda_\Theta^{\mathrm{prox}}(H) = \overline{\{x \in \mathscr{F}(\mathfrak{g},\alpha) \mid \exists g \in H,\ g \text{ proximal with attracting fixed point } x\}}.$$

By definition, we have $\Lambda_\Theta^{\mathrm{prox}}(H) \subset \Lambda_\Theta(H)$. We then have:

Proposition 7.2.12. *Let $(\mathfrak{g},\alpha)$ be an irreducible Nagano pair of real type, and $G \in \mathscr{G}_{\{\alpha\}}(\mathfrak{g})$. Let $\Omega \subset \mathscr{F}(\mathfrak{g},\alpha)$ be a proper domain. If $H \leq \mathsf{Aut}_G(\Omega)$ acts almost-homogeneously on Ω, i.e. if*

$$\{a \in \partial\Omega \mid \exists (h_n) \in H^{\mathbb{N}},\ \exists x \in \Omega,\ h_n \cdot x \to a\} = \partial\Omega,$$

then H is $\{\alpha\}$-proximal and

$$\Lambda_{\{\alpha\}}^{\mathrm{prox}}(H) = \overline{\mathrm{Extr}_{\mathscr{R}}(\Omega)} = \Lambda_{\{\alpha\}}(H).$$

In the notation of Proposition 7.2.12, if H acts cocompactly on Ω, then it acts almost-homogeneously. If Ω is almost-homogeneous, then the conditions of Proposition 7.2.12 are satisfied for $H = \mathsf{Aut}_G(\Omega)$.

Proof of Proposition 7.2.12. Let us first prove that $\mathrm{Extr}_{\mathscr{R}}(\Omega) \subset \Lambda_\Theta^{\mathrm{prox}}(H)$. To this end we use the strategy of the proof of [Bla21, Prop. 2.3.15]. Let $a \in \mathrm{Extr}_{\mathscr{R}}(\Omega)$, and let $(g_k) \in H^{\mathbb{N}}$, $x \in \Omega$ such that $g_k \cdot x \to a$. By Lemma 7.2.10, there exists $b \in \mathscr{F}(\mathfrak{g}, \alpha)^-$ such that $g_k \cdot x \to a$ uniformly on compact subsets of $\mathscr{F}(\mathfrak{g}, \alpha) \smallsetminus \mathrm{Z}_b$.

Let (G, ρ, V) be a Plücker triple of $(\mathfrak{g}, \alpha)$. Then the sequence $(\rho(g_k))$ is $\{\alpha_1\}$-contracting in $\mathrm{SL}(V)$. Thus there exists a rank-one endomorphism $\pi_1 \in \mathrm{End}(V)$ such that $\frac{\rho(g_k)}{||\rho(g_k)||} \to \pi_1$. Then $\mathrm{Im}(\pi_1) = \iota_\rho(a)$ and $\ker(\pi_1) = \iota_\rho^-(b)$. By Lemma 7.2.5, we do not have $\iota_\rho(\mathrm{Extr}_{\mathscr{R}}(\Omega)) \subset \iota_\rho^-(b)$. Thus there exists $a_2 \in \mathrm{Extr}_{\mathscr{R}}(\Omega)$ such that $\iota_\rho(a_2) \notin \ker(\pi_1)$. Denoting $a_1 := a$, by induction there exist $\dim(V)$ points $a_1, a_2, \dots, a_{\dim(V)} \in \mathrm{Extr}_{\mathscr{R}}(\Omega)$ and rank-one endomorphisms

$$\pi_1, \dots, \pi_{\dim(V)} \in \mathrm{End(V)} \cap \overline{\rho(H)}^{\mathrm{End}(V)}$$

such that

$$\begin{aligned} \iota_\rho(a_i) &= \mathrm{Im}(\pi_i) \quad \forall \leq i \leq \dim(V); \\ \iota_\rho(a_{i+1}) &\notin \ker(\pi_i) \quad \forall 1 \leq i \leq \dim(V) - 1. \end{aligned} \tag{7.2.2}$$

Thus $\iota_\rho(a_i) = \pi_i(a_{i+1}) \in \overline{\rho(H) \cdot \iota_\rho(a_{i+1})}$ and

$$a_i \in \overline{H \cdot a_{i+1}} \quad \forall 1 \leq i \leq \dim(V) - 1. \tag{7.2.3}$$

By induction, we thus have $a \in \overline{H \cdot a_{i+1}}$.

Assume for a contradiction that $a \notin \Lambda_{\{\alpha\}}^{\mathrm{prox}}(H)$. Since $\Lambda_{\{\alpha\}}^{\mathrm{prox}}(H)$ is closed and H-invariant, by Equation (7.2.3) for all $1 \leq i \leq \dim(V)$ we have $a_i \notin \Lambda_{\{\alpha\}}^{\mathrm{prox}}(H)$. By Equation (7.2.2), for all $1 \leq i \leq j \leq \dim(V)$, we have $\iota_\rho(a_i) = \mathrm{Im}(\pi_i \circ \pi_{i+1} \circ \cdots \circ \pi_j)$. Thus $\pi_i \circ \pi_{i+1} \circ \cdots \circ \pi_j$ is not proximal. Thus $\iota_\rho(a_i) \in \ker(\pi_i \circ \pi_{i+1} \circ \cdots \circ \pi_j) = \ker(\pi_j)$.

We have just proved that for all $1 \leq i \leq j \leq \dim(V)$, we have $\iota_\rho(a_i) \in \ker(\pi_j)$. In paricular, for all $1 \leq i \leq \dim(V) - 1$, we have

$$\iota_\rho(a_{i+1}) \in (\ker(\pi_{i+1}) \cap \cdots \cap \ker(\pi_{\dim(V)})) \smallsetminus \ker(\pi_i).$$

Thus the sequence $(\ker(\pi_i) \cap \cdots \cap \ker(\pi_{\dim(V)}))_{1 \leq i \leq \dim(V)}$ is an increasing sequence of nonempty vector subspaces of V. Thus $\dim(\ker(\pi_{\dim(V)})) \geq \dim(V)$. This is in contradiction with the fact that $\pi_{\dim(V)} \neq 0$. Thus $a \in \Lambda_{\{\alpha\}}^{\mathrm{prox}}(H)$.

By closedness of $\Lambda_{\{\alpha\}}^{\mathrm{prox}}(H)$, we have $\overline{\mathrm{Extr}_{\mathscr{R}}(\Omega)} \subset \Lambda_{\{\alpha\}}^{\mathrm{prox}}(H)$.

Now let us prove that $\Lambda_{\{\alpha\}}(H) \subset \overline{\mathrm{Extr}_{\mathscr{R}}(\Omega)}$. Let $x \in \Lambda_{\{\alpha\}}(H)$. Let $(g_k) \in H^{\mathbb{N}}$ and $y \in \mathscr{F}(\mathfrak{g}, \alpha)^-$ such that (g_k) is $\{\alpha\}$-contracting with respect to (x, y). By Lemma 7.2.5, there exists $z \in \mathrm{Extr}_{\mathscr{R}}(\Omega)$ such that $\iota_\rho(z) \notin \iota_\rho^-(y)$. Thus $z \notin \mathrm{Z}_y$, and $g_k \cdot z \to x$. By $\mathsf{Aut}_G(\Omega)$-invariance of $\mathrm{Extr}_{\mathscr{R}}(\Omega)$, we then have $x \in \overline{\mathrm{Extr}_{\mathscr{R}}(\Omega)}$. We have proven that $\Lambda_{\{\alpha\}}(H) \subset \overline{\mathrm{Extr}_{\mathscr{R}}(\Omega)}$. We thus have

$$\Lambda_{\{\alpha\}}(H) \subset \overline{\mathrm{Extr}_{\mathscr{R}}(\Omega)} \subset \Lambda_{\{\alpha\}}^{\mathrm{prox}}(H) \subset \Lambda_{\{\alpha\}}(H).$$

Hence these inclusions are equalities. $\square$

Chapter 8

Divisible convex sets and rigidity

In this chapter, we study Question 1.2.1. First, we examine the general case in Section 8.1, where we strengthen Facts 1.2.2 and 1.2.3, stated for quasi-homogeneous domains, to the case of proper almost-homogeneous domains. Then, in Section 8.2, we focus on the case of Nagano spaces of real type and prove Theorem 8.2.2 and its Corollary 8.2.3. In Sections 8.3, 8.4, 8.5 and 8.6, we conduct an in-depth study of proper almost-homogeneous domains in the three key families of examples from Sections 2.4.2 to 2.4.4. The main theorems are 8.3.1, 8.4.1, and 8.5.1. In Section 8.7, we investigate the rigidity of proper divisible domains in a flag manifold that is *not* an irreducible Nagano space, see Proposition 8.7.1.Finally, in the last Section 8.8, we study closed proper manifolds locally modeled on flag manifolds. Theorems 8.3.1 and 8.4.1 allow us to classify these manifolds when the flag manifold is either a causal flag manifold or Einstein universe.

The results of this chapter will be interpreted and explained in Section 8.9, where we will be able refine Question 1.2.1, see Conjecture 8.9.1.

Notation 8.0.1. For all this chapter, whenever we consider an irreducible Nagano pair $(\mathfrak{g}, \alpha)$, we will use Notation 5.1.2.

8.1 The general case

Let us first note that we can strengthen Facts 1.2.2 and 1.2.3, using the formalism of Section 3.1.1.1:

Lemma 8.1.1. *Let $\mathfrak{g}$ be a semisimple Lie algebra with no compact factors, and write $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$, where $\mathfrak{g}_i$ is a simple Lie algebra of noncompact type for all $1 \leq i \leq k$. For any subset Θ of the simple restricted roots of $\mathfrak{g}$, there exist subsets Θ_i of the simple restricted roots of $\mathfrak{g}_i$ for all $1 \leq i \leq k$ such that $\Theta = \Theta_1 \cup \cdots \cup \Theta_k$. We then have an $\mathsf{Aut}_\Theta(\mathfrak{g})$-equivariant identification*

$$\mathscr{F}(\mathfrak{g}, \Theta) \simeq \mathscr{F}(\mathfrak{g}_1, \Theta_1) \times \cdots \times \mathscr{F}(\mathfrak{g}_k, \Theta_k). \tag{8.1.1}$$

Now let $\Omega \subset \mathscr{F}(\mathfrak{g}, \Theta)$ be a proper almost-homogeneous domain. Then there exist proper almost-homogeneous domains $\Omega_i \subset \mathscr{F}(\mathfrak{g}_i, \Theta_i)$ such that $\Omega = \Omega_1 \times \cdots \times \Omega_k$.

Proof. Equation (8.1.1) is just a general fact on flag manifolds. For all $1 \leq i \leq k$, let $G_i \in \mathscr{G}_{\Theta_i}(\mathfrak{g}_i)$. Then $G := G_1 \times \cdots \times G_k \in \mathscr{G}_\Theta(\mathfrak{g})$.

Let us denote by $\Omega_i \subset \mathscr{F}(\mathfrak{g}_i, \Theta_i)$ the image of Ω by the canonical G_i-equivariant projection $\Pi_i : \mathscr{F}(\mathfrak{g}, \Theta) \to \mathscr{F}(\mathfrak{g}_i, \Theta_i)$ for $1 \leq i \leq k$, and let $\Omega' := \Omega_1 \times \cdots \times \Omega_k$. Then Ω' is a proper $\mathsf{Aut}_G(\Omega)$-invariant domain of $\mathscr{F}(\mathfrak{g}, \Theta)$ containing Ω. Then by Lemma 3.1.7, we have $\Omega = \Omega'$.

Moreover, note that for all $1 \leq i \leq k$, the domain Ω_i is almost-homogeneous, since $\partial\Omega_i \subset \Pi_i(\partial\Omega)$ and Π_i is G_i-equivariant. ☐

Lemma 8.1.2. *Let $\mathfrak{g}$ be a real simple Lie algebra of noncompact type and Θ a subset of the simple restricted roots of $\mathfrak{g}$, with $|\Theta| \geq 2$. Then there are no proper almost-homogeneous domains in $\mathscr{F}(\mathfrak{g}, \Theta)$.*

We do not give the proof of Lemma 8.1.1, as it is exactly the same as in [Zim18a], replacing "quasi-homogeneous" by "almost-homogeneous"

8.2 Rigidity in Nagano spaces of real type

In this section, we analyze two properties of proper almost-homogeneous domains in Nagano spaces of real type and of rank ≥ 2 (i.e. those different from real projective space).

As already mentioned in the introduction, in convex projective geometry, a famous result by Benoist states the following:

Fact 8.2.1. *[Ben01] Let $\Gamma \leq \mathrm{PGL}(n, \mathbb{R})$ be a discrete subgroup acting cocompactly on a proper strictly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^n)$. Then the following assertions are equivalent:*

1. *The group Γ is Gromov-hyperbolic.*
2. *The geodesic metric space $(\Omega, \mathsf{H}_\Omega)$ is Gromov-hyperbolic, where H_Ω is the Hilbert metric on Ω (see Section 2.1.1.3).*
3. *The domain Ω is strictly convex.*

In the case where $\mathscr{F}(\mathfrak{g}, \alpha)$ is a Nagano space of real type of rank ≥ 2, if $\Omega \subset \mathscr{F}(\mathfrak{g}, \alpha)$ is a proper domain, divisible (and even just almost-homogeneous) by a discrete subgroup $\Gamma \leq G$ (where $G \in \mathscr{G}_{\{\alpha\}}(\mathfrak{g})$), it cannot be "strictly convex", in the sense that not all points of $\partial\Omega$ can be $\mathscr{R}$-extremal (see Remark 7.2.7.(2)). However, a similar phenomenon to that of Fact 8.2.1 occurs: Theorem 8.2.2 and Corollary 8.2.3, stated and proven in Section 8.2.1 below, express that the geodesic metric space (Ω, K_Ω) (geodesic by Corollary 6.4.12 since Ω is divisible and hence dually convex by Proposition 3.1.11) cannot be Gromov-hyperbolic, and nor can Γ. This is therefore a higher-rank phenomenon: the divisible convex sets of $\mathscr{F}(\mathfrak{g}, \alpha)$ obey the same principle as those of $\mathbb{P}(\mathbb{R}^n)$ stated in Fact 8.2.1, but excluding the hyperbolic behavior (which is a rank-one behavior).

In fact, strict convexity of Ω and the hyperbolicity of its Hilbert metric are not the correct interpretations of the rank-one behavior in convex projective geometry: more generally, a divisible convex set $\Omega \subset \mathbb{P}(\mathbb{R}^n)$ is said to be *rank-one* if there exists $a \in \partial\Omega$ such that $[a, b] \cap \Omega \neq \emptyset$ for all $b \in \partial\Omega$. Such divisible convex sets were introduced by Islam [Isl25] and have properties analogous to real hyperbolic space [Ben03, Ben06, Cra09, CLT15, Zim23], although they are in general not strictly convex and hence not Gromov-hyperbolic (equipped with their Hilbert metric). A. Zimmer [Zim23] proved that all irreducible non-symmetric divisible convex sets are rank-one. Thus, all non-symmetric divisible convex sets exhibit this “rank-one behavior”. We think this phenomenon is related to Question 1.2.1, and discuss it further in Section 8.9.2. We will see in Section 8.2.2, in particular in Proposition 8.2.7, that one of the main properties of rank-one divisible convex sets in $\mathbb{P}(\mathbb{R}^n)$, proven by Blayac [Bla24], is not shared with proper divisible domains of Nagano spaces of real type and of higher rank (in the sense of Definition 5.1.5).

8.2.1 Non-hyperbolicity of the Kobayashi metric

The analysis conducted in Chapter 7 on the boundary of proper almost-homogeneous domains allows us to prove Theorem 1.4.9 below. Recall that if Ω is a proper almost-homogeneous domain in an irreducible Nagano space of real type, then by Proposition 3.1.11 it is dually convex, and thus the metric space (Ω, K_Ω) is proper and geodesic by Corollary 6.4.12.

Theorem 8.2.2. *Let $(\mathfrak{g}, \alpha)$ be an irreducible Nagano pair of real type and of higher rank (in the sense of Definition 5.1.5). Let $\Omega \subset \mathscr{F}(\mathfrak{g}, \alpha)$ be a proper almost-homogeneous domain. Then (Ω, K_Ω) is not Gromov-hyperbolic.*

An immediate corollary of Theorem 8.2.2 is the following:

Corollary 8.2.3 (see Corollary 1.4.10)**.** *Let $(\mathfrak{g}, \alpha)$ be an irreducible Nagano pair of real type and of rank ≥ 2, and $G \in \mathscr{G}_{\{\alpha\}}(\mathfrak{g})$. Let $\Omega \subset \mathscr{F}(\mathfrak{g}, \alpha)$ be a proper domain. If there exists $\Gamma \leq \mathsf{Aut}_G(\Omega)$ dividing Ω, then Γ is not Gromov hyperbolic.*

Proof of Corollary 8.2.3. Since Ω is divisible, it is almost-homogeneous. Now since Γ acts cocompactly and properly discontinuously (Fact 3.1.3) on the proper geodesic metric space (Ω, K_Ω), by Svark–Milnor's Lemma, by Theorem 8.2.2 the group Γ is not Gromov hyperbolic. ☐

The aim of this section is to prove Theorem 8.2.2. This theorem and its Corollary 8.2.3 are proven in [Zim15] for $(\mathfrak{g}, \alpha) = (\mathfrak{sl}(p+q, \mathbb{R}), \alpha_p)$, with $p, q \geq 2$. With the formalism on Nagano spaces of real type that we have introduced, the proof given in [Zim15] generalizes verbatim to any Nagano pair of real type and of higher rank. We give this proof for convenience.

Remark 8.2.4. For the Nagano pairs $(\mathfrak{g}, \alpha)$ of the form $(\mathfrak{g}, \alpha_r)$, where $\mathfrak{g}$ is a HTT Lie algebra of rank $r \geq 2$ and α_r the unique long root of $\mathfrak{g}$, and $(\mathfrak{so}(p+1, q+1), \alpha_1)$ with $p, q \geq 1$, Theorems 8.3.1 and 8.4.1 will imply that proper almost-homogeneous domains of $\mathscr{F}(\mathfrak{g}, \alpha)$ are realizations of the higher-rank symmetric space $\mathbb{X}(\mathfrak{g}, \alpha)$. Hence Corollary 8.2.3 is just a consequence of the fact that the rank of $\mathbb{X}(\mathfrak{g}, \alpha)$ is ≥ 2. Since Corollary 8.2.3 is already proven in [Zim15], the cases at issue here are are the remaining higher-rank Nagano pairs of real type, see Tables 8.1 and 8.2: $(\mathfrak{so}(n, n), \alpha_n)$ for $n \geq 2$, $(\mathfrak{e}_{6(-4)}, \alpha_2)$ and $(\mathfrak{e}_{7(-5)}, \alpha_2)$.

We fix a higher-rank Nagano pair $(\mathfrak{g}, \alpha)$ of real type, a Plücker triple (G, ρ, V) of $(\mathfrak{g}, \alpha)$, and $\mathsf{v}_0 \in V^{\omega_\alpha} \smallsetminus \{0\}$. We moreover take Notation 5.1.2. Let $\Omega \subset \mathscr{F}(\mathfrak{g}, \alpha)$ be a proper almost-homogeneous domain. We may assume that Ω is contained in the standard affine chart $\mathbb{A}_{\mathsf{std}}$ and that $P \in \Omega$. Then $\Lambda_{\mathsf{std}} \cap \Omega \neq \emptyset$, so by properness of Ω there exists some $t \in \mathbb{R} \smallsetminus \{0\}$ such that $\exp(tv^-) \cdot \mathfrak{p}^+ \in \partial\Omega$. Up to dilating in $\mathbb{A}_{\mathsf{std}}$ (see Section 5.1.1), we may assume that $a := \exp(v^-) \cdot \mathfrak{p}^+ \in \partial\Omega$.

If a was $\mathscr{R}$-extremal, then by Theorem 7.2.6 there would exist $\xi \in \Omega^*$ such that $\mathrm{id} \in \mathsf{pos}^{(\{\alpha\},\{i(\alpha)\})}(a, \xi)$. Then by Lemma 6.3.9 and the fact that $(\mathfrak{g}, \alpha)$ has higher rank, we would have $P \in \Lambda_{\mathsf{std}} \subset \mathrm{Z}_\xi$, contradicting the fact that $\xi \in \Omega^*$. Thus a is not $\mathscr{R}$-extremal, i.e. its $\mathscr{R}$-face is nontrivial. Thus by Lemma 6.3.3.(2), there exists $w := \mathrm{Ad}(\ell) \cdot v^-$, with $\ell \in L$ and $\delta > 0$, such that $\{\exp(v^- + sw) \cdot \mathfrak{p}^+ \mid -\delta \leq s \leq \delta\} \subset \partial\Omega$. Up to considering $\mathrm{Ad}(\ell_0(\frac{1}{\delta})) \cdot w$ instead of w, we may assume that $\delta = 1$, and

$$\{\exp(v^- + sw) \cdot \mathfrak{p}^+ \mid -\delta \leq s \leq \delta\} \subset \partial\Omega.$$

By Corollary 7.1.8, there exists $\varepsilon > 0$ such that

$$\{\exp(tv^- + sw) \cdot \mathfrak{p}^+ \mid 1 - \varepsilon \leq t < 1,\ -\varepsilon < s < \varepsilon\} \subset \Omega.$$

Since Ω is dually convex, there exists $\xi \in \Omega^*$ such that $a \in \mathrm{Z}_\xi$. But then, since $\mathscr{F}_\Omega^{\mathscr{R}}(a) \subset \mathrm{Z}_\xi$ by Corollary 7.2.3, we also have

$$\exp(v^- + sw) \cdot \mathfrak{p}^+ \in \mathrm{Z}_\xi \qquad \forall s \in (-1, 1). \tag{8.2.1}$$

Let $f \in V^* \smallsetminus \{0\}$ be the unique lift of $\iota_\rho^-(\xi)$ such that $f(\mathsf{v}_0) = 1$. By Equation (8.2.1) and Fact 2.3.4, we have

$$f(\rho(\exp(v^- + sw)) \cdot \mathsf{v}_0) = 0 \qquad \forall s \in (-1, 1), \tag{8.2.2}$$

and since this is a polynomial, we have $f(\rho(\exp(v^- + sw) \cdot \mathsf{v}_0) \cdot x_0) = 0$ for all $s \in \mathbb{R}$. Note that for all $s, t \in \mathbb{R}$, we have

$$f(\rho(\exp(tv^- + sw) \cdot \mathsf{v}_0) \cdot x_0) = 1 + sf(\rho_*(w) \cdot \mathsf{v}_0) + tf(\rho_*(v^-) \cdot \mathsf{v}_0) + tsf(\rho_*(v^-)\rho_*(w) \cdot \mathsf{v}_0). \tag{8.2.3}$$

for $t = 1$, by Equation (8.2.2), this gives:

$$0 = 1 + sf(\rho_*(w) \cdot \mathsf{v}_0) + f(\rho_*(v^-) \cdot \mathsf{v}_0) + sf(\rho_*(v^-)\rho_*(w) \cdot \mathsf{v}_0) \qquad \forall s \in \mathbb{R}.$$

This implies:

$$\begin{cases} 1 + f(\rho_*(v^-) \cdot \mathsf{v}_0) & = 0 \\ f(\rho_*(w) \cdot \mathsf{v}_0) + f(\rho_*(v^-)\rho_*(w) \cdot \mathsf{v}_0) & = 0. \end{cases}$$

This system allows us to simplify Equation (8.2.3):

$$f(\rho(\exp(tv^- + sw) \cdot \mathsf{v}_0) \cdot x_0) = (1 - t)(1 + s\lambda),$$

with $\lambda := f(\rho_*(w) \cdot \mathsf{v}_0)$.

