COMPACT ANTI-DE SITTER 3-MANIFOLDS AND FOLDED HYPERBOLIC STRUCTURES ON SURFACES

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Abstract. We prove that any non-Fuchsian representation ρ of a surface group into $\mathrm{PSL}(2,\mathbb{R})$ is the holonomy of a folded hyperbolic structure on the surface, unless the image of ρ is virtually abelian. Using similar ideas, we establish that any non-Fuchsian representation ρ is strictly dominated by some Fuchsian representation j, in the sense that the hyperbolic translation lengths for j are uniformly larger than for ρ. Conversely, any Fuchsian representation j strictly dominates some non-Fuchsian representation ρ, whose Euler class can be prescribed. This has applications to the theory of compact anti-de Sitter 3-manifolds.

1. Introduction

Let Σ_g be a closed, connected, oriented surface of genus g, with fundamental group $\Gamma_g = \pi_1(\Sigma_g)$, and let $\mathsf{Rep}_g^{\mathrm{fd}}$ (resp. $\mathsf{Rep}_g^{\mathrm{nfd}}$) be the set of conjugacy classes of Fuchsian (resp. non-Fuchsian) representations of Γ_g into $\mathrm{PSL}(2,\mathbb{R})$. The letters "fd" stand for "faithful, discrete". By work of Goldman [Go2], the space $\mathrm{Hom}(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$ of representations of Γ_g into $\mathrm{PSL}(2,\mathbb{R})$ has $4g-3$ connected components, indexed by the values of the Euler class

$$\mathsf{eu}: \ \mathrm{Hom}(\Gamma_g, \mathrm{PSL}(2,\mathbb{R})) \longrightarrow \{2-2g, \dots, -1, 0, 1, \dots, 2g-2\}.$$

In the quotient, $\mathsf{Rep}_g^{\mathrm{fd}}$ consists of the two connected components of $\mathrm{Hom}(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$ of extremal Euler class, and $\mathsf{Rep}_g^{\mathrm{nfd}}$ of all the other components.

1.1. Strictly dominating representations.

For any $g \in \mathrm{PSL}(2,\mathbb{R})$, let

$$\lambda(g) := \inf_{p \in \mathbb{H}^2} d(p, g\cdot p) \geq 0 \tag{1.1}$$

be the translation length of g in the hyperbolic plane $\mathbb{H}^2$. The function $\lambda : \mathrm{PSL}(2,\mathbb{R}) \to \mathbb{R}^+$ is invariant under conjugation. We say that an element $[j] \in \mathsf{Rep}_g^{\mathrm{fd}}$ *strictly dominates* an element $[\rho] \in \mathsf{Rep}_g^{\mathrm{nfd}}$ if

$$\sup_{\gamma \in \Gamma_g \smallsetminus \{1\}} \frac{\lambda(\rho(\gamma))}{\lambda(j(\gamma))} < 1. \tag{1.2}$$

Note that (1.2) can never hold when j and ρ are both Fuchsian [T2]. In this paper we prove the following.

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Theorem 1.1. *Any* $[\rho] \in \mathsf{Rep}_g^{\mathrm{nfd}}$ *is strictly dominated by some* $[j] \in \mathsf{Rep}_g^{\mathrm{fd}}$. *Any* $[j] \in \mathsf{Rep}_g^{\mathrm{fd}}$ *strictly dominates some* $[\rho] \in \mathsf{Rep}_g^{\mathrm{nfd}}$*, whose Euler class can be prescribed.*

The first statement of Theorem 1.1 has been simultaneously and independently obtained by Deroin–Tholozan [DT], using more analytical methods. Their paper deals, more generally, with representations of Γ_g into the isometry group of any complete, simply connected Riemannian manifold with sectional curvature ≤ -1. They also announce a version for general CAT(-1) spaces. The present methods, relying as they do on the Toponogov theorem (see Lemma 2.2 below), could likely extend to this general setting as well.

Our approach is constructive, using folded (or pleated) hyperbolic surfaces, as we now explain.

1.2. **Folded hyperbolic surfaces.** Pleated hyperbolic surfaces were introduced by Thurston [T1] and play an important role in the theory of hyperbolic 3-manifolds. A *folded hyperbolic surface* is a pleated surface with all angles equal to 0 or π, whose holonomy takes values in $\mathrm{PSL}(2,\mathbb{R})$ (see Section 2.2). It is easy to check (see [T2, Prop. 2.1]) that the holonomy of a (nontrivially) folded hyperbolic structure on Σ_g belongs to $\mathsf{Rep}_g^{\mathrm{nfd}}$. In order to establish Theorem 1.1, we prove that the converse holds for representations whose image is not virtually abelian.

Theorem 1.2. *An element of* $\mathsf{Rep}_g^{\mathrm{nfd}}$ *is the holonomy of a folded hyperbolic structure on* Σ_g *if and only if its image is not virtually abelian.*

As usual, *virtually abelian* means that there is an abelian subgroup of finite index. Besides abelian representations, Theorem 1.2 rules out dihedral representations, which preserve a geodesic line of $\mathbb{H}^2$ and contain order-two symmetries of that line.

This result seems to have been known to experts since the work of Thurston [T1], but to our knowledge it is not stated nor proved in the literature.

We construct the folded hyperbolic structures of Theorem 1.2 explicitly, folding along geodesic laminations that are the union of simple closed curves and of maximal laminations of some pairs of pants (Proposition 3.1). More precisely, given a non-Fuchsian representation ρ whose image is not virtually abelian, we use a result of Gallo–Kapovich–Marden [GKM] to find a pants decomposition of Σ_g such that the restriction of ρ to any pair of pants P is nonabelian and maps any cuff to a hyperbolic element. (The term *cuff*, always specific to a pair of pants, will in the sequel denote indifferently the homotopy class of a boundary component, or the geodesic in that class, or its length.) Folding along a certain maximal lamination in P then gives a simple dictionary between the representations of the fundamental group of P that have Euler class 0 and those that have Euler class ± 1 (Lemma 3.6). The converse direction in Theorem 1.2 is elementary (Observation 2.7).

1.3. **Idea of the proof of Theorem 1.1.** If $[\rho] \in \mathsf{Rep}_g^{\mathrm{nfd}}$ is the holonomy of a folded hyperbolic structure on Σ_g, then the holonomy $[j_0] \in \mathsf{Rep}_g^{\mathrm{fd}}$ of the corresponding unfolded hyperbolic structure clearly dominates $[\rho]$ in the sense that $\lambda(\rho(\gamma)) \leq \lambda(j_0(\gamma))$ for all $\gamma \in \Gamma_g$. In fact,

$$\sup_{\gamma\in\Gamma_g\smallsetminus\{1\}} \frac{\lambda(\rho(\gamma))}{\lambda(j_0(\gamma))} = 1$$

since any minimal component of the folding lamination can be approximated by simple closed curves. In order to prove Theorem 1.1 we need to make the domination *strict*.

To establish the first statement, the idea is, for $[\rho] \in \mathsf{Rep}_g^{\mathrm{nfd}}$, to consider the holonomy $[j_0] \in \mathsf{Rep}_g^{\mathrm{fd}}$ of the unfolded hyperbolic structure given by Theorem 1.2, and to lengthen the closed curves (close to being) contained in the folding lamination while simultaneously not shortening too much the other curves. To do this, we work independently in each "folded subsurface" of Σ_g, which is a compact surface with boundary, endowed with a hyperbolic structure induced by j_0. In each such subsurface we use a *strip deformation* construction due to Thurston [T2], which consists in adding hyperbolic strips to obtain a new hyperbolic metric with longer boundary components. We then glue back along the boundaries, after making sure that the lengths agree.

The second statement is easier in that it does not rely on Theorem 1.2. Starting with an element $[j] \in \mathsf{Rep}_g^{\mathrm{fd}}$, we choose a pants decomposition of Σ_g along which to fold. To make sure that the cuffs of the pairs of pants will get contracted, we first deform j slightly by *negative strip deformations* into another element $[j_0] \in \mathsf{Rep}_g^{\mathrm{fd}}$ with shorter cuffs, in such a way that the other curves do not get much longer. Folding j_0 then gives an element $[\rho] \in \mathsf{Rep}_g^{\mathrm{nfd}}$ which is strictly dominated by $[j]$.

1.4. **An application to compact anti-de Sitter 3-manifolds.** Theorem 1.1 has consequences on the theory of compact *anti-de Sitter* 3-manifolds. These are the compact Lorentzian 3-manifolds of constant negative curvature, i.e. the Lorentzian analogues of the compact hyperbolic 3-manifolds. They are locally modeled on the 3-dimensional anti-de Sitter space

$$\mathrm{AdS}^3 = \mathrm{PO}(2,2)/\mathrm{PO}(2,1),$$

which identifies with $\mathrm{PSL}(2,\mathbb{R})$ endowed with the natural Lorentzian structure induced by the Killing form of its Lie algebra. The identity component of the isometry group of AdS^3 is $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$, acting on $\mathrm{PSL}(2,\mathbb{R}) \simeq \mathrm{AdS}^3$ by right and left multiplication: $(g_1, g_2) \cdot g = g_2 g g_1^{-1}$. By [Kl], all compact anti-de Sitter 3-manifolds are geodesically complete. By [KR] and the Selberg lemma [Se, Lem. 8], they are quotients of $\mathrm{PSL}(2,\mathbb{R})$ by torsion-free discrete subgroups $\mathbf{\Gamma}$ of $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ acting properly discontinuously, up to a finite covering; moreover, the groups $\mathbf{\Gamma}$ are graphs of the form

$$\mathbf{\Gamma} = (\Gamma_g)^{j,\rho} := \{(j(\gamma), \rho(\gamma)) \mid \gamma \in \Gamma_g\},$$

for some $g \geq 2$, where $j, \rho \in \mathrm{Hom}(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$ are representations and j is Fuchsian, up to switching the two factors of $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$. In particular, $\mathbf{\Gamma}\backslash\mathrm{AdS}^3$ is Seifert fibered over a hyperbolic base (see [Sa1, § 3.4.2]).

Following [Sa2], we shall say that a pair $(j, \rho) \in \mathrm{Hom}(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))^2$ with j Fuchsian is *admissible* if the action of $(\Gamma_g)^{j,\rho}$ on AdS^3 is properly discontinuous. Clearly, (j, ρ) is admissible if and only if its conjugates under $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ are. Therefore, in order to understand the moduli space of compact anti-de Sitter 3-manifolds, we need to understand, for any $g \geq 2$, the space

$$\mathrm{Adm}_g \subset \mathsf{Rep}_g^{\mathrm{fd}} \times \mathrm{Hom}(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$$

of conjugacy classes of admissible pairs (j, ρ) with j Fuchsian.

Examples of admissible pairs are readily obtained by taking ρ to be constant, or more generally with bounded image. The corresponding quotients of AdS^3 are called *standard*. The first nonstandard examples were constructed by Goldman [Go1] by deformation of standard ones — a technique later generalized by Kobayashi [Ko]. Salein [Sa2] constructed the first examples of admissible pairs (j, ρ) with $\mathsf{eu}(\rho) \neq 0$. He actually constructed examples where $\mathsf{eu}(\rho)$ can take any nonextremal value. A necessary and sufficient condition for admissibility was given in [Ka2]: a pair (j, ρ) with j Fuchsian is admissible if and only if ρ is strictly dominated by j in the sense of (1.2). In particular, by [T2],

$$\mathrm{Adm}_g \subset \mathsf{Rep}_g^{\mathrm{fd}} \times \mathsf{Rep}_g^{\mathrm{nfd}}.$$

This properness criterion was extended in [GK] to quotients of $\mathrm{PO}(n,1) = \mathrm{Isom}(\mathbb{H}^n)$ by discrete subgroups of $\mathrm{PO}(n,1) \times \mathrm{PO}(n,1)$ acting by left and right multiplication, for arbitrary $n \geq 2$ (recall that $\mathrm{PSL}(2,\mathbb{R}) \simeq \mathrm{PO}(2,1)_0$), and in [GGKW] to quotients of any simple Lie group G of real rank 1.

By completeness [Kl] of compact anti-de Sitter manifolds, the Ehresmann–Thurston principle (see [T1]) implies that Adm_g is open in $\mathsf{Rep}_g^{\mathrm{fd}} \times \mathsf{Rep}_g^{\mathrm{nfd}}$. Moreover, Adm_g has at least $4g-5$ connected components, as Salein's examples show. Using the fact that the two connected components of $\mathsf{Rep}_g^{\mathrm{fd}}$ are conjugate under $\mathrm{PGL}(2,\mathbb{R})$, we can reformulate Theorem 1.1 as follows.

Corollary 1.3. *The projections of* Adm_g *to* $\mathsf{Rep}_g^{\mathrm{fd}}$ *and to* $\mathsf{Rep}_g^{\mathrm{nfd}}$ *are both surjective. Moreover, for any connected components* $\mathcal{C}_1$ *of* $\mathsf{Rep}_g^{\mathrm{fd}}$ *and* $\mathcal{C}_2$ *of* $\mathsf{Rep}_g^{\mathrm{nfd}}$*, the projections of* $\mathrm{Adm}_g \cap (\mathcal{C}_1 \times \mathcal{C}_2)$ *to* $\mathcal{C}_1$ *and to* $\mathcal{C}_2$ *are both surjective.*

The topology of Adm_g is still unknown, but we believe that Corollary 1.3 (and the ideas behind its proof) could be used to prove that Adm_g is homeomorphic to $\mathsf{Rep}_g^{\mathrm{fd}} \times \mathsf{Rep}_g^{\mathrm{nfd}}$. Using the work of Hitchin [H, Th. 10.8 & Eq. 10.6], this would give the homeomorphism type of the connected components of Adm_g corresponding to $\mathsf{eu}(\rho) \neq 0$.

Furthermore, it would be interesting to obtain a geometric and combinatorial description of the fibers of the second projection $\mathrm{Adm}_g \to \mathsf{Rep}_g^{\mathrm{nfd}}$. Such a description is given in [DGK], in terms of the arc complex, in the different case that j and ρ are the holonomies of two convex cocompact hyperbolic structures on a given *noncompact* surface.

1.5. **Organization of the paper.** In Section 2 we recall some facts about Lipschitz maps, folded hyperbolic structures, and the Euler class. Section 3 is devoted to the proof of Theorem 1.2, and Section 4 to that of Theorem 1.1.

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2. Reminders and useful facts

2.1. **Lipschitz maps and their stretch locus.** In the whole paper, we denote by d the metric on the real hyperbolic plane $\mathbb{H}^2$. For a Lipschitz map $f : \mathbb{H}^2 \to \mathbb{H}^2$ and a point $p \in \mathbb{H}^2$, we set

- $\mathrm{Lip}(f) := \sup_{q \neq q'} d(f(q), f(q'))/d(q, q') \geq 0$ (Lipschitz constant);
- $\mathrm{Lip}_p(f) := \inf_{\mathcal{U}} \mathrm{Lip}(f|_{\mathcal{U}}) \geq 0$, where $\mathcal{U}$ ranges over all neighborhoods of p in $\mathbb{H}^2$ (local Lipschitz constant).

