PROPER AFFINE ACTIONS FOR RIGHT-ANGLED COXETER GROUPS

JEFFREY DANCIGER, FRANÇOIS GUÉRITAUD, AND FANNY KASSEL

Abstract. For any right-angled Coxeter group Γ on k generators, we construct proper actions of Γ on $\mathrm{O}(p, q+1)$ by right and left multiplication, and on the Lie algebra $\mathfrak{o}(p, q+1)$ by affine transformations, for some $p, q \in \mathbb{N}$ with $p+q+1 = k$. As a consequence, any virtually special group admits proper affine actions on some $\mathbb{R}^n$: this includes e.g. surface groups, hyperbolic 3-manifold groups, examples of word hyperbolic groups of arbitrarily large virtual cohomological dimension, etc. We also study some examples in cohomological dimension two and four, for which the dimension of the affine space may be substantially reduced.

1. Introduction

Tiling space with regular shapes is an old endeavor, both practical and ornamental. It is also at the heart of crystallography, and Hilbert, prompted by recent progress in that discipline, asked in his 18th problem for a better understanding of regular tilings of Euclidean space $\mathbb{R}^n$. In 1910, Bieberbach [Bi] gave a partial answer by showing that a discrete group Γ acting properly by affine isometries on $\mathbb{R}^n$ has a finite-index subgroup acting as a lattice of translations on some affine subspace $\mathbb{R}^m$. Moreover, $m = n$ if and only if the quotient $\Gamma\backslash\mathbb{R}^n$ is compact, and the number $\mathscr{N}_n$ of such cocompact examples Γ up to affine conjugation is finite for fixed n. Crystallographers had known since 1891 that $\mathscr{N}_2 = 17$ and $\mathscr{N}_3 = 219$ (or 230 if chiral meshes are counted twice), a result due independently to Schoenflies and Fedorov.

The picture for affine actions becomes much less familiar in the absence of an invariant Euclidean metric. The Auslander conjecture [Au] states that if Γ acts properly discontinuously and cocompactly on $\mathbb{R}^n$ by affine transformations, then Γ should be virtually (i.e. up to finite index) solvable,

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J.D. was partially supported by an Alfred P. Sloan Foundation fellowship, and by the National Science Foundation under grants DMS 1103939 and DMS 1510254. F.G. and F.K. were partially supported by the Agence Nationale de la Recherche under grants DiscGroup (ANR-11-BS01-013) and DynGeo (ANR-16-CE40-0025-01), and through the Labex CEMPI (ANR-11-LABX-0007-01). F.K. was partially supported by the European Research Council under ERC Starting Grant 715982 (DiGGeS). The authors also acknowledge support from the GEAR Network, funded by the National Science Foundation under grant numbers DMS 1107452, 1107263, and 1107367 ("RNMS: GEometric structures And Representation varieties"). Part of this work was completed while J.D. and F.K. were in residence at the MSRI in Berkeley, California, for the program *Dynamics on Moduli Spaces of Geometric Structures* (Spring 2015) supported by NSF grant DMS 1440140.

or equivalently [Mi], virtually polycyclic. This conjecture has been proved up to dimension six [FG, AMS5] and in certain special cases [GoK, T, AMS4], but remains wide open in general.

In 1983, Margulis [M1, M2] constructed the first examples of proper actions of nonabelian *free* groups Γ on $\mathbb{R}^3$, answering a question of Milnor [Mi]. These actions do not violate the Auslander conjecture because they are not cocompact. They preserve a natural flat Lorentzian structure on $\mathbb{R}^3$ and the corresponding affine 3-manifolds are now known as *Margulis spacetimes*. Drumm [Dr] constructed more examples of Margulis spacetimes by building explicit fundamental domains in $\mathbb{R}^3$ bounded by polyhedral surfaces called *crooked planes*; it is now known [CDG, DGK2, DGK6] that all Margulis spacetimes are obtained in this way. Abels–Margulis–Soifer [AMS2, AMS3] have studied proper affine actions by free groups whose linear part is Zariski-dense in an indefinite orthogonal group, showing that such actions exist if and only if the signature is, up to sign, of the form $(2m, 2m-1)$. Recently, Smilga [S1] generalized Margulis's construction and showed that for any noncompact real semisimple Lie group G there exist proper actions, on the Lie algebra $\mathfrak{g} \simeq \mathbb{R}^{\dim(G)}$, of nonabelian free discrete subgroups of $G \ltimes \mathfrak{g}$ acting affinely via the adjoint action, with Zariski-dense linear part; Margulis spacetimes correspond to $G = \mathrm{PSL}(2,\mathbb{R}) \simeq \mathrm{SO}(2,1)_0$. More recently, he gave a sufficient condition [S2] (also conjectured to be necessary) for an algebraic subgroup of $\mathrm{GL}(n,\mathbb{R}) \ltimes \mathbb{R}^n$ to admit a Zariski-dense nonabelian free discrete subgroup acting properly on $\mathbb{R}^n$.