For $s, t \in \mathbb{R}$, we write $x_{t,s} := \exp(tv^- + sw) \cdot \mathfrak{p}^+$. By Proposition 6.4.10, if $x_{t_1,s_1}, x_{t_2,s_2} \in \Omega$, then:

$$\begin{aligned} K_\Omega(x_{t_1,s_1}, x_{t_2,s_2}) &\geq C^\rho_\Omega(x_{t_1,s_1}, x_{t_2,s_2}) \\ &\geq \Big| \log \Big| [\xi : x_{t_1,s_1} : x_{t_2,s_2} : \mathfrak{p}^-]_\rho \Big| \Big| \\ &= \Big| \log \Big| \frac{f(\rho(\exp(t_1 v^- + s_1 w)) \cdot \mathsf{v}_0)}{f(\rho(\exp(t_2 v^- + s_2 w)) \cdot \mathsf{v}_0)} \Big| \Big| \\ &= \Big| \log \Big| \frac{(1-t_1)(1+s_1\lambda)}{(1-t_2)(1+s_2\lambda)} \Big| \Big|. \end{aligned} \tag{8.2.4}$$

By Corollary 7.1.8, for all $0 \leq t < 1$ there exist $m_t < 0 < M_t$ such that

$$\Omega \cap \{\exp(tv^- + sw) \cdot \mathfrak{p}^+ \mid s \in \mathbb{R}\} = \{\exp(tv^- + sw) \cdot \mathfrak{p}^+ \mid m_t < s < M_t\}.$$

Now let $m := \liminf_{s \to 1} m_s$. Then $\{\exp(v^- + sw) \cdot \mathfrak{p}^+ \mid m \leq s \leq 0\} \subset \partial\Omega$. By Corollary 7.1.8, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\{\exp(tv^- + sw) \cdot \mathfrak{p}^+ \mid m + \varepsilon \leq s \leq 0,\ 1 - \delta \leq t < 1\} \subset \Omega.$$

Hence $m + \varepsilon \geq \limsup_{s \to 1} m_s$. This is true for all $\varepsilon > 0$, so $m = \limsup_{s \to 1} m_s$. Hence $m_t \to m$ as $t \to 1$. Similarly, there exists $M \in \mathbb{R}_{\geq 0}$ such that $M_t \to M$ as $t \to 1$.

Up to replacing w with $-w$, we may assume that $M \leq \frac{-1}{\lambda}$. Hence $\lambda < 0$ and $1+m\lambda > 1$. The following lemma ends the proof of Theorem 8.2.2. Lemma 8.2.5 below is proven in [Zim15] for $(\mathfrak{g}, \alpha) = (\mathfrak{sl}(p+q, \mathbb{R}), \alpha_p)$ with $p, q \geq 2$, and the proof generalizes verbatim to any higher-rank Nagano space of real type. We reproduce the proof for convenience.

Lemma 8.2.5. *[Zim15] For any $A > 0$, there exists a geodesic rectangle in (Ω, K_Ω) which is not A-thin.*

Proof. Let $R > 2A + \log(1 + m\lambda) + 2$, $s_0 \in (m, 0)$ so that $(m : s_0 : 0 : M) > R + 1$, and $t_0 \in [e^{-1}, 1)$ such that $s_0 \in (m_t, 0)$ and $(m_t : s_0 : 0 : M_t) > R$ for every $t > t_0$. Note that $|\log(t_0)| \leq 1$.

Now for $t_1 \in (t_0, 1)$, consider the (closed) segments of photon

$$\gamma_1 = [x_{t_0,s_0}, x_{t_0,0}],\ \gamma_2 = [x_{t_0,0}, x_{t_1,0}],\ \gamma_3 = [x_{t_1,s_0}, x_{t_1,0}],\ \gamma_4 = [x_{t_0,s_0}, x_{t_1,s_0}].$$

By Proposition 6.4.10, each of the γ_i is a geodesic for K_Ω. The concatenation of these segments forms thus a geodesic rectangle in (Ω, K_Ω).

Let us determine $t_1 \in (t_0, 1)$ such that γ is not A-thin. By properness of K_Ω, there exists $u_0 \in (t_0, 1)$ so that $K_\Omega(x_{u_0,s_0}, \gamma_1) \geq A$.

Let $t \in (t_0, t_1)$.

1. Let us first assume that $\log \frac{1-u_0}{1-t} \leq R - A - |\log(t_0)|$.

Then since $|\log(t/u_0)| \leq |\log(t_0)|$, one has $(0 : u_0 : t : 1) = \log \frac{(1-u_0)t}{(1-t)u_0} \leq R - A$. Thus one has

$$\begin{aligned} K_\Omega(x_{u_0,s_0}, x_{t,0}) &\geq K_\Omega(x_{u_0,s_0}, x_{u_0,0}) \\ &\quad - K_\Omega(x_{u_0,0}, x_{t,0}) \\ &\geq (m_{u_0} : s_0 : 0 : M_{u_0}) - (0 : u_0 : t : 1) \\ &\geq R - (R - A) = A. \end{aligned}$$

Thus the geodesic rectangle γ is not A-thin.

2. Now let us assume that $\log \frac{1-u_0}{1-t} \geq R - A - |\log(t_0)|$. By Equation (8.2.4) and definition of R one has

$$\begin{aligned} K_\Omega(x_{u_0,s_0}, x_{t,0}) &\geq \log \frac{1-u_0}{1-t} + \log(1 + s_0\lambda) \\ &\geq \log \frac{1-u_0}{1-t} - |\log(t_0)| - |\log(1 + s_0\lambda)| \\ &\geq R - A - 2|\log(t_0)| - \log(1 + m\lambda) \geq A \end{aligned}$$

since $|\log(t_0)| \leq 1$. Thus the geodesic rectangle γ is not A-thin.

In any case, the rectangle γ is not A-thin. $\square$

8.2.2 Almost-homogeneous domains are not of rank one

In [Bla24], Blayac proved the following: if $\Omega \subset \mathbb{P}(\mathbb{R}^n)$ $(n \geq 2)$ is an irreducible convex domain *of rank one*, divisible by some discrete subgroup $\Gamma \leq \mathrm{PGL}(n, \mathbb{R})$, then one has

$$\Lambda^{\mathrm{prox}}_{\{\alpha_1\}}(\Gamma) = \partial\Omega.$$

On the other hand, A. Zimmer proved [Zim23] that any nonsymmetric irreducible divisible convex domain is of rank one. This gives:

Theorem 8.2.6. *[Bla24, Zim23] Let $n \geq 2$. If $\Omega \subset \mathbb{P}(\mathbb{R}^n)$ is a nonsymmetric irreducible convex domain, divisible by some discrete subgroup $\Gamma \leq \mathrm{PGL}(n, \mathbb{R})$, then one has*

$$\Lambda^{\mathrm{prox}}_{\{\alpha_1\}}(\Gamma) = \partial\Omega.$$

Given Lemmas 7.2.11 and 6.3.9, as well as Proposition 7.2.12, we observe that the situation is different whenever $\mathscr{F}(\mathfrak{g}, \alpha)$ is of higer rank, in the sens of Definition 5.1.5. In this case, if $\Omega \subset \mathscr{F}(\mathfrak{g}, \alpha)$ is a proper almost-homogeneous domain, then by Proposition 7.2.12, we have

$$\Lambda^{\mathrm{prox}}_{\{\alpha\}}(\mathsf{Aut}_G(\Omega)) = \Lambda_{\{\alpha\}}(\mathsf{Aut}_G(\Omega)).$$

Let $x \in \Omega$, and Λ be a photon through x. Then there exists $a \in \Lambda \cap \partial\Omega$. If $a \in \Lambda_{\{\alpha\}}(\mathsf{Aut}_G(\Omega))$, then by Lemma 7.2.11, there exists $b \in \Omega^*$ such that

$$\mathsf{pos}^{(\{\alpha\}, \{i(\alpha)\})}(a, b) = \overline{\mathrm{id}}$$

Then, by Lemma 6.3.9 and since

$$|W_{\Delta \smallsetminus \{\alpha\}} \backslash W / W_{\Delta \smallsetminus \{\alpha\}}| \geq 3,$$

we must have $x \in \mathrm{Z}_b$, which contradicts the fact that $b \in \Omega^*$. Thus, we have just proved:

Proposition 8.2.7. *Let $(\mathfrak{g}, \alpha)$ be an irreducible Nagano pair of real type and of higher rank. Then, for any proper almost-homogeneous domain $\Omega \subset \mathscr{F}(\mathfrak{g}, \alpha)$, we have*

$$\partial\Omega \smallsetminus \Lambda_{\{\alpha\}}(\mathsf{Aut}_G(\Omega)) \neq \emptyset.$$

In particular, if Ω is divisible by some $\Gamma \leq G$, then it is almost-homogeneous, and we have $\partial\Omega \smallsetminus \Lambda_{\{\alpha\}}(\Gamma) \neq \emptyset$. We see that this situation is very different than the real projective case described in Theorem 8.2.6.

8.3 Rigidity in causal flag manifolds

If $\mathfrak{g}$ is a HTT Lie algebra of real rank $r \geq 1$, then recall that diamonds in $\mathbf{Sb}(\mathfrak{g})$ have been defined in Section 3.5.1 and are the realizations of the noncompact dual $\mathbb{X}(\mathfrak{g}, \alpha_r)$ of the Nagano space $\mathbf{Sb}(\mathfrak{g})$. In this section, we prove:

Theorem 8.3.1 (see Theorem 1.4.14)**.** *Let $\mathfrak{g}$ be a* HTT *Lie algebra. Any almost-homogeneous domain of $\mathbf{Sb}(\mathfrak{g})$ is a diamond.*

Since diamonds of $\mathbf{Sb}(\mathfrak{g})$ are symmetric domains (see Fact 3.5.5), Theorem 8.3.1 implies that any proper almost-homogeneous domain in $\mathbf{Sb}(\mathfrak{g})$ is a diamond, providing a positive answer to Question 1.2.1 for any HTT Lie group G of rank $r \geq 1$ and $G/P = G/P_{\{\alpha_r\}} = \mathbf{Sb}(\mathfrak{g})$.

Corollary 1.4.15 then directly follows from Theorem 8.3.1, and Lemmas 8.1.1.

Let us give an outline of the proof of Theorem 8.3.1. Let $\mathfrak{g}$ be a HTT Lie algebra of real rank $r \geq 1$ and $G \in \mathscr{G}_{\{\alpha_r\}}(\mathfrak{g})$. Let $\Omega \subset \mathbf{Sb}(\mathfrak{g})$ be a proper almost-homogeneous domain. By almost-homogeneity, the domain Ω is dually convex (see Proposition 3.1.11), that is, for any $a \in \partial\Omega$ there is a supporting hypersurface to Ω at a of the form Z_z. However, we know by Theorem 7.2.6 that if a is $\mathscr{R}$-extremal, then this supporting hypersurface Z_z can actually be taken to be Z_a itself. This Theorem, applied to two *strongly $\mathscr{R}$-extremal* points $a_0, b_0 \in \partial\Omega$ (see Section 8.3.2), which are the candidates for the extremities of Ω (since we want to prove that Ω is a diamond), implies that Ω is contained in the diamond $\mathbf{D}(a_0, b_0)$ (Section 8.3.3). In Section 8.3.3, we prove that $\Omega = \mathbf{D}(a_0, b_0)$. The key point is the inclusion $\mathsf{Aut}_G(\Omega) \leq \mathsf{Aut}_G(\mathbf{D}(a_0, b_0))$, which implies, by almost-homogeneity, that Ω is closed in (and hence equal to) the diamond $\mathbf{D}(a_0, b_0)$. This inclusion holds because any automorphism of Ω preserves the pair $\{a_0, b_0\}$; this fact is proven in Proposition 8.3.8, and essentially characterizes $\mathbf{D}(a_0, b_0)$ (Fact 3.5.5).

Remark 8.3.2. Let us recall that in our joint work with Chalumeau [CG24], we prove Theorem 1.4.17, i.e. we establish the rigidity of proper almost-homogeneous domains in $\mathrm{Ein}^{p,q}$ $(p, q \geq 1)$. In the case where $q = 1$, we give in [CG24, Sect. 6.3] a different proof from the general case, relying on causality arguments; a key argument is [CG24, Lem. 6.4]. In Section 8.3.2 below, we extend this argument to arbitrary causal flag manifolds, see Lemma 8.3.5, in order to prove Theorem 8.3.1.

8.3.1 Proper domains in Θ-positive flag manifolds

Before starting the proof of Theorem 8.3.1, let us develop on its main corollary, namely Corollary 8.3.4 below. To this end, in this section, we recall the notion of Θ-positivity of Guichard–Wienhard, and state their classification of Θ-positive flag manifolds (see Fact 8.3.3).

Let $\mathfrak{g}$ be a real simple Lie algebra of noncompact type, and let Θ be a subset of the simple restricted roots of $\mathfrak{g}$. For all $\alpha \in \Theta$, let

$$\Sigma_\alpha^+ := \{\beta \in \Sigma_{\{\alpha\}}^+ \mid \beta - \alpha \in \mathrm{Span}(\Delta \smallsetminus \{\alpha\})\}.$$

By [Kos10], the Levi subgroup L_Θ acts irreducibly on $\mathfrak{u}_\alpha^- := \bigoplus_{\beta \in \Sigma_\alpha^+} \mathfrak{g}_{-\beta}$.

As defined in [GW25], the flag manifold $\mathscr{F}(\mathfrak{g}, \Theta)$ admits a *positive structure* (resp. the Lie algebra $\mathfrak{g}$ admits a Θ-positive structure) if for all $\alpha \in \Theta$, there exists a properly convex L_Θ^0-invariant open cone $c_\alpha \subset \mathfrak{u}_\alpha^-$. More generally, a real semisimple Lie algebra $\mathfrak{g}$ with no compact factors admits a Θ-positive structure, where Θ is a subset of the simple restricted roots of $\mathfrak{g}$, if there exist two decompositions $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_N$ and $\Theta = \Theta_1 \sqcup \cdots \sqcup \Theta_N$, where the $\mathfrak{g}_i$ are real simple Lie algebras of noncompact type, Θ_i a subset of the simple restricted roots of $\mathfrak{g}_i$ such that $\mathfrak{g}_i$ admits a Θ_i-positive structure. Thus, the study of positivity reduces to the case where $\mathfrak{g}$ is simple. Guichard–Wienhard have established the flag manifolds admitting a positive structure:

Fact 8.3.3. *[GW25] Let $\mathfrak{g}$ be a real simple Lie aglebra of noncompact type, and let Θ be a subset of the simple restricted roots of $\mathfrak{g}$. The flag manifold $\mathscr{F}(\mathfrak{g}, \Theta)$ admits a positive structure if and only if the pair $(\mathfrak{g}, \alpha)$ satisfies one of the following conditions:*

1. *The Lie algebra $\mathfrak{g}$ is split, and $\Theta = \Delta$.*
2. *The Lie algebra $\mathfrak{g}$ is* HTT *of real rank $r \geq 1$, and $\Theta = \alpha_r$.*
3. *One has $\mathfrak{g} = \mathfrak{so}(p+1, p+k)$ with $p, k > 1$, and $\Theta = \{\alpha_1, \ldots, \alpha_p\}$ (in the notation of Example 2.2.1.(2)).*
4. *The Lie algebra $\mathfrak{g}$ is the real form of $\mathfrak{e}_{4,\mathbb{C}}$, $\mathfrak{e}_{6,\mathbb{C}}$, $\mathfrak{e}_{7,\mathbb{C}}$ or $\mathfrak{e}_{8,\mathbb{C}}$ whose root system is F_4, and $\Theta = \{\alpha_1, \alpha_2\}$, where α_1 and α_2 are the unique long simple restricted roots.*

Theorem 8.3.1, Lemma 8.1.1 and Fact 8.3.3 give us a complete classification of proper almost-homogeneous domains in positive flag manifolds:

Corollary 8.3.4 (see Corollary 1.4.16)**.** *Let $\mathfrak{g}$ be a real semisimple Lie algebra with no compact factors, and Θ be a subset of the simple restricted roots of $\mathfrak{g}$ such that $\mathfrak{g}$ admits a Θ-positive structure. Then all proper almost-homogeneous domains of $\mathscr{F}(\mathfrak{g}, \Theta)$ are symmetric. More precisely, we have the following description of the proper almost-homogeneous domains in $\mathscr{F}(\mathfrak{g}, \Theta)$: write $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_N$ and $\Theta = \Theta_1 \sqcup \cdots \sqcup \Theta_N$, where the $\mathfrak{g}_i$ are real simple Lie algebras of noncompact type, Θ_i a subset of the simple restricted roots of $\mathfrak{g}_i$ such that $\mathfrak{g}_i$ admits a Θ_i-positive structure. Then:*

1. *If for all $1 \leq i \leq N$, one has $|\Theta_i| = 1$, then for all i the Lie algebra $\mathfrak{g}_i$ is* HTT *and $\mathscr{F}(\mathfrak{g}_i, \Theta_i) = \mathbf{Sb}(\mathfrak{g}_i)$. The proper almost-homogeneous domains of $\mathscr{F}(\mathfrak{g}, \Theta)$ are the products of diamonds of the $\mathscr{F}(\mathfrak{g}_i, \Theta_i)$ for all $1 \leq i \leq N$.*

2. *If there exists $1 \leq i \leq N$ such that $|\Theta_i| \geq 2$, then there are no proper almost-homogeneous domains in $\mathscr{F}(\mathfrak{g}, \Theta)$.*

Proof. Let us prove point (1). By Lemma 8.1.1, it reduces to the case where $\mathfrak{g}$ is simple. Fact 8.3.3 tells us that the only case where $|\Theta| = 1$ and $\mathfrak{g}$ admits a Θ-positive structure is when $\mathfrak{g}$ is HTT and $\mathscr{F}(\mathfrak{g}, \Theta) = \mathbf{Sb}(\mathfrak{g})$. Point (1) then follows from Theorem 8.3.1.

To prove point (2), it suffices to notice that if $\mathscr{F}(\mathfrak{g}, \Theta)$ contained a proper almost-homogeneous domain, then by Lemma 8.1.1, the flag manifold $\mathscr{F}(\mathfrak{g}_i, \Theta_i)$ would also contained one, which is impossible by Lemma 8.1.2. $\square$

8.3.2 Strongly $\mathscr{R}$-extremal points

Let $\mathfrak{g}$ be a HTT Lie algebra. In this section, we define a specific type of $\mathscr{R}$-extremal points of a proper domain $\Omega \subset \mathbf{Sb}(\mathfrak{g})$. In the case where Ω is almost-homogeneous, we will want to prove that it is a diamond, and these points will be the candidates for the extremities of Ω.

We say that a point $a \in \partial\Omega$ is *strongly $\mathscr{R}$-extremal* if either $\mathbf{C}^-(a) \cap \overline{\Omega} = \{a\}$ or $\mathbf{C}^+(a) \cap \overline{\Omega} = \{a\}$.

In general, there are less strongly $\mathscr{R}$-extremal points than $\mathscr{R}$-extremal points. However, the next lemma shows that strongly $\mathscr{R}$-extremal points always exist.

Lemma 8.3.5. *For any $x \in \Omega$ there exist at least two strongly $\mathscr{R}$-extremal points $a \in \mathbf{J}^-(x)$ and $b \in \mathbf{J}^+(x)$.*

Proof. Up to translating Ω in $\mathbb{A}_{\mathsf{std}}$, we may assume that $x = \mathfrak{p}^+$. Since c^0 is a properly convex cone of $\mathfrak{u}^-$, there is a nonzero linear form f of $\mathfrak{u}^-$ such that $\overline{c^0} \smallsetminus \{0\}$ is contained in $\{f > 0\}$. Let $X \in \mathfrak{u}^-$ be the element of $\overline{\varphi_{\mathsf{std}}^{-1}(\Omega)} \cap (-\overline{c^0})$ such that $f(X) \in \mathbb{R}$ is minimal. Then $a := \exp(X) \cdot \mathfrak{p}^+$ lies in $\partial\Omega$.

Let us show that a is strongly $\mathscr{R}$-extremal. Let $y \in \mathbf{C}^-(a)$. Write $y = \exp(Y) \cdot \mathfrak{p}^+$ with $Y \in X - \partial c^0$. Then one has

$$f(X - Y) \geq 0, \text{ with equality if and only if } y = a. \tag{8.3.1}$$

If moreover $y \in \partial\Omega$, then $y \in \mathbf{J}^-(x) \cap \overline{\Omega}$, so $f(Y) \geq f(X)$. Then, by Equation (8.3.1), one has $y = a$. Hence a is strongly $\mathscr{R}$-extremal. $\square$

Remark 8.3.6. 1. In Lemma 8.3.5, we do not need Ω to be almost-homogeneous.

2. By Lemma 6.3.7, the fact that the trace in $\mathbb{A}_{\mathsf{std}}$ of any photon through a point $x \in \mathbb{A}_{\mathsf{std}}$ is contained in $\mathbf{C}(x)$ (see Section 6.3.4.2) and the fact that c^0 is a properly convex cone in $\mathfrak{u}^-$, strongly $\mathscr{R}$-extremal points are always $\mathscr{R}$-extremal, but the converse is false in general. For instance, for $G = \mathrm{SO}(n, 2)$, take $x, y \in \mathbb{A}_{\mathsf{std}}$ with $y \in \mathbf{I}^+(x)$. Then $\mathbf{D}(x, y)$ has exactly two strongly $\mathscr{R}$-extremal points, namely x and y. The points of $\mathbf{C}^+(x) \cap \mathbf{C}^-(y)$ are $\mathscr{R}$-extremal but not strongly $\mathscr{R}$-extremal. See also Figure 8.1.

3. Contrary to the notion of $\mathscr{R}$-extremality, that of strong $\mathscr{R}$-extremality is here only defined for a domain Ω which is proper in $\mathbb{A}_{\mathsf{std}}$. It is not clear at first that this second notion is invariant under $\mathsf{Aut}_G(\Omega)$. We will only need this invariance in the *almost-homogeneous case*; see Lemma 8.3.8.

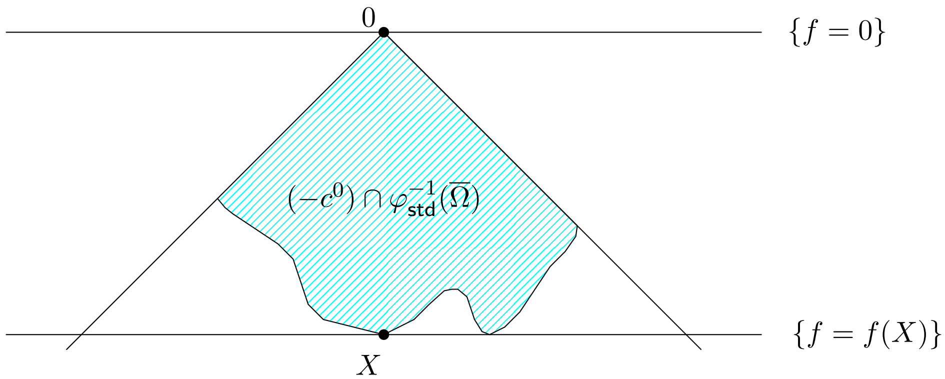

Figure 8.1 – Existence of strongly $\mathscr{R}$-extremal points (see the proof of Lemma 8.3.5). In blue is the intersection of $\varphi_{\mathsf{std}}(\overline{\Omega})$ with $-c^0$. Note that the point X of the proof of Lemma 8.3.5 is not necessarily unique if Ω is any proper domain of $\mathbb{A}_{\mathsf{std}}$.

8.3.3 End of the proof of Theorem 8.3.1

In this section, we finish the proof of Theorem 8.3.1. We fix a HTT Lie aglebra $\mathfrak{g}$ and $G \in \mathscr{G}_{\{\alpha_r\}}(\mathfrak{g})$. We fix $\Omega \subset \mathbf{Sb}(\mathfrak{g})$ a proper almost-homogeneous domain. By Section 2.2.6.1, for all $a \in \partial\Omega$ there exist $(g_k) \in \mathsf{Aut}_G(\Omega)^{\mathbb{N}}$ and $x \in \Omega$ such that $g_k \cdot x \to a$.