The function $p \mapsto \mathrm{Lip}_p(f)$ is upper semicontinuous:

$$\mathrm{Lip}_p(f) \geq \limsup_{n \to +\infty} \mathrm{Lip}_{p_n}(f)$$

for any sequence $(p_n)_{n \in \mathbb{N}}$ converging to p. The following is straightforward.

Remark 2.1. For any rectifiable path $\mathcal{L} \subset \mathbb{H}^2$,

$$\mathrm{length}(f(\mathcal{L})) \leq \sup_{p \in \mathcal{L}} \mathrm{Lip}_p(f) \cdot \mathrm{length}(\mathcal{L}).$$

In particular, if $\mathrm{Lip}_p(f) \leq C$ for all p in a convex set K, then $\mathrm{Lip}(f|_K) \leq C$.

2.1.1. *The stretch locus.* The following result is contained in [GK, Th. 5.1]. It relies on the Toponogov theorem, a comparison theorem relating the curvature to the divergence rate of geodesics (see [BH, Lem. II.1.13]).

Lemma 2.2 [GK]**.** *Let Γ be a torsion-free, finitely generated, discrete group and $(j, \rho) \in \mathrm{Hom}(\Gamma, \mathrm{PSL}(2, \mathbb{R}))^2$ a pair of representations with j convex cocompact. Suppose the infimum of Lipschitz constants for all (j, ρ)-equivariant maps $f : \mathbb{H}^2 \to \mathbb{H}^2$ is 1, and the space $\mathcal{F}$ of maps achieving this infimum is nonempty. Then there exists a nonempty, $j(\Gamma)$-invariant geodesic lamination $\widetilde{\Lambda}$ of $\mathbb{H}^2$ such that*

- *any leaf of $\widetilde{\Lambda}$ is isometrically preserved by all maps $f \in \mathcal{F}$;*
- *any connected component of $\mathbb{H}^2 \smallsetminus \widetilde{\Lambda}$ is either isometrically preserved by all $f \in \mathcal{F}$, or consists entirely of points p at which $\mathrm{Lip}_p(f) < 1$ for some $f \in \mathcal{F}$ (independent of p).*

Definition 2.3. The union of $\widetilde{\Lambda}$ and of the connected components of $\mathbb{H}^2 \smallsetminus \widetilde{\Lambda}$ that are isometrically preserved by all $f \in \mathcal{F}$ is called the *stretch locus* of (j, ρ).

By *convex cocompact* we mean that j is injective and discrete and that the group $j(\Gamma)$ does not contain any parabolic element. By (j, ρ)*-equivariant* we mean that $f(j(\gamma) \cdot p) = \rho(\gamma) \cdot f(p)$ for all $\gamma \in \Gamma$ and $p \in \mathbb{H}^2$. The space $\mathcal{F}$ is always nonempty, except possibly if $\rho(\Gamma)$ admits a unique fixed point in the boundary at infinity $\partial_\infty \mathbb{H}^2$ of $\mathbb{H}^2$ [GK, Lem. 4.11]. If j and ρ are conjugate under $\mathrm{PGL}(2, \mathbb{R})$, then the stretch locus of (j, ρ) is the preimage of the convex core of $j(\Gamma) \backslash \mathbb{H}^2$. (This preimage is by definition the smallest nonempty $j(\Gamma)$-invariant convex subset of $\mathbb{H}^2$.)

2.1.2. *Averaging Lipschitz maps.* We now describe a technical tool for understanding the stretch locus. It is a procedure for averaging Lipschitz maps (see [GK, § 2.5]), under which Lip_p behaves as it would for the barycenter of maps between affine Euclidean spaces. In Section 3.4, we shall use this procedure with a partition of unity, as follows.

Let $\psi_0, \dots, \psi_n : \mathbb{H}^2 \to [0,1]$ be Lipschitz functions inducing a partition of unity on a subset X of $\mathbb{H}^2$, subordinated to an open covering $B_0 \cup \ldots \cup B_n \supset X$. For $0 \leq i \leq n$, let $\varphi_i : B_i \to \mathbb{H}^2$ be a Lipschitz map. For $p \in X$, let $I(p)$ be the collection of indices i such that $p \in B_i$. Let $\sum_{i=0}^n \psi_i \, \varphi_i : X \to \mathbb{H}^2$ be the map sending any $p \in X$ to the minimizer in $\mathbb{H}^2$ of

$$\sum_{i \in I(p)} \psi_i(p) \, d(\,\cdot\,, \varphi_i(p))^2.$$

Then the following holds.

Lemma 2.4 [GK, Lem. 2.13]**.** *The averaged map $\varphi := \sum_{i=0}^n \psi_i \, \varphi_i$ satisfies the "Leibniz rule"*

$$\mathrm{Lip}_p(\varphi) \leq \sum_{i \in I(p)} \big(\mathrm{Lip}_p(\psi_i) \, R(p) + \psi_i(p) \, \mathrm{Lip}_p(\varphi_i) \big)$$

for all $p \in X$, where $R(p)$ is the diameter of the set $\{\varphi_i(p) \,|\, i \in I(p)\}$.

2.1.3. *Admissibility.* For any discrete group Γ (not necessarily of the form Γ_g), we say that a pair of representations $(j,\rho) \in \mathrm{Hom}(\Gamma, \mathrm{PSL}(2,\mathbb{R}))^2$ is *admissible* if the group $\Gamma^{j,\rho} = \{(j(\gamma), \rho(\gamma)) \,|\, \gamma \in \Gamma\}$ acts properly discontinuously on AdS^3. In this case, at least one of j or ρ is injective and discrete [Ka1].

Understanding the stretch locus has led to the following necessary and sufficient conditions for admissibility. We denote by Γ_s the set of nontrivial elements of Γ corresponding to simple closed curves on the surface $j(\Gamma)\backslash\mathbb{H}^2$.

Theorem 2.5 [Ka2, GK]**.** *Let Γ be a torsion-free, finitely generated, discrete group. For $(j,\rho) \in \mathrm{Hom}(\Gamma, \mathrm{PSL}(2,\mathbb{R}))^2$ with j injective and discrete, the following conditions are equivalent:*

(i) *There exists a (j,ρ)-equivariant map $f : \mathbb{H}^2 \to \mathbb{H}^2$ with $\mathrm{Lip}(f) < 1$;*

(ii) *The representation ρ is strictly dominated by j:*

$$\sup_{\gamma \in \Gamma \text{ with } \lambda(j(\gamma))>0} \frac{\lambda(\rho(\gamma))}{\lambda(j(\gamma))} < 1.$$

If j is convex cocompact, then (i) and (ii) are also equivalent to:

(iii) *The representation ρ is strictly dominated by j in restriction to simple closed curves:*

$$\sup_{\gamma \in \Gamma_s} \frac{\lambda(\rho(\gamma))}{\lambda(j(\gamma))} < 1.$$

In general, the pair (j,ρ) is admissible if and only if (i) and (ii) hold up to switching j and ρ.

The implication $(iii) \Rightarrow (i)$ is nontrivial and relies on Lemma 2.2. The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are immediate modulo the following easy remark (see [GK, Lem. 4.5]).

Remark 2.6. Let Γ be a discrete group and $(j,\rho) \in \mathrm{Hom}(\Gamma, \mathrm{PSL}(2,\mathbb{R}))^2$ a pair of representations. For any $\gamma \in \Gamma$ and any (j,ρ)-equivariant Lipschitz map $f : \mathbb{H}^2 \to \mathbb{H}^2$,

$$\lambda(\rho(\gamma)) \leq \mathrm{Lip}(f) \, \lambda(j(\gamma)).$$

2.2. **Pleated and folded hyperbolic structures.** Let Σ be a connected, oriented surface of negative Euler characteristic, possibly with boundary, and let $\Gamma = \pi_1(\Sigma)$ be its fundamental group. Recall from [B, § 7] that a *pleated hyperbolic structure* on Σ is a quadruple (j, ρ, Υ, f) where

- $j \in \mathrm{Hom}(\Gamma, \mathrm{PSL}(2,\mathbb{R}))$ is the holonomy of a hyperbolic structure on Σ;
- $\rho \in \mathrm{Hom}(\Gamma, \mathrm{PSL}(2,\mathbb{C}))$ is a representation;
- Υ is a geodesic lamination on Σ;
- $f : \mathbb{H}^2 \to \mathbb{H}^3$ is a (j,ρ)-equivariant, continuous map whose restriction to any connected component of $\mathbb{H}^2 \smallsetminus \widetilde{\Upsilon}$ is an isometric embedding. (Here we denote by $\widetilde{\Upsilon} \subset \mathbb{H}^2$ the preimage of $\Upsilon \subset \Sigma \simeq j(\Gamma)\backslash\mathbb{H}^2$.)

The representation ρ is called the *holonomy* of the pleated hyperbolic structure. The closures of the connected components of $\mathbb{H}^2 \smallsetminus \widetilde{\Upsilon}$ are called the *plates.* Note that f is 1-Lipschitz. For any $g, h \in \mathrm{PGL}(2,\mathbb{R})$,

$$\big(gj(\cdot)g^{-1}, h\rho(\cdot)h^{-1}, \Upsilon, h \circ f \circ g^{-1}\big)$$

is still a pleated hyperbolic structure on Σ.

Observation 2.7. *Suppose that Σ is compact. If (j, ρ, Υ, f) is a pleated hyperbolic structure on Σ, then the group $\rho(\Gamma)$ is not virtually abelian.*

Proof. We see Σ as the convex core of the hyperbolic surface $j(\Gamma)\backslash\mathbb{H}^2$. Consider a nondegenerate ideal triangle T of $\mathbb{H}^2$ which is entirely contained in the intersection of one plate with the preimage of Σ in $\mathbb{H}^2$. Let $(p_n)_{n\in\mathbb{N}}$ be a sequence of points of T going to infinity. Since Σ is compact, there exist $R > 0$ and a sequence $(\gamma_n)_{n\in\mathbb{N}}$ of elements of Γ such that $d(j(\gamma_n)\cdot p_0, p_n) \leq R$ for all $n \in \mathbb{N}$. Since f is (j,ρ)-equivariant and 1-Lipschitz,

$$d(\rho(\gamma_n) \cdot f(p_0), f(p_n)) \leq d(j(\gamma_n) \cdot p_0, p_n) \leq R$$

for all $n \in \mathbb{N}$. Applying this to sequences (p_n) converging to the three ideal vertices of T, and using the fact that the restriction of f to T is an isometry, we see that the limit set of $\rho(\Gamma)$ contains at least three points. In particular, $\rho(\Gamma)$ is not virtually abelian. $\square$

We shall also use the following elementary remark.

Remark 2.8. Let (j, ρ, Υ, f) be a pleated hyperbolic structure on Σ. If some leaf of Υ spirals to a boundary component of Σ corresponding to an element $\gamma \in \Gamma$, then $\lambda(j(\gamma)) = \lambda(\rho(\gamma))$, where $\lambda : \mathrm{PSL}(2,\mathbb{C}) \to \mathbb{R}^+$ is the translation length function in $\mathbb{H}^3$ extending (1.1).

Any pleated hyperbolic structure (j, ρ, Υ, f) on Σ defines a *bending cocycle*, i.e. a map β from the set of pairs of plates to $\mathbb{R}/2\pi\mathbb{Z}$ which is symmetric and additive:

$$\beta(P,Q) = \beta(Q,P) \quad \text{and} \quad \beta(P,Q) + \beta(Q,R) = \beta(P,R)$$

for all plates P, Q, R. Intuitively, $\beta(P,Q)$ is the total angle of pleating encountered when traveling from $f(P)$ to $f(Q)$ along $f(\mathbb{H}^2)$ in $\mathbb{H}^3$. Conversely, to any bending cocycle, Bonahon associates a pleated surface (see [B, § 8]).

In this paper we consider a special case of pleated surfaces (j, ρ, Υ, f), namely those for which f takes values in a copy of $\mathbb{H}^2$ inside $\mathbb{H}^3$ (i.e. a totally geodesic plane) and ρ takes values in $\mathrm{Isom}^+(\mathbb{H}^2) = \mathrm{PSL}(2,\mathbb{R})$. In this case, we speak of a *folded hyperbolic structure* on Σ, and say that ρ is a *folding* of j. The map f defines a *coloring* of $\Sigma \smallsetminus \Upsilon$, i.e. a $j(\Gamma)$-invariant function $\widetilde{c}$ from the set of plates to $\{-1,1\}$. Namely, we set $\widetilde{c}(P)=-1$ if the restriction of f to P is orientation-preserving, and $\widetilde{c}(P)=1$ otherwise. Note that the bending cocycle of a folded hyperbolic structure is valued in $\{0,\pi\}$: for all plates P and Q,

$$\beta(P,Q)=\frac{1-\widetilde{c}(P)\,\widetilde{c}(Q)}{2}\,\pi\ \in\{0,\pi\}. \tag{2.1}$$

The coloring $\widetilde{c}$ descends to a continuous, locally constant function c from $\Sigma\smallsetminus\Upsilon$ to $\{-1,1\}$. Conversely, any such function, after lifting to a coloring $\widetilde{c}$ from the set of connected components of $\mathbb{H}^2\smallsetminus\widetilde{\Upsilon}$ to $\{-1,1\}$, defines a bending cocycle on $\mathbb{H}^2\smallsetminus\widetilde{\Upsilon}$ by the formula (2.1). This bending cocycle, in turn, defines a folded hyperbolic structure on Σ by the work of Bonahon [B].

2.3. **The Euler class.** We now give a brief introduction to the Euler class, along the lines of [W, § 2.3.3]. For details and complements we refer to [Gh] or [C, § 2].

As in the introduction, let Σ_g be a closed, connected, oriented surface of genus $g\geq 2$, with fundamental group Γ_g. The Euler class of a representation $\rho\in\mathrm{Hom}(\Gamma_g,\mathrm{PSL}(2,\mathbb{R}))$ measures the obstruction to lifting ρ to the universal cover $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ of $\mathrm{PSL}(2,\mathbb{R})$, and its parity measures the obstruction to lifting ρ to $\mathrm{SL}(2,\mathbb{R})$. To define the Euler class, choose a set-theoretic section s of the covering map $\widetilde{\mathrm{PSL}}(2,\mathbb{R})\to\mathrm{PSL}(2,\mathbb{R})$. Consider a triangulation of Σ_g with a vertex at the basepoint x_0 defining $\Gamma_g=\pi_1(\Sigma_g,x_0)$, and choose an orientation on every edge of the triangulation. Choose a maximal tree in the 1-skeleton of the triangulation, and for every oriented edge σ in this tree, set $\rho(\sigma):=1\in\mathrm{PSL}(2,\mathbb{R})$. Any other oriented edge σ' corresponds (by closing up in the unique possible way along the rooted tree) to an element $\gamma\in\Gamma_g$, and we set $\rho(\sigma'):=\rho(\gamma)\in\mathrm{PSL}(2,\mathbb{R})$. The boundary of any oriented triangle τ of the triangulation can be written as $\sigma_1^{\varepsilon_1}\sigma_2^{\varepsilon_2}\sigma_3^{\varepsilon_3}$ where $\sigma_1,\sigma_2,\sigma_3$ are edges with the chosen orientation, and $\varepsilon_i\in\{\pm 1\}$. We set

$$\mathsf{eu}(\rho)(\tau):=s(\rho(\sigma_1))^{\varepsilon_1}\,s(\rho(\sigma_2))^{\varepsilon_2}\,s(\rho(\sigma_3))^{\varepsilon_3}.$$

Summing over triangles τ, this defines an element of $H^2(\Sigma_g,\pi_1(\mathrm{PSL}(2,\mathbb{R})))$, hence an element of $H^2(\Sigma_g,\mathbb{Z})$ under the identification $\pi_1(\mathrm{PSL}(2,\mathbb{R}))\simeq\mathbb{Z}$. This element $\mathsf{eu}(\rho)\in H^2(\Sigma_g,\mathbb{Z})$ is called the *Euler class* of ρ. Its evaluation on the fundamental class in $H_2(\Sigma_g,\mathbb{Z})$ is an integer, which we still call the Euler class of ρ. It is invariant under conjugation by $\mathrm{PSL}(2,\mathbb{R})$, and changes sign under conjugation by $\mathrm{PGL}(2,\mathbb{R})\smallsetminus\mathrm{PSL}(2,\mathbb{R})$.