1.1. **New examples of proper affine actions.** The existence of proper affine actions by nonabelian free groups suggests the possibility that other finitely generated groups which are not virtually solvable might also admit proper affine actions. However, in the more than thirty years since Margulis's discovery, very few examples have appeared. In particular, until now, all known examples of word hyperbolic groups acting properly by affine transformations on $\mathbb{R}^n$ were virtually free groups. In this paper, we give many new examples, both word hyperbolic and not, by establishing the following.

Theorem 1.1. *Any right-angled Coxeter group on k generators admits proper affine actions on $\mathbb{R}^{k(k-1)/2}$.*

We note that right-angled Coxeter groups, while simple to describe in terms of generators and relations, have a rich structure and contain many interesting subgroups. As a first example, the fundamental group of any closed orientable surface of negative Euler characteristic embeds as a finite-index subgroup in the right-angled pentagon group. Since any right-angled Artin group embeds into a right-angled Coxeter group [DJ], we obtain the following answer to a question of Wise [W2, Problem 13.47].

Corollary 1.2. *Any right-angled Artin group admits proper affine actions on $\mathbb{R}^n$ for some $n \geq 1$.*

See e.g. [BB] for interesting subgroups of right-angled Artin groups, for which Corollary 1.2 provides proper affine actions. Haglund–Wise [HW1] proved that the fundamental group of any *special* nonpositively curved cube complex embeds into a right-angled Artin group. Thus we obtain:

Corollary 1.3. *Any virtually special group admits proper affine actions on $\mathbb{R}^n$ for some $n \geq 1$.*

Virtually special groups include:

- all Coxeter groups (not necessarily right-angled) [HW2];
- all cubulated word hyperbolic groups, using Agol's virtual specialness theorem [Ag];
- all fundamental groups of closed hyperbolic 3-manifolds, using [Sag, KM]: see [BW];
- the fundamental groups of many other 3-manifolds, see [W1, L, PW].

Januszkiewicz–Świątkowski [JS] proved the existence of word hyperbolic right-angled Coxeter groups of arbitrarily large virtual cohomological dimension; see also [O] for another construction. Hence another consequence of Theorem 1.1 is:

Corollary 1.4. *There exist proper affine actions by word hyperbolic groups of arbitrarily large virtual cohomological dimension.*

The Auslander conjecture is the statement that a group acting properly discontinuously by affine transformations on $\mathbb{R}^n$ is either virtually solvable, or has virtual cohomological dimension $< n$. In the examples from Theorem 1.1, the dimension $n = k(k-1)/2$ of the affine space grows quadratically in the number of generators k, while the virtual cohomological dimension of the Coxeter group acting is naively bounded above by k (and is even much smaller in the examples above [JS, O]). Hence, Corollary 1.4 is far from giving counterexamples to the Auslander Conjecture.

1.2. **Proper actions on Lie algebras.** The proper affine actions in Theorem 1.1 are obtained in the following way. Let G be a real Lie group. It acts linearly on its Lie algebra $\mathfrak{g} \simeq \mathbb{R}^{\dim(G)}$ by the adjoint action, and $G \ltimes \mathfrak{g}$ acts affinely on $\mathfrak{g}$ by $(g, w) \cdot v = \mathrm{Ad}(g)v + w$. Let Γ be a discrete group. A group homomorphism from Γ to $G \ltimes \mathfrak{g}$ is given by a group homomorphism $\rho : \Gamma \to G$ and a map $u : \Gamma \to \mathfrak{g}$ which is a *ρ-cocycle*, i.e. satisfies

$$u(\gamma_1\gamma_2) = u(\gamma_1) + \mathrm{Ad}(\rho(\gamma_1))\, u(\gamma_2) \tag{1.1}$$