We may assume that Ω is proper in $\mathbb{A}_{\mathsf{std}}$. Let $x \in \Omega$ and let $a_0 \in \partial\Omega \cap \mathbf{J}^-(x)$ and $b_0 \in \partial\Omega \cap \mathbf{J}^+(x)$ be two strongly $\mathscr{R}$-extremal points of $\partial\Omega$ given by Lemma 8.3.5. Then in particular $a_0, b_0 \in \mathrm{Extr}_{\mathscr{R}}(\Omega)$ (Remark 8.3.6.(2)), so by Theorem 7.2.6, one has

$$\Omega \cap \mathrm{Z}_{a_0} = \Omega \cap \mathrm{Z}_{b_0} = \emptyset. \tag{8.3.2}$$

By reflexivity, one has $x \in \mathbf{J}^+(a_0)$. By Equation (8.3.2), we also know that $x \notin \mathrm{Z}_{a_0}$. Then, by Fact 2.4.5.(1), one has $x \notin \mathbf{C}(a_0)$, and hence $x \in \mathbf{I}^+(a_0)$. Similarly, one has $x \in \mathbf{I}^-(b_0)$. Hence $x \in \mathbf{D}(a_0, b_0)$. By connectedness of Ω, we then have the inclusion

$$\Omega \subset \mathbf{D}(a_0, b_0). \tag{8.3.3}$$

The goal of the rest of this section is to prove the converse inclusion. First observe that a_0 and b_0 are characterized among $\mathscr{R}$-extremal points of $\partial\Omega$ by a geometric property:

Lemma 8.3.7. *Let $a \in \mathrm{Extr}_{\mathscr{R}}(\Omega)$ be such that $\mathbf{I}^+(a) \cap \Omega \neq \emptyset$ (resp. $\mathbf{I}^-(a) \cap \Omega \neq \emptyset$). Then $a = a_0$ (resp. $a = b_0$).*

Proof. Let us prove the Lemma for $\mathbf{I}^+(a) \cap \Omega \neq \emptyset$, the proof being similar for the case where $\mathbf{I}^-(a) \cap \Omega \neq \emptyset$. Since a is $\mathscr{R}$-extremal, by Theorem 7.2.6, one has $\mathrm{Z}_a \cap \Omega = \emptyset$, so by connectedness, the set Ω is included in one of the connected components of $\mathbb{A}_{\mathsf{std}} \setminus \mathrm{Z}_a$. Since $\mathbf{I}^+(a) \cap \Omega \neq \emptyset$, one has $\Omega \subset \mathbf{I}^+(a)$. But then one has $a_0 \in \overline{\Omega} \subset \mathbf{J}^+(a)$. By Equation (8.3.3), we also have $a \in \overline{\Omega} \subset \mathbf{J}^+(a_0)$. By antisymmetry this implies $a = a_0$. $\square$

Using Lemma 2.4.6, we can now prove:

Proposition 8.3.8. *Let $g \in \mathsf{Aut}_G(\Omega)$. Then $g \cdot a_0, g \cdot b_0 \in \{a_0, b_0\}$.*

Proof. Let us prove the proposition only for a_0, the case of b_0 being similar. Up to translating Ω in $\mathbb{A}_{\mathsf{std}}$, one can assume that $a_0 = \mathfrak{p}^+$. Since $g \cdot \mathfrak{p}^+ = g \cdot a_0 \in \overline{\Omega} \subset \mathbb{A}_{\mathsf{std}}$, by Lemma 2.4.6 there is a neighborhood $\mathscr{U}$ of P such that $g \cdot (\mathscr{U} \cap \mathbf{I}^+(\mathfrak{p}^+)) \subset \mathbf{I}^\delta(g \cdot \mathfrak{p}^+)$ for some $\delta \in \{-, +\}$. Since $P \in \partial\Omega$, there exists $z \in \mathscr{U} \cap \Omega$. Since $g \in \mathsf{Aut}_G(\Omega)$, one has $g \cdot z \in \Omega$. Hence $g \cdot z \in \Omega \cap \mathbf{I}^\delta(g \cdot \mathfrak{p}^+) \neq \emptyset$. Then, by Lemma 8.3.7, we must have $g \cdot a_0 = b_0$ if $\delta = +$, and $g \cdot a_0 = b_0$ if $\delta = -$. □

We have shown in Proposition 8.3.8 that for any $g \in \mathsf{Aut}_G(\Omega)$, the element g stabilizes the pair $\{p_0, q_0\}$. Then $\mathsf{Aut}_G(\Omega)$ preserves the set $\mathbf{Sb}(\mathfrak{g}) \smallsetminus (\mathrm{Z}_{a_0} \cup \mathrm{Z}_{b_0})$, and hence permutes its connected component. Since $\Omega \subset \mathbf{D}(a_0, b_0)$ is $\mathsf{Aut}_G(\Omega)$-invariant, the group $\mathsf{Aut}_G(\Omega)$ preserves the connected component $\mathbf{D}(a_0, b_0)$ of $\mathbf{Sb}(\mathfrak{g}) \smallsetminus (\mathrm{Z}_{a_0} \cup \mathrm{Z}_{b_0})$. Then:

$$\mathsf{Aut}_G(\Omega) \leq \mathsf{Aut}_G(\mathbf{D}(a_0, b_0)).$$

Since $\mathbf{D}(a_0, b_0)$ is proper, Lemma 3.1.7 implies that $\Omega = \mathbf{D}(a_0, b_0)$. This concludes the proof of Theorem 8.3.1.

8.4 Rigidity in Einstein universe

Let $p, q \geq 1$. Recall that diamonds of $\mathrm{Ein}^{p,q}$ have been defined in Secyion 3.4.2, and are realizations of the $\mathbb{X}(\mathfrak{so}(p+1, q+1), \alpha_1)$ in $\mathrm{Ein}^{p,q}$. In this section, we provide the proof of Theorem 8.4.1 below, coming from a collaboration with Adam Chalumeau [CG24]:

Theorem 8.4.1 (with Chalumeau, see Theorem 1.4.17)**.** *Let $p, q \geq 1$. Every almost-homogeneous domain of $\mathscr{F}(\mathfrak{so}(p+1, q+1), \alpha_1) = \mathrm{Ein}^{p,q}$ is a diamond.*

Since diamonds of $\mathrm{Ein}^{p,q}$ are symmetric domains (see Section 3.4.2 — so also Fact 5.1.12), Theorem 8.4.1 implies that any proper almost-homogeneous domain in $\mathrm{Ein}^{p,q}$ is symmetric, providing a positive answer to Question 1.2.1 for any Lie group G locally isomorphic to $\mathrm{SO}(p+1, q+1)$ and $G/P = G/P_{\{\alpha_1\}} = \mathrm{Ein}^{p,q}$.

8.4.1 Notation and outline of the proof

Let us fix some notation for the proof of Theorem 8.4.1. We fix once and for all a proper almost-homogeneous domain $\Omega \subset \mathrm{Ein}^{p,q}$. Identifying $G := \mathrm{PO}(p+1, q+1)$ with its image under Ad, we have $G \in \mathscr{G}_{\{\alpha_1\}}(\mathfrak{so}(p+1, q+1))$ to be $G := \mathrm{PO}(p+1, q+1)$. According to Section 3.1.1.1, for all $a \in \partial\Omega$ there exists $(g_k) \in \mathsf{Aut}_G(\Omega)^{\mathbb{N}}$ and $x \in \Omega$ such that $g_k \cdot x \to a$.

We consider the Plücker triple $(\mathrm{PO}(p+1, q+1), \rho_1, \mathbb{R}^{p+q+2})$ of $(\mathfrak{so}(p+1, q+1), \alpha_1)$ defined in Equation (2.4.11). Recall that $\iota_{\rho_1}, \iota_{\rho_1}^-$ are described in Equation (2.4.12) and that ι_{ρ_1} is just the inclusion map $\mathrm{Ein}^{p,q} \subset \mathbb{P}(\mathbb{R}^{p+q+2})$ here.

Let $\Omega \subset \mathrm{Ein}^{p,q}$ be a proper almost-homogeneous domain. We will consider its convex hull

$$\mathscr{O}_\Omega := \mathscr{O}_\Omega^{\rho_1} \subset \mathbb{P}(\mathbb{R}^{p+q+2})$$

in $\mathbb{P}(\mathbb{R}^{p+q+2})$ as in Definition 7.1.6, and $\widetilde{\mathscr{O}_\Omega} \subset \mathbb{R}^{p+q+2}$ any properly convex open cone lifting $\mathscr{O}_\Omega^{\rho_1}$. For all $x \in \overline{\mathscr{O}_\Omega}$, we will denote by $\widetilde{a}$ any lift of a in $\overline{\widetilde{\mathscr{O}_\Omega}}$. For any $g \in \mathsf{Aut}_G(\Omega)$, we will denote by $\widetilde{g}$ the unique lift of g in $\mathrm{O}(p+1, q+1)$ preserving $\widetilde{\mathscr{O}_\Omega}$.

By Theorem 7.2.6, for any extremal point $a \in \partial\Omega$, one has $\mathrm{Z}_a \cap \Omega = \emptyset$. In this case, the pseudo-Riemannian conformal structure allows us to split $\mathscr{R}$-extremal points into two $\mathsf{Aut}_G(\Omega)$-invariant subsets: *spacelike-extremal* and *timelike-extremal* points (Section 8.4.2). This will impose that the group $\mathsf{Aut}_G(\Omega)$ preserves a splitting $\mathbb{R}^{p+1,q+1} = V_+ \oplus V_-$. The study of the signature of V_+ and V_- will show that $\mathsf{Aut}_G(\Omega) = \mathrm{PO}(p,1) \times \mathrm{PO}(1,q)$ and that Ω is a diamond, see Section 8.4.3.

8.4.2 Spacelike and timelike-extremal points

We will see in the present section that almost-homogeneity makes it possible to separate the set of $\mathscr{R}$-extremal points into two $\mathsf{Aut}_G(\Omega)$-invariant families: namely, the one of *spacelike-extremal* points and the one of *timelike-extremal* points.

Given some point $z \in \mathrm{Extr}_{\mathscr{R}}(\Omega)$, we know by Theorem 7.2.6 that $z \in \Omega^*$. Thus $\iota_{\rho_0}^-(z) = \mathbb{P}(z^\perp)$ does not intersect $\iota_{\rho_0}(\Omega)$. By definition of $\mathscr{O}_\Omega$ we thus have $\mathscr{O}_\Omega \cap \mathbb{P}(z^\perp) = \emptyset$ and $\widetilde{\mathscr{O}_\Omega} \cap z^\perp = \emptyset$. Hence we either have

$$\mathbf{b}(v, \widetilde{z}) > 0 \quad \forall v \in \widetilde{\mathscr{O}_\Omega},$$

in which case we say that z is *timelike extremal*, or

$$\mathbf{b}(v, \widetilde{z}) < 0 \quad \forall v \in \widetilde{\mathscr{O}_\Omega},$$

in which case we say that z is *spacelike extremal*. We denote by $E^+(\Omega)$ (resp. $E^-(\Omega)$) the set of timelike (resp. spacelike) extremal points of Ω.

Lemma 8.4.2. *The sets $E^+(\Omega)$ and $E^-(\Omega)$ satisfy the following properties:*

1. *They are $\mathsf{Aut}_G(\Omega)$-invariant.*
2. *They are both nonempty.*
3. *If $a \in E^+(\Omega)$ and $b \in E^-(\Omega)$, then $a \in \mathrm{Z}_b$.*

Proof. (1) Let us prove for instance the invariance of $E^+(\Omega)$. Let $a \in E^+(\Omega)$. Since $\mathrm{Extr}_{\mathscr{R}}(\Omega)$ is $\mathsf{Aut}_G(\Omega)$-invariant, it suffices to notice that $\mathbf{b}(v, \widetilde{g} \cdot \widetilde{a}) > 0$ for all $v \in \widetilde{\mathscr{O}_\Omega}$ and $g \in \mathsf{Aut}_G(\Omega)$. By $\mathrm{O}(p+1, q+1)$-invariance of $\mathbf{b}$, this is just a consequence of the $\mathsf{Aut}_G(\Omega)$-invariance of $\mathscr{O}_\Omega$.

(2) Assume for example that $E^+(\Omega) = \emptyset$. Then for any two points $a, b \in \mathrm{Extr}_{\mathscr{R}}(\Omega)$, one has $\mathbf{b}(\widetilde{a}, \widetilde{b}) \leq 0$, by definition of $E^-(\Omega)$. By bilinearity of $\mathbf{b}$ and the fact that $\mathscr{O}_\Omega$ is contained in the convex hull of $\mathrm{Extr}_{\mathscr{R}}(\Omega)$ (by Lemma 7.2.5), one has $\mathbf{b}(v, w) \leq 0$ for all $v, w \in \overline{\widetilde{\mathscr{O}_\Omega}}$. This is impossible by openness of $\widetilde{\mathscr{O}_\Omega}$. Hence $E^+(\Omega) \neq \emptyset$, and similarly $E^-(\Omega) \neq \emptyset$.

(3) Assume $a \in E^+(\Omega)$ and $b \in E^-(\Omega)$. Then we get both $\mathbf{b}(\widetilde{a}, \widetilde{b}) \geq 0$ and $\mathbf{b}(\widetilde{a}, \widetilde{b}) \leq 0$. Thus $\mathbf{b}(\widetilde{a}, \widetilde{b}) = 0$. Therefore $a \in \mathrm{Z}_b$. $\square$

8.4.3 End of the proof of Theorem 8.4.1

In this section, we end the proof of Theorem 8.4.1. We will make use of the following lemma, which is already proven in [DGK24, Lem. 3.3] (and stated with a discreteness assumption which is not necessary) and will also be used in Section 8.5.5.2. We give its proof for the reader's convenience.

Lemma 8.4.3. *Let V be a finite-dimensional real vector space. We fix $|.|$ any norm on V. Let $U \subset V$ be a properly convex open cone, and let $H \leq \mathrm{SL}^{\pm}(V)$ be a subgroup preserving U. Let $v \in U$ and $(h_k) \in H^{\mathbf{N}}$ such that there exists $a \in \partial\mathbb{P}(U)$ satisfying $\mathbb{P}(h_k \cdot v) \to a$. Then $|h_k \cdot v| \to +\infty$.*

Proof. We still denote by $|.|$ the operator norm associated with the norm $|.|$ on V. Let us first show that $|h_k| \to +\infty$. Assume by contradiction that (h_k) admits a subsequence with bounded norm. We still denote this subsequence by (h_k). Then, up to extracting, we may assume that (h_k) converges in $\mathrm{End}(V)$ to some $h \in \mathrm{SL}^{\pm}(V)$ and preserving U. Thus $[h_k] \to [h]$ in $\mathrm{PGL}(V)$. Since $[h_k \cdot v] \to [h \cdot v]$, one has $a = [h \cdot v] \in \mathbb{P}(U)$, contradiction. Hence $|h_k| \to +\infty$.

Let φ be a linear form on V such that $\overline{U} \smallsetminus \{0\} \subset \{\varphi > 0\}$. We may assume that $\varphi(v) = 1$. The set $U \cap \{\varphi = 1\}$ is bounded; let K be its boundary. Since K is compact, there exists some $0 < \varepsilon < 1$ such that for all $w \in \mathsf{K}$, the line through v and w intersects K in a $w' \neq w$ such that $v = tw + (1-t)w'$ for some $t \geq \varepsilon$. Then for all $k \in \mathbb{N}$ one has

$$\varphi(h_k \cdot v) \geq \varepsilon \max_{\mathsf{K}} (\varphi \circ h_k).$$

Thus it is sufficient to see that the maximum of $\varphi \circ h_k$ over K tends to infinity with k. Since $U \cap \{\varphi = 1\}$ is the convex hull of K, it suffices to show that the maximum of $\varphi \circ h_k$ over $U \cap \{\varphi = 1\}$ tends to infinity with k. Now since U is a cone, it suffices to show that the supremum of $\varphi \circ h_k$ over $U \cap \{\varphi < 1\}$ tends to infinity with k. Since $\overline{U} \subset \{\varphi > 0\}$, there exists some $\alpha > 0$ such that $\varphi(u) \geq \alpha |u|$ for all $u \in \overline{U}$. By openness, there exists some $\beta > 0$, such that for all $k \in \mathbb{N}$ there exists $u_k \in U \cap \{\varphi < 1\}$ such that $|h_k \cdot u_k| > \beta |h_k|$. Then one has:

$$\max_{U \cap \{\varphi < 1\}} (\varphi \circ h_k) \geq \varphi(h_k \cdot u_k) \geq \alpha |h_k \cdot u_k| \geq \alpha\beta |h_k| \to +\infty.$$

$\square$

Let $V_+ := \mathrm{Span}(E^+(\Omega)) \subset \mathbb{R}^{p+1,q+1}$ and $V_- := \mathrm{Span}(E^-(\Omega)) \subset \mathbb{R}^{p+1,q+1}$. By Lemma 7.2.5, one has $\mathbb{R}^{p+1,q+1} = V_+ + V_-$. Moreover, by Lemma 8.4.2.(3), the two spaces V_+ and V_- are orthogonal. It follows easily that $V_+ \cap V_- = \{0\}$, so $\mathbb{R}^{p+1,q+1} = V_+ \oplus V_-$. For the same reason, the spaces V_+, V_- are nondegenerate, meaning that the restriction of $\mathbf{b}$ to V_i for $i \in \{+,-\}$ has no kernel and is of signature (p_i, q_i), with $p_+ + p_- = p + 1$ and $q_+ + q_- = q + 1$.

By Lemma 8.4.2.(1), each V_i is $\mathsf{Aut}_G(\Omega)$-invariant. Therefore we get a $\mathsf{Aut}_G(\Omega)$-invariant orthogonal decomposition

$$\mathbb{R}^{p+1,q+1} = V_+ \oplus V_-.$$

Let us recall the notation of Section 3.4.2 and of the proof of Proposition 3.4.3; we write

$$J = \mathbb{P}\left\{v_+ + v_- \in V_+ \oplus V_- \mid \mathbf{b}(v_i, v_i) = 0 \text{ for } i \in \{+,-\}\right\}$$

and

$$U_i = \mathbb{P}\left\{v_+ + v_- \in V_+ \oplus V_- \mid -\mathbf{b}(v_+, v_+) = \mathbf{b}(v_-, v_-) = i\right\},$$

for $i \in \{-,+\}$. Recall that $\mathrm{Ein}^{p,q} = U_- \sqcup J \sqcup U_+$.

Let $\Omega_+ := \Omega \cap U_+$, $\Omega_- := \Omega \cap U_-$, and $\Omega_J := \Omega \cap J$. These sets are $\mathsf{Aut}_G(\Omega)$-invariant. One has either $\Omega_+ \neq \emptyset$ or $\Omega_- \neq \emptyset$, because Ω_J has empty interior in $\mathrm{Ein}^{p,q}$. Let us assume for example that $\Omega_+ \neq \emptyset$. Let $a \in \partial\Omega_+ \subset J \cup \partial\Omega$. If $a \in J$, then $a \in \partial U_+$. If $a \in \partial\Omega$, then by almost-homogeneity there exist $(g_k) \in \mathsf{Aut}_G(\Omega)^{\mathbf{N}}$ and $x \in \Omega$ such that $g_k \cdot x \to a$.

Since $x \in \Omega \subset \mathscr{O}_\Omega$, there exists $(v_+, v_-) \in V_+ \times V_-$ such that $v_+ + v_- \in \widetilde{\mathscr{O}}_\Omega$ and $x = \mathbb{P}(v_+ + v_-)$. Since for all $k \in \mathbb{N}$ the operator $\widetilde{g}_k \in \mathrm{O}(p+1, q+1)$ preserves $\widetilde{\mathscr{O}}_\Omega$, by Lemma 8.4.3, one has

$$|\widetilde{g}_k \cdot (v_+ + v_-)| \to +\infty,$$

for any fixed norm $|.|$ on V.

On the other hand, up to extracting, there exist $w_+ \in V_+$ and $w_- \in V_-$ such that $|w_+ + w_-| = 1$ and $\frac{\widetilde{g}_k \cdot (v_+ + v_-)}{|\widetilde{g}_k \cdot (v_+ + v_-)|} \to w_+ + w_-$. This implies $a = [w_+ + w_-]$. But

$$\mathbf{b}(w_+, w_+) = \lim_{k\to+\infty} \mathbf{b}\Big(\frac{\widetilde{g}_k \cdot v_+}{|\widetilde{g}_k \cdot (v_+ + v_-)|}, \frac{\widetilde{g}_k \cdot v_+}{|\widetilde{g}_k \cdot (v_+ + v_-)|}\Big) = \lim_{k\to+\infty} \frac{\mathbf{b}(v_+, v_+)}{|\widetilde{g}_k \cdot (v_+ + v_-)|^2} = 0,$$

and the same computation holds for w_-, meaning that $a \in J$. Since $J \subset \partial U_+$, we have $a \in \partial U_+$.

We have proven that $\partial\Omega_+ \subset \partial U_+$. Hence Ω_+ is closed in U_+. Since it is also open, it is a union of connected components of U_+. But as soon as $p_+ \geq 2$ or $q_- \geq 2$, by Proposition 3.4.3, the open set U_+ has no proper connected components. Since $\Omega_+ \subset \Omega$ is proper, this implies that $(p_+, q_+) = (1, q)$ and $(p_-, q_-) = (p, 1)$. Hence again by Proposition 3.4.3, the set Ω_+ is a diamond.

If $\Omega_- \neq \emptyset$, then by Proposition 3.4.3, it has to be the diamond dual to Ω_+. But then $\Omega_+ \cup \Omega_- \subset \Omega$ is not proper. Hence necessarily $\Omega_- = \emptyset$. By openness of Ω, one thus have $\Omega_J = \emptyset$, hence $\Omega = \Omega_+$ is a diamond. This concludes the proof of Theorem 8.4.1.

8.4.4 Exceptional isomorphisms in low dimensions

This section comes from a collaboration with Adam Chalumeau [CG24]. We use exceptional isomorphisms in low dimensions with $\mathfrak{so}(p,p)$, $p \in \{3,4\}$, to deduce, from Theorem 8.4.1, more information on proper almost-homogeneous domains in certain Grassmannians and other flag manifolds of $\mathfrak{so}(p,p)$; see Corollary 8.5.2 below.