We can also define the Euler class for representations of the fundamental group of a compact, connected, oriented surface Σ with boundary, of negative Euler characteristic, provided that the boundary curves are sent to hyperbolic elements. Indeed, any hyperbolic element $g\in\mathrm{PSL}(2,\mathbb{R})$ has a canonical lift to $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$, because it belongs to a unique one-parameter subgroup of $\mathrm{PSL}(2,\mathbb{R})$, which defines a path from the identity to g. Choose a section s of the projection $\widetilde{\mathrm{PSL}}(2,\mathbb{R})\to\mathrm{PSL}(2,\mathbb{R})$ that maps any hyperbolic element to its canonical lift. Then the construction above, using triangulations of Σ containing exactly one vertex on each boundary component, defines an Euler class, independent of all choices.

For instance, let Σ be an oriented pair of pants with fundamental group $\Gamma = \langle \alpha, \beta, \gamma \,|\, \alpha\beta\gamma = 1\rangle$, where α, β, γ correspond to the three boundary curves, endowed with the orientation induced by the surface. For any representation $\rho \in \mathrm{Hom}(\Gamma, \mathrm{PSL}(2,\mathbb{R}))$ with $\rho(\alpha), \rho(\beta), \rho(\gamma)$ hyperbolic,

$$\mathsf{eu}(\rho) = s(\rho(\alpha))\, s(\rho(\beta))\, s(\rho(\gamma)) \in Z(\widetilde{\mathrm{PSL}}(2,\mathbb{R})) \simeq \mathbb{Z}. \tag{2.2}$$

In particular, $\mathsf{eu}(\rho) \in \{-1, 0, 1\}$, and $|\mathsf{eu}(\rho)| = 1$ if and only if ρ is the holonomy of a hyperbolic structure on Σ, after possibly reversing the orientation. If s' is a section of the projection $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R})$ that maps any hyperbolic element to its lift of positive trace, then (2.2) implies

$$s'(\rho(\alpha))\, s'(\rho(\beta))\, s'(\rho(\gamma)) = (-\mathrm{Id})^{\mathsf{eu}(\rho)}. \tag{2.3}$$

By construction, the Euler class is *additive*: if Σ is the union of two subsurfaces Σ' and Σ'' glued along curves γ_i, and if $\rho \in \mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}(2,\mathbb{R}))$ is a representation sending all the curves γ_i (and the boundary curves of Σ, if any) to hyperbolic elements of $\mathrm{PSL}(2,\mathbb{R})$, then $\mathsf{eu}(\rho)$ is the sum of the Euler classes of the restrictions of ρ to the fundamental groups of Σ' and Σ''. This implies that a folded hyperbolic structure defined by a coloring c from the set $\mathcal{P}$ of connected components of $\Sigma \smallsetminus \Upsilon$ to $\{-1, 1\}$ has Euler class $\frac{1}{2\pi}\sum_{P\in\mathcal{P}} c(P)\,\mathcal{A}(P)$ where $\mathcal{A}(P)$ is the area of P.

We shall use the following terminology.

Definition 2.9. A representation $\rho \in \mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}(2,\mathbb{R}))$ is *geometric* if it maps the boundary curves of Σ to hyperbolic elements of $\mathrm{PSL}(2,\mathbb{R})$ and has extremal Euler class or, equivalently, if it is the holonomy of a hyperbolic structure on Σ, after possibly reversing the orientation.

2.4. **Laminations in a pair of pants.** A hyperbolic pair of pants Σ carries only finitely many geodesic laminations, because only 21 geodesics are simple — namely 3 closed geodesics (the boundary components), 6 geodesics spiraling from a boundary component to itself, and 12 geodesics spiraling from a boundary component to another. It admits 32 ideal triangulations, of which 24 contain a geodesic spiraling from a boundary component to itself, and the other 8 do not (see Figure 1). We shall call the laminations corresponding to these 8 triangulations the *triskelion* laminations of Σ. They differ by the spiraling directions of the spikes of the triangles at each boundary component.

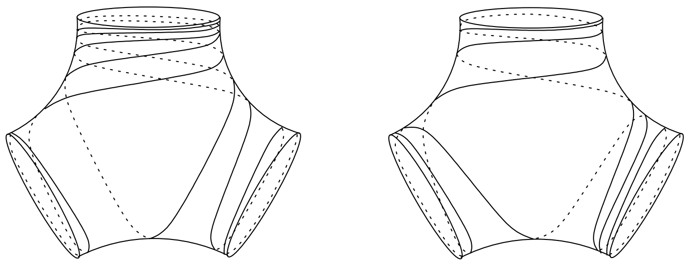

FIGURE 1. A pair of pants carries 24 maximal geodesic laminations containing a geodesic spiraling from a boundary component to itself (left), and 8 triskelion laminations (right).

3. Holonomies of folded hyperbolic structures

Let $\lambda : \mathrm{PSL}(2,\mathbb{R}) \to \mathbb{R}^+$ be the translation length function (1.1). For any representation $\rho \in \mathrm{Hom}(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$, we set

$$\lambda_\rho := \lambda \circ \rho : \ \Gamma_g \longrightarrow \mathbb{R}^+.$$

The function λ_ρ is identically zero if and only if the group $\rho(\Gamma_g)$ is unipotent or bounded. The goal of this section is to prove the following.

Proposition 3.1. *For any $[\rho] \in \mathsf{Rep}_g^{\mathrm{nfd}}$ with $\lambda_\rho \not\equiv 0$, there exist elements $[j_0], [j_0'] \in \mathsf{Rep}_g^{\mathrm{fd}}$ and a decomposition of Σ_g into pairs of pants, each labeled -1, 0, or 1, with the following properties:*

1. *for any representations j_0, ρ in the respective classes $[j_0], [\rho]$, there is a 1-Lipschitz, (j_0, ρ)-equivariant map $f : \mathbb{H}^2 \to \mathbb{H}^2$ that is an orientation-preserving (resp. orientation-reversing) isometry in restriction to any connected subset of $\mathbb{H}^2$ projecting to a union of pants labeled -1 (resp. 1) in $j_0(\Gamma_g)\backslash\mathbb{H}^2 \simeq \Sigma_g$, and that satisfies $\mathrm{Lip}_p(f) < 1$ for any $p \in \mathbb{H}^2$ projecting to the interior of a pair of pants labeled 0;*
2. *for any representations j_0', ρ in the respective classes $[j_0'], [\rho]$, if the group $\rho(\Gamma_g)$ is not virtually abelian, then ρ is a folding of j_0' along a lamination Υ of Σ_g consisting of all the cuffs together with a triskelion lamination inside each pair of pants labeled 0, with the coloring $c : \Sigma_g \smallsetminus \Upsilon \to \{-1, 1\}$ taking the value -1 (resp. 1) on each pair of pants labeled -1 (resp. 1), and both values on each pair of pants labeled 0;*
3. *$[j_0]$ and $[j_0']$ only differ by earthquakes along the cuffs of the pairs of pants of the decomposition.*

Property (1) is used to prove Theorem 1.1 in Section 4, while (2) is a more precise statement of Theorem 1.2. We refer to Section 2.1 for the notation $\mathrm{Lip}_p(f)$ and to Section 2.4 for triskelion laminations. By additivity (see Section 2.3), the Euler class of ρ is the sum of the labels of the pairs of pants.

Proposition 3.1 is proved by choosing an appropriate pants decomposition (Section 3.1) and understanding the representations of the fundamental group of a pair of pants (Section 3.2). These ingredients are brought together in Section 3.3. In Section 3.4 we present a variation on Proposition 3.1.(1), which is later used to prove the second statement of Theorem 1.1.

3.1. Pants decompositions.

Our first ingredient is the following.

Lemma 3.2. *For any $[\rho] \in \mathsf{Rep}_g^{\mathrm{nfd}}$ with $\lambda_\rho \not\equiv 0$, there is a pants decomposition of Σ_g such that ρ maps any cuff to a hyperbolic element. If $\rho(\Gamma_g)$ is not virtually abelian, then we may assume that the restriction of ρ to the fundamental group of any pair of pants of the decomposition is nonabelian.*

Recall that $[\rho] \in \mathsf{Rep}_g^{\mathrm{nfd}}$ is said to be *elementary* if the group $\rho(\Gamma_g)$ admits a finite orbit in $\mathbb{H}^2$ or in $\partial_\infty\mathbb{H}^2$. In the case that $[\rho]$ is *not* elementary, Lemma 3.2 is contained in the following result of Gallo–Kapovich–Marden.

Lemma 3.3 [GKM, part A]**.** *For any nonelementary $[\rho] \in \mathsf{Rep}_g^{\mathrm{nfd}}$, there is a pants decomposition of Σ_g such that the fundamental group of any pair of pants maps injectively to a 2-generator Schottky group under ρ.*

We now treat the case that ρ is elementary.

Proof of Lemma 3.2 when ρ is elementary. By induction, Lemma 3.2 is a consequence of the following two claims.

Claim 3.4. *Let Σ be a connected compact surface of genus $g \geq 1$ with $k \geq 0$ boundary components, such that $\chi(\Sigma) = 2 - 2g - k < 0$, and let $\rho : \pi_1(\Sigma) \to \mathrm{PSL}(2, \mathbb{R})$ be an elementary representation with $\lambda_\rho \not\equiv 0$, sending each boundary curve of Σ (if any) to a hyperbolic element. Then we can cut Σ open along some nonseparating simple closed curve whose image by ρ is a hyperbolic element, yielding a new surface Σ' of genus $g-1$ and an induced representation $\rho' : \pi_1(\Sigma') \to \mathrm{PSL}(2, \mathbb{R})$ sending all $k+2$ boundary curves of Σ' to hyperbolic elements. If the image of ρ is not virtually abelian, then the image of ρ' is not virtually abelian.*

Claim 3.5. *Let Σ be a connected compact surface of genus $g = 0$ with $k \geq 4$ boundary components, and let $\rho : \pi_1(\Sigma) \to \mathrm{PSL}(2, \mathbb{R})$ be an elementary representation sending each boundary curve of Σ to a hyperbolic element. Then we can cut Σ along some simple closed curve of Σ, not freely homotopic to a boundary component, whose image by ρ is a hyperbolic element, yielding two new surfaces Σ_1 and Σ_2 with lower complexity and two induced representations $\rho_i : \pi_1(\Sigma_i) \to \mathrm{PSL}(2, \mathbb{R})$ sending each boundary curve to a hyperbolic element. If the image of ρ is nonabelian, then we can do this in such a way that the images of the ρ_i are nonabelian.*

Proof of Claim 3.4. We first observe that $\pi_1(\Sigma)$ is generated by elements representing nonseparating simple closed curves on Σ. Indeed, consider a standard presentation

$$\pi_1(\Sigma) = \big\langle a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_k \,|\, [a_1, b_1] \cdots [a_g, b_g]\, c_1 \cdots c_k = 1 \big\rangle \tag{3.1}$$

of $\pi_1(\Sigma)$ by generators and relations, where a_i, b_i represent nonseparating simple closed curves and c_i a curve freely homotopic to a boundary component. Either $a_1 c_i$ represents a nonseparating simple closed curve for all i, or $a_1^{-1} c_i$ represents a nonseparating simple closed curve for all i. Thus we may take the generating set $\{a_1, b_1, \dots, a_g, b_g, a_1^{\varepsilon} c_1, \dots, a_1^{\varepsilon} c_k\}$ for some $\varepsilon \in \{-1, 1\}$.

Let us show that ρ sends some nonseparating simple closed curve of Σ to a hyperbolic element. Since $\lambda_\rho \not\equiv 0$, two mutually exclusive situations are possible:

- (T) the group $\rho(\pi_1(\Sigma))$ has a fixed point ξ in $\partial_\infty \mathbb{H}^2$; it is then conjugate to a group of triangular (possibly diagonal) matrices in $\mathrm{PSL}(2, \mathbb{R})$;
- (VA) the group $\rho(\pi_1(\Sigma))$ preserves a geodesic line ℓ of $\mathbb{H}^2$, and contains both translations along ℓ and order-two symmetries of ℓ reversing its orientation; it is then virtually abelian but not abelian.

Consider a system F of generators of $\pi_1(\Sigma)$ representing nonseparating simple closed curves. In case (T), some element of F is necessarily sent by ρ to a hyperbolic element: otherwise the group $\rho(\pi_1(\Sigma))$ would contain only parabolic elements and the identity, which would contradict the fact that $\lambda_\rho \not\equiv 0$. Suppose we are in case (VA) and ρ does not send any element of F to a hyperbolic element; it then sends some element $\gamma \in F$ to an order-two symmetry of ℓ (because it is not the constant homomorphism). We may complete γ into a new standard presentation of the form (3.1) with $\gamma = a_1$. Consider the generating set

$$F' = \{b_1,\, a_1 b_1,\, a_2^{-1} b_1,\, b_2 b_1,\, \dots,\, a_g^{-1} b_1,\, b_g b_1,\, c_1^{\varepsilon} b_1,\, \dots,\, c_k^{\varepsilon} b_1\},$$

where $\varepsilon \in \{-1, 1\}$. If ε is suitably chosen, then every $\gamma' \in F'$ represents a nonseparating simple closed curve, and γ' and $\gamma = a_1$ are standard generators of a one-holed torus embedded in Σ; it follows that $\gamma\gamma'$ is a nonseparating simple closed curve as well. Necessarily, there exists $\gamma' \in F'$ such that $\rho(\gamma')$ does not commute with $\rho(\gamma)$: otherwise the group $\rho(\pi_1(\Sigma))$ would be contained in the centralizer of $\rho(\gamma)$, which is compact, and this would contradict the fact that $\lambda_\rho \not\equiv 0$. Either this $\rho(\gamma')$ is hyperbolic, or it is an order-two symmetry whose center is different from that of $\rho(\gamma)$, in which case $\rho(\gamma\gamma')$ is hyperbolic. In either case we have found a nonseparating simple closed curve mapped by ρ to a hyperbolic element.