for all $\gamma_1, \gamma_2 \in \Gamma$. For instance, for any smooth path $(\rho_t)_{t\in I}$ in $\mathrm{Hom}(\Gamma, G)$ (where I is an open interval) and any $t \in I$, the map $u_t : \Gamma \to \mathfrak{g}$ given by $u_t(\gamma) = \frac{\mathrm{d}}{\mathrm{d}\tau}\big|_{\tau=t} \rho_\tau(\gamma)\rho_t(\gamma)^{-1}$ is a ρ_t-cocycle; it is the unique ρ_t-cocycle such that for all $\gamma \in \Gamma$,

$$\rho_{t+s}(\gamma) = \mathrm{e}^{su_t(\gamma)+o(s)}\rho_t(\gamma) \quad \text{as } s \to 0. \tag{1.2}$$

The cocycles in this paper will all be constructed in this way. (In general there may exist cocycles not of this form: see [LM, §2].) We prove the following.

Theorem 1.5. *For any irreducible right-angled Coxeter group Γ on k generators, there exist $p, q \in \mathbb{N}$ with $p+q+1 = k$ and a smooth path $(\rho_t)_{t\in I}$ in $\mathrm{Hom}(\Gamma, G)$ of faithful and discrete representations into $G := \mathrm{O}(p, q+1)$ (where I is a nonempty open interval) such that for any $t \in I$, the affine action of Γ on $\mathfrak{g} \simeq \mathbb{R}^{k(k-1)/2}$ via $(\rho_t, \frac{\mathrm{d}}{\mathrm{d}\tau}\big|_{\tau=t} \rho_\tau \rho_t^{-1})$ is properly discontinuous.*

Full properness criteria for actions on $\mathfrak{o}(2,1) \simeq \mathfrak{psl}(2,\mathbb{R})$ with convex cocompact linear part were provided in [GLM] in terms of the so-called Margulis invariant, and in [DGK1] in terms of uniform contraction in the hyperbolic plane $\mathbb{H}^2$ (see Theorem 3.3.(2) and Remark 3.4). Here we establish sufficient conditions for properness on $\mathfrak{o}(p, q+1)$ in terms of uniform spacelike contraction in the pseudo-Riemannian analogue $\mathbb{H}^{p,q}$ of hyperbolic space in signature (p, q) (Theorem 3.6.(2)). In order to prove Theorem 1.5, we then construct explicit representations and uniformly contracting cocycles for irreducible right-angled Coxeter groups.

Since any right-angled Coxeter group is a direct product of irreducible ones, we obtain Theorem 1.1 by applying Theorem 1.5 to each irreducible factor and then taking the direct sum of the resulting affine actions.

We also use the same techniques as in Theorem 1.5 to construct, in some specific cases, examples of proper affine actions on $\mathfrak{g} = \mathfrak{o}(p, q+1)$ where $p+q+1$ is smaller than the number k of generators of Γ.

Proposition 1.6. *(a) For any even $k \geq 6$, the group Γ generated by reflections in the sides of a convex right-angled k-gon of $\mathbb{H}^2$ admits proper affine actions on $\mathfrak{g} = \mathfrak{o}(3,1) \simeq \mathbb{R}^6$.*

(b) The group Γ generated by reflections in the faces of a 4-dimensional regular right-angled 120-cell admits proper affine actions on $\mathfrak{g} = \mathfrak{o}(8,1) \simeq \mathbb{R}^{36}$.

The group Γ is virtually the fundamental group of a closed surface of genus ≥ 2 in (a), and of a closed hyperbolic 4-manifold in (b). Both examples follow from a general face-coloring method explained in Proposition 5.1.

Whereas the examples of proper affine actions of free groups of Margulis, Drumm, Abels–Margulis–Soifer, and Smilga all relied to some degree on the idea of free groups playing ping pong on $\mathbb{R}^n$, here we use a metric method in the spirit of [DGK1, DGK2, DGK6] to prove Theorem 1.5, which we explain in Section 3.

1.3. Proper actions on Lie groups.

Following the strategy of [DGK1, DGK2], we also view the proper affine actions on the Lie algebra $\mathfrak{g}$ in Theorem 1.5 as "infinitesimal analogues" of proper actions on the corresponding Lie group G. Here we consider the action of $G \times G$ by right and left multiplication: $(g_1, g_2) \cdot g = g_2 g g_1^{-1}$. Given a discrete group Γ, not all pairs $(\rho, \rho') \in \mathrm{Hom}(\Gamma, G)^2 = \mathrm{Hom}(\Gamma, G\times G)$ give rise to proper actions of Γ on G:

for instance, if $\rho = \rho'$, then the action has a global fixed point in G. We prove the following "macroscopic versions" of Theorem 1.5 and Proposition 1.6.