In [LZ19], Limbeek–Zimmer prove that any proper divisible domain of Grassmannian $\mathrm{Gr}_p(\mathbb{R}^{2p})$ of p-planes of $\mathbb{R}^{2p}$ which is *convex in some affine chart* is a realization of $\mathbb{X}(\mathfrak{sl}(2p,\mathbb{R}), \alpha_p)$ (see Fact 1.4.11). The exceptional isomorphism $\mathrm{PGL}(4,\mathbb{R})^0 \simeq \mathrm{PO}(3,3)^0$ allows us to strengthen this rigidity result in the case where $p = 2$, not making any convexity assumption and only asking for almost-homogeneity, see Corollary 8.4.4 below. Another exceptionnal isomorphism, called *triality*, allows us to study new flag manifolds of $\mathfrak{so}(4,4)$. In this section, we provide the proof of the following corollary of Theorem 8.4.1:

Corollary 8.4.4 (with Chalumeau, see Corollary 1.4.18)**.** (1) *Let $\Omega \subset \mathrm{Gr}_2(\mathbb{R}^4)$ be a proper almost-homogeneous domain. Then Ω is a realization of $\mathbb{X}(\mathfrak{sl}(4,\mathbb{R}),\alpha_2)$. In other words, there exists $g \in \mathrm{PGL}(4,\mathbb{R})$ such that $\Omega = g\cdot\mathbb{B}_{2,2}$.*

(2) *Let $\mathscr{F}_i$, $i \in \{3,4\}$, be one of the two connected components of the space of maximal isotropic subspaces of $\mathbb{R}^{4,4}$. Let $\Omega \subset \mathscr{F}_i$ be a proper almost-homogeneous domain. Then Ω is a realization of $\mathbb{X}(\mathfrak{so}(4,4),\alpha_i)$.*

8.4.4.1 The Lie algebra isomorphism $\mathfrak{so}(3,3) \simeq \mathfrak{sl}(4,\mathbb{R})$. Let us recall the Plücker triple $(\mathrm{PGL}(4,\mathbb{R}),\rho_0,\bigwedge^2\mathbb{R}^4)$ defined in Section 2.4.2.2 (see also Example 7.1.1). The bilinear form ω defined on $\bigwedge^2\mathbb{R}^4$ by

$$\omega(x,y) = x\wedge y \quad \forall x,y \in \bigwedge^2\mathbb{R}^4$$

is nondegenerate, symmetric, of signature $(3,3)$. Since $\rho_0(\mathrm{PGL}(4,\mathbb{R}))$ preserves this bilinear form, one has $\rho_0(\mathrm{PGL}(4,\mathbb{R})) \subset \mathrm{PO}(\omega) \simeq \mathrm{PO}(3,3)$,so $\mathfrak{sl}(4,\mathbb{R}) \hookrightarrow \mathfrak{so}(3,3)$. For a reason of dimension, this embedding is an isomorphism of Lie algebras, which gives an equality between the identity components of $\rho_0(\mathrm{PGL}(4,\mathbb{R}))$ and $\mathrm{PO}(\omega)$. Then

$$\iota_{\rho_0}(\mathrm{Gr}_2(\mathbb{R}^4)) = \{[x] \in \bigwedge^2\mathbb{R}^4 \mid \omega(x,x)=0\} \simeq \mathrm{Ein}^{2,2}\,.$$

Thus there is a ρ_0-equivariant diffeomorphism

$$\mathrm{Gr}_2(\mathbb{R}^4) = \mathscr{F}(\mathfrak{sl}(4,\mathbb{R}),\alpha_2) \simeq \mathscr{F}(\mathfrak{so}(3,3),\alpha_1) = \mathrm{Ein}^{2,2}\,.$$

8.4.4.2 Triality and $\mathfrak{so}(4,4)$. Another exceptional isomorphism arising in low dimension appears for $\mathrm{Ein}^{3,3}$. The set of maximal totally isotropic subspaces of $\mathbb{R}^{4,4}$ has two connected components, denoted by $\mathscr{F}_3$ and $\mathscr{F}_4$. They are both flag manifolds, corresponding to two extremal roots of the Dynkin diagram of $\mathfrak{so}(4,4)$. The root system of $\mathfrak{so}(4,4)$ is D_4 (see Figure 8.2). It is a tripod and has automorphism group the symmetric group $\mathfrak{S}_3$. The extremal roots correspond to the flag manifolds $\mathrm{Ein}^{3,3}$, $\mathscr{F}_3$ and $\mathscr{F}_4$, and are permuted by the automorphism group of D_4. In particular, there exists an automorphism σ of order 3, sending the root corresponding to $\mathrm{Ein}^{3,3}$ to the one corresponding to $\mathscr{F}_3$, called *triality* (see Figure 8.2). The automorphism σ induces an outer automorphism φ of order 3 of $\mathfrak{so}(4,4)$. Then φ induces a φ-equivariant diffeomorphism $\mathrm{Ein}^{3,3} \simeq \mathscr{F}_3$. The notion of transversality, and hence of properness, is preserved by this diffeomorphism, as all flag manifolds of $\mathfrak{so}(4,4)$ are self-opposite (the opposition involution of D_4 is trivial). The same construction holds for $\mathscr{F}_4$, considering φ^2 instead of φ.

8.4.4.3 Proof of Corollary 8.4.4. We can now prove Corollary 8.4.4. Let us first prove (1). Let $\Omega \subset \mathrm{Gr}_2(\mathbb{R}^4)$ be a proper almost-homogeneous domain. By properness, there exists $y \in \mathrm{Gr}_2(\mathbb{R}^4)$ such that $y \cap x = \{0\}$ for all $x \in \overline{\Omega}$. But this is equivalent to saying that $\iota_{\rho_0}(x) \notin \iota_{\rho_0}(y)^{\perp_\omega}$ for all $x \in \overline{\Omega}$. Hence $\mathrm{Z}_{\iota_{\rho_0}(y)} \cap \overline{\iota_{\rho_0}(\Omega)} = \emptyset$, so $\iota_{\rho_0}(\Omega)$ is a proper domain of $\mathrm{Ein}^{2,2}$. Moreover, by ρ_0-equivariance of ι_{ρ_0}, the domain $\iota_{\rho_0}(\Omega)$ is almost-homogeneous. Then by Theorem 8.4.1, it is a diamond. By the equality $\rho_0(\mathrm{PGL}(4,\mathbb{R})^0) = \mathrm{PO}(3,3)^0$,

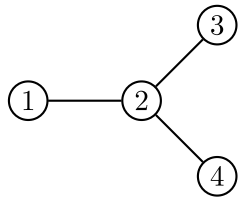

Figure 8.2 – Dynkin diagram of D_4. The triality is the automorphism σ of D_4 that stabilizes 2, and maps 3 to 1, 1 to 4, and 4 to 3. The existence of triality implies that each of the extremal vertices (1, 3, and 4) corresponds to a restricted root defining the Einstein universe $\mathrm{Ein}^{2,2}$.

we thus know that all proper almost-homogeneous domains of $\mathrm{Gr}_2(\mathbb{R}^4)$ are $\mathrm{PGL}(4,\mathbb{R})$-translates of each other. Since the domain $\mathbf{B}_{2,2}(\mathbf{b}_{\mathsf{std}})$ (in the notation of Equation (3.3.1)) is part of them, they are all $\mathrm{PGL}(4,\mathbb{R})$-translates of it. Point (1) then follows by Proposition 3.3.1(see also Example 5.1.13.(2)).

Point (2) of Corollary 8.4.4 is a straightforward consequence of the φ-equivariance (resp. φ^2-equivariance) of the diffeomorphism $\mathrm{Ein}^{3,3} \simeq \mathscr{F}_1$ (resp. $\mathrm{Ein}^{3,3} \simeq \mathscr{F}_2$) preserving the notions of transversality and properness. □

8.5 The centralizer of a group dividing a proper domain in Grassmannians

In this section, we investigate Question 1.2.1 for the Nagano pair $(\mathfrak{sl}(p+q,\mathbb{R}), \alpha_p)$, with $p, q \in \mathbb{N}_{\geq 2}$.

Given a proper domain $\Omega \subset \mathrm{Gr}_p(\mathbb{R}^{p+q})$, the boundary $\partial\Omega$ of Ω is a C^0-subanifold of $\mathrm{Gr}_p(\mathbb{R}^{p+q})$ if for all $x \in \partial\Omega$ there exists a neighborhood $\mathscr{V}$ of x in $\mathrm{Gr}_p(\mathbb{R}^{p+q})$, an open subset $\mathscr{U}$ of $\mathbb{R}^{pq}$ containing 0 and a homeomorphism $f : \mathscr{V} \to \mathscr{U}$ such that $\partial\Omega \cap \mathscr{V} = f^{-1}(\mathscr{U} \cap (\mathbb{R}^{pq-1} \times \{0\}))$ (recall that the manifold $\mathrm{Gr}_p(\mathbb{R}^{p+q})$ has dimension pq). This is the case for instance if Ω is properly convex in some affine chart of $\mathrm{Gr}_p(\mathbb{R}^{p+q})$, but not guaranteed by dual convexity. In this section, we prove:

Theorem 8.5.1 (see Theorem 1.4.12)**.** *Let $2 \leq p \leq q$. Let $\Omega \subset \mathrm{Gr}_p(\mathbb{R}^{p+q})$ be a proper domain whose boundary is a C^0-subanifold of $\mathrm{Gr}_p(\mathbb{R}^{p+q})$. Assume that there exists a discrete subgroup $\Gamma \leq \mathrm{PGL}(p+q,\mathbb{R})$ acting cocompactly on Ω. Then any Γ-invariant decomposition of $\mathbb{R}^{p+q}$ is trivial.*

Theorem 8.5.1 is proved in [LZ19] under the additional assumptions that $p = q$ and Ω is properly convex in an affine chart.

The main consequence of Theorem 8.5.1 is:

Corollary 8.5.2 (see Corollary 1.4.13)**.** *Let $2 \leq p \leq q$. Let $\Omega \subset \mathrm{Gr}_p(\mathbb{R}^{p+q})$ be a proper domain whose boundary is a C^0-subanifold of $\mathrm{Gr}_p(\mathbb{R}^{p+q})$. Assume that there exists a discrete subgroup $\Gamma \leq \mathrm{PGL}(p+q,\mathbb{R})$ acting cocompactly on Ω. Then the centralizer of Γ in $\mathrm{PGL}(p+q,\mathbb{R})$ is finite.*

Proof of Corollary 8.5.2. Let

$$H := \{h \in \mathrm{GL}(p+q,\mathbb{R}) \mid [g] \in \Gamma\};$$
$$C_\Gamma := \{g \in \mathrm{Mat}_{p+q}(\mathbb{R}) \mid gh = hg\ \forall h \in H\}$$

By [LZ19, Thm 9.3] (which is stated for domains that are properly convex in an affine chart but whose proof only uses the properness assumption), there exist Γ-invariant subspaces $V_1, \dots, V_N$ of $\mathbb{R}^{p+q}$ such that

$$C_\Gamma = \sum_{i=1}^{N} \mathbb{R}\,\mathrm{id}_{V_i}\,.$$

By Theorem 8.5.1, one has $N = 1$ and $C_\Gamma = \mathbb{R}\,\mathrm{id}$. Since the centralizer C is of Γ in $\mathrm{PGL}(p+q,\mathbb{R})$ is an algebraic group, its identity component C^0 has finite index in C. Thus it suffices to prove that C^0 is trivial.

Let gH. Now let $h \in C^0$ and $\mathsf{h} \in \mathrm{GL}(p+q,\mathbb{R})$ such that $h = [\mathsf{h}]$.

Then $[g\mathsf{h}g^{-1}\mathsf{h}^{-1}] = [\mathrm{id}]$. Thus there exists $\lambda_h \in \mathbb{R}$ such that $g\mathsf{h}g^{-1}\mathsf{h}^{-1} = \lambda_h\,\mathrm{id}$. Since $g\mathsf{h}g^{-1}\mathsf{h}^{-1}$ has determinant 1, we have $\lambda_h = 1$ or -1. Since $h \mapsto \lambda_h$ is continuous, it is constant on C^0. Since $\lambda_{\mathrm{id}} = 1$, this map is constant equal to 1. Hence we have $g\mathsf{h}g^{-1}\mathsf{h}^{-1} = \mathrm{id}$ for all $\mathsf{h} \in \mathrm{GL}(p+q,\mathbb{R})$ such that $[\mathsf{h}] \in C^0$. This is true for all $g \in H$. Hence $\mathsf{h} \in C_\Gamma$ for all $h \in C^0$. Since $C_\Gamma = \mathbb{R}\,\mathrm{id}$, it implies that $C^0 = \{\mathrm{id}\}$. $\square$

As explained in Section 1.4.4.2, Theorem 8.5.1 and its Corollary 8.5.2 point towards a form of rigidity for proper divisible domains in the Grassmannians $\mathrm{Gr}_p(\mathbb{R}^{p+q})$: whenever $p, q \geq 2$, one cannot produce proper divisible domains in $\mathrm{Gr}_p(\mathbb{R}^{p+q})$ by *joining* two proper divisible domains of smaller Grassmannians.

8.5.1 Notation and reminders on cohomological dimension

Let us fix some notation for the proof of Theorem 8.5.1 : in this section, we consider the Plücker triple $(\mathrm{PGL}(p+q,\mathbb{R}), \rho_0, \bigwedge^p \mathbb{R}^{p+q})$ of $(\mathfrak{sl}(p+q,\mathbb{R}), \alpha_p)$ defined in Equation (2.4.2.2). Recall that the associate embeddings by Fact 2.3.4 are the classical Plücker embeddings, given in Equation (2.4.5).

8.5.1.1 Special subvarieties of Grassmannians. For any vector subspace L of $\mathbb{R}^{p+q}$ of dimension ℓ, we will consider the algebraic subvarieties

$$\mathrm{Z}_L := \{x \in \mathrm{Gr}_p(\mathbb{R}^{p+q}) \mid \dim(x \cap L) > \max(0, \ell - q)\};$$
$$\mathrm{Z}_L^- := \{\xi \in \mathrm{Gr}_q(\mathbb{R}^{p+q}) \mid \dim(\xi \cap L) > \max(0, \ell - p)\}.$$

We say that L is a *proper* subspace of $\mathbb{R}^{p+q}$ if $L \notin \{\{0\}, \mathbb{R}^{p+q}\}$. If L is proper, then Z_L (resp. Z_L^-) is a nonempty proper subvariety of $\mathrm{Gr}_p(\mathbb{R}^{p+q})$ (resp. $\mathrm{Gr}_q(\mathbb{R}^{p+q})$).

If $\xi \in \mathrm{Gr}_q(\mathbb{R}^{p+q})$, then we recover the set $\mathrm{Z}_\xi \subset \mathrm{Gr}_p(\mathbb{R}^{p+q})$ defined in Section 2.2.6, see Example 2.2.1.(1).

8.5.1.2 Reminders on the virtual cohomological dimension. The *cohomological dimension* of a group Γ is defined as

$$\mathrm{cd}(\Gamma) := \sup\{k \in \mathbb{N} \mid H^k(\Gamma, M) \neq 0 \text{ for } M \text{ a } \Gamma\text{-module}\},$$

where $H^k(\Gamma, M)$ is the k-th *cohomology group of* Γ, see [Hat02] for more details. If $\Gamma' \leq \Gamma$, then $\mathrm{cd}(\Gamma') \leq \mathrm{cd}(\Gamma)$. The *virtual cohomological dimension* $\mathrm{vcd}(\Gamma)$ of Γ is then the infimum of $\mathrm{cd}(\Gamma')$ as Γ' ranges over finite-index subgroups of Γ.

We shall not develop further on the definition of cohomological dimension and virtual cohomological dimension. What we are interested in is their property of "encoding the dimension of a space" when Γ is finitely generated:

1. $\mathrm{vcd}(\Gamma) > 0$ as soon as Γ is infinite.
2. If Γ acts properly discontinuously on a contractible manifold X, then one has $\mathrm{vcd}(\Gamma) \leq \dim(X)$, with equality if and only if the action is cocompact.
3. If $\Gamma = \Gamma_1 \times \Gamma_2$, then $\mathrm{vcd}(\Gamma) \leq \mathrm{vcd}(\Gamma_1) + \mathrm{vcd}(\Gamma_2)$.

8.5.2 Outline of the proof

In this section, we briefly outline the key steps of the proof of Theorem 8.5.1. Let $\Omega \subset \mathrm{Gr}_p(\mathbb{R}^{p+q})$ be a proper domain whose boundary is a C^0-subanifold of $\mathrm{Gr}_p(\mathbb{R}^{p+q})$, divisible by a discrete subgroup $\Gamma \leq \mathrm{PGL}(p+q, \mathbb{R})$. We assume, for contradiction, that there exists a non-trivial Γ-invariant decomposition $\mathbb{R}^{p+q} = V_1 \oplus V_2$. We may assume that $\dim(V_1) \geq \dim(V_2)$. By the assumption on $p \leq q$, we necessarily have $p \leq \dim(V_1)$. Thus there are three cases to consider:

1. *If* $\dim(V_1), \dim(V_2) > p$. The idea is then to construct two Γ-equivariant projections $P_1 : \mathrm{Gr}_p(\mathbb{R}^{p+q}) \smallsetminus F_1 \to \mathrm{Gr}_p(V_1)$ and $P_2 : \mathrm{Gr}_p(\mathbb{R}^{p+q}) \smallsetminus F_1 \to \mathrm{Gr}_p(V_2)$ (see Section 8.5.5.3), where F_1 and F_2 are subsets of $\mathrm{Gr}_p(\mathbb{R}^{p+q})$ with empty interior that do not intersect Ω, such that $P_i(\Omega) \subset \mathrm{Gr}_p(V_i)$ is a proper (proved in Section 8.5.5.3.2), quasi-homogeneous (proved in Section 8.5.5.3.1) domain, invariant under the action of Γ. Note that Γ does not necessarily act properly discontinuously on $P_i(\Omega)$; however, it does have a properly discontinuous action on the product $P_1(\Omega) \times P_2(\Omega) \times \mathbb{R}$ (considered as a "join"), as we prove in Proposition 8.5.16.

The fact that the open set $P_i(\Omega)$ is proper follows from the fact that it is equal to the intersection of $\partial\Omega$ with a submanifold of $\mathrm{Gr}_p(\mathbb{R}^{p+q})$ identified with $\mathrm{Gr}_p(V_i)$, as proven in Lemma 8.5.12. This is a consequence of the divisibility of Ω and a lemma of Vey and Limbeek–Zimmer (see Lemma 8.5.11).

If our open sets satisfy the appropriate contractibility conditions, then we have (see Section 8.5.5.4)

$$\begin{aligned} pq &= \dim(\mathrm{Gr}_p(\mathbb{R}^{p+q})) = \dim(\Omega) \\ &= \mathrm{vcd}(\Gamma) \leq \dim(P_1(\Omega) \times P_2(\Omega) \times \mathbb{R}) \\ &= \dim(P_1(\Omega)) + \dim(P_2(\Omega)) + 1 = \dim(\mathrm{Gr}_p(V_1)) + \dim(\mathrm{Gr}_p(V_2)) + 1 < pq, \end{aligned}$$

which is a contradiction. By "appropriate contractibility conditions," we mean that Ω, $P_1(\Omega)$, and $P_2(\Omega)$ are all contractible, see Section 8.5.1.2. We show in Proposition 8.5.14 that this holds as soon as $\partial\Omega$ is a C^0-subanifold of $\mathrm{Gr}_p(\mathbb{R}^{p+q})$.

Here, contractibility is mainly used to reduce the argument to considerations on the virtual cohomological dimension of Γ. We believe that this assumption can be removed with a finer understanding of the structure of the domains $P_1(\Omega)$ and $P_2(\Omega)$. This case is treated in Section 8.5.5.4.2.

2. $\dim(V_1) > p$ *and* $\dim(V_2) \leq p$. The space $\mathrm{Gr}_p(V_2)$ is either empty or a singleton. Thus, we can no longer construct a well-behaved proper domain $P_2(\Omega)$. Instead of considering the action of Γ on a space of the form $P_1(\Omega) \times P_2(\Omega) \times \mathbb{R}$, we consider a new space of the form $P_1(\Omega) \times \mathrm{SL}(V_2)/\,\mathrm{SO}(V_2) \times \mathbb{R}$, which again leads to a contradiction on the virtual cohomological dimensions. This case is trated in Section 8.5.5.4.1.

3. *If* $p = q = \dim(V_1) = \dim(V_2)$. In this case, a completely different argument, using Lemma 8.4.3, allows us to conclude. This case is treated in Section 8.5.5.2.

8.5.3 Preliminary lemmas and notations

Before starting the proof of Theorem 8.5.1, we first establish in this section several lemmas that, while directly useful for the proof, are slightly more general.

8.5.3.1 Proper domains in subgrassmannians. Given a vector subspace L of $\mathbb{R}^{p+q}$ of dimension ℓ, let we define

$$\begin{aligned}\mathscr{A}_L &:= \{x \in \mathrm{Gr}_p(\mathbb{R}^{p+q}) \mid \dim(x \cap L) = \min(p, \ell)\}\\ \mathscr{A}_L^- &:= \{\xi \in \mathrm{Gr}_q(\mathbb{R}^{p+q}) \mid \dim(\xi \cap L) = \min(q, \ell)\}.\end{aligned}$$

If $\ell \geq p$, then $\mathscr{A}_L$ is exactly the set of p-planes that are contained in L. If $\ell \leq p$, these are the p-planes that contain L. The same analysis holds for $\ell \geq q$ and $\ell \leq q$.

Remark 8.5.3. In the notation of Section 2.2.6.3, one has $\mathscr{A}_L = \mathbf{C}_{\mathrm{id}}^{(\{\alpha_p\},\{\alpha_\ell\})}(L)$.

If $\ell \geq p$, we identify $\mathrm{Gr}_p(L)$ with $\mathscr{A}_L$ via the canonical homeomorphism

$$\begin{array}{rccc}\delta_L : & \mathrm{Gr}_p(L) & \longrightarrow & \mathscr{A}_L \\ & x & \longmapsto & x.\end{array} \tag{8.5.1}$$

Lemma 8.5.4. *Let $\Omega \subset \mathrm{Gr}_p(\mathbb{R}^{p+q})$ be a proper open subset, and let $L \leq \mathbb{R}^{p+q}$ be a vector subspace of dimension larger than p. Then $\overline{\Omega} \cap \mathscr{A}_L$ is a proper subset of $\mathrm{Gr}_p(L)$.*

Proof. By openness of Ω, recall that its dual Ω^* has nonempty interior. Then, by density of $\mathrm{Gr}_q(\mathbb{R}^{p+q}) \smallsetminus \mathrm{Z}_L^-$ in $\mathrm{Gr}_q(\mathbb{R}^{p+q})$, there exists $\xi \in \Omega^* \smallsetminus \mathrm{Z}_L^-$. Then by definition of Z_L^-:

$$\dim(\xi \cap L) \leq \ell - p.$$

On the other hand, one always has:

$$\dim(\xi \cap L) = \dim(\xi) + \dim(L) - \dim(\xi + L) = q + \ell - \dim(\xi + L) \geq \ell - p.$$

Thus $\dim(\xi \cap L) = \ell - p$. Then $\xi \cap L$ defines an element of $\mathrm{Gr}_{\ell-p}(L)$. By definition of Ω^*, one has $x \cap (\xi \cap L) = \{0\}$ for all $x \in \overline{\Omega} \cap \mathscr{A}_L$. Hence $\overline{\Omega} \cap \mathscr{A}_L$ is contained in the affine chart $\mathrm{Gr}_p(L) \smallsetminus \mathrm{Z}_{\xi \cap L}$ of $\mathrm{Gr}_p(L)$. ∎

8.5.3.2 Contractibility. Lemma 7.2.10, proven in Chapter 7, allows us to deduce the topology of a properly homogeneous domain from the topology near its boundary:

Proposition 8.5.5. *Let $\Omega \subset \mathrm{Gr}_p(\mathbb{R}^{p+q})$ be a proper almost-homogeneous domain, such that any element $x \in \partial\Omega$ admits a neighborhood $\mathscr{V}_x$ such that $\mathscr{V}_x \cap \Omega$ is contractible. Then Ω is contractible.*

Proof. By Lemma 7.2.5, there exists an $\mathscr{R}$-extremal point $x \in \partial\Omega$. Let $\mathscr{V}_x$ be defined as in the Proposition.

Let $x_0 \in \Omega$. Let $k \geq 1$ and let $\gamma : \mathbb{S}^k \to \Omega$ be a continuous map such that $\gamma(s_0) = x_0$ (s_0 being the south of $\mathbb{S}^k$). Then $\gamma(\mathbb{S}^k)$ is compact so by divisibility and Lemma 7.2.10 there exists $g \in \Gamma$ such that $g \cdot \gamma(\mathbb{S}^k) \subset \mathscr{V}_x$. Then $s \mapsto g \cdot \gamma(s)$ is a continuous map from $\mathbb{S}^k$ to $\mathscr{V}_x \cap \Omega$ such that $g \cdot \gamma(s_0) = g \cdot x_0$. Since the k-th homotopy group $\pi_k(\mathscr{V}_x \cap \Omega, g \cdot x_0)$ of $\mathscr{V}_x \cap \Omega$ pointed at $g \cdot x_0$ is trivial (by contractibility), the path $g \cdot \gamma$ is homotopic to $g \cdot x_0$, and hence γ is homotopic to x_0. Thus $\pi_k(\Omega, x_0)$ is trivial.