Let Σ' be obtained by cutting Σ open along such a simple closed curve. If the image of the induced representation $\rho' : \pi_1(\Sigma') \to \mathrm{PSL}(2, \mathbb{R})$ is virtually abelian, then so is the image of ρ. Indeed, $\pi_1(\Sigma)$ is generated by $\pi_1(\Sigma')$ together with an element γ' that conjugates two elements of $\pi_1(\Sigma')$ with hyperbolic images under ρ'. If the image of ρ' is virtually abelian, preserving some geodesic line ℓ of $\mathbb{H}^2$, then $\rho(\gamma')$ has to preserve ℓ, and so does the whole image of ρ. Thus the image of ρ is virtually abelian. $\square$

Proof of Claim 3.5. Since the boundary curves of Σ generate $\pi_1(\Sigma)$ and since they all have hyperbolic image under the elementary representation ρ, the group $\rho(\pi_1(\Sigma))$ has a fixed point ξ in $\partial_\infty \mathbb{H}^2$ (case (T) above). Choose a geodesic line ℓ of $\mathbb{H}^2$ with endpoint ξ. For any $\gamma \in \Gamma$ we may write in a unique way $\rho(\gamma) = a_\gamma u_\gamma$ where a_γ belongs to the stabilizer A of ξ and ℓ in $\mathrm{PSL}(2, \mathbb{R})$ and $u_\gamma \in \mathrm{PSL}(2, \mathbb{R})$ is unipotent or trivial. The map $\gamma \mapsto a_\gamma$ can be seen as a nonzero element ω of $H^1(\Sigma_g, \mathbb{R})$ after identifying A with $(\mathbb{R}, +)$. Consider a standard presentation

$$\pi_1(\Sigma) = \langle c_1, \dots, c_k \,|\, c_1 \cdots c_k = 1 \rangle$$

of $\pi_1(\Sigma)$ by generators and relations, where $c_1, \dots, c_k$ represent curves freely homotopic to the boundary components of Σ, and $c_i c_j$ represents a simple curve for any $i < j$. We claim that ρ sends one of the $c_i c_j$ to a hyperbolic element. Indeed, otherwise we would have $\omega(c_i) + \omega(c_j) = 0$ for all $i \neq j$; solving this linear system gives $\omega(c_i) = 0$ for all i, which would contradict the assumption that $\rho(c_i)$ is hyperbolic.

For $1 \leq i \leq k$, let $\xi_i \in \partial_\infty \mathbb{H}^2$ be the fixed point of $\rho(c_i)$ that is different from ξ. If the image of ρ is not abelian, then there exists i such that $\xi_i \neq \xi_{i+1}$ (with the convention that $\xi_{k+1} = \xi_1$). Precomposing ρ by a Dehn twist along a curve freely homotopic to $c_i c_{i+1}$ corresponds to conjugating $\rho(c_i)$ and $\rho(c_{i+1})$ by $\rho(c_i c_{i+1})$ while leaving all the other $\rho(c_j)$ unchanged. Applying a large enough power of this Dehn twist, with the appropriate sign if $\rho(c_i c_{i+1})$ is hyperbolic, pushes ξ_i and ξ_{i+1} to two distinct points arbitrarily close to ξ; in particular, we can make ξ_i and ξ_{i+1} distinct from the other points ξ_j. We then proceed similarly with the new point ξ_{i+1} and ξ_{i+2}, and so on, until all the points ξ_i are pairwise distinct. We then conclude as above: one of the

$c_i c_j$ (with $i \neq j$) has hyperbolic image under ρ. It cuts Σ into two smaller surfaces on which ρ induces nonabelian representations. $\square$

To prove Lemma 3.2, just make repeated use of Claim 3.4 to reduce to a surface of genus 0, then of Claim 3.5 to decompose it into pairs of pants. $\square$

3.2. **Representations of the fundamental group of a pair of pants.** The following lemma gives a dictionary between the geometric and nongeometric representations (Definition 2.9) of the fundamental group of a pair of pants.

Lemma 3.6. *Let $\Gamma = \langle \alpha, \beta, \gamma \,|\, \alpha\beta\gamma = 1\rangle$ be the fundamental group of a pair of pants Σ, with α, β, γ corresponding to the boundary loops.*

- *For any $a, b, c > 0$ such that none is the sum of the other two, there are exactly two representations $\tau \in \mathrm{Hom}(\Gamma, \mathrm{PSL}(2, \mathbb{R}))$ satisfying*
$$(\lambda_\tau(\alpha), \lambda_\tau(\beta), \lambda_\tau(\gamma)) = (a, b, c), \tag{3.2}$$
up to conjugation under $\mathrm{PGL}(2, \mathbb{R})$. One of them is geometric (with $|\mathsf{eu}(\tau)| = 1$). The other is nongeometric (with $\mathsf{eu}(\tau) = 0$), and is obtained from the geometric one by folding along any of the eight triskelion laminations of Σ.
- *For any $a, b, c > 0$ such that one is the sum of the other two, there are exactly four representations $\tau \in \mathrm{Hom}(\Gamma, \mathrm{PSL}(2, \mathbb{R}))$ satisfying (3.2), up to conjugation under $\mathrm{PGL}(2, \mathbb{R})$. One of them is geometric (with $|\mathsf{eu}(\tau)| = 1$). The other three are elementary (with $\mathsf{eu}(\tau) = 0$): two have an image that is not virtually abelian, the third one is their abelianization. Each of the two nonabelian elementary representations is obtained from the geometric one by folding along any of four different triskelion laminations of Σ.*

When one of a, b, c is the sum of the other two, the images of the two nonabelian elementary representations $\tau \in \mathrm{Hom}(\Gamma, \mathrm{PSL}(2, \mathbb{R}))$ are conjugate to triangular matrices; their abelianization is by definition their projection to the group of diagonal matrices.

Proof. Fix $a, b, c > 0$. We first determine the number of conjugacy classes of representations τ satisfying (3.2). Set $(A, B, C) := (e^{a/2}, e^{b/2}, e^{c/2})$, and let $\tau \in \mathrm{Hom}(\Gamma, \mathrm{PSL}(2, \mathbb{R}))$ satisfy (3.2). Up to conjugating τ by $\mathrm{PGL}(2, \mathbb{R})$, we can find lifts $\overline{\tau}(\alpha) \in \mathrm{SL}(2, \mathbb{R})$ of $\tau(\alpha)$ and $\overline{\tau}(\beta) \in \mathrm{SL}(2, \mathbb{R})$ of $\tau(\beta)$ of the form
$$\overline{\tau}(\alpha) = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \quad \text{and} \quad \overline{\tau}(\beta) = \begin{pmatrix} B + x & y \\ z & B^{-1} - x \end{pmatrix}$$
with $x, y, z \in \mathbb{R}$. Since α and β freely generate Γ, this determines a lift $\overline{\tau} \in \mathrm{Hom}(\Gamma, \mathrm{SL}(2, \mathbb{R}))$ of τ. The sign $\varepsilon \in \{\pm 1\}$ of $\mathrm{Tr}(\overline{\tau}(\alpha))\, \mathrm{Tr}(\overline{\tau}(\beta))\, \mathrm{Tr}(\overline{\tau}(\gamma))$ does not depend on the choice of $\overline{\tau}(\alpha)$ and $\overline{\tau}(\beta)$. By (2.2), we have $\mathsf{eu}(\tau) \in \{-1, 0, 1\}$, with $|\mathsf{eu}(\tau)| = 1$ if and only if τ is geometric, and by (2.3)
$$\varepsilon = (-1)^{\mathsf{eu}(\tau)}.$$
The trace of $\overline{\tau}(\gamma) = \overline{\tau}(\alpha\beta)^{-1}$ is
$$A(B + x) + A^{-1}(B^{-1} - x) = \varepsilon(C + C^{-1}),$$

hence
$$x = \frac{\varepsilon(C + C^{-1}) - AB - (AB)^{-1}}{A - A^{-1}}$$
is uniquely determined by A, B, C and ε. Let $\nu := (B + x)(B^{-1} - x)$. Since $\overline{\tau}(\beta) \in \mathrm{SL}(2, \mathbb{R})$, we have $yz = \nu - 1$. If $\nu \neq 1$, then any pair (y, z) of reals with product $\nu - 1$ can be obtained by conjugating $\overline{\tau}(\alpha)$ and $\overline{\tau}(\beta)$ by a diagonal matrix in $\mathrm{PGL}(2, \mathbb{R})$ (which does not change x). Thus τ is unique up to conjugation once we fix $\varepsilon \in \{-1, 1\}$. If $\nu = 1$, then $\overline{\tau}(\beta)$ is either upper or lower triangular, or both, hence three conjugacy classes for τ, with $\tau(\Gamma)$ consisting respectively of upper triangular, lower triangular, and diagonal matrices. The condition $\nu = 1$ amounts to $(B^{-1} - B - x)x = 0$, or equivalently to
$$\left(\frac{BC}{A} - \varepsilon\right)\left(\frac{AC}{B} - \varepsilon\right) \cdot \left(\frac{AB}{C} - \varepsilon\right)(ABC - \varepsilon) = 0 :$$
in other words, $\varepsilon = 1$ and one of a, b, c is the sum of the other two.

Let $j \in \mathrm{Hom}(\Gamma, \mathrm{PSL}(2, \mathbb{R}))$ be geometric (Definition 2.9). For any folding ρ of j along a triskelion lamination Υ of Σ, the functions λ_j and λ_ρ agree on $\{\alpha, \beta, \gamma\}$ (Remark 2.8), and ρ is not conjugate to j under $\mathrm{PGL}(2, \mathbb{R})$ because the folding map f is not an isometry (see Section 2.1). Therefore, $\mathsf{eu}(\rho) = 0$ by the above discussion.

If none of a, b, c is the sum of the other two, then ρ belongs to the unique conjugacy class of representations τ satisfying (3.2) and $\mathsf{eu}(\tau) = 0$.

If one of a, b, c is the sum of the other two, then ρ belongs to one of the two conjugacy classes of representations τ whose image is not virtually abelian and that satisfy (3.2) and $\varepsilon = 1$ (Observation 2.7). The representation ρ' obtained from j by folding along the image of Υ under the natural involution of the pair of pants belongs to the other conjugacy class of such representations. The abelianization of ρ or ρ' is not conjugate to j, hence satisfies (3.2) and $\varepsilon = 1$ as well. ☐

Corollary 3.7. *Let $\Gamma = \langle \alpha, \beta, \gamma \,|\, \alpha\beta\gamma = 1 \rangle$ be the fundamental group of a pair of pants Σ, with α, β, γ corresponding to the boundary loops. Consider two representations $j, \rho \in \mathrm{Hom}(\Gamma, \mathrm{PSL}(2, \mathbb{R}))$ with j geometric (Definition 2.9), with ρ nongeometric, and with*
$$(\lambda_j(\alpha), \lambda_j(\beta), \lambda_j(\gamma)) = (\lambda_\rho(\alpha), \lambda_\rho(\beta), \lambda_\rho(\gamma)).$$
Then there exists a 1-Lipschitz, (j, ρ)-equivariant map $f : \mathbb{H}^2 \to \mathbb{H}^2$ such that $\mathrm{Lip}_p(f) < 1$ for any $p \in \mathbb{H}^2$ projecting to a point of $j(\Gamma)\backslash\mathbb{H}^2$ off the boundary of the convex core.

Note that in this setting any (j, ρ)-equivariant map $f : \mathbb{H}^2 \to \mathbb{H}^2$ satisfies $\mathrm{Lip}(f) \geq 1$ (Remark 2.6), and if $\mathrm{Lip}(f) = 1$ then f is an isometry in restriction to the translation axes of $j(\alpha), j(\beta), j(\gamma)$ in $\mathbb{H}^2$. The convex core of $j(\Gamma)\backslash\mathbb{H}^2$ naturally identifies with Σ.

Proof. We first assume that the group $\rho(\Gamma)$ is nonabelian. By Lemma 3.6, the representation ρ is obtained from j by folding along any of at least four of the eight triskelion laminations of Σ. Let ℓ be an injectively immersed geodesic that spirals between two boundary components.

If the two boundary components are different, then ℓ is contained in only two triskelion laminations, and intersects the others transversely.

If the two boundary components are the same, then ℓ intersects transversely all triskelion laminations of Σ.

In both cases we see that a lift of ℓ to $\mathbb{H}^2$ cannot be isometrically preserved by all 1-Lipschitz, (j,ρ)-equivariant maps $f:\mathbb{H}^2\to\mathbb{H}^2$ (such maps exist since ρ is a folding of j). This holds for any ℓ, hence shows that the lamination $\widetilde{\Lambda}\subset\mathbb{H}^2$ of Lemma 2.2 is contained in (in fact, equal to) the preimage of the boundary of the convex core of $j(\Gamma)\backslash\mathbb{H}^2$, which identifies with the boundary of Σ. By Lemma 2.2, this means that there exists a 1-Lipschitz, (j,ρ)-equivariant map $f:\mathbb{H}^2\to\mathbb{H}^2$ such that $\mathrm{Lip}_p(f)<1$ for any $p\in\mathbb{H}^2$ projecting to a point of $j(\Gamma)\backslash\mathbb{H}^2$ off the boundary of the convex core.

We now assume that $\rho(\Gamma)$ is abelian. By Lemma 3.6, the representation ρ is the abelianization of some representation ρ' that is a folding of j. The group $\rho'(\Gamma)$ fixes a point $\xi\in\partial_\infty\mathbb{H}^2$, and $\rho(\Gamma)$ preserves a geodesic line ℓ of $\mathbb{H}^2$ with endpoint ξ. By postcomposing any 1-Lipschitz, (j,ρ')-equivariant map with the projection onto ℓ along the horospheres centered at ξ, we obtain a 1-Lipschitz, (j,ρ)-equivariant map. Moreover, since 1 is the optimal Lipschitz constant (Remark 2.6), this shows that the stretch locus (Definition 2.3) of (j,ρ) is contained in that of (j,ρ'), and we conclude as above. $\square$

Remark 3.8. The nonabelian, nongeometric representations in Lemma 3.6 can also be obtained by folding along a nonmaximal geodesic lamination consisting of a unique leaf spiraling from a boundary component to itself. Folding along a maximal lamination which is not a triskelion gives a representation with values in $\mathrm{PGL}(2,\mathbb{R})$ and not $\mathrm{PSL}(2,\mathbb{R})$.

3.3. **Proof of Proposition 3.1.** By Lemma 3.2, there is a pants decomposition Π of Σ_g such that ρ maps any cuff to a hyperbolic element, and such that if $\rho(\Gamma_g)$ is not virtually abelian then the restriction of ρ to the fundamental group of each pair of pants is nonabelian. Let $j\in\mathrm{Hom}(\Gamma_g,\mathrm{PSL}(2,\mathbb{R}))$ be a Fuchsian representation such that $\lambda_j(\gamma)=\lambda_\rho(\gamma)$ for all $\gamma\in\Gamma_g$ corresponding to cuffs of pants of Π. The twist parameters along the cuffs will be adjusted later, so for the moment we choose them arbitrarily.