Theorem 1.7. *For any irreducible right-angled Coxeter group Γ on k generators, there exist $p, q \in \mathbb{N}$ with $p + q + 1 = k$ and a smooth path $(\rho_t)_{t\in I}$ in $\mathrm{Hom}(\Gamma, G)$ of faithful and discrete representations into $G := \mathrm{O}(p, q + 1)$ (where I is a nonempty open interval) such that for any $t < t'$ in I, the action of Γ on G by right and left multiplication via $(\rho_{t'}, \rho_t)$ is properly discontinuous.*

In general, it is easy to obtain proper actions on G by right and left multiplication by considering a discrete group Γ, a representation $\rho \in \mathrm{Hom}(\Gamma, G)$ with finite kernel and discrete image, and a representation $\rho' \in \mathrm{Hom}(\Gamma, G)$ with bounded image (for instance the constant representation, with image $\{e\} \subset G$): such proper actions are often called *standard*. The point of Theorem 1.7 is to build nonstandard proper actions on G, where both factors are faithful and discrete — and in fact, can be arbitrarily close to each other.

Full properness criteria for proper actions on $\mathrm{O}(n, 1)$ via (ρ, ρ') with ρ geometrically finite were provided in [K2, GuK] in terms of uniform contraction in $\mathbb{H}^n$ (see Theorem 3.3.(2) and Remark 3.4). Here, in order to prove Theorem 1.7, we establish sufficient conditions for properness on $\mathrm{O}(p, q + 1)$ in terms of uniform contraction in $\mathbb{H}^{p,q}$ (Theorem 3.6.(1)).

We also construct examples of proper actions on $G = \mathrm{O}(p, q + 1)$ where $p + q + 1$ is smaller than the number k of generators of Γ, in the same cases as for Proposition 1.6.

Proposition 1.8. *(a) For any even $k \geq 6$, the group Γ generated by reflections in the sides of a convex right-angled k-gon in $\mathbb{H}^2$ admits proper actions on $G = \mathrm{O}(3, 1)$ by right and left multiplication via pairs $(\rho, \rho') \in \mathrm{Hom}(\Gamma, G)^2$ with ρ, ρ' both faithful and discrete.*

(b) The group Γ generated by reflections in the faces of a 4-dimensional regular right-angled 120-cell admits proper actions on $G = \mathrm{O}(8, 1)$ by right and left multiplication via pairs $(\rho, \rho') \in \mathrm{Hom}(\Gamma, G)^2$ with ρ, ρ' both faithful and discrete.

Remark 1.9. For $p \geq 1$, the group $G = \mathrm{O}(p, q + 1)$ has four connected components. The proper actions on G constructed in Theorem 1.7 and Proposition 1.8 all yield proper actions on the identity component G_0.

For $p = 2$ and $q = 0$, the identity component $G_0 = \mathrm{O}(2, 1)_0 \simeq \mathrm{PSL}(2, \mathbb{R})$ is the so-called *anti-de Sitter* 3-space AdS^3, a Lorentzian analogue of $\mathbb{H}^3$. The group of orientation-preserving isometries of AdS^3 identifies with the quotient of the four diagonal components of $G \times G$ by $\{\pm(I, I)\}$, acting on G_0 by right and left multiplication. Many examples of proper actions on AdS^3 were constructed since the 1980s, see in particular [KR, Sal, K2, GuK, GKW, DT].

Examples of nonstandard cocompact proper actions on $\mathrm{O}(n,1)$ by right and left multiplication for $n > 2$ were constructed in [Gh, K], using deformation techniques. After we announced the results of this paper, Lakeland–Leininger [LL] found examples of nonstandard cocompact proper actions on $\mathrm{O}(3,1)$ and $\mathrm{O}(4,1)$ by right-angled Coxeter groups which cannot be obtained from standard proper actions by deformation. Note that for cocompact proper actions on $\mathrm{O}(n,1)$ by right and left multiplication via (ρ,ρ'), exactly one of ρ or ρ' has finite kernel and discrete image [K1, GuK]. On the other hand, in the noncompact proper actions that we construct in Theorem 1.7 and Proposition 1.8, both ρ,ρ' have finite kernel and discrete image.