Since Ω is connected, all its homotopy groups are trivial. Since Ω is a differentiable manifold, it is contractible. □

8.5.3.3 Continuous boundary. The goal of this section is to prove Corollary 8.5.7 below. It will follow from Proposition 8.5.6 below and Proposition 8.5.5 from the previous section. We first prove:

Proposition 8.5.6. *Let $\Omega \subset \mathrm{Gr}_p(\mathbb{R}^{p+q})$ be a proper dually convex domain whose boundary is a C^0-subanifold of $\mathrm{Gr}_p(\mathbb{R}^{p+q})$. Then for all $x \in \partial\Omega$ there exist a neighborhood $\mathscr{V}$ of x, an open subset $\mathscr{U}$ of $\mathbb{R}^{pq}$ containing 0 and a homeomorphism $f : \mathscr{V} \to \mathscr{U}$ such that*

$$\Omega \cap \mathscr{V} = f^{-1}\left((\mathbb{R}^{pq-1} \times \mathbb{R}_{>0}) \cap \mathscr{U}\right).$$

Proof. By assumtion on the boundary of Ω, we know that there exist a neighborhood $\mathscr{V}$ of x, an open subset $\mathscr{U}$ of $\mathbb{R}^{pq}$ and a homeomorphism $f : \mathscr{V} \to \mathscr{U}$ such that $\partial\Omega \cap \mathscr{V} = f^{-1}(\mathscr{U} \cap (\mathbb{R}^{pq-1} \times \{0\}))$. Up to shrinking, we may assume that $\mathscr{U}$ is convex.

Since $x \in \partial\Omega$, there exists $y_0 \in \mathscr{U} \cap f(\Omega)$. Up to post composing by the symmetry $(x_1, \ldots, x_{pq}) \mapsto (x_1, \ldots, -x_{pq})$, we may assume that $y \in \mathscr{U} \cap (\mathbb{R}^{pq-1} \times \mathbb{R}_{>0})$.

Assume for a contradiction that there exists $z \in \mathscr{U} \cap (\mathbb{R}^{pq-1} \times \mathbb{R}_{>0}) \smallsetminus f(\Omega \cap \mathscr{V})$. Then the segment $[z, y]$ must hit the boundary of $f(\mathscr{V} \cap \Omega)$ in a point w. But the condition $w \in [z, y]$ and the convexity of $\mathscr{U}$ imply that $w \in \mathscr{U} \cap (\mathbb{R}^{pq-1} \times \mathbb{R}_{>0})$. This is in contradiction with the fact that $\partial f(\Omega \cap \mathscr{V}) = f(\partial\Omega \cap \mathscr{V}) \subset \mathbb{R}^{pq-1} \times \{0\}$. Then $\mathscr{U} \cap (\mathbb{R}^{pq-1} \times \mathbb{R}_{>0}) \subset f(\Omega \cap \mathscr{V})$.

Now let us assume that the exists $z \in f(\Omega \cap \mathscr{V}) \smallsetminus \mathscr{U} \cap (\mathbb{R}^{pq-1} \times \mathbb{R}_{>0})$. Then we have $z \in \mathscr{U} \cap (\mathbb{R}^{pq-1} \times \mathbb{R}_{\leq 0})$, so as in the previous paragraph, one has $\mathscr{U} \cap (\mathbb{R}^{pq-1} \times \mathbb{R}_{<0}) \subset f(\Omega \ \cap \mathscr{V})$. But then,

$$f(\Omega \cap \mathscr{V}) = \mathscr{U} \smallsetminus \mathbb{R}^{pq-1} \times \{0\}. \tag{8.5.2}$$

Since Ω is proper and divisible, and thus almost-homogeneous, it has to be dually convex by Proposition 3.1.11. Then by Proposition 7.1.9 (since the Grassmannian is an irreducible Nagano space of real type), it is equal to the interior of its closure. Equation (8.5.2) gives:

$$f(\overline{\Omega} \cap \mathscr{V}) = \overline{f(\Omega \cap \mathscr{V})} = \mathscr{U}.$$

Taking the interior gives:

$$f(\Omega \cap \mathscr{V}) = f(\mathrm{int}(\overline{\Omega}) \cap \mathscr{V}) = \mathrm{int}(f(\overline{\Omega} \cap \mathscr{V})) = \mathscr{U},$$

contradicting Equation (8.5.2). Hence $f(\Omega \cap \mathscr{V}) \subset \mathscr{U} \cap (\mathbb{R}^{pq-1} \times \mathbb{R}_{>0})$.

We have proven that $\Omega \cap \mathscr{V} = f^{-1}(\mathscr{U} \cap (\mathbb{R}^{pq-1} \times \mathbb{R}_{>0}))$. □

Propositions 8.5.5 and 8.5.6 imply directly:

Corollary 8.5.7. *Let $\Omega \subset \mathrm{Gr}_p(\mathbb{R}^{p+q})$ be a proper almost-homogeneous domain whose boundary is a C^0-subanifold of $\mathrm{Gr}_p(\mathbb{R}^{p+q})$. Then for all $x \in \partial\Omega$ there exists a neighborhood $\mathscr{V}_x$ of x such that $\mathscr{V}_x \cap \Omega$ is contractible. In particular, the domain Ω is contractible.*

8.5.4 Invariant subspaces

Let $L \subset \mathbb{R}^{p+q}$ be a vector subspace. Note that we have

$$\mathrm{Z}_L = \bigcap_{\xi \in \mathscr{A}_L^-} \mathrm{Z}_\xi .$$

Thus $\iota_{\rho_0}(\mathrm{Z}_L) \subset \bigcap_{\xi \in \mathscr{A}_L^-} \iota_{\rho_0}(\mathrm{Z}_\xi) \subset \bigcap_{\xi \in \mathscr{A}_L^-} \iota_{\rho_0}^-(\xi)$. Thus, if L is a proper subspace of $\mathbb{R}^{p+q}$, then $\iota_{\rho_0}(\mathrm{Z}_L)$ is contained in a projective hyperplane of $\mathbb{P}(\bigwedge^p \mathbb{R}^{p+q})$. This implies the following lemma:

Lemma 8.5.8. *Let $\Omega \subset \mathrm{Gr}_p(\mathbb{R}^{p+q})$ be a proper domain, disible by some discrete subgroup $\Gamma \leq \mathrm{PGL}(p+q, \mathbb{R})$. If L is a proper Γ-invariant subspace of $\mathbb{R}^{p+q}$, then $\mathrm{Z}_L \cap \Omega$ is empty.*

Proof. Assume for a contradiction that there exists $y \in \mathrm{Z}_L \cap \Omega$. Let $a \in \mathrm{Extr}_{\mathscr{R}}(\Omega)$. By almost-homogeneity and Lemma 7.2.10, there exists a sequence (g_k) such that $g_k \cdot y \to a$. Since L is Γ-invariant, for any $k \in \mathbb{N}$, we have $g_k \cdot y \in \mathrm{Z}_L$. Since Z_L is closed in $\mathrm{Gr}_p(\mathbb{R}^{p+q})$, we have $a \in \mathrm{Z}_L$. This is true for any $\mathscr{R}$-extremal point, so $\mathrm{Extr}_{\mathscr{R}}(\Omega) \subset \mathrm{Z}_L$. Thus the image $\iota_{\rho_0}(\mathrm{Extr}_{\mathscr{R}}(\Omega))$ is contained in the proper projective subspace $\bigcap_{\xi \in \mathscr{A}_L^-} \iota_{\rho_0}^-(\xi)$ of $\mathbb{P}(\bigwedge^p \mathbb{R}^{p+q})$, which contradicts Lemma 7.2.5. Thus $\mathrm{Z}_L \cap \Omega$ has to be empty. □

8.5.5 Proof of Theorem 8.5.1

In this section, we end the proof of Theorem 8.5.1.

8.5.5.1 Notation. Let us first introduce some additional notation. We fix once and for all a proper domain $\Omega \subset \mathrm{Gr}_p(\mathbb{R}^{p+q})$ whose boundary is a C^0-subanifold of $\mathrm{Gr}_p(\mathbb{R}^{p+q})$, and assume that there exists a discrete subgroup $\Gamma \leq \mathrm{PGL}(p+q, \mathbb{R})$ dividing Ω. Note that, by Proposition 3.1.11, the domain Ω is dually convex, and thus, by Corollary 6.4.12 and Svarc–Milnor's Lemma, we have:

Fact 8.5.9. *The group Γ is finitely generated.*

Until the end of the section, we assume *for a contradiction* that there exist two non-trivial Γ-invariant subspaces V_1, V_2 of $\mathbb{R}^{p+q}$ such that

$$\mathbb{R}^{p+q} = V_1 \oplus V_2.$$

We denote by d_i the dimension of V_i for $i \in \{1, 2\}$. By assumption, we have $d_1, d_2 \geq 1$. Up to translating Ω by an element of $\mathrm{PGL}(p+q, \mathbb{R})$, we may assume that

$$V_1 = \mathrm{Span}(e_1, \dots, e_{d_1}); \quad V_2 = \mathrm{Span}(e_{d_1+1}, \dots, e_{p+q}),$$

where $(e_1, \dots, e_{p+q})$ is the canonical basis of $\mathbb{R}^{p+q}$ (recall Section 2.4.2). Then

$$\Gamma \leq \Big\{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \mid A \in \mathrm{GL}(d_1, \mathbb{R}), B \in \mathrm{GL}(d_2, \mathbb{R}) \Big\}.$$

We define the group homomorphism

$$\begin{array}{rccc} F : & \Gamma & \longrightarrow & \mathrm{PGL}(d_1, \mathbb{R}) \times \mathrm{PGL}(d_2, \mathbb{R}) \times \mathbb{R}_{>0} \\ & \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} & \longmapsto & \Big([A],\ [B],\ |\det(A)|^{d_2} |\det(B)|^{-d_1} \Big). \end{array} \tag{8.5.3}$$

The map F has finite kernel, and the image $F(\Gamma)$ is a discrete subgroup of $\mathrm{PGL}(d_1, \mathbb{R}) \times \mathrm{PGL}(d_2, \mathbb{R}) \times \mathbb{R}_{>0}$. We denote by p_i, $i \in \{1, 2\}$, the canonical projection from $\mathrm{PGL}(d_1, \mathbb{R}) \times \mathrm{PGL}(d_2, \mathbb{R}) \times \mathbb{R}_{>0}$ to $\mathrm{PGL}(d_i, \mathbb{R})$, and

$$\Gamma_i := \{ \mathsf{p}_i(F(g)) \mid g \in \Gamma \} \leq \mathrm{PGL}(d_i, \mathbb{R}).$$

The group Γ_i identifies with $\{ [g|_{V_i}] \mid [g] \in \Gamma \} \leq \mathrm{PGL}(V_i)$.

From now on, we will make a disjonction of cases, depending on the relative values of p, q, d_1, d_2.

8.5.5.2 Proof of Theorem 8.5.1 when $p = q = d_1 = d_2$**.** In this case, the two vector spaces V_1, V_2 are in $\mathrm{Gr}_p(\mathbb{R}^{2p})$.

Let $\mathscr{O}_\Omega := \mathscr{O}_\Omega^{\rho_0}$ be the convex hull of Ω in $\mathbb{P}(\bigwedge^p \mathbb{R}^{2p})$ of Definition 7.1.6, associated with the Plücker triple $(\mathrm{PGL}(2p, \mathbb{R}), \rho_0, \bigwedge^p \mathbb{R}^{2p})$ of $(\mathfrak{sl}(2p, \mathbb{R}), \alpha_p)$ defined in Equation (2.4.2.2). Let $\widetilde{\mathscr{O}_\Omega}$ be any properly convex open cone of $\bigwedge^p \mathbb{R}^{2p}$ lifting $\mathscr{O}_\Omega$.

Let $i \in \{1, 2\}$. By Lemma 7.2.5, there exists $a_i \in \mathrm{Extr}_{\mathscr{R}}(\Omega) \smallsetminus \mathrm{Z}_{V_i}$. By divisibility, there exist $(g_k) \in \Gamma^{\mathbb{N}}$ and $x_0 \in \Omega$ such that $g_k \cdot x_0 \to a_i$. Then by Lemma 7.2.10, up to extracting, we may assume that there exists $b_i \in \mathrm{Gr}_p(\mathbb{R}^{2p})$ such that $g_k \cdot x \to a_i$ whenever $x \notin \mathrm{Z}_{b_i}$. Thus by Fact 2.3.1.(2), we have $g_k^{-1} \cdot x \to b_i$ whenever $x \notin \mathrm{Z}_{a_i}$. Thus $V_i = g_k^{-1} \cdot V_i \to b_i$. Thus $V_i \in \partial \Omega$. Hence by Lemma 7.1.7, we have $\iota_{\rho_0}(V_i) \in \partial \mathscr{O}_\Omega$. Let

$$v_i \in \overline{\widetilde{\mathscr{O}_\Omega}} \smallsetminus \{0\}$$

be a lift of $\iota_{\rho_0}(V_i)$. For all $g \in \Gamma$, let $\widetilde{g} \in \mathrm{SL}^{\pm}(2p, \mathbb{R})$ be a lift of g, and let $\widetilde{\rho_0(g)} \in \mathrm{SL}^{\pm}(\bigwedge^p \mathbb{R}^{2p})$ be the unique lift of $\rho_0(g)$ preserving $\mathscr{O}_\Omega$. Since V_i is invariant, there exists $\lambda^i(g) \in \mathbb{R} \smallsetminus \{0\}$ such that $\widetilde{\rho_0(g)} \cdot v_i = \lambda^i(g) v_i$. Since

$$(\widetilde{\rho_0(g)} \cdot v_1 \wedge \widetilde{\rho_0(g)} \cdot v_2) = \det(\widetilde{g})(v_1 \wedge v_2),$$

one has $\lambda^1(g)\lambda^2(g) = \pm 1$.

Now let $x_\infty \in \partial\Omega$. By divisibility, there exists $x \in \Omega$ and $(g_k) \in \Gamma^{\mathbb{N}}$ such that $g_k \cdot x \to x_\infty$. Let $v \in \widetilde{\overline{\mathscr{O}_\Omega}} \smallsetminus \{0\}$ be any lift of $\iota_{\rho_0}(x)$. Then the sequence

$$\Big(\frac{\widetilde{\rho_0(g_k)} \cdot v}{|\widetilde{\rho_0(g_k)} \cdot v|}\Big)_{k\in\mathbb{N}}$$

converges to a point $v_\infty \in \iota_{\rho_0}(x_\infty) \smallsetminus \{0\}$. For all $k \in \mathbb{N}$ we have $\lambda^1(g_k^{-1})\lambda^2(g_k^{-1}) = \pm 1$, so up to extracting, we may assume that one of the two sequences $(\lambda^1(g_k^{-1}))$ and $(\lambda^2(g_k^{-1}))$ is bounded in $\mathbb{R}$, for instance assume that $(\lambda^1(g_k^{-1}))$ is bounded.

On the other hand, since $x \in \iota_{\rho_0}(\Omega) \subset \mathscr{O}_\Omega$ (by Proposition 7.1.7) and $x_\infty \in \partial\mathscr{O}_\Omega$, applying Lemma 8.4.3 we get that $|\widetilde{\rho_0(g_k)} \cdot v| \to +\infty$. Hence we have

$$v_1 \wedge \frac{\widetilde{\rho_0(g_k)} \cdot v}{|\widetilde{\rho_0(g_k)} \cdot v|} = \det(\widetilde{g_k})\Big(\widetilde{\rho_0(g_k^{-1})} \cdot v_1\Big) \wedge \frac{v}{|\widetilde{\rho_0(g_k)} \cdot v|} = \pm\frac{\lambda^1(g_k^{-1})}{|\widetilde{\rho_0(g_k)} \cdot v|}(v_1 \wedge v) \longrightarrow 0.$$

But one also has

$$v_1 \wedge \frac{\widetilde{\rho_0(g_k)} \cdot v}{|\widetilde{\rho_0(g_k)} \cdot v|} \longrightarrow v_1 \wedge v_\infty.$$

Hence $v_1 \wedge v_\infty = 0$. Recall from Equation (2.4.5) and Fact 2.3.4 that this is equivalent to $x_\infty \in \mathrm{Z}_{V_1}$. If we had chosen that $(\lambda^2(g_k^{-1}))$ was bounded in $\mathbb{R}$, we would have found $x_\infty \in \mathrm{Z}_{V_2}$.

Hence for all $x_\infty \in \partial\Omega$, we find $x_\infty \in \mathrm{Z}_{V_1} \cup \mathrm{Z}_{V_2}$, so $\partial\Omega \subset \mathrm{Z}_{V_1} \cup \mathrm{Z}_{V_2}$. Then Ω is equal to a connected component of $\mathrm{Gr}_p(\mathbb{R}^{2p}) \smallsetminus (\mathrm{Z}_{V_1} \cup \mathrm{Z}_{V_2})$. This is not possible because none of these connected components are proper. This concludes the proof in the case where $p = q = d_1 = d_2$.

Remark 8.5.10. In this section, we did not use the assumption that $\partial\Omega$ is a C^0-submanifold of $\mathrm{Gr}_p(\mathbb{R}^{2p})$.

8.5.5.3 Construction of a projection when p, q, d_1, d_2 are not all equal. In this section, we assume p, q, d_1, d_2 are not all equal. Up to exchanging d_1 and d_2, we may thus assume that $d_1 > p$. Then $d_2 = p + q - d_1 < q$. For $i \in \{1, 2\}$, let

$$\mathbf{Proj}_i : V \longrightarrow V_i \tag{8.5.4}$$

be the projection on V_i parallel to V_{3-i}.

For any $x \in \mathrm{Gr}_p(\mathbb{R}^{p+q})$, the set $\mathbf{Proj}_i(x) := \{\mathbf{Proj}_i(v) \mid v \in x\}$ is a vector subspace of V_i, not necessarily of dimension p. However, for a "generic" x, the set $\mathbf{Proj}_1(x)$ is p-dimensional: indeed, if $x \in \mathrm{Gr}_p(\mathbb{R}^{p+q}) \smallsetminus \mathrm{Z}_{V_2}$, since $d_2 < q$, by the definition of Z_{V_2}, we have $\dim(x \cap V_2) = 0$. Since $\ker(\mathbf{Proj}_{|x}) = x \cap V_2$, we have $\mathrm{rk}(p_1|_x) = p$, so $\mathbf{Proj}_1(x) \in \mathrm{Gr}_p(V_1)$. Hence the map

$$\begin{array}{cccc} P_1: & \mathrm{Gr}_p(\mathbb{R}^{p+q}) \smallsetminus \mathrm{Z}_{V_2} & \longrightarrow & \mathrm{Gr}_p(V_1) \\ & x & \longmapsto & \mathbf{Proj}_1(x) \end{array}$$

is well defined. By Lemma 8.5.8, one has $\Omega \subset \mathrm{Gr}_p(\mathbb{R}^{p+q}) \smallsetminus \mathrm{Z}_{V_2}$. Hence the set

$$P_1(\Omega) := \{P_1(x) \mid x \in \Omega\} \subset \mathrm{Gr}_p(V_1)$$

is a well-defined subset of $\mathrm{Gr}_p(V_1)$. It is moreover Γ_1-stable. We will prove in this section that $P_1(\Omega)$ is proper, connected, open in $\mathrm{Gr}_p(V_1)$, contractible, and quasi-homogeneous under Γ_1.

8.5.5.3.1 Openness and connectedness. In this section, we prove that the set $P_1(\Omega)$ is a domain of $\mathrm{Gr}_p(V_1)$. For any $M \in M_{p+q,p}(\mathbb{R})$ and $i \in \{1, \dots, p+q\}$, we denote $L_i(M)$ the i-th line of M. Let us define the continuous and open map

$$\begin{array}{cccc} \phi: & M_{p+q,p}(\mathbb{R}) & \longrightarrow & M_{d_1,p}(\mathbb{R}) \\ & M & \longmapsto & \begin{pmatrix} L_1(M) \\ \vdots \\ L_{d_1}(M) \end{pmatrix}. \end{array}$$

Let $\mathscr{U} := \{M \in M_{p+q,p}(\mathbb{R}) \mid \mathrm{rk}(\phi(M)) = p\}$. The set $\mathscr{U}$ is an open subset of $M_{p+q,p}(\mathbb{R})$, hence the restriction of ϕ to $\mathscr{U}$ is still open and continuous. Its image is the open subset $\mathscr{U}' := \{N \in M_{d_1,p}(\mathbb{R}) \mid \mathrm{rk}(N) = p\}$ of $M_{d_1,p}$. Note that the two sets $\mathscr{U}$ and $\mathscr{U}'$ are invariant under the action of $\mathrm{GL}(p,\mathbb{R})$ by right multiplication defined in Section 2.4.2.1. The map $\phi|_{\mathscr{U}}^{\mathscr{U}'}$ is equivariant under these actions. Then by factorization one gets an open and continuous map $\Psi : \mathscr{U} / \mathrm{GL}(p,\mathbb{R}) \to \mathscr{U}' / \mathrm{GL}(p,\mathbb{R}) = \mathrm{Gr}_p(V_1)$.

Recall the open set $\widetilde{\mathscr{U}} = \{M \in \mathrm{Mat}_{p+q,p}(\mathbb{R}) \mid \mathrm{rk}(M) = p\}$ defined in Section 2.4.2.1. With the identification $\mathrm{Gr}_p(\mathbb{R}^{p+q}) \simeq \widetilde{\mathscr{U}} / \mathrm{GL}(p,\mathbb{R})$ of Section 2.4.2.1, the set $\mathscr{U} / \mathrm{GL}(p,\mathbb{R})$ corresponds to $\mathrm{Gr}_p(\mathbb{R}^{p+q}) \smallsetminus \mathrm{Z}_{V_2}$. Moreover, the image $\Psi(x)$ of a point $x \in \mathrm{Gr}_p(\mathbb{R}^{p+q}) \smallsetminus \mathrm{Z}_{V_2}$ is the p-plane $\mathbf{Proj}_1(x)$ of $\mathrm{Gr}_p(V_1)$. Hence Ψ and P_1 coincide on $\mathrm{Gr}_p(\mathbb{R}^{p+q}) \smallsetminus \mathrm{Z}_{V_2}$, which proves that P_1 is open and continuous.

Since Ω is an open subset of $\mathrm{Gr}_p(\mathbb{R}^{p+q}) \smallsetminus \mathrm{Z}_{V_2}$, its image $P_1(\Omega)$ is an open subset of $\mathrm{Gr}_p(V_1)$. Since Ω is connected and P_1 is continuous, the set $P_1(\Omega)$ is moreover connected.

8.5.5.3.2 Properness. In this section, we prove that $P_1(\Omega)$ is proper. Let

$$C_\Gamma := \{h \in \mathrm{End}(\mathbb{R}^{p+q}) \mid hg = gh \quad \forall g \in \Gamma\},$$

and let C_Γ^0 be the identity component of the intersection $C_\Gamma \cap \mathrm{GL}(p+q,\mathbb{R})$. Since V_1 and V_2 are Γ-invariant, the two projections $\mathbf{Proj}_1$ and $\mathbf{Proj}_2$ defined in Equation (8.5.4) belong to C_Γ. We will use the following lemma:

Lemma 8.5.11. *([Vey70, LZ19]) The projection $[C_\Gamma^0]$ of C_Γ^0 in $\mathrm{PGL}(p+q,\mathbb{R})$ is contained in the identity component $\mathsf{Aut}_{\mathrm{PGL}(p+q,\mathbb{R})}(\Omega)^0$ of $\mathsf{Aut}_{\mathrm{PGL}(p+q,\mathbb{R})}(\Omega)$.*

In [LZ19], Lemma 8.5.11 is stated for domains which are convex in an affine chart, but the proof does not use the convexity assumption.

Recall the map δ_{V_1} introduced in Equation (8.5.1).

Lemma 8.5.12. *One has* $\overline{P_1(\Omega)} = \delta_{V_1}(\overline{\Omega} \cap \mathscr{A}_{V_1}) = \delta_{V_1}(\partial\Omega \cap \mathscr{A}_{V_1})$.

Proof. Note that $[\mathbf{Proj}_1 + \mathbf{Proj}_2] = [\mathrm{id}]$. Thus for all $t \in \mathbb{R}$, we have $[\mathbf{Proj}_1 + e^t\mathbf{Proj}_2] \in C^0_\Gamma$. By Lemma 8.5.11, we then have $[\mathbf{Proj}_1 + e^t\mathbf{Proj}_2] \in \mathsf{Aut}_{\mathrm{PGL}(p+q,\mathbb{R})}(\Omega)$. For $x \in \Omega$, we then have:

$$\mathbf{Proj}_1(x) = \lim_{t\to-\infty}(\mathbf{Proj}_1 + e^t\mathbf{Proj}_2)(x) \in \overline{\Omega}.$$

Thus $P_1(\Omega) \subset \delta_{V_1}(\overline{\Omega} \cap \mathscr{A}_{V_1})$. Since every element of $P_1(\Omega)$ is contained in V_1, by Lemma 8.5.8, we have $P_1(\Omega) \subset \delta_{V_1}(\partial\Omega \cap \mathscr{A}_{V_1})$. Hence $\overline{P_1(\Omega)} \subset \partial\Omega$ in $\mathrm{Gr}_p(\mathbb{R}^{p+q})$. Since the map δ_{V_1} is closed, taking the closure, we have $\overline{P_1(\Omega)} \subset \partial\Omega \cap \mathscr{A}_{V_1}$.