Let $\mathcal{C}$ be the $j(\Gamma_g)$-invariant (disjoint) union of all geodesics of $\mathbb{H}^2$ projecting to the cuffs in $j(\Gamma_g)\backslash\mathbb{H}^2\simeq\Sigma_g$. For each pair of pants P in Π, choose a subgroup Γ^P of Γ_g which is conjugate to $\pi_1(P)$. Then $j|_{\Gamma^P}$ is the holonomy of a hyperbolic metric on P with cuff lengths given by λ_ρ. Choose a lift $\widetilde{P}\subset\mathbb{H}^2$ of the convex core of $j(\Gamma^P)\backslash\mathbb{H}^2$. This lift is the closure of a connected component of $\mathbb{H}^2\smallsetminus\mathcal{C}$. If the restrictions of j and ρ to Γ^P are conjugate by some isometry f^P of $\mathbb{H}^2$, then we give P the label -1 or 1, depending on whether f^P preserves the orientation or not. If the restrictions of j and ρ to Γ^P are not conjugate, then we give P the label 0. In this case,

- by Corollary 3.7, there is a 1-Lipschitz, $(j|_{\Gamma^P},\rho|_{\Gamma^P})$-equivariant map $f^P:\widetilde{P}\to\mathbb{H}^2$ with $\mathrm{Lip}_p(f^P)<1$ for all $p\notin\partial\widetilde{P}$;
- by Lemma 3.6, if $\rho(\Gamma_g)$ is not virtually abelian then $\rho|_{\Gamma^P}$ is a folding of $j|_{\Gamma^P}$ along some triskelion lamination of P; we denote by $F^P:\widetilde{P}\to\mathbb{H}^2$ the folding map.

Note that in restriction to any connected component of $\partial\widetilde{P}$ (a line), the maps f^P and F^P are both isometries, but they may disagree by a constant shift.

The collection of all maps f^P, extended (j,ρ)-equivariantly, piece together to yield a map $f^* : \mathbb{H}^2 \smallsetminus \mathcal{C} \to \mathbb{H}^2$.

The obstruction to extending f^* by continuity on each geodesic $\ell \subset \mathcal{C}$ is that the maps on either side of ℓ may disagree by a constant shift along ℓ. This discrepancy $\delta(\ell) \in \mathbb{R}$ is the same on the whole $j(\Gamma_g)$-orbit of ℓ. To correct it, we postcompose j with an earthquake supported on the cuff associated with ℓ, of length $-\delta(\ell)$.

We repeat for each $j(\Gamma_g)$-orbit in $\mathcal{C}$, and eventually obtain a new Fuchsian representation j_0. By construction, there is a 1-Lipschitz, (j_0,ρ)-equivariant map $f : \mathbb{H}^2 \to \mathbb{H}^2$, obtained simply by gluing together isometric translates of the f^P. This extension f satisfies Proposition 3.1.(1).

If $\rho(\Gamma_g)$ is not virtually abelian, then similarly the maps f^P for P labeled ± 1 and F^P for P labeled 0 piece together to yield a map $F^* : \mathbb{H}^2 \smallsetminus \mathcal{C} \to \mathbb{H}^2$. As above, we can modify j by earthquakes into a new Fuchsian representation j'_0, and F^* by piecewise isometries into a (j'_0,ρ)-equivariant, continuous map $F : \mathbb{H}^2 \to \mathbb{H}^2$ which is a folding map. This proves Proposition 3.1.(2).

Proposition 3.1.(3) is satisfied by construction.

3.4. **Uniform Lipschitz bounds.** In order to prove the second statement of Theorem 1.1 in Section 4.4, we shall use the following result, which gives Lipschitz bounds analogous to Proposition 3.1.(1) but uniform.

Proposition 3.9. *For any decomposition Π of Σ_g into pairs of pants labeled $-1,0,1$ and any continuous family $(j_t)_{t\geq 0} \subset \mathrm{Hom}(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$ of Fuchsian representations, there exist a family $(\rho_t)_{t\geq 0} \subset \mathrm{Hom}(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$ of non-Fuchsian representations and, for any t in a small interval $[0,t_0]$, a 1-Lipschitz, (j_t,ρ_t)-equivariant map $\varphi_t : \mathbb{H}^2 \to \mathbb{H}^2$, with the following properties:*

- *φ_t is an orientation-preserving (resp. orientation-reversing) isometry in restriction to any connected subset of $\mathbb{H}^2$ projecting to a union of pants labeled -1 (resp. 1) in $j_t(\Gamma_g)\backslash\mathbb{H}^2 \simeq \Sigma_g$;*
- *for any $\eta > 0$ there exists $C < 1$ such that $\mathrm{Lip}_p(\varphi_t) \leq C$ for all $t \in [0,t_0]$ and all $p \in \mathbb{H}^2$ whose image in $j_t(\Gamma_g)\backslash\mathbb{H}^2 \simeq \Sigma_g$ lies inside a pair of pants P labeled 0, at distance $\geq \eta$ from the boundary of P.*

Proposition 3.9 is based on the following uniform version of Corollary 3.7.

Lemma 3.10. *Let $\Gamma = \langle \alpha,\beta,\gamma \,|\, \alpha\beta\gamma = 1\rangle$ be the fundamental group of a pair of pants Σ, with α,β,γ corresponding to the boundary loops. Consider two continuous families $(j_t)_{t\geq 0}, (\rho_t)_{t\geq 0} \subset \mathrm{Hom}(\Gamma, \mathrm{PSL}(2,\mathbb{R}))$ of representations with j_t geometric (Definition 2.9), ρ_t nongeometric, and*

$$\big(\lambda_{j_t}(\alpha), \lambda_{j_t}(\beta), \lambda_{j_t}(\gamma)\big) = \big(\lambda_{\rho_t}(\alpha), \lambda_{\rho_t}(\beta), \lambda_{\rho_t}(\gamma)\big)$$

for all $t \geq 0$. Then there exists a family of 1-Lipschitz, (j_t,ρ_t)-equivariant maps $\varphi_t : \mathbb{H}^2 \to \mathbb{H}^2$, defined for all t in a small interval $[0,t_0]$, with the following property: for any $\eta > 0$ there exists $C < 1$ such that $\mathrm{Lip}_p(\varphi_t) \leq C$ for any $t \in [0,t_0]$ and any $p \in \mathbb{H}^2$ whose image in $j_t(\Gamma)\backslash\mathbb{H}^2$ lies at distance $\geq \eta$ from the boundary of the convex core.

Proof of Lemma 3.10. By Corollary 3.7, there exists a 1-Lipschitz, (j_0, ρ_0)-equivariant map $f_0 : \mathbb{H}^2 \to \mathbb{H}^2$ such that $\mathrm{Lip}_p(f_0) < 1$ for any $p \in \mathbb{H}^2$ whose image in $j_0(\Gamma)\backslash\mathbb{H}^2$ does not belong to the boundary of the convex core. If $(j_t, \rho_t) = (j_0, \rho_0)$ for all t, then we may take $\varphi_t = f_0$. In the general case, we shall build φ_t as a small deformation of f_0 in restriction to the preimage of the convex core of $j_t(\Gamma)\backslash\mathbb{H}^2$.

Choose $\Delta > 0$ so that for all small $t \geq 0$, the 2Δ-neighborhoods of the boundary components of the convex core of the hyperbolic surface $j_t(\Gamma)\backslash\mathbb{H}^2$ are disjoint. Choose a small $\delta \in (0, \Delta/2)$ and let $\sigma_\delta : \mathbb{R}^+ \to \mathbb{R}^+$ be the function that satisfies

$$\sigma_\delta(\eta) = \begin{cases} 0 & \text{for} \quad 0 \leq \eta \leq 2\delta, \\ \Delta - 2\delta & \text{for} \quad \eta = \Delta, \\ \eta & \text{for} \quad \eta \geq 2\Delta \end{cases}$$

and is affine on $[2\delta, \Delta]$ and $[\Delta, 2\Delta]$ (Figure 2). Note that σ_δ is $(1 + o(1))$-

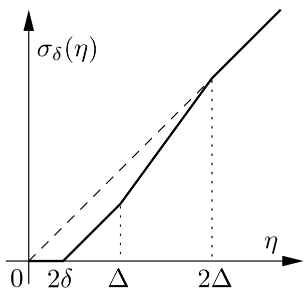

FIGURE 2. The function σ_δ

Lipschitz as $\delta \to 0$, and 1-Lipschitz away from $[\Delta, 2\Delta]$. For any $t \geq 0$, let $N_t \subset \mathbb{H}^2$ be the preimage of the convex core of $j_t(\Gamma)\backslash\mathbb{H}^2$, and let $\pi_t : \mathbb{H}^2 \to N_t$ be the closest-point projection, which is 1-Lipschitz. We set

$$\varphi_0 := f_0 \circ J_\delta \circ \pi_0,$$

where J_δ is the homotopy of $\mathbb{H}^2$ taking any point at distance $\eta \leq 2\Delta$ from a boundary component ℓ_0 of N_0, to the point at distance $\sigma_\delta(\eta)$ from ℓ_0 on the same perpendicular ray to ℓ_0, leaving other points unchanged. By construction, in restriction to the 2δ-neighborhood of ∂N_0, the map φ_0 factors through the closest-point projection onto ∂N_0. The function $p \mapsto \mathrm{Lip}_p(f_0)$ is $j_0(\Gamma)$-invariant, upper semicontinuous, and < 1 on $\mathbb{H}^2 \smallsetminus \partial N_0$, hence bounded away from 1 when $p \in N_0$ stays at distance $\geq \Delta - 2\delta$ from ∂N_0. This implies that if we have chosen δ small enough (which we shall assume from now on), then $\mathrm{Lip}(\varphi_0) = 1$ and $\mathrm{Lip}_p(\varphi_0) < 1$ for all p in the interior of N_0. For $t > 0$, we construct φ_t as a deformation of φ_0 via a partition of unity, as follows.

Let $\mathcal{U}_t^\delta \subset N_t$ be the δ-neighborhood of ∂N_t and $N_t^\delta := N_t \smallsetminus \mathcal{U}_t^\delta$ its complement in N_t; we define $\mathcal{U}_t^{2\delta}$ similarly. Choose a 1-Lipschitz, (j_t, ρ_t)-equivariant map $\varphi_t^0 : \mathcal{U}_t^{2\delta} \to \mathbb{H}^2$ factoring through the closest-point projection onto ∂N_t and taking any boundary component ℓ_t of N_t, stabilized by a cyclic subgroup $j_t(S)$ of $j_t(\Gamma)$, isometrically to the translation axis of $\rho_t(S)$ in $\mathbb{H}^2$. Up to postcomposing each φ_t^0 with an appropriate shift along the axis of $\rho_t(S)$, we may assume that $\varphi_t^0(p) \to \varphi_0(p)$ for any $p \in \mathcal{U}_0^{2\delta}$ as $t \to 0$ (recall that the restriction of φ_0 to any boundary component of N_0 is an isometry).

Let $B^1, \dots, B^n \subset N_0$ be balls of $\mathbb{H}^2$, each projecting injectively to $j_0(\Gamma)\backslash\mathbb{H}^2$, disjoint from a neighborhood of ∂N_0, and such that

$$N_0^\delta \subset j_0(\Gamma) \cdot \bigcup_{i=1}^n B^i.$$

For $1 \leq i \leq n$, let $\varphi_t^i : j_t(\Gamma) \cdot B^i \to \mathbb{H}^2$ be the (j_t, ρ_t)-equivariant map that agrees with φ_0 on B^i. By construction, for all $1 \leq i \leq n$ (resp. for $i = 0$) and for all $p \in j_0(\Gamma) \cdot B^i$ (resp. $p \in \mathcal{U}_0^{2\delta}$) we have $\varphi_t^i(p) \to \varphi_0(p)$ as $t \to 0$, uniformly for p in any compact set. However, the maps φ_t^i, for $0 \leq i \leq n$, may not agree at points where their domains overlap. The goal is to paste them together by the procedure described in Section 2.1, using a $j_t(\Gamma)$-invariant partition of unity $(\psi_t^i)_{0 \leq i \leq n}$ that we now construct.

Let $\psi_t^0 : \mathbb{H}^2 \to [0,1]$ be the function supported on $\mathcal{U}_t^{2\delta}$ that takes any point at distance η from ∂N_t to $\tau(\eta) \in [0,1]$, where $\tau([0,\delta]) = 1$, where $\tau([2\delta, +\infty)) = 0$, and where τ is affine on $[\delta, 2\delta]$. Let $\psi^1, \dots, \psi^n : \mathbb{H}^2 \to [0,1]$ be $j_0(\Gamma)$-invariant Lipschitz functions inducing a partition of unity on a neighborhood of N_0^δ, with ψ^i supported in $j_0(\Gamma) \cdot B^i$. Since N_t has a compact fundamental domain for $j_t(\Gamma)$ that varies continuously in t (for instance a right-angled octagon), for small enough t we have

$$N_t^\delta \subset j_t(\Gamma) \cdot \bigcup_{i=1}^n B^i.$$

For $1 \leq i \leq n$ and $t \geq 0$, let $\hat{\psi}_t^i : \mathbb{H}^2 \to [0,1]$ be the $j_t(\Gamma)$-invariant function supported on $j_t(\Gamma) \cdot B^i$ that agrees with ψ^i on B^i. Then $\sum_{i=1}^n \hat{\psi}_t^i = 1 + o(1)$ as $t \to 0$, with an error term uniform on N_t^δ. Therefore the functions

$$\psi_t^0 \qquad \text{and} \qquad \psi_t^i := (1 - \psi_t^0)\, \frac{\hat{\psi}_t^i}{\sum_{k=1}^n \hat{\psi}_t^k} \;:\; \mathbb{H}^2 \longrightarrow [0,1]$$

for $1 \leq i \leq n$ form a $j_t(\Gamma)$-invariant partition of unity of N_t, subordinated to the covering $\mathcal{U}_t^{2\delta} \cup j_t(\Gamma) \cdot B^1 \cup \dots \cup j_t(\Gamma) \cdot B^n \supset N_t$, and are all L-Lipschitz for some $L > 0$ independent of i and t.

For $t \geq 0$, let $\varphi_t := \sum_{i=0}^n \psi_t^i \varphi_t^i : N_t \to \mathbb{H}^2$ be the averaged map defined in Section 2.1. This map is (j_t, ρ_t)-equivariant by construction. We extend it to a map $\varphi_t : \mathbb{H}^2 \to \mathbb{H}^2$ by precomposing with the closest-point projection $\pi_t : \mathbb{H}^2 \to N_t$. We claim that the maps φ_t satisfy the conclusion of Lemma 3.10. Indeed, by Lemma 2.4, for any $t \geq 0$ and p in the interior of N_t,

$$\mathrm{Lip}_p(\varphi_t) \leq \sum_{i \in I_t(p)} \big(\mathrm{Lip}_p(\psi_t^i)\, R_t(p) + \psi_t^i(p)\, \mathrm{Lip}_p(\varphi_t^i)\big), \tag{3.3}$$

where $I_t(p)$ is the set of indices $0 \leq i \leq n$ such that p belongs to the support of ψ_t^i, and $R_t(p) \geq 0$ is the diameter of the set $\{\varphi_t^i(p) \,|\, i \in I_t(p)\}$. Let $\eta > 0$ be the distance from p to ∂N_t.