1.4. **Plan of the paper.** In Section 2 we recall some background on properly convex sets and pseudo-Riemannian hyperbolic spaces $\mathbb{H}^{p,q}$. In Section 3 we state some sufficient criteria for properness, expressed in terms of uniform contraction in $\mathbb{H}^{p,q}$. In Section 4 we prove these criteria for the Riemannian case $q = 0$ (Theorem 3.3) and in Section 5 we give examples in $\mathbb{H}^3$ and $\mathbb{H}^8$, establishing Propositions 1.6 and 1.8. In Section 6 we prove the criteria for general $\mathbb{H}^{p,q}$ (Theorem 3.6). In Section 7 we prove Theorems 1.5 and 1.7 (hence also 1.1) by constructing appropriate families of representations $(\rho_t)_{t\in I}$ to which we can apply Theorem 3.6.

Acknowledgements. We would like to thank Ian Agol, Yves Benoist, Suhyoung Choi, Bill Goldman, Gye-Seon Lee, Ludovic Marquis, Vivien Ripoll, and Anna Wienhard for interesting discussions. We also thank Piotr Przytycki for providing several references and Dani Wise for pointing out [W2, Problem 13.47] and encouraging us. We gratefully acknowledge the hospitality and excellent working conditions provided by the MSRI in Berkeley, where most of this work was done in the Spring 2015, and the CNRS-Pauli Institute in Vienna.

2. Notation and reminders

In this section we briefly set up some notation and recall some useful definitions and basic facts on properly convex domains in projective space and on the pseudo-Riemannian hyperbolic spaces $\mathbb{H}^{p,q}$.

2.1. **Properly convex domains in projective space.** Let V be a real vector space of dimension ≥ 2. Recall that an open subset Ω of $\mathbb{P}(V)$ is said to be *properly convex* if it convex and bounded in some affine chart of $\mathbb{P}(V)$. There is a natural metric d_Ω on Ω, the *Hilbert metric*:

$$d_\Omega(x,y) := \frac{1}{2}\log\,[a,x,y,b]$$

for all distinct $x,y\in\Omega$, where $[\cdot,\cdot,\cdot,\cdot]$ is the cross-ratio on $\mathbb{P}^1(\mathbb{R})$, normalized so that $[0,1,y,\infty] = y$, and where a,b are the intersection points of $\partial\Omega$ with the projective line through x and y, with a,x,y,b in this order. The metric space (Ω, d_Ω) is proper (i.e. closed balls are compact) and complete, and the group

$$\mathrm{Aut}(\Omega) := \{g \in \mathrm{PGL}(V) \mid g \cdot \Omega = \Omega\}$$

acts on Ω by isometries for d_Ω. As a consequence, any discrete subgroup of $\mathrm{Aut}(\Omega)$ acts properly discontinuously on Ω.

By definition, the *dual convex set* of Ω is

$$\Omega^* := \mathbb{P}\big(\{\varphi \in V^* \mid \mathbb{P}(\mathrm{Ker}\,\varphi) \cap \overline{\Omega} = \emptyset\}\big),$$

where $\overline{\Omega}$ is the closure Ω. The set Ω^* is a nonempty properly convex open subset of $\mathbb{P}(V^*)$ preserved by the dual action of $\mathrm{Aut}(\Omega)$ on $\mathbb{P}(V^*)$. We can use any nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on V to view Ω^* as a properly convex subset of $\mathbb{P}(V)$: if $\widetilde{\Omega} \subset V \smallsetminus \{0\}$ denotes a convex lift of Ω,

$$\Omega^* \simeq \mathrm{Int}\,\mathbb{P}(\{\tilde{y} \in V \mid \langle \tilde{y}, \tilde{x} \rangle < 0 \quad \forall \tilde{x} \in \widetilde{\Omega}\}). \tag{2.1}$$

Remark 2.1. It follows from the definition that if $\Omega' \subset \Omega$ are nonempty properly convex open subsets of $\mathbb{P}(\mathbb{R}^{n+1})$, then the corresponding Hilbert metrics satisfy $d_{\Omega'}(x, y) \geq d_\Omega(x, y)$ for all $x, y \in \Omega$.