For the converse inclusion, let $x \in \overline{\Omega} \cap \mathscr{A}_{V_1}$. Then $x \in \mathscr{A}_{V_1} \subset \mathrm{Gr}_p(\mathbb{R}^{p+q}) \smallsetminus \mathrm{Z}_{V_2}$. Let $(x_k) \in \Omega^{\mathbb{N}}$ such that $x_k \to x$. By continuity of the map P_1, we have $P_1(x_k) \to P_1(x) = \delta_{V_1}(x)$ in $\mathrm{Gr}_p(V_1)$. Hence $\delta_{V_1}(x) \in \overline{P_1(\Omega)}$.

We have proved

$$\delta_{V_1}(\overline{\Omega} \cap \mathscr{A}_{V_1}) \subset \overline{P_1(\Omega)} \subset \delta_{V_1}(\partial\Omega \cap \mathscr{A}_{V_1}).$$

Since the right-hand term is trivially included in the left-hand term, all these inclusions are equalities. This proves the lemma. □

Lemma 8.5.13. *The set* $P_1(\Omega)$ *is a proper domain of* $\mathrm{Gr}_p(V_1)$.

Proof. We already know from Section 8.5.5.3.1 that $P_1(\Omega)$ is a domain of $\mathrm{Gr}_p(V_1)$. For the properness, one has $P_1(\Omega) \subset \overline{P_1(\Omega)} = \delta_{V_1}(\overline{\Omega} \cap \mathscr{A}_{V_1})$ by Lemma 8.5.12. By Lemma 8.5.4, and since the map δ_{V_1} sends transversse points to transverse points, the domain $P_1(\Omega)$ is proper in $\mathrm{Gr}_p(V_1)$. □

8.5.5.3.3 Contractibility. Note that, until now, we have not used the continuous-boundary assumption. In this section, we use it to prove:

Proposition 8.5.14. *The domain* $P_1(\Omega)$ *is contractible.*

We will use the following Lemma:

Lemma 8.5.15. *The group* Γ_1 *acts cocompactly on* $P_1(\Omega)$*. In particular the domain* $P_1(\Omega)$ *is quasi-homogeneous.*

Note that Γ_1 is not necessarily discrete, so we cannot deduce that $P_1(\Omega)$ is divisible.

Proof of Lemma 8.5.15. Let $\mathsf{K} \subset \Omega$ be a compact subset such that $\Omega = \Gamma\cdot\mathsf{K}$. By continuity of P_1 on $\mathrm{Gr}_p(\mathbb{R}^{p+q}) \smallsetminus \mathrm{Z}_{V_2}$, the set $P_1(\mathsf{K})$ is a compact subset of $\mathrm{Gr}_p(V_1)$. By $\mathsf{p}_1 \circ F$-equivariance of the map P_1, we have $P_1(\Omega) = \Gamma_1 \cdot P_1(\mathsf{K})$. Thus Γ_1 acts cocompactly on $P_1(\Omega)$, and $P_1(\Omega) \subset \mathrm{Gr}_p(V_1)$ is quasi-homogeneous. □

Proof of Proposition 8.5.14. We fix a point $x_0 \in \mathscr{A}_{V_1}$ such that $\delta_{V_1}^{-1}(x_0) \in P_1(\Omega)$ for this proof. We will prove that all homotpy groups $\pi_k(P_1(\Omega), \delta_{V_1}(x_0))$ of $P_1(\Omega)$ pointed at $\delta_{V_1}(x_0)$ are trivial, for all $k \in \mathbb{N}$. Since $P_1(\Omega)$ is a manifold, this will imply that $P_1(\Omega)$ is contractible.

For $k = 0$, we know from Section 8.5.5.3.1 that the open set $P_1(\Omega)$ is connected and $\pi_0(P_1(\Omega), \delta_{V_1}(x_0))$ is trivial.

Let $k \geq 1$. We will now prove that $\pi_k(\delta_{V_1}(P_1(\Omega)), x_0)$ is trivial (which is equivalent to proving that $\pi_k(P_1(\Omega), \delta_{V_1}^{-1}(x_0))$ is). To this end, let us take $a \in \mathscr{A}_{V_1}$ such that $\delta_{V_1}(a) \in \mathrm{Extr}_{\mathscr{R}}(P_1(\Omega))$. Recall that such a point always exists by Lemma 7.2.5, since $\mathrm{Gr}_p(V_1)$ is an irreducible Nagano space of real type. By Lemma 8.5.12, we know that $a \in \partial\Omega$. Then by Proposition 8.5.6 there exist a neighborhood $\mathscr{V}$ of a in $\mathrm{Gr}_p(\mathbb{R}^{p+q})$, an open subset $\mathscr{U}$ of $\mathbb{R}^{pq}$ and a homeomorphism $f : \mathscr{V} \to \mathscr{U}$ such that $\mathscr{V} \cap \Omega = f^{-1}(\mathbb{R}^{pq-1} \times \mathbb{R}_{>0} \cap \mathscr{U})$. Since $a \in \mathscr{A}_{V_1}$ and $\mathscr{A}_{V_1} \cap \mathrm{Z}_{V_2} = \emptyset$, we can also assume that $\mathscr{V} \cap \mathrm{Z}_{V_2} = \emptyset$, so P_1 is well-defined on $\mathscr{V}$. Up to further shrinking, and post-composing f by a dilation and a translation, we may assume that

$$\mathscr{U} =]-3, 3[^{pq},$$

so $f(\Omega \cap \mathscr{V}) =]-3, 3[^{pq-1} \times]0, 3[$. We denote $\mathscr{U}^+$ the contractible set $]-2, 2[^{pq-1} \times]0, 2[$. One has $f^{-1}(\mathscr{U}^+) \subset \Omega$.

We see the sphere $\mathbb{S}^k$ as the set $\mathbb{S}^k = \{(s_0, \dots, s_k) \in \mathbb{R}^{k+1} \mid s_0^2 + \cdots + s_k^2 = 1\}$ and set $\mathsf{s} = (0, \dots, 0, -1)$. For $x \in \mathbb{S}^k$, write $s = (s_0, \dots, s_k)$.

Let $\gamma_0 : \mathbb{S}^k \to \delta_{V_1}(P_1(\Omega))$ be a continuous map such that $\gamma_0(\mathsf{s}) = x_0$. Since $\gamma_0(\mathbb{S}^k)$ is compact and since Γ_1 acts cocompactly on $P_1(\Omega)$ (by Lemma 8.5.15), by Lemma 7.2.10, there exists $g \in \Gamma_1$ such that $g \cdot \gamma_0(\mathbb{S}^k) \subset \mathscr{V}$. Let us set

$$p_0 := g \cdot x_0 \text{ and } \gamma := g \cdot \gamma_0.$$

Then $\gamma : \mathbb{S}^k \to \mathscr{V} \cap \delta_{V_1}(P_1(\Omega))$ is a continuous map such that $\gamma(\mathsf{s}) = p_0$.

For the simplicity of the proof, we assume that $f(p_0) = 0$. Then $f \circ \gamma : \mathbb{S}^k \to \mathscr{U} \cap \mathbb{R}^{pq-1} \times \{0\}$ is continuous, and satisfies $f \circ \gamma(\mathsf{s}) = 0$.

Note that for all $t \in [0, \frac{1}{2}]$ and $x \in \mathbb{S}^k \smallsetminus \{\mathsf{s}\}$ one has

$$f \circ \gamma(x) + \big(0, \dots, 0, 2t(1 + x_k)\big) \in \mathscr{U}^+,$$

i.e. we have $f^{-1}(f \circ \gamma(x) + (0, \dots, 0, t(1 + x_k))) \in \Omega$. Since $\mathscr{U}^+$ is convex, for all $t \in [\frac{1}{2}, 1]$ one has

$$2(1-t) f \circ \gamma(x) + \big(0, \dots, 0, 2(1-t)(1 + x_k)\big) \in \mathscr{U}^+,$$

so that $f^{-1}\big((2t-1) f \circ \gamma(x) + (0, \dots, 0, (2t-1)(1 + x_k))\big) \in \Omega$. On the other hand, for all $t \in [0, 1]$ one has $\delta_{V_1} \circ P_1(f^{-1}(f \circ \gamma(\mathsf{s}))) = \delta_{V_1} \circ P_1(\gamma(\mathsf{s})) = p_0$. Hence the map

$$\begin{array}{rccc} H: & [0,1] \times \mathbb{S}^k & \longrightarrow & \delta_{V_1}(P_1(\Omega)) \\ & (t, x) & \longmapsto & \begin{cases} \delta_{V_1} \circ P_1 \circ f^{-1}\big(f \circ \gamma(x) + (0, \dots, 0, 2t(1 + x_k))\big) \text{ if } t \in [0, \frac{1}{2}] \\ \delta_{V_1} \circ P_1 \circ f^{-1}\big(2(1-t) f \circ \gamma(x) + (0, \dots, 0, 2(1-t)(1 + x_k))\big) \\ \hfill \text{if } t \in [\frac{1}{2}, 1] \end{cases} \end{array}$$

is well defined, continuous, and satisfies

$$\begin{cases} H(0,\cdot) & = \gamma \\ H(1,\cdot) & = p_0 \\ H(t,\mathsf{s}) & = p_0 \quad \forall t \in [0,1] \end{cases}.$$

Hence $\gamma = g \cdot \gamma_0$ is homotopic to the point $p_0 = g \cdot x_0$ in $\delta_{V_1}(P_1(\Omega))$. Since $g \in \Gamma_1$ preserves $\delta_{V_1}(P_1(\Omega))$, the map γ_0 is homotopic to x_0 in $\delta_{V_1}(P_1(\Omega))$.

This is true for all continuous map $\gamma_0 : \mathbb{S}^k \to \delta_{V_1}(P_1(\Omega))$ such that $\gamma_0(\mathsf{s}) = p_0$. Hence the group $\pi_k(\delta_{V_1}(P_1(\Omega)), x_0)$ is trivial. This is true for all $k \geq 1$. Since $\delta_{V_1}(P_1(\Omega))$ is open in $\mathrm{Gr}_p(V_1)$ (recall Section 8.5.5.3.1), it is contractible. □

8.5.5.4 The proof of Theorem 8.5.1 when p, q, d_1, d_2 are not all equal. In this section, we prove Theorem 8.5.1 assuming that p, q, d_1, d_2 are not all equal. Together with Section 8.5.5.2, this section ends the proof of Theorem 8.5.1. As in Section 8.5.5.3, we may assume that $d_1 > p$. We make a disjonction of cases, depending on the relative values of p and d_2.

8.5.5.4.1 Case 1: $d_2 \leq p$. In this section, we consider the case where $d_2 \leq p$. Let

$$M := \mathrm{Gr}_p(V_1) \times (\mathrm{PGL}(d_2,\mathbb{R})/\,\mathrm{PO}(d_2)) \times \mathbb{R}_{>0}.$$

The group $\mathrm{PGL}(d_1,\mathbb{R}) \times \mathrm{PGL}(d_2,\mathbb{R}) \times \mathbb{R}_{>0}$ acts on M via the diagonal action. This induces an action of Γ on M via the group homomorphism F defined in Equation (8.5.3):

$$\forall g \in \Gamma, \quad \forall x \in X, \quad g \cdot x = F(g) \cdot x.$$

The set $X := P_1(\Omega) \times (\mathrm{PGL}(d_2,\mathbb{R})/\,\mathrm{PO}(d_2)) \times \mathbb{R}_{>0}$ is stable under this action.

Proposition 8.5.16. *The group Γ acts properly discontinuously on X.*

Proof. Let $\mathsf{K} \subset X$ be a compact subset. We may assume that K is of the form $\mathsf{K}_1 \times \mathsf{K}_2 \times \mathsf{K}_3$, where each K_i is a compact subset of the corresponding factor of X. Let us assume that there exists a sequence $(g_k = [\mathrm{diag}(A_k, B_k)]) \in \Gamma^{\mathbb{N}}$ and $(x_k, y_k, \lambda_k) \in \mathsf{K}$ such that $F(g_k) \cdot (x_k, y_k, \lambda_k) \in \mathsf{K}$ for all $k \in \mathbb{N}$. We may assume that $|\det(A_k)| = 1$ for all $k \in \mathbb{N}$.

Since K_3 is a compact subset of $\mathbb{R}_{>0}$, up to extracting we may assume that there exists $\lambda \in \mathbb{R}_{>0}$ such that $\lambda_k \to \lambda$. Since $|\det(B_k)|^{-d_1} \lambda_k \in \mathsf{K}_3$ for all $k \in \mathbb{N}$, we may also assume that there exists $t \in \mathbb{R}_{>0}$ such that $|\det(B_k)|^{-d_1} \lambda_k \to t$. Then

$$|\det(B_k)|^{-d_1} \longrightarrow \mu := \frac{t}{\lambda} \in \mathbb{R}_{>0} \tag{8.5.5}$$

The set K_1 is a compact subset of $P_1(\Omega)$ and for all $k \in \mathbb{N}$, we have

$$[A_k] \cdot x_k = [\mathsf{p}_1(F(g_k))] \cdot x_k \in [\mathsf{p}_1(F(g_k))] \cdot \mathsf{K}_1.$$

Since the action of $\Gamma_1 \leq \mathsf{Aut}_{\mathrm{PGL}(V_1)}(\delta_{V_1}(P_1(\Omega)))$ on $P_1(\Omega)$ is proper by Lemma 8.5.13 and Fact 3.1.3, up to extracting, we may assume that there exists $A \in \mathrm{SL}^{\pm}(d_1, \mathbb{R})$ such that $A_k \to A$.

By the same argument with the action of Γ_2 on $\mathrm{PGL}(d_2, \mathbb{R})/\mathrm{PO}(d_2)$, which is proper, we may assume that there exists $B \in \mathrm{SL}^{\pm}(d_2, \mathbb{R})$ such that $\frac{B_k}{|\det(B_k)|^{1/d_2}} \longrightarrow B$.

Moreover, we have $|\det(B_k)|^{-d_1} \to \mu$, so $B_k \to \pm\mu^{\frac{1}{d_2 d_1}} B$.

Then $g_k \to \begin{bmatrix} A & 0 \\ 0 & \pm\mu^{\frac{1}{d_2 d_1}} B \end{bmatrix}$. By discreteness of Γ, the sequence (g_k) is stationnary. This proves that Γ acts properly discontinuously on X. □

Since Γ is finitely generated (Fact 8.5.9) and acts properly discontinuously on the contractible manifold X (Proposition 8.5.16), by Section 8.5.1.2 we have $\mathrm{vcd}(\Gamma) \leq \dim(X)$. On the other hand, the group Γ acts properly discontinuously and cocompactly on Ω, so $\mathrm{vcd}(\Gamma) = \dim(\Omega) = \dim(\mathrm{Gr}_p(\mathbb{R}^{p+q})) = pq$ by Section 8.5.1.2. This gives:

$$\begin{aligned} pq &= \mathrm{vcd}(\Gamma) \leq \dim(X) \\ &= \dim(P_1(\Omega)) + \dim \mathrm{PGL}(d_2, \mathbb{R})/\mathrm{PO}(d_2) + \dim \mathbb{R}_{>0} \\ &= p(d_1 - p) + \frac{d_2(d_2+1)}{2} - 1 + 1 \\ &= pq - pd_2 + \frac{d_2(d_2+1)}{2}, \end{aligned}$$

Then we must have $pd_2 = d_2(d_2+1)/2$. This is only possible when $d_2 = p = 1$. Since we have assumed that $p \geq 2$, we get a contradiction.

8.5.5.4.2 Case 2: $d_2 > p$. In this section, we consider the case where $d_2 > p$. In this case, one can construct the contractible domain

$$P_2(\Omega) := \{\mathbf{Proj}_2(x) \mid x \in \Omega\}$$

of $\mathrm{Gr}_p(V_2)$, proceeding the same way as in Section 8.5.5.3 but exchanging the roles of V_1 and V_2. The same analysis as for $P_1(\Omega)$ gives that $P_2(\Omega)$ is a proper contractible domain of $\mathrm{Gr}_p(V_2)$, on which Γ_2 acts properly and cocompactly.

As in Section 8.5.5.3, we define a contractible manifold X on which Γ acts properly discontinuously and freely: here $X := P_1(\Omega) \times P_2(\Omega) \times \mathbb{R}_{>0}$. The group $\mathrm{PGL}(d_1, \mathbb{R}) \times \mathrm{PGL}(d_2, \mathbb{R}) \times \mathbb{R}_{>0}$ acts on X the natural way, which induces an action of Γ on X via F:

$$\forall g \in \Gamma, \quad \forall x \in X, \quad g \cdot x = F(g) \cdot x.$$

By the same proof as in Proposition 8.5.16, this action is properly discontinuous, so by Section 8.5.1.2 we have $\mathrm{vcd}(\Gamma) \leq \dim(X)$. On the other hand, as in the previous Section 8.5.5.4.1, we have $\mathrm{vcd}(\Gamma) = pq$. This gives:

$$\begin{aligned} pq &= \mathrm{vcd}(\Gamma) \leq \dim(X) = \dim(P_1(\Omega)) + \dim(P_2(\Omega)) + \dim \mathbb{R}_{>0} \\ &= p(d_1 - p) + p(d_2 - p) + 1 \\ &= pq - p^2 + 1. \end{aligned}$$

This is possible only if $p = 1$. Since we have assumed that $p \geq 2$, we get a contradiction.

This ends the proof of Theorem 8.5.1.

8.6 Extremal points of proper almost-homogeneous domains in Grassmannians

In this section, we establish Proposition 8.6.1 below, which is an elementary consequence of Propositions 4.1.5 and 7.2.12. Proposition 8.6.1 goes in the direction of a rigidity phenomenon in self-opposite Grassmannians, as it tells us that, if p is odd, then proper almost-homogeneous domain of $\mathrm{Gr}_p(\mathbb{R}^{2p})$ share a restrictive property with the realizations of the noncompact dual of $\mathrm{Gr}_p(\mathbb{R}^{2p})$.

Let $p \in \mathbb{N}_{>0}$. We know from Example 7.2.2.(2) that $\mathrm{Extr}_{\mathscr{R}}(\mathbb{B}_{p,p})$ is the set of totally isotropic subspaces of $\mathbb{R}^{p,p}$. This set has two connected components $\mathscr{F}_1$ and $\mathscr{F}_2$, both being naturally identified with flag manifolds of $\mathfrak{so}(p,p)$. If $x \in \mathscr{F}_1$, then $x \cap y \neq \{0\}$, for all $y \in \mathscr{F}_2$. Thus, if we chose $(x,y) \in \mathscr{F}_1 \times \mathscr{F}_2$, we have $\mathrm{Extr}_{\mathscr{R}}(\mathbb{B}_{p,p}) \subset \mathrm{Z}_x \cup \mathrm{Z}_y$. This is a quite restrictive property for the domain $\Omega := \mathbb{B}_{p,p}$. Propositions 4.1.5 and 7.2.12 allow us to extend this property to *any* proper almost-homogeneous domain $\Omega \subset \mathrm{Gr}_p(\mathbb{R}^{2p})$, in the case where p is odd.

Proposition 8.6.1. *Let p be an odd integer, and let $\Omega \subset \mathrm{Gr}_p(\mathbb{R}^{2p})$ be a proper almost-homogeneous domain. Then there exist $\xi_1, \xi_2 \in \mathrm{Extr}_{\mathscr{R}}(\Omega)$ such that*

$$\overline{\mathrm{Extr}_{\mathscr{R}}(\Omega)} \subset \mathrm{Z}_{\xi_1} \cup \mathrm{Z}_{\xi_2}.$$

Proof. By Lemma 7.2.5 applied to the Nagano pair $(\mathfrak{sl}(2p,\mathbb{R}), \alpha_p)$ of real type, there exist two transverse points $\xi_1, \xi_2 \in \mathrm{Extr}_{\mathscr{R}}(\Omega)$. Assume for a contradiction that there exists $z \in \overline{\mathrm{Extr}_{\mathscr{R}}(\Omega)} \smallsetminus (\mathrm{Z}_{\xi_1} \cup \mathrm{Z}_{\xi_2})$. By Proposition 7.2.12, we have $z, \xi_1, \xi_2 \in \Lambda_{\{\alpha_p\}}(\mathsf{Aut}_{\mathrm{PGL}(2p,\mathbb{R})}(\Omega))$. Applying Proposition 4.1.5 to the group $H := (\mathsf{Aut}_{\mathrm{PGL}(2p,\mathbb{R})}(\Omega)$ preserving the proper domain Ω, in the notations of Section 4.1.1, we get that the set $\mathscr{E}_{\{\alpha_p\}}$ of connected components of $\mathrm{Gr}_p(\mathbb{R}^{2p}) \smallsetminus (\mathrm{Z}_{\mathfrak{p}^+} \cup \mathrm{Z}_{\mathfrak{p}^-})$ should contain an element fixed by $\mathsf{s}_{\{\alpha_p\}}$, where $\mathsf{s}_{\{\alpha_p\}}$ is defined in Equation (4.1.1). But we know from Remark 4.1.1 that this is not the case. Thus, we have $\mathrm{Extr}_{\mathscr{R}}(\Omega) \subset \mathrm{Z}_{\xi_1} \cup \mathrm{Z}_{\xi_2}$, and the proposition holds taking the closure. ☐

8.7 An example in a flag manifold which is not a Nagano space

In Sections from 8.2 to 8.5, we focused on Question 1.2.1 for Nagano pairs of real type. So far, we have not considered flag manifolds $\mathscr{F}(\mathfrak{g}, \Theta)$ defined by a single root (i.e. such that $|\Theta| = 1$) but which are not Nagano spaces. This is the purpose of this section; see Proposition 8.7.1 below.

Recall from Section 2.5 that we have a $\mathrm{PSp}(2n,\mathbb{R})$-equivariant identification, denoted by $\mathscr{I}$:

$$\mathscr{F}(\mathfrak{sp}(2n,\mathbb{R}), \alpha_1) \overset{\mathscr{I}}{\simeq} \mathbb{P}(\mathbb{R}^{2n}) = \mathscr{F}(\mathfrak{sl}(2n,\mathbb{R}), \alpha_1). \tag{8.7.1}$$

Recall from Remark 5.1.1 that the pair $(\mathfrak{sp}(2n,\mathbb{R}),\alpha_1)$ is *not* an irreducible Nagano pair, even though $\mathbb{P}(\mathbb{R}^{2n})$ is a Nagano space (associated with the pair $(\mathfrak{sl}(2n,\mathbb{R}),\alpha_1)$). This is due to the fact that $\mathrm{PSp}(2n,\mathbb{R})$ is not a transformation group of the compact symmetric space $(\mathbb{P}(\mathbb{R}^{2n}), g_{\mathfrak{sl}(2n,\mathbb{R}),\alpha_1})$ (in the sense of Section 5.1.3). By Lemma 5.4.2, the flag manifold $\mathscr{F}(\mathfrak{sp}(2n,\mathbb{R}),\alpha_1)$ does not contain any symmetric domain. By the positive answers to Question 1.2.1, we can expect that it does not contain any proper divisible domain. This is established in Proposition 8.7.1 below.