If $\eta < \delta$, then φ_t coincides on a neighborhood of p with φ_t^0, hence with the closest-point projection onto ∂N_t postcomposed with an isometry of $\mathbb{H}^2$, and the right-hand side of (3.3) reduces to

$$\mathrm{Lip}_p(\varphi_t^0) = \frac{1}{\cosh \eta} < 1$$

(see [GK, (A.9)] for instance).

If $\eta \geq \delta$, then the bound on $\mathrm{Lip}_p(\varphi_t^0)$ still holds, and $\mathrm{Lip}_p(\varphi_t^i)$ for $1 \leq i \leq n$ can also be uniformly bounded away from 1. Indeed, $\sup_{q \in B^i} \mathrm{Lip}_q(\varphi_t^i) < 1$ since B^i is disjoint from a neighborhood of ∂N_0 and the local Lipschitz constant is upper semicontinuous, and we argue by equivariance. Moreover, all the other contributions to (3.3) are small: $R_t(p) \to 0$ as $t \to 0$, uniformly in p, and $\mathrm{Lip}_p(\psi_t^i)$ is bounded independently of p, i, t (by L). Therefore, for small t there exists $C < 1$, independent of p and t, such that $\mathrm{Lip}_p(\varphi_t) \leq C$.

This treats the case when $p \in N_t$. To conclude, we note that on a neighborhood of any $p \in \mathbb{H}^2 \smallsetminus N_t$ the map φ_t coincides with the closest-point projection onto ∂N_t postcomposed with an isometry of $\mathbb{H}^2$, hence $\mathrm{Lip}_p(\varphi_t) = 1/\cosh \eta < 1$ where $\eta = d(p, \partial N_t)$. ☐

Proof of Proposition 3.9. Let Υ be a lamination of Σ_g consisting of all the cuffs of Π together with a triskelion lamination inside each pair of pants labeled 0. Let $c : \Sigma_g \smallsetminus \Upsilon \to \{-1, 1\}$ be a coloring taking the value -1 (resp. 1) on each pair of pants labeled -1 (resp. 1), and both values on each pair of pants labeled 0. For any $t \geq 0$, let ρ'_t be the folding of j_t along Υ with coloring c.

We now argue similarly to the proof of Proposition 3.1 in Section 3.3. For each pair of pants P in Π, choose a subgroup Γ^P of Γ_g which is conjugate to $\pi_1(P)$, and for any $t \geq 0$ a lift $\widetilde{P}_t \subset \mathbb{H}^2$ of the convex core of $j_t(\Gamma^P)\backslash\mathbb{H}^2$.

If P is labeled -1 (resp. 1), then for any $t \geq 0$ the restrictions of j_t and ρ'_t to Γ^P are conjugate by some orientation-preserving (resp. orientation-reversing) isometry φ_t^P of $\mathbb{H}^2$.

If P is labeled 0, then by Lemma 3.10 there is a family of 1-Lipschitz, $(j_t|_{\Gamma^P}, \rho'_t|_{\Gamma^P})$-equivariant maps $\varphi_t^P : \mathbb{H}^2 \to \mathbb{H}^2$, defined for all t in a small interval $[0, t_0]$, with the following property: for any $\eta > 0$ there exists $C < 1$ such that $\mathrm{Lip}_p(\varphi_t^P) \leq C$ for all $t \in [0, t_0]$ and all $p \in \widetilde{P}_t$ at distance $\geq \eta$ from $\partial \widetilde{P}_t$.

The collection of all maps φ_t^P, extended (j_t, ρ'_t)-equivariantly, piece together to yield a map $\varphi_t^* : \mathbb{H}^2 \smallsetminus \mathcal{C}_t \to \mathbb{H}^2$, where $\mathcal{C}_t$ is the union of all geodesics of $\mathbb{H}^2$ projecting to cuffs of Π in $j_t(\Gamma_g)\backslash\mathbb{H}^2 \simeq \Sigma_g$.

The obstruction to extending φ_t^* by continuity on each geodesic $\ell_t \subset \mathcal{C}_t$ is that the maps on either side of ℓ_t may disagree by a constant shift along ℓ_t if ℓ_t separates two pairs of pants labeled $(\pm 1, 0)$ or $(0, 0)$. This discrepancy $\delta(\ell_t) \in \mathbb{R}$ is the same on the whole $j_t(\Gamma_g)$-orbit of ℓ_t. To correct it, we precompose the folding ρ'_t of j_t with an earthquake supported on the cuff associated with ℓ_t (in the j_t-metric), of length $-\delta(\ell_t)$.

We repeat for each $j_t(\Gamma_g)$-orbit in $\mathcal{C}_t$, and eventually obtain a new folded representation ρ_t. By construction, there is a family of 1-Lipschitz, (j_t, ρ_t)-equivariant maps $\varphi_t : \mathbb{H}^2 \to \mathbb{H}^2$ satisfying Proposition 3.9, obtained simply by gluing together isometric translates of the φ_t^P. ☐

4. Surjectivity of the two projections

In this section we prove Theorem 1.1. We first construct uniformly lengthening deformations of surfaces with boundary (Section 4.1), then glue these together according to combinatorics given by Proposition 3.1 (Sections 4.2 and 4.4). Section 4.3 is devoted to the proof of a technical lemma.

4.1. **Uniformly lengthening deformations of compact hyperbolic surfaces with boundary.** Our two main tools to prove Theorem 1.1 are Proposition 3.1 and the following lemma.

Lemma 4.1. *Let Γ be the fundamental group and $j_0 \in \mathrm{Hom}(\Gamma, \mathrm{PSL}(2,\mathbb{R}))$ the holonomy of a compact, connected, hyperbolic surface Σ with nonempty geodesic boundary. Then there exist $t_0 > 0$ and a continuous family of representations $(j_t)_{0\leq t\leq t_0}$ with the following properties:*

(*a*) $\lambda_{j_0}(\gamma) = (1-t)\,\lambda_{j_t}(\gamma)$ *for any $t \in [0, t_0]$ and any $\gamma \in \Gamma$ corresponding to a boundary component of Σ;*
(*b*) $\sup_{\gamma\in\Gamma\smallsetminus\{1\}} \frac{\lambda_{j_0}(\gamma)}{\lambda_{j_t}(\gamma)} < 1$ *for any $t \in (0, t_0]$;*
(*c*) $j_t(\gamma) = j_0(\gamma) + O(t)$ *for any $\gamma \in \Gamma$ as $t \to 0$, where both sides are seen as 2×2 real matrices with determinant 1;*
(*d*) *for any compact subset K of $\mathbb{H}^2$ projecting to the interior of the convex core of $j_0(\Gamma)\backslash\mathbb{H}^2$, there exists $L > 0$ such that*
$$d(p, f_t(p)) \leq Lt$$
for any $p \in K$, any $t \in [0, t_0]$, and any 1-Lipschitz, (j_t, j_0)-equivariant map $f_t : \mathbb{H}^2 \to \mathbb{H}^2$.

As in Section 3.2, the convex core of $j_0(\Gamma)\backslash\mathbb{H}^2$ naturally identifies with Σ. The idea is to construct the representations j_t as holonomies of hyperbolic surfaces obtained from $j_0(\Gamma)\backslash\mathbb{H}^2$ by *strip deformations*. This type of deformation was first introduced by Thurston [T2, proof of Lem. 3.4]. We refer to [PT] and [DGK] for more details.

Proof. We first explain how to lengthen one boundary component β of Σ. Choose a finite collection of disjoint, biinfinite geodesic arcs $\alpha_1, \dots, \alpha_n \subset j_0(\Gamma)\backslash\mathbb{H}^2$, each crossing β orthogonally twice, and subdividing the convex core Σ into right-angled hexagons and one-holed right-angled bigons. Along each arc α_i, following [T2], slice $j_0(\Gamma)\backslash\mathbb{H}^2$ open and insert a strip A_i of $\mathbb{H}^2$, bounded by two geodesics, with narrowest cross section at the midpoint of $\alpha_i \cap \Sigma$ (see Figure 3).

This yields a new complete hyperbolic surface, with a compact convex core, equipped with a natural 1-Lipschitz map ς_t^β to $j_0(\Gamma)\backslash\mathbb{H}^2$ obtained by collapsing the strips A_i back to lines. Note that the image under ς_t^β of the new convex core is *strictly contained* in Σ (see Figure 3). The geodesic corresponding to β is longer in the new surface than in Σ. By adjusting the widths of the strips A_i, we may assume that the ratio of lengths is $\frac{1}{1-t}$. Note that the appropriate widths for this ratio are in $O(t)$ as $t \to 0$. All lengths of geodesics corresponding to boundary components other than β are unchanged.

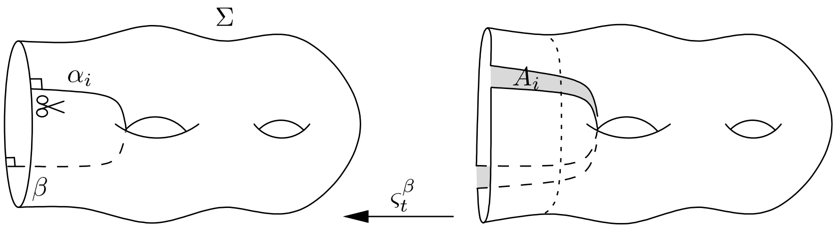

FIGURE 3. A strip deformation. In the source of the collapsing map ς_t^β we show the new peripheral geodesic, dotted.

Repeat the construction, iteratively, for all boundary components $\beta_1, \ldots, \beta_r$ of Σ, in some arbitrary order. We thus obtain a new complete hyperbolic surface $j_t(\Gamma)\backslash\mathbb{H}^2$, with a compact convex core Σ_t, such that j_t satisfies (a).

We claim that j_t also satisfies (b). Indeed, consider the 1-Lipschitz map $\varsigma_t := \varsigma_t^{\beta_r} \circ \cdots \circ \varsigma_t^{\beta_1}$ from Σ_t to Σ. If 1 were its optimal Lipschitz constant, then by Lemma 2.2 there would exist a geodesic lamination of Σ_t whose leaves are isometrically preserved by ς_t. But this is not the case here since for every i, the map $\varsigma_t^{\beta_i}$ does not isometrically preserve any geodesic lamination except the boundary components other than β_i. Therefore ς_t has Lipschitz constant < 1, which implies (b) by Remark 2.6.

Up to replacing each j_t with a conjugate under $\mathrm{PSL}(2,\mathbb{R})$, we may assume that (c) holds. Indeed, it is well known that there exist elements $\gamma_1, \ldots, \gamma_n \in \Gamma$ whose length functions form a smooth coordinate system for $\mathrm{Hom}(\Gamma, \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$ near $[j_0]$ (see [GX, Th. 2.1] for instance). For any i, the preimage under ς_t of the closed geodesic of Σ associated with γ_i is obtained by expanding finitely many strips of width $O(t)$, hence $\lambda_{j_t}(\gamma_i) \leq \lambda_{j_0}(\gamma_i) + O(t)$ as $t \to 0$. On the other hand, $\lambda_{j_t}(\gamma_i) \geq \lambda_{j_0}(\gamma_i)$ due to the existence of the 1-Lipschitz map ς_t. Therefore, $d'(j_0, j_t) = O(t)$ for any smooth metric d' on a neighborhood of $[j_0]$ in $\mathrm{Hom}(\Gamma, \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$.

To check (d), we use a perturbative version of the argument that a $j_0(\Gamma)$-invariant, 1-Lipschitz map must be the identity on the preimage $N_0 \subset \mathbb{H}^2$ of the convex core Σ of $j_0(\Gamma)\backslash\mathbb{H}^2$. For any hyperbolic element $h \in \mathrm{PSL}(2,\mathbb{R})$, with translation axis $\mathcal{A}_h \subset \mathbb{H}^2$, and for any $p \in \mathbb{H}^2$, a classical formula gives

$$\sinh\Big(\frac{d(p, h\cdot p)}{2}\Big) = \sinh\Big(\frac{\lambda(h)}{2}\Big) \cdot \cosh d(p, \mathcal{A}_h) \tag{4.1}$$

(see Figure 4, left). Consider $p \in \mathbb{H}^2$ in the interior of N_0. We can find three translation axes $\mathcal{A}_{j_0(\gamma_1)}, \mathcal{A}_{j_0(\gamma_2)}, \mathcal{A}_{j_0(\gamma_3)} \subset \partial N_0$ of elements of $j_0(\Gamma)$ such that if q_i denotes the projection of p to $\mathcal{A}_{j_0(\gamma_i)}$, then p belongs to the interior of the triangle $q_1q_2q_3$. For any $t \geq 0$ and any 1-Lipschitz, (j_t, j_0)-equivariant map $f_t : \mathbb{H}^2 \to \mathbb{H}^2$,

$$d\big(f_t(p), j_0(\gamma_i) \cdot f_t(p)\big) \leq d(p, j_t(\gamma_i) \cdot p),$$

which by (4.1) may be written as

$$\sinh\Big(\frac{\lambda_{j_0}(\gamma_i)}{2}\Big) \cdot \cosh d\big(f_t(p), \mathcal{A}_{j_0(\gamma_i)}\big) \leq \sinh\Big(\frac{\lambda_{j_t}(\gamma_i)}{2}\Big) \cdot \cosh d\big(p, \mathcal{A}_{j_t(\gamma_i)}\big).$$

Since $\lambda_{j_0}(\gamma_i) = \lambda_{j_t}(\gamma_i) + O(t)$ and $d(p, \mathcal{A}_{j_t(\gamma_i)}) = d(p, \mathcal{A}_{j_0(\gamma_i)}) + O(t)$ by (c), this implies

$$\cosh d\big(f_t(p), \mathcal{A}_{j_0(\gamma_i)}\big) \leq \cosh d\big(p, \mathcal{A}_{j_0(\gamma_i)}\big) + O(t),$$

where the error term does not depend on the choice of the map f_t. Since $d(p, \mathcal{A}_{j_0(\gamma_i)}) > 0$, we may invert the hyperbolic cosine:

$$d\big(f_t(p), \mathcal{A}_{j_0(\gamma_i)}\big) \leq d\big(p, \mathcal{A}_{j_0(\gamma_i)}\big) + O(t).$$

Applied to $i = 1, 2, 3$, this means that $f_t(p)$ belongs to a curvilinear triangle around p bounded by three hypercycles (curves at constant distance from a geodesic line) expanding at rate $O(t)$ as t becomes positive, hence $d(p, f_t(p)) = O(t)$ (see Figure 4, right). All estimates $O(t)$ are robust under small perturbations of p, hence can be made uniform (and still independent of f_t) for p in a compact set K, yielding (d). ☐

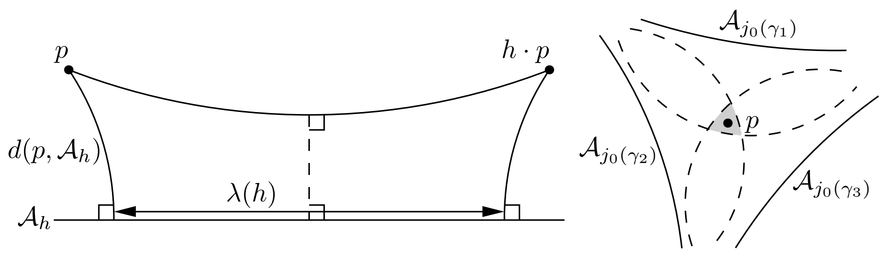

FIGURE 4. Left: A hyperbolic quadrilateral with two right angles. Right: The point $f_t(p)$ belongs to the shaded region.