2.2. **The pseudo-Riemannian space $\mathbb{H}^{p,q}$.** For $p, q \in \mathbb{N}$ with $p \geq 1$, let $\mathbb{R}^{p,q+1}$ be $\mathbb{R}^{p+q+1}$ endowed with a symmetric bilinear form $\langle \cdot, \cdot \rangle_{p,q+1}$ of signature $(p, q+1)$. We set

$$\mathbb{H}^{p,q} := \left\{[v] \in \mathbb{P}(\mathbb{R}^{p,q+1}) \;\middle|\; \langle v, v \rangle_{p,q+1} < 0\right\}.$$

The form $\langle \cdot, \cdot \rangle_{p,q+1}$ induces a pseudo-Riemannian metric $\mathsf{g}^{p,q}$ of signature (p, q) on $\mathbb{H}^{p,q}$. Explicitly, the metric $\mathsf{g}^{p,q}$ at a point $[v]$ is obtained from the restriction of $\langle \cdot, \cdot \rangle_{p,q+1}$ to the tangent space at $v/\sqrt{-\langle v, v \rangle_{p,q+1}}$ to the hypersurface

$$\widehat{\mathbb{H}}^{p,q} := \{v \in \mathbb{R}^{p,q+1} \mid \langle v, v \rangle_{p,q+1} = -1\},$$

a double cover of $\mathbb{H}^{p,q}$ with covering group $\{\pm I\}$. The sectional curvature of $\mathsf{g}^{p,q}$ is constant negative, hence $\mathbb{H}^{p,q}$ can be seen as a pseudo-Riemannian analogue of the real hyperbolic space $\mathbb{H}^p = \mathbb{H}^{p,0}$ in signature (p, q).

The isometry group of the pseudo-Riemannian space $\mathbb{H}^{p,q}$ is $\mathrm{PO}(p, q+1) = \mathrm{O}(p, q+1)/\{\pm 1\}$. The point stabilizers are conjugate to $\mathrm{O}(p, q)$, hence $\mathbb{H}^{p,q} \simeq \mathrm{PO}(p, q+1)/\mathrm{O}(p, q)$.

The set $\mathbb{H}^p = \mathbb{H}^{p,0}$ is a properly convex open subset of $\mathbb{P}(\mathbb{R}^{p,1})$, and the Hilbert metric $d_{\mathbb{H}^p}$ on $\mathbb{H}^p$ coincides with the standard hyperbolic metric. On the other hand, for $q \geq 1$ the space $\mathbb{H}^{p,q}$ is *not* convex in $\mathbb{P}(\mathbb{R}^{p,q+1})$. The boundary of $\mathbb{H}^{p,q}$ in $\mathbb{P}(\mathbb{R}^{p,q+1})$, given by

$$\partial \mathbb{H}^{p,q} = \left\{[v] \in \mathbb{P}(\mathbb{R}^{p,q+1}) \mid \langle v, v \rangle_{p,q+1} = 0\right\},$$

is a quadric which at each point has $p-1$ positive and q negative principal curvature directions.

For any $x \in \mathbb{H}^{p,q}$, a nonzero vector $v \in T_x\mathbb{H}^{p,q}$ and the geodesic line $\mathcal{L}$ it generates are called *spacelike* (resp. *lightlike*, resp. *timelike*) if $\mathsf{g}_x^{p,q}(v, v)$ is positive (resp. zero, resp. negative). The line $\mathcal{L}$ is the intersection of $\mathbb{H}^{p,q}$ with a projective line meeting $\partial \mathbb{H}^{p,q}$ in two (resp. one, resp. zero) points:

see Figure 1. In general, the totally geodesic subspaces of $\mathbb{H}^{p,q}$ are exactly the intersections of $\mathbb{H}^{p,q}$ with projective subspaces of $\mathbb{P}(\mathbb{R}^{p,q+1})$. As in [GM],

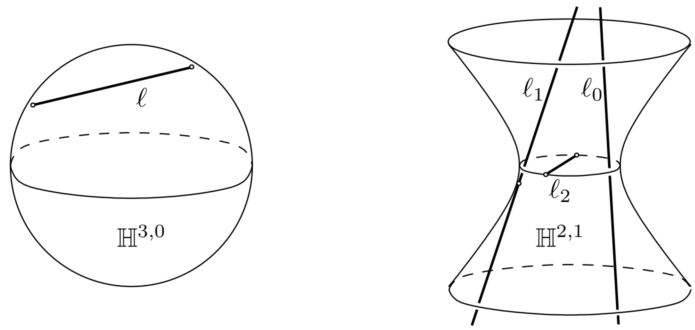

FIGURE 1. Left: $\mathbb{H}^3 = \mathbb{H}^{3,0}$ with a geodesic line ℓ (necessarily spacelike). Right: $\mathbb{H}^{2,1}$ with three geodesic lines ℓ_2 (spacelike), ℓ_1 (lightlike), and ℓ_0 (timelike).

we shall use the following convention.