The flag manifold $\mathscr{F}(\mathfrak{sp}(2n,\mathbb{R}),\alpha_1)$ is self-opposite, and a domain Ω of the flag manifold $\mathscr{F}(\mathfrak{sp}(2n,\mathbb{R}),\alpha_1)$ is proper if and only if there exists $x \in \mathscr{F}(\mathfrak{sp}(2n,\mathbb{R}),\alpha_1)$ such that $\overline{\Omega} \cap \mathbb{P}(x^{\perp}) = \emptyset$. Thus, if a domain $\Omega \subset \mathscr{F}(\mathfrak{sp}(2n,\mathbb{R}),\alpha_1)$ is proper, then the domain $\mathscr{I}(\Omega)$ is *a fortiori* proper in $\mathbb{P}(\mathbb{R}^{2n}) = \mathscr{F}(\mathfrak{sl}(2n,\mathbb{R}),\alpha_1)$ via the identification $\mathscr{I}$. Moreover, since $\mathfrak{sp}(2n,\mathbb{R}) \subset \mathfrak{sl}(2n,\mathbb{R})$, if the domain Ω is divisible (resp. quasi-homogeneous, resp. almost-homogeneous) in $\mathscr{F}(\mathfrak{sp}(2n,\mathbb{R}),\alpha_1)$, then $\mathscr{I}()$ is also in $\mathbb{P}(\mathbb{R}^{2n}) = \mathscr{F}(\mathfrak{sl}(2n,\mathbb{R}),\alpha_1)$. The identification $\mathscr{I}$ is only $\mathrm{PSp}(2n,\mathbb{R})$-equivariant, and not $\mathrm{PGL}(2n,\mathbb{R})$-equivariant. This is why the automorphism group $\mathsf{Aut}_{\mathrm{PSp}(2n,\mathbb{R})}(\Omega)$ of Ω (as a subset of $\mathscr{F}(\mathfrak{sp}(2n,\mathbb{R}),\alpha_1)$) is in general strictly contained in its group of automorphisms $\mathsf{Aut}_{\mathrm{PGL}(2n,\mathbb{R})}(\mathscr{I}(\Omega))$ as a subset of $\mathscr{F}(\mathfrak{sl}(2n,\mathbb{R}),\alpha_1)$. Using the classical theory of divisible convex sets in projective space, we obtain:

Proposition 8.7.1. *For $n \geq 2$, the flag manifold $\mathscr{F}(\mathfrak{sp}(2n,\mathbb{R}),\alpha_1)$ does not contain any proper divisible domains.*

Proof. Suppose, for contradiction, that there exists a proper domain Ω of the flag manifold $\mathscr{F}(\mathfrak{sp}(2n,\mathbb{R}),\alpha_1)$ that is divisible by a discrete subgroup $\Gamma \leq \mathrm{PSp}(2n,\mathbb{R})$.

The proper divisible domain $\mathscr{I}(\Omega)$ is thus convex, by Proposition 3.1.11. If it is irreducible, then by [Vey70, Ben03], either it is symmetric (in $\mathscr{F}(\mathfrak{sl}(2n,\mathbb{R}),\alpha_1)$), or Γ is Zariski-dense in $\mathrm{PGL}(2n,\mathbb{R})$. Since $\Gamma \leq \mathrm{PSp}(2n,\mathbb{R})$, we must necessarily be in the first case. It is then known from [Koe99, Ben00] that the Zariski closure $\overline{\Gamma}^Z \subset \mathrm{PSp}(2n,\mathbb{R})$ of Γ in $\mathrm{PGL}(2n,\mathbb{R})$ preserves $\mathscr{I}(\Omega)$ and is a connected real semisimple projective-linear Lie group. However, the representation induced by the natural inclusion $\overline{\Gamma}^Z \hookrightarrow \mathrm{PSp}(2n,\mathbb{R})$ is irreducible and proximal, so by [Ben00, Thm 1.5], the group $\overline{\Gamma}^Z$ cannot preserve any proper domain in $\mathbb{P}(\mathbb{R}^{2n})$, contradiction.

From now on, we no longer assume $\mathscr{I}(\Omega)$ to be necessarily irreducible. Let ω be a non-degenerate antisymmetric bilinear form on $\mathbb{R}^{2n}$ such that $\mathrm{PSp}(2n,\mathbb{R})$ identifies with $\mathrm{PSp}(\omega)$. By [Vey70, Ben01], to reduce to the irreducible case, it suffices to prove that $\omega_{|V_i\times V_i}$ is non-degenerate for any Γ-invariant decomposition $\mathbb{R}^{2n} = V_1 \oplus V_2$. Suppose such a decomposition exists. Then, there exist properly convex open cones $\Omega_i \subset V_i$ ($i = 1, 2$) such that

$$\mathscr{I}(\Omega) = \mathbb{P}(v_1 + v_2 \mid [v_i] \in \Omega_i \ \forall i = 1, 2).$$

Up to finite index, we have $\Gamma = \Gamma_1 \times \Gamma_2$, where Γ_i is a discrete subgroup of $\mathrm{PSp}(2n,\mathbb{R}) \cap \mathrm{PGL}(V_i)$ acting cocompactly on Ω_i. Since Γ_i acts cocompactly on Ω_i, by Theorem 7.2.6, we have

$$\mathbb{P}(x^{\perp}) \cap \mathscr{I}(\Omega) = \emptyset \tag{8.7.2}$$

for every extremal point x of $\mathscr{I}(\Omega)$.

By Lemma 7.2.10, since the decomposition $\mathbb{R}^{2n} = V_1 \oplus V_2$ is Γ-stable, we have $\mathrm{Extr}(\mathscr{I}(\Omega)) \subset (V_1 \cup V_2) \cap \partial\Omega = \overline{\Omega}_2 \cup \overline{\Omega}_2$.

Let $a \in \mathrm{Extr}(\mathscr{I}(\Omega)) \cap \overline{\Omega}_1$. Then a is extremal in Ω_1. Since Γ_1 acts cocompactly on Ω_1, by Lemma 7.2.10, there exist $(g_k) \in \Gamma_1^{\mathbb{N}}$ and $a' \in \partial\Omega \cap V_1$ such that (g_k) is P_1-contracting with respect to (a, a'). Now let $b \in \mathrm{Extr}(\mathscr{I}(\Omega)) \cap \overline{\Omega}_2$. If $b \notin \mathbb{P}(a^\perp)$, then $b = g_k^{-1} \cdot b \to a' \in V_1$, absurd. Hence $(\mathrm{Extr}(\mathscr{I}(\Omega)) \cap \overline{\Omega}_2) \subset \mathbb{P}(a^\perp)$. We have just proven that for all $a, b \in \mathrm{Extr}(\mathscr{I}(\Omega))$, we have

$$a \in \overline{\Omega}_1, b \in \overline{\Omega}_2 \Longrightarrow b \in \mathbb{P}(a^\perp).$$

Since $\mathrm{Extr}(\mathscr{I}(\Omega)) \cap \overline{\Omega}_i$ generates V_i for $i = 1, 2$, this implies that $\mathbb{P}(V_2) \subset \mathbb{P}(x^\perp)$ for all $x \in \mathbb{P}(V_1)$. Since ω is non-degeneratee, this imposes that $\omega_{|V_i \times V_i}$ is non-degenerate for $i = 1, 2$, and concludes the proof. □

A corollary of Proposition 8.7.1, coming from low-dimensional Lie groups isomorphisms, is the following. It answers Question 1.2.1 for any flag manifold of $\mathfrak{so}(3, 2)$:

Corollary 8.7.2. *Let Θ be a nonempty subset of the simple restricted roots of $\mathfrak{so}(2, 3)$. Then any proper divisible domain of $\mathscr{F}(\mathfrak{so}(2, 3), \Theta)$ is symmetric. In particular, we have the following case dichotomy:*

1. *If $\Theta = \{\alpha_1\}$, then any proper divisible domains of $\mathscr{F}(\mathfrak{so}(2, 3), \Theta)$ is a diamond.*
2. *If $\Theta = \{\alpha_2\}$ or $\{\alpha_1, \alpha_2\} = \Delta$, then there are no proper divisible domains in $\mathscr{F}(\mathfrak{so}(2, 3), \Theta)$.*

Proof. Recall that the restricted root system of $\mathfrak{so}(2, 3)$ is given in Example 2.2.1. There exists an exceptional isomorphism $\mathfrak{so}(2, 3) \simeq \mathfrak{sp}(4, \mathbb{R})$, mapping the root α_2 of $\mathfrak{so}(2, 3)$ to the root α_1 of $\mathfrak{sp}(4, \mathbb{R})$. The case where $\Theta = \{\alpha_2\}$ thus follows from Proposition 8.7.1. The case where $\Theta = \Delta$ is a direct consequence of Fact 1.2.3. Finally, the case where $\Theta = \{\alpha_1\}$ follows from Theorem 8.4.1. □

8.8 Application: closed proper manifolds locally modelled on flag manifolds

In this section, we discuss the consequences of the various results of this chapter on the classification of *proper* (G, X)-manifolds; see Corollaries 8.8.4, 8.8.5 and 8.8.6 below.

Let G be a real semisimple Lie group of noncompact type and X be a manifold on which G acts smoothly and transitively. A manifold M is a (G, X)-*manifold*, if there exists a (maximal) atlas of charts $(U_i, \psi_i)_{i \in \mathscr{A}}$ on M with values in X, such that for any $i, j \in \mathscr{A}$ with $U_i \cap U_j \neq \emptyset$, the map $\psi_j \circ \psi_i^{-1}$ is locally the restriction of an element of G to $\varphi_i(U_i \cap U_j)$. In this case there exists a map $\mathsf{dev} : \widetilde{M} \to X$, called the *developing map* (unique up to postcomposition by elements of G), where $\widetilde{M}$ is the universal cover of M. The map dev is a local diffeomorphism, but it is in general not injective.

Given a (G, X)-manifold M, any representative loop γ of on element of $\pi_1(M)$ can be lifted to a curve $\widetilde{\gamma} : [0, 1] \to \widetilde{M}$. Since $\gamma(0) = \gamma(1)$, there exists an element $g := \mathsf{hol}([\gamma]) \in G$

such that $\mathsf{dev}(\widetilde{\gamma}(0)) = g \cdot \mathsf{dev}(\widetilde{\gamma}(1))$. The map $\mathsf{hol} : \pi_1(M) \to G$ is then a well-defined homomorphism, called the *holonomy*, making dev a hol-equivariant map. It is uniquely defined by the choice of developing map dev, but in general only well defined up to conjugation.

A manifold M is *Kleinian* if dev is a diffeomorphism onto its image. In this case, the holonomy hol is discrete and the manifold M identifies with the quotient $\mathsf{dev}(\widetilde{M})/\mathsf{hol}(\pi_1(M))$.

We say that M is *proper* if $\mathsf{dev}(\widetilde{M})$ is a proper open subset of X. This property is independent on the choice of the developing map for M.

In this section, we will use the following elementary lemmas:

Lemma 8.8.1. *Let G be a real semisimple Lie group and Θ a subset of the simple restricted roots of G. Let M be a closed proper $(G, \mathscr{F}(\mathfrak{g}, \Theta))$-manifold. Then $\mathsf{dev}(\widetilde{M})$ is a proper almost-homogeneous (and even quasi-homogeneous) domain.*

Proof. Since M is closed, there is a compact fundamental domain $\mathsf{K} \subset \widetilde{M}$ for the action of $\pi_1(M)$. Since the developing map is hol-equivariant, the compact set $\mathsf{dev}(\mathsf{K}) \subset \mathsf{dev}(\widetilde{M})$ intersects any $\mathsf{hol}(\pi_1(M))$-orbit, that is, $\mathsf{dev}(\widetilde{M})$ is quasi-homogeneous. Since $\mathsf{dev}(\widetilde{M})$ is proper, it is also almost-homogeneous. □

Lemma 8.8.2. *Let $(\mathfrak{g}, \alpha)$ be an irreducible Nagano pair and M be a closed proper manifold such that $\mathsf{dev}(\widetilde{M})$ is a realization of $\mathbb{X}(\mathfrak{g}, \alpha)$. Then M is Kleinian. In particular, there exists a cocompact lattice $\Gamma \leq \mathrm{Isom}(\mathbb{X}(\mathfrak{g}, \alpha))$ such that $M = \Omega/\mathsf{F}_{(\mathfrak{g},\alpha)}(\Gamma)$, and M is $\mathsf{F}_{(\mathfrak{g},\alpha)}$-equivariantly identified with a compact quotient of $\mathbb{X}(\mathfrak{g}, \alpha)$.*

Proof. Let $\Omega := \mathsf{dev}(\widetilde{M})$. Let g be a Riemannian metric on Ω, induced by the one on $\mathbb{X}(\mathfrak{g}, \alpha)$. It is $\mathsf{hol}(\pi_1(M))$-invariant. Then $\mathsf{dev}^* g$ is a $\pi_1(M)$-invariant metric on $\widetilde{M}$. It induces a Riemannian metric on M, which has to be complete because M is closed. Thus, the metric g is complete. Hence, the map dev is a local isometry between two complete Riemannian manifolds, and thus a covering map. Since Ω is simply connected (because it is diffeomorphic to $\mathbb{X}(\mathfrak{g}, \alpha)$), the map dev is a diffeomorphism onto its image and M is a compact quotient of Ω. □

Remark 8.8.3. To prove that dev is a covering map in the proof of Lemma 8.8.2, one could also directly apply the following result due to A. Zimmer [Zim18a]: Let G be a connected semisimple Lie group with trivial center and no compact factors, and let $\Theta \subset \Delta$ be a subset of the simple restricted roots of G. Let M be a closed proper $(G, \mathscr{F}(\mathfrak{g}, \Theta))$-manifold. Then any developing map dev of M is a covering map.

8.8.1 Closed proper conformally flat manifolds

This section is devoted to the proof of Corollary 1.4.20. Recall the notation and notions introduced in Section 2.4.3.3. We say that a conformal manifold $(M, [g])$ is *conformally flat* if it is locally conformally equivalent to an open subset of the Minkowski space. A consequence of Fact 2.4.1 is that any conformally flat pseudo-Riemannian manifold inherits a canonical $(\mathrm{PO}(p+1, q+1), \mathrm{Ein}^{p,q})$-structure. Theorem 8.4.1 then implies the following.

Corollary 8.8.4 (with Chalumeau, see Corollary 1.4.20). *Let $p, q \geq 1$ be two integers, and let M be a closed proper connected pseudo-Riemannian conformally flat manifold of signature (p,q). Then M is conformally equivalent to a quotient D/Γ, where D is a diamond of $\mathrm{Ein}^{p,q}$ and $\Gamma \leq \mathsf{Aut}_{\mathrm{PO}(p+1,q+1)}(D)$ is a cocompact lattice. Moreover, if $p \neq q$ and $\{p,q\} \neq \{2,3\}$, then up to a finite cover, the manifold M is conformally equivalent to*

$$\Sigma^p \times (-\Sigma^q),$$

where Σ^p and Σ^q are closed hyperbolic Riemannian manifolds. In Lorentzian signature, i.e. for $q = 1$, the manifold M is (up to a finite cover) conformally equivalent to the product $\Sigma \times (-\mathbb{S}^1)$, where Σ is a closed hyperbolic Riemannian manifold.

Proof. Let $\Omega := \mathsf{dev}(\widetilde{M})$. By Lemma 8.8.1, the domain Ω is almost-homogeneous. Since it is also proper, by Theorem 8.4.1, the domain Ω is a diamond. It is thus a realization of $\mathbb{X}(\mathfrak{so}(p+1,q+1),\alpha_1)$. By Lemma 8.8.2, the manifold M is the quotient of $\Omega \simeq \mathbb{H}^p \times \mathbb{H}^q$ by a cocompact lattice of $\mathsf{Aut}_{\mathrm{PO}(p+1,q+1)}(\Gamma) \simeq \mathrm{PO}(p,1) \times \mathrm{PO}(1,q)$.

When $p \neq q$ and $\neq p, q\} \neq \{2,3\}$, the group $\mathrm{SO}(p,1) \times \mathrm{SO}(1,q)$ is non-isotypic and it is a general fact that its cocompact lattices are virtually products (see e.g. [Mor15, Thm. 5.6.2] for a proof using Margulis Arithmeticity when $p, q \geq 2$). This fact completes the proof of the theorem. ☐

When $p = q$, the group $\mathrm{PO}(p,1) \times \mathrm{PO}(1,p)$ is isotypic, hence it admits irreducible cocompact lattices (see [Mor15, Cor. 18.7.4]). The case $(p,q) = (2,3)$ is also special, as $\mathrm{PO}(1,2) \times \mathrm{PO}(1,3)$ admits cocompact lattices which are not virtually products of cocompact lattices of $\mathrm{PO}(1,2)$ and $\mathrm{PO}(1,3)$, since $\mathfrak{so}(3,\mathbb{C})$ is an irreducible factor of both $\mathfrak{so}(1,2)_{\mathbb{C}} = \mathfrak{so}(3,\mathbb{C})$ and $\mathfrak{so}(1,3)_{\mathbb{C}} \simeq \mathfrak{so}(4,\mathbb{C}) \simeq \mathfrak{so}(3,\mathbb{C}) \oplus \mathfrak{so}(3,\mathbb{C})$.

8.8.2 Closed proper manifolds locally modelled on causal flag manifolds

In this section, we prove the following corollary of Theorem 8.3.1. We use Notation 2.4.2.

Corollary 8.8.5 (see Corollary 1.4.19). *Let G be a* HTT *Lie group, and let M be a closed proper connected $(G, \mathbf{Sb}(\mathfrak{g}))$-manifold. Then the manifold M identifies, as a $(G, \mathbf{Sb}(\mathfrak{g}))$-manifold, to a quotient D/Γ, where D is a diamond of $\mathbf{Sb}(\mathfrak{g})$ and Γ is a cocompact lattice of $\mathsf{Aut}_G(D)$. Thus, the manifold M is, up to finite cover, a quotient*

$$(\mathbb{X}_{L_s}/\Gamma') \times \mathbb{S}^1,$$

where $\mathbb{X}_{L_s}$ is the symmetric space of the semisimple part L_s of the Levi subgroup L, and Γ' is a cocompact lattice of L_s.

Proof. The corollary is straightforward if G is of real rank 1, because then we have $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$, hence we may assume that the real rank of G is $r \geq 2$.

By Lemma 8.8.1, the domain $\mathsf{dev}(\widetilde{M})$ is almost-homogeneous. Since it is also proper, by Theorem 8.3.1 , it is a diamond. It is thus a realization of $\mathbb{X}(\mathfrak{g},\alpha_r)$. By Lemma 8.8.2, the manifold M is then a quotient of a diamond by a cocompact lattice of $\mathrm{Isom}(\mathbb{X}(\mathfrak{g},\alpha_r))$. Recall that diamonds are $\mathsf{F}_{(\mathfrak{g},\alpha)}$-equivariantly diffeomorphic to the symmetric space $\mathbb{X}_{L_s} \times \mathbb{R}$. Thus M is thus a quotient of $\mathbb{X}_{L_s} \times \mathbb{R}$ by a cocompact lattice of $L_s \times \mathrm{Isom}(\mathbb{R}) = L_s \times (\mathbb{R} \rtimes (\mathbb{Z}/2\mathbb{Z}))$. It is a classical fact that the cocompact lattices of a product $G \times \mathbb{R}$, where G is a simple Lie group, are virtually products of a cocompact lattice of $\mathbb{R}$ by a cocompact lattice of G. This fact applied to $L_s \times \mathbb{R}$ completes the proof of the corollary. □

8.8.3 Nonexistence of some closed proper manifolds

In this section, we prove a corollary of Proposition 8.7.1:

Corollary 8.8.6. *Let M be a closed $(\mathrm{Sp}(2n,\mathbb{R}), \mathscr{F}(\mathfrak{sp}(2n,\mathbb{R}),\alpha_1))$-manifold. Then M is not proper.*

Proof. Since M is compact, up to considering a connected component of M, we may assume that M is connected. Assume for a contradiction that $\mathsf{dev}(\widetilde{M})$ is proper. Then by [Zim18a] (see also Remark 2.4.3), the map dev is a covering map. Since M is compact, the group $\mathsf{hol}(\pi_1(M))$ acts cocompactly on $\mathsf{dev}(\widetilde{M})$. Thus $\Omega := \mathscr{I}(\mathsf{dev}(\widetilde{M}))$ is a proper quasi-homogeneous domain of $\mathbb{P}(\mathbb{R}^{2n})$, where $\mathscr{I}$ is defined in Equation (8.7.1). It is thus properly convex in $\mathbb{P}(\mathbb{R}^{2n})$, by Proposition 3.1.11. In particular, it is simply connected. Thus dev is a diffeomorphism, and hol is discrete. Then $\mathsf{dev}(\widetilde{M})$ is a proper divisible (by $\mathsf{hol}(\pi_1(M))$) domain of $\mathscr{F}(\mathfrak{sp}(2n,\mathbb{R}),\alpha_1)$. This is in contradiction with Proposition 8.7.1. □

8.9 Perspectives on the rigidity of proper almost-homogeneous domains in flag manifolds

We conclude this chapter with a discussion of the rigidity results for proper divisible and almost-homogeneous domains obtained in this memoir.

8.9.1 Higher-rank rigidity of almost-homogeneous domains

In view of Lemma 5.4.2, Question 1.2.1, and Proposition 8.7.1, we can conjecture that if G is a simple Lie group and $\alpha \in \Delta$ is a simple restricted root root such that $\mathscr{F}(\mathfrak{g},\alpha)$ admits proper almost-homogeneous domains, then $(\mathfrak{g},\alpha)$ is an irreducible Nagano pair.

Now, as seen in Observation 5.1.10, if $(\mathfrak{g},\alpha)$ is an irreducible Nagano pair, then the rank of $\mathbb{X}(\mathfrak{g},\alpha)$ is determined by the number of incidence degrees between two points of $\mathscr{F}(\mathfrak{g},\alpha)$, i.e. the cardinality of $W_{\Delta\smallsetminus\{\alpha\}}\backslash W/W_{\Delta\smallsetminus\{\alpha\}}$ (see Remark 2.2.7). If $(\mathfrak{g},\alpha)$ is an irreducible Nagano pair of real type, then the rank of $\mathbb{X}(\mathfrak{g},\alpha)$ is 1 if and only if the Nagano space $\mathscr{F}(\mathfrak{g},\alpha)$ is either a real projective space or its dual. In this case, there exist proper non-symmetric almost-homogeneous domains in $\mathscr{F}(\mathfrak{g},\alpha)$, as already mentioned in the introduction. In other cases, as highlighted in Theorems 8.3.1 and 8.4.1 and Corollary 8.2.3, a certain rigidity appears. As seen in the proofs, this rigidity is related to the stratified structure of Schubert cells, in other words, to the number of incidence degrees (again, the integer $|W_{\Delta\smallsetminus\{\alpha\}}\backslash W/W_{\Delta\smallsetminus\{\alpha\}}|$, see Observation 5.1.10) being strictly greater than 2. This structure implies, for instance, that all photons passing through a given point are contained in the maximal proper Schubert cell defined by a point in the dual flag manifold (see Lemma 6.3.9), which is not the case in projective space. Since this number of degrees is intrinsically related to the rank of $(\mathfrak{g},\alpha)$ (see Observation 5.1.10), we interpret this rigidity as a *higher-rank rigidity* phenomenon.

According to the classification given in Table 8.1, if $(\mathfrak{g},\alpha)$ is an irreducible Nagano pair of *complex* or *quaternionic* type (i.e. $\dim(\mathfrak{g}_\alpha)=2$ or 4), then $\mathscr{F}(\mathfrak{g},\alpha)$ has a complex, resp. quaternionic, structure. In this case, a certain form of rigidity has already been observed in [Zim18b, Fra89]; for instance, as highlighted in [LZ19], the argument of Frankel [Fra89, Sect. 6] implies that $\mathsf{Aut}_G(\Omega)$ is non-discrete (with $G\in\mathscr{G}_{\{\alpha\}}(\mathfrak{g})$). We thus believe that the rigidity of proper almost-homogeneous domains should hold in this case as well, this time due to complex and quaternionic structural reasons. The Kobayashi pseudo-metric discussed in Section 6.5.3 might be an interesting tool to study proper almost-homogeneous domains in these Nagano spaces.