4.2. **Gluing surfaces with boundary.** We now prove the first statement of Theorem 1.1. Namely, given $[\rho] \in \mathsf{Rep}_g^{\mathrm{nfd}}$, we construct $[j] \in \mathsf{Rep}_g^{\mathrm{fd}}$ that strictly dominates $[\rho]$.

If $\lambda_\rho \equiv 0$, then any $[j] \in \mathsf{Rep}_g^{\mathrm{fd}}$ strictly dominates $[\rho]$. We now suppose that $\lambda_\rho \not\equiv 0$. Proposition 3.1.(1) then gives us an element $[j_0] \in \mathsf{Rep}_g^{\mathrm{fd}}$, a labeled pants decomposition Π of Σ_g, and, for any $j_0, \rho \in \mathrm{Hom}(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$ in the respective classes $[j_0], [\rho]$ (which we now fix), a 1-Lipschitz, (j_0, ρ)-equivariant map $f : \mathbb{H}^2 \to \mathbb{H}^2$ that is an orientation-preserving (resp. orientation-reversing) isometry in restriction to any connected subset of $\mathbb{H}^2$ projecting to a union of pants labeled -1 (resp. 1) in $j_0(\Gamma_g)\backslash\mathbb{H}^2 \simeq \Sigma_g$, and that satisfies $\mathrm{Lip}_p(f) < 1$ for any $p \in \mathbb{H}^2$ projecting to the interior of a pair of pants labeled 0.

Not all pairs of pants are labeled -1, and not all 1, since j_0 and ρ are not conjugate under $\mathrm{PGL}(2,\mathbb{R})$. By Remark 2.6, the class $[j_0]$ dominates $[\rho]$ in the sense that $\lambda(\rho(\gamma)) \leq \lambda(j_0(\gamma))$ for all $\gamma \in \Gamma_g$. Our goal is to use Lemma 4.1 to modify j_0 into a representation j such that $[j]$ *strictly* dominates $[\rho]$. For this purpose, we erase all the cuffs that separate two pairs of pants of Π with labels $(-1,-1)$ or $(1,1)$, and write

$$\Sigma_g = \Sigma^1 \cup \cdots \cup \Sigma^m,$$

where Σ^i, for any $1 \leq i \leq m$, is a compact surface with boundary that is

- either a pair of pants labeled 0,

- or a full connected component of the subsurface of Σ_g made of pants labeled -1,
- or a full connected component of the subsurface of Σ_g made of pants labeled 1

(see Figure 5). The boundary components of the Σ^i are the cuffs that separated two pairs of pants of Π with labels $(-1, 1)$, $(\pm 1, 0)$, or $(0, 0)$.

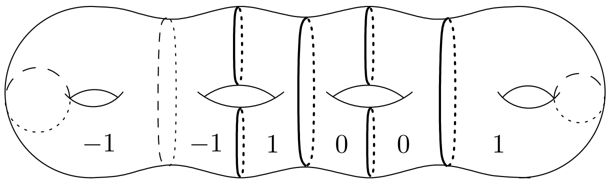

FIGURE 5. A labeled pants decomposition with $m = 5$. The boundary components of the Σ^i, $1 \leq i \leq 5$, are in bold.

Choose a small $\delta > 0$ such that in all hyperbolic metrics on Σ_g which are close enough to that defined by j_0, any simple geodesic entering the δ-neighborhood of the geodesic representative of a cuff of Π crosses it. Let $\mathcal{C}_0 \subset \mathbb{H}^2$ be the union of all geodesic lines of $\mathbb{H}^2$ projecting to boundary components of the Σ^i in $j_0(\Gamma_g)\backslash\mathbb{H}^2 \simeq \Sigma_g$, let $N_0^\delta \subset \mathbb{H}^2$ be the complement of the δ-neighborhood of $\mathcal{C}_0$, and let $K \subset \mathbb{H}^2 \smallsetminus \mathcal{C}_0$ be a compact set whose interior contains a fundamental domain of N_0^δ for the action of $j_0(\Gamma_g)$, with m connected components projecting respectively to $\Sigma^1, \dots, \Sigma^m$.

We apply Lemma 4.1 to $\Gamma^i := \pi_1(\Sigma^i)$ and $j_0^i := j_0|_{\Gamma^i}$ and obtain continuous families $(j_t^i)_{0 \leq t \leq t_0} \subset \mathrm{Hom}(\Gamma^i, \mathrm{PSL}(2, \mathbb{R}))$ of representations, for $1 \leq i \leq m$, satisfying properties (a),(b),(c),(d) of Lemma 4.1, with a uniform constant $L > 0$ for the compact set $K \subset \mathbb{H}^2 \smallsetminus \mathcal{C}_0$. For any $t \in [0, t_0]$, using (a), we can glue together the (compact) convex cores of the $j_t^i(\Gamma^i)\backslash\mathbb{H}^2$ following the same combinatorics as the Σ^i. This gives a closed hyperbolic surface of genus g, hence a holonomy representation $j_t \in \mathrm{Hom}(\Gamma_g, \mathrm{PSL}(2, \mathbb{R}))$. By (c), up to adjusting the twist parameters, we may assume that

$$j_t(\gamma) = j_0(\gamma) + O(t) \tag{4.2}$$

for any $\gamma \in \Gamma_g$ as $t \to 0$, where both sides are seen as 2×2 real matrices with determinant 1. To complete the proof of the first statement of Theorem 1.1, it is sufficient to prove that for small enough $t > 0$,

$$\sup_{\gamma \in (\Gamma_g)_s} \frac{\lambda_\rho(\gamma)}{\lambda_{j_t}(\gamma)} < 1, \tag{4.3}$$

where $(\Gamma_g)_s$ is the set of nontrivial elements of Γ_g corresponding to simple closed curves on Σ_g: then $[j] := [j_t]$ will strictly dominate $[\rho]$ by Theorem 2.5. Note that $\lambda(j_t(\gamma)) = \lambda(j_t^i(\gamma))$ for all γ in Γ^i, seen as a subgroup of Γ_g. Thus (b) gives the control required in (4.3) for simple closed curves *contained in one of the* Σ^i. We now explain why the lengths of the other simple closed curves also decrease uniformly, based on (c) and (d).

For any $t \in (0, t_0]$, let $\mathcal{C}_t \subset \mathbb{H}^2$ be the union of the lifts to $\mathbb{H}^2$ of the simple closed geodesics of $j_t(\Gamma_g)\backslash\mathbb{H}^2 \simeq \Sigma_g$ corresponding to $\mathcal{C}_0$ and let N_t^δ be the complement of the δ-neighborhood of $\mathcal{C}_t$ in $\mathbb{H}^2$. For t small enough, we can find a fundamental domain K_t of N_t^δ for the action of $j_t(\Gamma_g)$ that is contained in K and has m connected components. By (b) and Theorem 2.5, for any $1 \leq i \leq m$ and $t \in (0, t_0]$ there exists a $(j_t|_{\Gamma^i}, j_0|_{\Gamma^i})$-equivariant map $f_t^i : \mathbb{H}^2 \to \mathbb{H}^2$ with $\mathrm{Lip}(f_t^i) < 1$. For small $t > 0$, we choose a (j_t, j_0)-equivariant map $f_t : (N_t^\delta \cup \mathcal{C}_t) \to \mathbb{H}^2$ such that

- $f_t = f_t^i$ on the component of K_t projecting to Σ^i, for all $1 \leq i \leq m$;
- f_t takes any geodesic line in $\mathcal{C}_t$ to the corresponding line in $\mathcal{C}_0$, multiplying all distances on it by the uniform factor $(1-t)$.

We choose the f_t so that, in addition, for any compact set $K' \subset \mathbb{H}^2$ there exists $L_1 \geq 0$ such that $d(x', f_t(x')) \leq L_1 t$ for all small enough $t > 0$ and all $x' \in \mathcal{C}_t \cap K'$. Consider the (j_t, ρ)-equivariant map

$$F_t := f \circ f_t : \ (N_t^\delta \cup \mathcal{C}_t) \longrightarrow \mathbb{H}^2,$$

where $f : \mathbb{H}^2 \to \mathbb{H}^2$ is the (j_0, ρ)-equivariant map from the beginning of the proof. In order to prove (4.3), it is sufficient to establish the following.

Lemma 4.2. *For small enough $t > 0$, there exists $C < 1$ such that for all $p, q \in \partial N_t^\delta$ lying at distance δ from a line $\ell_t \subset \mathcal{C}_t$, on opposite sides of ℓ_t,*

$$d(F_t(p), F_t(q)) \leq C\, d(p, q).$$

Indeed, fix a small $t > 0$. Any geodesic segment $I = [p, q]$ of $\mathbb{H}^2$ projecting to a closed geodesic of $j_t(\Gamma_g)\backslash\mathbb{H}^2 \simeq \Sigma_g$ may be decomposed into subsegments $I_1, \dots, I_n$ contained in N_t^δ alternating with subsegments $I'_1, \dots, I'_n$ crossing connected components of $\mathbb{H}^2 \smallsetminus N_t^\delta$ (indeed, any simple closed curve that enters one of these components crosses it, by choice of δ). By construction, the map F_t has Lipschitz constant < 1 on each connected component of N_t^δ, hence moves the endpoints of each I_k closer together by a uniform factor (independent of I). Lemma 4.2 ensures that the same holds for the I'_k. Thus the ratio $d(F_t(p), F_t(q))/d(p, q)$ is bounded by some factor $C' < 1$ independent of I, and the corresponding element $\gamma \in \Gamma_g$ satisfies $\lambda(\rho(\gamma)) \leq C' \lambda(j_t(\gamma))$. This proves (4.3), hence completes the proof of the first statement of Theorem 1.1.

4.3. **Proof of Lemma 4.2.** In this section we give a proof of Lemma 4.2. We first make the following observation.

Observation 4.3. *There exists $L' \geq 0$ such that for any small enough $t > 0$, any $p \in \partial N_t^\delta$ at distance δ from a geodesic $\ell_t \subset \mathcal{C}_t$, and any $x \in \ell_t$,*

$$d(f_t(p), f_t(x)) \leq (1-t)\, d(p, x) + L' t.$$

Proof of Observation 4.3. Since f_t is (j_t, j_0)-equivariant and $\mathcal{C}_0$ has only finitely many connected components modulo $j_0(\Gamma_g)$, we may fix a geodesic $\ell_0 \subset \mathcal{C}_0$ and prove the observation only for the geodesics $\ell_t \subset \mathcal{C}_t$ corresponding to ℓ_0. For any $t > 0$, the map f_t takes ℓ_t linearly to ℓ_0, multiplying all distances by the uniform factor $1-t$. Let $h_t : \mathbb{H}^2 \to \mathbb{H}^2$ be the orientation-preserving map that coincides with f_t on ℓ_t, takes any line orthogonal to ℓ_t to a line orthogonal to ℓ_0, and multiplies all distances by $1-t$ on such lines. At distance η from ℓ_t, the differential of h_t has principal values $1-t$ and $(1-t)\cosh((1-t)\eta)/\cosh\eta \leq 1-t$ (see [GK,(A.9)]), hence $\mathrm{Lip}(h_t) \leq 1-t$ and

$$d(f_t(x), h_t(p)) = d(h_t(x), h_t(p)) \leq (1-t)\, d(x, p)$$

for all $x \in \ell_t$ and $p \in \mathbb{H}^2$. By the triangle inequality, it is enough to find $L' \geq 0$ such that $d(h_t(p), f_t(p)) \leq L't$ for all small enough $t > 0$ and all $p \in \partial N_t^\delta$ at distance δ from ℓ_t. Since f_t and h_t are both (j_t, j_0)-equivariant under the stabilizer S of ℓ_0 in Γ_g, and $j_t(S)$ acts cocompactly on the set $\overline{\mathcal{U}}_t$ of points at distance $\leq \delta$ from ℓ_t, we may restrict to p in a compact fundamental domain of $\overline{\mathcal{U}}_t$ for $j_t(S)$. Let $K' \subset \mathbb{H}^2$ be a compact set containing such fundamental domains for all $t \in [0, t_0]$. By construction of f_t, there exists $L_1 \geq 0$ such that $d(x', f_t(x')) \leq L_1 t$ for all small enough $t > 0$ and all $x' \in \ell_t \cap K'$. By definition of h_t, this implies the existence of $L_2 \geq 0$ such that $d(p, h_t(p)) \leq L_2 t$ for all small enough $t > 0$ and all $p \in K'$. On the other hand, condition (d) of Lemma 4.1 (applied to the Γ^i and j_0^i as in Section 4.2) implies the existence of $L_3 \geq 0$ such that $d(p, f_t(p)) \leq L_3 t$ for all t and $p \in \partial N_t^\delta \cap K'$. By the triangle inequality, we may take $L' = L_2 + L_3$. $\square$

Proof of Lemma 4.2. As in the proof of Observation 4.3, we may fix a geodesic $\ell_0 \subset \mathcal{C}_0$ and restrict to the geodesics $\ell_t \subset \mathcal{C}_t$ corresponding to ℓ_0. Fix a small $t > 0$ and consider $p, q \in \partial N_t^\delta$ lying at distance δ from ℓ_t, on opposite sides of ℓ_t. The segment $[p, q]$ can be subdivided, at its intersection point x with ℓ_t, into two subsegments to which Observation 4.3 applies, yielding

$$\left\{ \begin{array}{lcl} d(f_t(p), f_t(x)) & \leq & (1-t)\, d(p, x) + L't, \\ d(f_t(x), f_t(q)) & \leq & (1-t)\, d(x, q) + L't. \end{array} \right. \tag{4.4}$$

Up to switching p and q, we may assume that either $[p, x]$ projects to a pair of pants labeled 0 in $j_t(\Gamma_g)\backslash\mathbb{H}^2 \simeq \Sigma_g$, or $[p, x]$ projects to a pair of pants labeled -1 and $[x, q]$ to a pair of pants labeled 1.

Suppose that $[p, x]$ projects to a pair of pants labeled 0 in $j_t(\Gamma_g)\backslash\mathbb{H}^2 \simeq \Sigma_g$. We first observe that if t is small enough (independently of p), then

$$d(f_t(p), \ell_0) \geq \frac{3\delta}{4}. \tag{4.5}$$

Indeed, as in the proof of Observation 4.3, the inequality is true for $p \in \partial N_t^\delta$ in a fixed compact set K' independent of t, by condition (d) of Lemma 4.1 and (4.2), and we then use the fact that f_t is (j_t, j_0)-equivariant under the stabilizer S of ℓ_0 in Γ_g, which acts cocompactly (by j_t) on the set of points at distance δ from ℓ_t. By (4.5), if t is small enough (independently of p), then the segment $[f_t(p), f_t(x)]$ spends at least $\delta/4$ units of length in the complement $N_0^{\delta/2}$ of the $\delta/2$-neighborhood of $\mathcal{C}_0$. The point is that $\mathrm{Lip}_y(f) < 1$ for all $y \in \mathbb{H}^2 \smallsetminus \mathcal{C}_0$ projecting to a pair of pants labeled 0 in $j_0(\Gamma_g)\backslash\mathbb{H}^2 \simeq \Sigma_g$, and this bound is uniform in restriction to $N_0^{\delta/2}$ since the function $p \mapsto \mathrm{Lip}_p(f)$ is upper semicontinuous and $j_0(\Gamma_g)$-invariant. Remark 2.1 thus implies the existence of a constant $\varepsilon > 0$, independent of t, ℓ_t, p, x, such that

$$d\big(f \circ f_t(p), f \circ f_t(x)\big) \leq d(f_t(p), f_t(x)) - \varepsilon. \tag{4.6}$$

Using the triangle inequality and the fact that f is 1-Lipschitz, together with (4.4) and (4.6), we find

$$\begin{array}{rcl} d(F_t(p), F_t(q)) & \leq & d\big(f \circ f_t(p), f \circ f_t(x)\big) + d\big(f \circ f_t(x), f \circ f_t(q)\big) \\ & \leq & (1-t)\, d(p, x) + L't - \varepsilon + (1-t)\, d(x, q) + L't, \end{array}$$

which is bounded by $(1-t)\, d(p, q)$ as soon as $t \leq \varepsilon/(2L')$.

Suppose that $[p,x]$ projects to a pair of pants labeled -1 and $[x,q]$ to a pair of pants labeled 1. We then use the fact that the continuous map f folds along $\ell_0 = f_t(\ell_t)$. In restriction to the connected component of $\mathbb{H}^2 \smallsetminus \mathcal{C}_0$ containing $f_t(p)$ (resp. $f_t(q)$), it is an isometry preserving (resp. reversing) the orientation. In particular, $d(F_t(p), F_t(q)) < d(f_t(p), f_t(q))$. Moreover, this inequality can be made uniform in the following sense: there exists $\varepsilon > 0$ such that

$$d(F_t(p), F_t(q)) \leq d(f_t(p), f_t(q)) - \varepsilon$$

whenever $f_t(p)$ and $f_t(q)$ lie at distance $\geq 3\delta/4$ from ℓ_0 (which is the case for t small enough by (4.5)) and at distance $\leq 3L'$ from each other. By (4.4),

$$d(f_t(p), f_t(q)) \leq (1-t)\, d(p,q) + 2L't, \tag{4.7}$$

which implies

$$d(F_t(p), F_t(q)) \leq (1-t)\, d(p,q)$$

for $d(p,q) \leq 3L'$ as soon as $t \leq \varepsilon/(2L')$ is small enough. If $d(p,q) \geq 3L'$, then applying the 1-Lipschitz map f to (4.7) directly gives

$$d(F_t(p), F_t(q)) \leq (1-t)\, d(p,q) + 2L't \leq \Big(1 - \frac{t}{3}\Big)\, d(p,q). \qquad \square$$

4.4. **Folding a given surface.** We now prove the second statement of Theorem 1.1. Namely, given $[j_0] \in \mathsf{Rep}_g^{\mathrm{fd}}$ and an integer $k \in (-2g+2, 2g-2)$, we construct $[\rho] \in \mathsf{Rep}_g^{\mathrm{nfd}}$ with $\mathsf{eu}(\rho) = k$ that is strictly dominated by $[j_0]$.

It is easy to find $[\rho]$ with $\mathsf{eu}(\rho) = k$ such that $\lambda_\rho(\gamma) \leq \lambda_{j_0}(\gamma)$ for all $\gamma \in \Gamma_g$: just decompose Σ_g into pairs of pants and attribute arbitrary values $0, 1, -1$ to each so that the sum is k. Consider a lamination Υ of Σ_g consisting of all the cuffs together with a triskelion lamination inside each pair of pants labeled 0, and let $c : \Sigma_g \smallsetminus \Upsilon \to \{-1, 1\}$ be a coloring taking the value -1 (resp. 1) on each pair of pants labeled -1 (resp. 1), and both values on each pair of pants labeled 0. Folding along Υ with the coloring c gives an element $[\rho] \in \mathsf{Rep}_g^{\mathrm{nfd}}$ with $\lambda_\rho(\gamma) \leq \lambda_{j_0}(\gamma)$ for all $\gamma \in \Gamma_g$. However, we need a *strict* domination. The idea is to obtain ρ by folding, not j_0, but a small deformation of j_0. For this purpose, we use the following result, which is analogous to Lemma 4.1.

Lemma 4.4. *Let Γ be the fundamental group and $j_0 \in \mathrm{Hom}(\Gamma, \mathrm{PSL}(2,\mathbb{R}))$ the holonomy of a compact, connected hyperbolic surface Σ with nonempty geodesic boundary. Then there exist $t_0 > 0$ and a continuous family of representations $(j_t)_{0 \leq t \leq t_0}$ with the following properties:*

- (a) *$\lambda_{j_t}(\gamma) = (1-t)\, \lambda_{j_0}(\gamma)$ for any $t \in [0, t_0]$ and any $\gamma \in \Gamma$ corresponding to a boundary component of Σ;*
- (b) *$\sup_{\gamma \in \Gamma \smallsetminus \{1\}} \frac{\lambda_{j_t}(\gamma)}{\lambda_{j_0}(\gamma)} < 1$ for any $t \in (0, t_0]$;*
- (c) *$j_t(\gamma) = j_0(\gamma) + O(t)$ for any $\gamma \in \Gamma$ as $t \to 0$, where both sides are seen as 2×2 real matrices with determinant 1;*
- (d) *for any compact subset K of $\mathbb{H}^2$ projecting to the interior of the convex core of $j_0(\Gamma)\backslash\mathbb{H}^2$, there exists $L > 0$ such that*

  $$d(p, f_t(p)) \leq Lt$$

  for any $p \in K$, any $t \in [0, t_0]$, and any 1-Lipschitz, (j_0, j_t)-equivariant map $f_t : \mathbb{H}^2 \to \mathbb{H}^2$.

As in the proof of Lemma 4.1, we construct the representations j_t as holonomies of hyperbolic surfaces obtained from $j_0(\Gamma)\backslash\mathbb{H}^2$ by deformation. Now the deformation needs to be shortening instead of lengthening, so we use *negative* strip deformations.

Proof of Lemma 4.4. We see Σ as the convex core of $j_0(\Gamma)\backslash\mathbb{H}^2$. To shorten one boundary component β of Σ, choose a finite collection of disjoint, bi-infinite geodesic arcs $\alpha_1, \dots, \alpha_n \subset j_0(\Gamma)\backslash\mathbb{H}^2$, each crossing β orthogonally twice, and subdividing Σ into right-angled hexagons and one-holed right-angled bigons. Near each α_i, choose a second geodesic arc α_i', also crossing β twice, such that α_i, α_i' approach each other closest at some points $p_i, p_i' \in \Sigma$. We take all arcs to be pairwise disjoint. For every i, delete the hyperbolic strip A_i bounded by α_i and α_i' and glue the arcs back together isometrically, identifying p_i with p_i'.

This yields a new complete hyperbolic surface, with a compact convex core, equipped with a natural 1-Lipschitz map ς_t^β *from* $j_0(\Gamma)\backslash\mathbb{H}^2$, obtained by collapsing the strips A_i to lines. The set $\varsigma_t^\beta(\Sigma)$ is strictly contained in the new convex core. The geodesic corresponding to β is shorter in the new surface than in Σ. By adjusting the widths of the strips A_i, we may assume that the ratio of lengths is $\frac{1}{1-t}$. Note that the appropriate widths for this ratio are in $O(t)$ as $t \to 0$. All lengths of geodesics corresponding to boundary components other than β are unchanged.

Repeat the construction, iteratively, for all boundary components $\beta_1, \dots, \beta_r$ of Σ, in some arbitrary order. We thus obtain a new complete hyperbolic surface $j_t(\Gamma)\backslash\mathbb{H}^2$, with a compact convex core Σ_t, such that j_t satisfies (a). As in the proof of Lemma 4.1, up to replacing each j_t with a conjugate under $\mathrm{PSL}(2, \mathbb{R})$, we may assume that (c) is satisfied. To see that (b) and (d) also hold, we use the 1-Lipschitz map $\varsigma_t := \varsigma_t^{\beta_r} \circ \cdots \circ \varsigma_t^{\beta_1}$ from Σ to Σ_t and argue as in the proof of Lemma 4.1, switching j_t and j_0. $\square$

As in Section 4.2, we write $\Sigma_g = \Sigma^1 \cup \cdots \cup \Sigma^m$, where Σ^i, for any $1 \leq i \leq m$, is a compact surface with boundary that is

- either a pair of pants labeled 0,
- or a full connected component of the subsurface of Σ_g made of pants labeled -1,
- or a full connected component of the subsurface of Σ_g made of pants labeled 1.

Choose a small $\delta > 0$ such that in all hyperbolic metrics on Σ_g which are close enough to that defined by j_0, any simple geodesic entering the δ-neighborhood of the geodesic representative of a cuff of our chosen pants decomposition crosses the cuff. We use again the notation $\mathcal{C}_0, N_0^\delta, K$ from Section 4.2. Applying Lemma 4.4 to $\Gamma^i := \pi_1(\Sigma^i)$ and $j_0^i := j_0|_{\Gamma^i}$, we obtain continuous families of representations $(j_t^i)_{0 \leq t \leq t_0}$ for $1 \leq i \leq m$ satisfying (a),(b),(c),(d), with a uniform constant $L > 0$ for the compact set $K \subset \mathbb{H}^2 \smallsetminus \mathcal{C}_0$. For any $t \geq 0$, using (a), we can glue together the convex cores of the $j_t^i(\Gamma^i)\backslash\mathbb{H}^2$ following the same combinatorics as the Σ^i. This gives a closed hyperbolic surface of genus g, hence a holonomy representation $j_t \in \mathrm{Hom}(\Gamma_g, \mathrm{PSL}(2, \mathbb{R}))$.

By (c), up to adjusting the twist parameters, we may assume that $j_t(\gamma) = j_0(\gamma) + O(t)$ for any $\gamma \in \Gamma_g$ as $t \to 0$.

Recall the notation $\mathcal{C}_t, N_t^\delta$ from Section 4.2. By Proposition 3.9, there exist a family $(\rho_t)_{0 \leq t \leq t_0} \subset \mathrm{Hom}(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$ of non-Fuchsian representations and, for any $t \in [0, t_0]$, a 1-Lipschitz, (j_t, ρ_t)-equivariant map $\varphi_t : \mathbb{H}^2 \to \mathbb{H}^2$ that is an orientation-preserving (resp. orientation-reversing) isometry in restriction to any connected subset of $\mathbb{H}^2$ projecting to a union of pants labeled -1 (resp. 1) in $j_t(\Gamma_g)\backslash\mathbb{H}^2 \simeq \Sigma_g$, such that

$$\mathrm{Lip}_p(\varphi_t) \leq C^* < 1 \tag{4.8}$$

for all $t \in [0, t_0]$ and all $p \in N_t^\delta$ projecting to a pair of pants labeled 0 in $j_t(\Gamma_g)\backslash\mathbb{H}^2 \simeq \Sigma_g$, for some $C^* < 1$ independent of p and t. We claim that for $t > 0$ small enough,

$$\sup_{\gamma \in (\Gamma_g)_s} \frac{\lambda_{\rho_t}(\gamma)}{\lambda_{j_0}(\gamma)} < 1, \tag{4.9}$$

which by Theorem 2.5 is enough to prove that $[\rho_t]$ is strictly dominated by $[j_0]$. Indeed, by (b) and Theorem 2.5, for any $1 \leq i \leq m$ and $t \in (0, t_0]$, there exists a $(j_t|_{\Gamma^i}, j_0|_{\Gamma^i})$-equivariant map $f_t^i : \mathbb{H}^2 \to \mathbb{H}^2$ with $\mathrm{Lip}(f_t^i) < 1$. Let $f_t : (N_0^\delta \cup \mathcal{C}_0) \to \mathbb{H}^2$ be a (j_0, j_t)-equivariant map such that

- $f_t = f_t^i$ on the component of K projecting to Σ^i, for all $1 \leq i \leq m$;
- f_t takes any geodesic line in $\mathcal{C}_0$ to the corresponding line in $\mathcal{C}_t$, multiplying all distances by the uniform factor $(1-t)$, and $d(x, f_t(x)) \leq L_1 t$ for all $x \in \mathcal{C}_0 \cap K$, for some $L_1 \geq 0$ independent of x and t.

Consider the (j_0, ρ_t)-equivariant map

$$G_t := \varphi_t \circ f_t : \ (N_0^\delta \cup \mathcal{C}_0) \longrightarrow \mathbb{H}^2.$$

Any geodesic segment $I = [p, q]$ of $\mathbb{H}^2$ projecting to a closed geodesic of $j_0(\Gamma_g)\backslash\mathbb{H}^2 \simeq \Sigma_g$ may be decomposed into subsegments $I_1, \dots, I_n$ contained in N_0^δ alternating with subsegments $I'_1, \dots, I'_n$ crossing connected components of $\mathbb{H}^2 \smallsetminus N_0^\delta$. By contractivity of f_t, the map G_t has Lipschitz constant < 1 on each connected component of N_0^δ, hence moves the endpoints of each I_k closer together by a uniform factor (independent of I). The subsegments I'_k are treated by the following lemma, which implies (4.9) and therefore completes the proof of the second statement of Theorem 1.1.

Lemma 4.5 (Analogue of Lemma 4.2)**.** *For small enough $t > 0$, there exists $C < 1$ such that for all $p, q \in \partial N_0^\delta$ lying at distance δ from a line $\ell_0 \subset \mathcal{C}_0$, on opposite sides of ℓ_0,*

$$d(G_t(p), G_t(q)) \leq C\, d(p, q).$$

The proof of Lemma 4.5 uses the following observation, which is identical to Observation 4.3 after exchanging j_0 and j_t.

Observation 4.6. *There exists $L' \geq 0$ such that for any small enough $t \geq 0$, any $p \in \partial N_0^\delta$ at distance δ from a geodesic $\ell_0 \subset \mathcal{C}_0$, and any $x \in \ell_0$,*

$$d(f_t(p), f_t(x)) \leq (1-t)\, d(p, x) + L't. \tag{4.10}$$

Proof of Lemma 4.5. We argue as in the proof of Lemma 4.2, but switch j_0 and j_t and use (4.8) to obtain the analogue

$$d\big(\varphi_t \circ f_t(p), \varphi_t \circ f_t(x)\big) \leq d(f_t(p), f_t(x)) - \varepsilon$$

of (4.6) when $[p, x]$ projects to a pair of pants labeled 0 in $j_0(\Gamma_g)\backslash\mathbb{H}^2 \simeq \Sigma_g$. $\square$

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