In summary, we can formulate the conjecture that if $(\mathfrak{g},\alpha)$ is an irreducible Nagano pair such that $(\mathfrak{g},\alpha)$ is neither $(\mathfrak{so}(n,1),\alpha_1)$ for some $n\geq 3$, nor $(\mathfrak{sl}(n,\mathbb{R}),\alpha_1)$ or $(\mathfrak{sl}(n,\mathbb{R}),\alpha_{n-1})$ for some $n\geq 3$, then any proper almost-homogeneous domain of $\mathscr{F}(\mathfrak{g},\alpha)$ is symmetric.

Considering Fact 1.2.3, stated in the introduction and strengthened to the almost-homogeneous case in Lemma 8.1.2, together with the positive answers to Question 1.2.1 and the discussion of this paragraph, we can reformulate Question 1.2.1 more precisely, in the following conjecture:

Conjecture 8.9.1. Let $\mathfrak{g}$ be a simple Lie algebra and let Θ be a subset of the simple restricted roots of $\mathfrak{g}$. If $(\mathfrak{g},\alpha)$ is neither $(\mathfrak{so}(n,1),\alpha_1)$ for some $n\geq 3$, nor $(\mathfrak{sl}(n,\mathbb{R}),\alpha_1)$ or $(\mathfrak{sl}(n,\mathbb{R}),\alpha_{n-1})$ for some $n\geq 3$, then any proper almost-homogeneous domain of $\mathscr{F}(\mathfrak{g},\Theta)$ is symmetric. More precisely:

1. If $\mathscr{F}(\mathfrak{g},\Theta)$ admits proper almost-homogeneous domains, then Θ is a singleton $\{\alpha\}$, and $(\mathfrak{g},\alpha)$ is an irreducible Nagano pair.
2. If $(\mathfrak{g},\alpha)$ is an irreducible Nagano pair such that $(\mathfrak{g},\alpha)\neq(\mathfrak{so}(n,1),\alpha_1)$ for some $n\geq 3$, or $(\mathfrak{sl}(n,\mathbb{R}),\alpha_1)$, $(\mathfrak{sl}(n,\mathbb{R}),\alpha_{n-1})$ for some $n\ \geq 3$, then any proper almost-homogeneous domain of $\mathscr{F}(\mathfrak{g},\alpha)$ is a realization of $\mathbb{X}(\mathfrak{g},\alpha)$.

Note that with Lemma 1.2.2, Conjecture 8.9.1 gives a complete picture of the situation when $\mathfrak{g}$ is semisimple instead of being simple.

8.9.2 Symmetric behavior

By Lemma 5.4.1, one can expect that the real projective spaces $\mathbb{P}(\mathbb{R}^n)$, for $n\geq 3$, and their duals, are the only Nagano spaces that contain symmetric divisible convex sets other than their non-compact dual. These proper symmetric domains are exactly the symmetric spaces of $\mathrm{SL}(k,\mathbb{K})$, where

$$\begin{aligned}
\mathbb{K}&=\mathbb{R}, \quad \text{if } n=\frac{k(k+1)}{2},\\
\mathbb{K}&=\mathbb{C}, \quad \text{if } n=k^2,\\
\mathbb{K}&=\mathbb{H}, \quad \text{if } n=2k^2-k,
\end{aligned}$$

and the one of $E_{6(-26)}$. As mentioned in the introduction of this memoir, irreducible divisible convex sets in the real projective space $\mathbb{P}(\mathbb{R}^n)$ that are nonsymmetric have geometric and dynamical properties analogous to those of the real hyperbolic space, see for instance [Ben01, Ben03, Ben06, Cra09, CLT15]. As mentioned in Section 8.2, A. Zimmer even proved that all such divisible convex sets are *of rank one* [Zim23]. The conclusion of this discussion is that divisible convex sets in $\mathbb{P}(\mathbb{R}^n)$ mimic the real hyperbolic space, with a certain flexibility that can likely be attributed to the fact that this symmetric space has real rank 1.

Thus, conjecturally (see Conjecture 8.9.1), all divisible convex sets in Nagano spaces (different from the conformal sphere) exhibit a symmetric-like behavior, including those in real projective space. This "symmetric" nature of divisible convex sets, in real projective space and in Nagano spaces in general, remains mysterious. As already mentioned in Section 8.9.1, the results of this memoir provide further insights into this phenomenon. We believe that developing further the general theory of proper divisible domains (or even just proper almost-homogeneous domains) in flag manifolds — particularly in Nagano spaces admitting a complex structure (i.e. those where $\dim(\mathfrak{g}_\alpha) = 2$) — could deepen our understanding of the projective case and the mechanisms driving the hyperbolic behavior of divisible convex sets in $\mathbb{P}(\mathbb{R}^n)$.

The case of the conformal sphere $\mathscr{F}(\mathfrak{so}(n,1), \alpha_1)$ also remains mysterious, as it is the only case where we have no idea of the possible dynamical or geometric constraints divisible convex sets could be subject to.

Tables

In this section, we provide two tables with informations on Nagano spaces. These informations come from [Nag65, Mak73, Tak88, OV12].

	$\mathfrak{g}$	α	$\mathscr{F}(\mathfrak{g},\alpha)$	$\dim(\mathfrak{g}_\alpha)$
Group manifolds	$\mathfrak{so}(n,n)$	$\alpha_i, i \in \{n-1,1\}$	$\mathrm{SO}(n)$	1
	$\mathfrak{sp}(n,n)$	$\alpha_n = \varepsilon_n$	$\mathrm{Sp}(n)$	3
	$\mathfrak{su}(n,n)$	$\alpha_n = 2\varepsilon_n$	$\mathrm{U}(n)$	1
Grassmannians	$\mathfrak{sl}(p+q,\mathbb{R})$	$\alpha_p = \varepsilon_p - \varepsilon_{p+1}$	$\mathrm{Gr}_p(\mathbb{R}^{p+q})$	1
	$\mathfrak{sl}(p+q,\mathbb{C})$	$\alpha_p = \varepsilon_p - \varepsilon_{p+1}$	$\mathrm{Gr}_p(\mathbb{C}^{p+q})$	2
	$\mathfrak{sl}(p+q,\mathbb{H})$	$\alpha_p = \varepsilon_p - \varepsilon_{p+1}$	$\mathrm{Gr}_p(\mathbb{H}^{p+q})$	4
	$\mathfrak{e}_{6(6)}$	$\alpha_i, i \in \{1,5\}$	$\mathrm{Sp}(4)/(\mathrm{Sp}(2)\times\mathrm{Sp}(2))$	1
Products of spheres	$\mathfrak{so}(p+1,q+1)$	$\alpha_1 = \varepsilon_1 - \varepsilon_2$	$\mathbb{S}^p \times \mathbb{S}^q$	1
	$\mathfrak{so}(n,1)$	$\alpha_1 = \varepsilon_1 - \varepsilon_2$	$\mathbb{S}^{n-1}$	$n-1$
manifold × sphere	$\mathfrak{so}^*(4n)$	$\alpha_n = 2\varepsilon_n$	$(\mathrm{SU}(2n)/\mathrm{Sp}(n))\times\mathbb{S}^1$	1
	$\mathfrak{sp}(2n,\mathbb{R})$	$\alpha_n = 2\varepsilon_n$	$(\mathrm{SU}(n)/\mathrm{SO}(n))\times\mathbb{S}^1$	1
	$\mathfrak{e}_{7(-25)}$	α_3	$(E_7/F_4)\times\mathbb{S}^1$	1
Hermitian sym. spaces	$\mathfrak{so}(n+2,\mathbb{C})$	$\alpha_1 = \varepsilon_1 - \varepsilon_2$	$\mathrm{SO}(n+2)/\mathrm{S}(\mathrm{O}(2)\times\mathrm{O}(n))$	2
	$\mathfrak{so}(2n,\mathbb{C})$	$\alpha_i \in \{n-1,n\}$	$\mathrm{SO}(2n)/\mathrm{U}(n)$	2
	$\mathfrak{sp}(2n,\mathbb{C})$	$\alpha_n = 2\varepsilon_n$	$\mathrm{Sp}(n)/\mathrm{U}(n)$	2
	$\mathfrak{e}_{6,\mathbb{C}}$	$\alpha_i,\, i \in \{1,5\}$	$E_6/(\mathrm{SO}(2)\times\mathrm{SO}(10))$	2
	$\mathfrak{e}_{7,\mathbb{C}}$	α_7	$E_7/(\mathrm{SO}(2)\times E_6)$	2
Others	$\mathfrak{e}_{6(-26)}$	$\alpha_i,\, i \in \{1,2\}$	F_4/B_4	8
	$\mathfrak{e}_{7(7)}$	α_7	$\mathrm{SU}(8)/\mathrm{Sp}(4)$	1

Table 8.1 – Nagano spaces. The numbering of restricted roots is the one of [OV12]. The numbers n, p, q are positive integers.

$(\mathfrak{g},\alpha)$	$\mathrm{Isom}(\mathscr{F}(\mathfrak{g},\alpha),\mathtt{g}_{\mathfrak{g},\alpha})$	$\mathrm{Isom}(\mathbb{X}(\mathfrak{g},\alpha))$	$\mathrm{rk}(\mathfrak{g},\alpha)$
$(\mathfrak{so}(n,n),\alpha_i),\ i\in\{n-1,n\}$	$\mathrm{SO}(n)\times\mathrm{SO}(n)$	$\mathrm{SO}(n,\mathbb{C})$	$\lfloor\frac{n}{2}\rfloor$
$(\mathfrak{sp}(n,n),\alpha_n)$	$\mathrm{Sp}(n)\times\mathrm{Sp}(n)$	$\mathrm{Sp}(2n,\mathbb{C})$	n
$(\mathfrak{su}(n,n),\alpha_n)$	$\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(n))$	$\mathrm{SL}(n,\mathbb{C})\times\mathbb{R}$	n
$(\mathfrak{sl}(p+q,\mathbb{R}),\alpha_p)$	$\mathrm{SO}(p+q)$	$\mathrm{SO}(p,q)$	$\min(p,q)$
$(\mathfrak{sl}(p+q,\mathbb{C}),\alpha_p)$	$\mathrm{SU}(p+q)$	$\mathrm{SU}(p,q)$	$\min(p,q)$
$(\mathfrak{sl}(p+q,\mathbb{H}),\alpha_p)$	$\mathrm{Sp}(p+q)$	$\mathrm{Sp}(p,q)$	$\min(p,q)$
$(\mathfrak{e}_{6(6)},\alpha_i),\ i\in\{1,5\}$	$\mathrm{Sp}(4)$	$\mathrm{Sp}(2,2)$	2
$(\mathfrak{so}(p+1,q+1),\alpha_1)$	$\mathrm{SO}(p+1)\times\mathrm{SO}(q+1)$	$\mathrm{SO}(p,1)\times\mathrm{SO}(1,q)$	2
$(\mathfrak{so}(n,1),\alpha_1)$	$\mathrm{SO}(n+1)$	$\mathrm{SO}(n-1,1)$	1
$(\mathfrak{so}^*(4n),\alpha_n)$	$\mathrm{U}(2n)$	$\mathrm{SL}(n,\mathbb{H})\times\mathbb{R}$	n
$(\mathfrak{sp}(2n,\mathbb{R}),\alpha_n)$	$\mathrm{U}(n)$	$\mathrm{SL}(n,\mathbb{R})\times\mathbb{R}$	n
$(\mathfrak{e}_{7(-25)},\alpha_3)$	$E_6\times\mathbb{S}^1$	$E_{6(-26)}\times\mathbb{R}$	3
$(\mathfrak{so}(n+2,\mathbb{C}),\alpha_1)$	$\mathrm{SO}(n+2)$	$\mathrm{SO}(n,2)$	$\min(n,2)$
$(\mathfrak{sp}(2n,\mathbb{C}),\alpha_n)$	$\mathrm{Sp}(n)$	$\mathrm{Sp}(2n,\mathbb{R})$	n
$(\mathfrak{so}(2n,\mathbb{C}),\alpha_i),\ i\in\{n-1,n\}$	$\mathrm{SO}(2n)$	$\mathrm{SO}^*(2n)$	$\lfloor\frac{n}{2}\rfloor$
$(\mathfrak{e}_{6,\mathbb{C}},\alpha_i),\ i\in\{1,5\}$	E_6	$E_{6(-14)}$	2
$(\mathfrak{e}_{7,\mathbb{C}},\alpha_7)$	E_7	$E_{7(-25)}$	3
$(\mathfrak{e}_{6(-26)},\alpha_i),\ i\in\{1,2\}$	F_4	$F_{4(-20)}$	1
$(\mathfrak{e}_{7(7)},\alpha_7)$	$\mathrm{SU}(8)$	$\mathrm{SL}(4,\mathbb{H})$	3

Table 8.2 – The isometry groups of Nagano spaces and of their noncompact dual. The groups are given up to finite index and finite cover. The numbers n,p,q are positive integers. Recall that $\mathrm{rk}(\mathfrak{g},\alpha)$ is introduced in Definition 5.1.5 and is equal to $|W_{\Delta\smallsetminus\{\alpha\}}\backslash W/W_{\Delta\smallsetminus\{\alpha\}}|-1$.

Principaux résultats du mémoire

Dans cette partie, nous listons les principaux résultats de ce mémoire, énoncés en introduction. Les notations utilisées sont celles de l'introduction.

Différentes notions de convexité

Proposition 1.4.1 (voir la proposition 3.5.24)**.** *Soit G un groupe de Lie hermitien de type tube et soit $\Omega \subset \mathbf{Sb}(\mathfrak{g})$ un domaine dualement convexe. Si $\Omega \neq \mathbf{Sb}(\mathfrak{g})$, alors Ω est causalement convexe (en particulier Ω est contenu dans au moins une carte affine).*

Groupes préservant des domaines propres

Proposition 1.4.2 (voir la proposition 4.1.5 et le corollaire 4.1.7)**.** *Soient G un groupe de Lie semi-simple réel et $P \leq G$ un sous-groupe parabolique auto-opposé. Soit $H \leq G$ un sous-groupe préservant un domaine propre $\Omega \subset G/P$ tel que l'ensemble limite $\Lambda_P(H)$ contienne au moins trois points deux à deux transverses. Alors il existe une composante connexe s-invariante $\mathscr{O}$ de $(G/P) \smallsetminus (\mathrm{Z}_P \cup \mathrm{Z}_{P^-})$ telle que le type $\mathsf{type}(a,b,c)$ d'un triplet $(a,b,c) \in \Lambda_P(H)^3$ de points deux à deux transverses soit égal à la $(P \cap P^-)$-orbite de $\mathscr{O}$.*

Dans le cas où G est un groupe de Lie HTT *de rang réel $r \geq 2$ et $G/P = \mathbf{Sb}(\mathfrak{g})$, alors r est pair et $\mathsf{idx}(x,y,z) = 0$ pour tout triplet de points deux à deux transverses $(x,y,z) \in \Lambda_P(H)^3$.*

Proposition 1.4.4 (voir la proposition 4.3.2)**.** *Soient G un groupe de Lie* HTT *et $\Gamma \leq G$ un sous-groupe discret. Soit $P \leq G$ un sous-groupe parabolique tel que $G/P = \mathbf{Sb}(\mathfrak{g})$. Les assertions suivantes sont équivalentes :*

1. *Le groupe Γ est de type fini, P-transverse, préserve un domaine propre $\Omega \subset \mathbf{Sb}(\mathfrak{g})$, et $\Lambda_P(\Gamma)$ contient au moins 3 points.*
2. *Il existe un domaine propre causalement convexe Γ-invariant $\Omega \subset \mathbf{Sb}(\mathfrak{g})$ tel que Γ agisse de manière cocompacte sur un cœur convexe $\mathscr{C}$ de (Ω, Γ) dont le bord idéal est transverse, et contient au moins trois points.*
3. *Il existe un domaine propre dualement convexe Γ-invariant $\Omega' \subset \mathbf{Sb}(\mathfrak{g})$ tel que Γ agisse de manière cocompacte sur un cœur convexe $\mathscr{C}'$ de (Ω', Γ) dont le bord idéal est transverse, et contient au moins trois points.*

Si ces assertions sont vérifiées, on a $\partial_i \mathscr{C} = \Lambda_P(\Gamma) = \Lambda^{\mathrm{orb}}_{\Omega}(\Gamma) = \Lambda^{\mathrm{orb}}_{\Omega'}(\Gamma) = \partial_i \mathscr{C}'$.

Proposition 1.4.5 (voir la proposition 4.4.2 et l'exemple 4.4.6). *Soit* $r = 2p$, *avec* $p \in \mathbb{N}^*$. *Si* G *est un groupe de Lie* HTT *de rang réel* r *et si* $P \leq G$ *est un sous-groupe parabolique tel que* $G/P = \mathbf{Sb}(\mathfrak{g})$, *alors il existe des groupes de surfaces* P*-anosoviens Zariski-denses dans* G *préservant un domaine propre dans* $\mathbf{Sb}(\mathfrak{g})$. *Si* p *est pair, alors il existe aussi de tels exemples qui ne sont ni virtuellement libres, ni des groupes de surface.*

Distance de Kobayashi

Théorème 1.4.6 (voir la proposition 6.4.8 et le corollaire 6.4.12). *Soit* G/P *un espace de Nagano irréductible de type réel et soit* $\Omega \subset G/P$ *un domaine propre. Alors* K_Ω *est une distance* $\mathsf{Aut}(\Omega)$*-invariante qui induit la topologie standard sur* Ω. *Si* Ω *est en outre dualement convexe, alors* K_Ω *est une distance propre et géodésique.*

Proposition 1.4.7. *[voir la proposition 6.4.10] Soit* G/P *un espace de Nagano irréductible de type réel et soit* (V, ρ) *une représentation linéaire réelle irréductible, proximale, de dimension finie de* G *de plus haut poids* $\chi = N\omega_\alpha$, *où* $N \in \mathbb{N}^*$ *et* ω_α *est le poids fondamental associé à* α. *Soit* $\Omega \subset G/P$ *un domaine propre dualement convexe et soit* C_Ω^ρ *la distance de Carathéodory sur* Ω *induite par* (V, ρ) *(voir l'équation (1.1.2)). Alors on a :*

$$K_\Omega \geq \frac{1}{N} C_\Omega^\rho.$$

En particulier, la distance K_Ω *est propre.*

Domaines propres divisibles dans les variétés de drapeaux

Non-hyperbolicté de la distance de Kobayashi en rang supérieur

Théorème 1.4.9 (voir le théorème 8.2.2). *Soit* G/P *un espace de Nagano irréductible de type réel de rang supérieur. Si* $\Omega \subset G/P$ *un domaine propre presque-homogène muni de sa distance de Kobayashi* K_Ω, *alors l'espace métrique géodésique* (Ω, K_Ω) *n'est pas Gromov-hyperbolique.*

Corollaire 1.4.10 (voir le corollaire 8.2.3). *Soient* G/P *un espace de Nagano irréductible de type réel de rang supérieur et* $\Gamma \leq G$ *un sous-groupe discret. Supposons que* Γ *divise un domaine propre de* G/P. *Alors* Γ *n'est pas Gromov-hyperbolique.*

Domaines propres divisibles dans les grassmanniennes

Théorème 1.4.12 (voir le théorème 8.5.1). *Soit* $2 \leq p \leq q$. *Soit* $\Gamma \leq \mathrm{PGL}(p+q, \mathbb{R})$ *un sous-groupe discret, agissant cocompactement sur un domaine propre* $\Omega \subset \mathrm{Gr}_p(\mathbb{R}^{p+q})$ *dont le bord est une hypersurface topologique de* $\mathrm{Gr}_p(\mathbb{R}^{p+q})$. *Alors toute décomposition* Γ*-invariante de* $\mathbb{R}^{p+q}$ *est triviale.*

Corollaire 1.4.13 (voir le corollaire 8.5.2). *Soit* $2 \leq p \leq q$. *Soit* $\Omega \subset \mathrm{Gr}_p(\mathbb{R}^{p+q})$ *un domaine propre dont le bord est une hypersurface topologique de* $\mathrm{Gr}_p(\mathbb{R}^{p+q})$. *Supposons qu'il existe un sous-groupe discret* $\Gamma \leq \mathrm{PGL}(p+q, \mathbb{R})$ *agissant cocompactement sur* Ω. *Alors le centralisateur de* Γ *dans* $\mathrm{PGL}(p+q, \mathbb{R})$ *est fini.*

Domaines propres presque-homogènes dans les variétés de drapeaux causales

Théorème 1.4.14 (voir le théorème 8.3.1)**.** *Soit G un groupe de Lie simple de hermitien de type tube. Alors tout domaine propre presque-homogène de $\mathbf{Sb}(\mathfrak{g})$ est un diamant.*

Corollaire 1.4.15. *Soit G un groupe de Lie semi-simple de type Hermitien de type tube, avec centre trivial et sans facteur compact. Écrivons $G = G_1 \times \cdots \times G_k$, où chaque G_i est un groupe de Lie simple non compact de type Hermitien de type tube pour tout $1 \leq i \leq k$. Alors, pour tout domaine propre presque homogène $\Omega \subset \mathbf{Sb}(\mathfrak{g})$, il existe des diamants $D_i \subset \mathbf{Sb}(\mathfrak{g}_i)$ pour $1 \leq i \leq k$ tels que $\Omega = D_1 \times \cdots \times D_k \subset \mathbf{Sb}(G_1) \times \cdots \times \mathbf{Sb}(G_k)$.*

Corollaire 1.4.16 (voir le corollaire 8.3.4)**.** *Soit G un groupe de Lie simple réel non compact et soit Θ un sous-ensemble des racines simples restreintes de G tel que G admette une structure Θ-positive. Alors on a la dichotomie suivante :*

1. Si $|\Theta| = 1$, alors G est hermitien de type tube et $G/P_\Theta = \mathbf{Sb}(\mathfrak{g})$ admet exactement un domaine propre presque-homogène à conjugaison par G près, qui est un diamant.

2. Si $|\Theta| \geq 2$, alors il n'existe aucun domaine propre presque-homogène dans G/P_Θ.

Domaines propres presque-homogènes dans les univers d'Einstein

Théorème 1.4.17 (avec Chalumeau, voir le théorème 8.4.1)**.** *Tout domaine propre presque-homogène de $\mathrm{Ein}^{p,q}$ est un diamant.*

Corollaire 1.4.18 (avec Chalumeau, voir le corollaire 8.4.4)**.** (1) *Soit $\Omega \subset \mathrm{Gr}_2(\mathbb{R}^4)$ un domaine propre presque-homogène. Alors Ω est une réalisation de $\mathbb{X}(\mathrm{Gr}_2(\mathbb{R}^4))$. Autrement dit, il existe $g \in \mathrm{PGL}(4, \mathbb{R})$ tel que $\Omega = g \cdot \mathbb{B}_{2,2}$.*

(2) *Soit $\mathscr{F}$ l'une des deux composantes connexes de l'espace des sous-espaces totalement isotropes maximaux de $\mathbb{R}^{4,4}$. Soit $\Omega \subset \mathscr{F}$ un domaine propre presque homogène. Alors Ω est une réalisation de $\mathbb{X}(\mathscr{F}(\mathfrak{g}, \alpha))$. En particulier, $\mathsf{Aut}(\Omega)$ est conjugué à $\mathrm{SO}(3,1) \times \mathrm{SO}(1,3)$ dans $\mathrm{SO}(4,4)$.*

Corollaire 1.4.20 (avec Chalumeau, voir le corollaire 8.8.4)**.** *Soient $p, q \geq 2$ deux entiers et M une variété pseudo-riemannienne conformément plate de signature (p, q) (où p est le nombre de $+$ et q le nombre de $-$), propre, compacte et connexe. Alors M est conformément équivalente à un quotient D/Γ, où D est un diamant de $\mathrm{Ein}^{p,q}$ et $\Gamma \leq \mathsf{Aut}(D)$ est un réseau cocompact. Si de plus $1 \leq p < q$ avec $(p, q) \neq (2, 3)$, alors à revêtement fini près, la variété M est conformément équivalente à*

$$\Sigma^p \times (-\Sigma^q),$$

où Σ^p et Σ^q sont des variétés hyperboliques compactes de dimensions respectives p et q. En signature lorentzienne, c'est-à-dire pour $q = 1$, la variété M est (à un revêtement fini près) conformément équivalente au produit $\Sigma \times (-\mathbb{S}^1)$, où Σ est une variété hyperbolique compacte.

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