Notation 2.2. If $x, y \in \mathbb{H}^{p,q}$ are distinct points belonging to a *spacelike* line, we denote by $d_{\mathbb{H}^{p,q}}(x,y) > 0$ the pseudo-Riemannian distance between x and y, obtained by integrating $\sqrt{\mathsf{g}^{p,q}}$ over the geodesic path from x to y. If $x, y \in \mathbb{H}^{p,q}$ are equal or belong to a lightlike or timelike line, we set $d_{\mathbb{H}^{p,q}}(x,y) := 0$.

Consider distinct points $x, y \in \mathbb{H}^{p,q}$ lying on a spacelike line. The distance can be computed directly from the formula:

$$d_{\mathbb{H}^{p,q}}(x,y) = \operatorname{arccosh} |\langle \tilde{x}, \tilde{y} \rangle_{p,q+1}| > 0 \tag{2.2}$$

where $\tilde{x}, \tilde{y} \in \mathbb{R}^{p,q+1}$ are respective lifts of x, y with $\langle \tilde{x}, \tilde{x} \rangle_{p,q+1} = \langle \tilde{y}, \tilde{y} \rangle_{p,q+1} = -1$. The following Hilbert geometry interpretation, well-known in the $\mathbb{H}^p$ setting, will also be useful:

$$d_{\mathbb{H}^{p,q}}(x,y) = \frac{1}{2} \log\, [a, x, y, b] > 0 \tag{2.3}$$

where $[\cdot,\cdot,\cdot,\cdot]$ is the cross-ratio on $\mathbb{P}^1(\mathbb{R})$, normalized so that $[0, 1, y, \infty] = y$, and where a, b are the two intersection points of $\partial\mathbb{H}^{p,q}$ with the projective line through x and y, with a, x, y, b in this order.

Note that when $q > 0$, the function $d_{\mathbb{H}^{p,q}}$ is *not* a distance function on $\mathbb{H}^{p,q}$ in the usual sense: for many triples it does not satisfy the triangle inequality. See [GM, § 3] for further discussion of this issue.

2.3. **The Lie algebra $\mathfrak{o}(p, q+1)$.** The Lie algebra $\mathfrak{o}(p, q+1)$ identifies with the set of *Killing vector fields* on $\mathbb{H}^{p,q}$, i.e. of vector fields whose flow is isometric: an element $Y \in \mathfrak{o}(p, q+1)$ corresponds to the Killing field

$$x \mapsto \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \exp(tY) \cdot x \in T_x\mathbb{H}^{p,q}.$$

On the other hand, the Lie algebra $\mathfrak{o}(p, q+1)$ identifies with $\mathbb{R}^{p',q'}$ where

$$(p', q') = \big(p(q+1)\ ,\ (p^2+q^2-p+q)/2\big). \tag{2.4}$$

Indeed, $\mathfrak{o}(p, q+1)$ is a real vector space of dimension $(p+q+1)(p+q)/2$ endowed with a natural nondegenerate symmetric bilinear form $\kappa_{p,q+1}$, the Killing form, of signature (p', q'). This form is invariant under the adjoint action of $\mathrm{O}(p, q+1)$ on $\mathfrak{o}(p, q+1)$. Using geometric properties of actions on $\mathbb{H}^{p,q}$, we shall construct proper affine actions on $\mathfrak{o}(p, q+1) \simeq \mathbb{R}^{p',q'}$.

3. A sufficient condition for properness: uniform contraction in (pseudo-Riemannian) hyperbolic spaces

In this section we state some sufficient conditions for the properness of actions of discrete groups on $\mathrm{O}(p, q+1)$ and $\mathfrak{o}(p, q+1)$. We shall use these conditions to prove Theorems 1.5 and 1.7, and Propositions 1.6 and 1.8.

In the whole section, we consider, for a Lie group G with Lie algebra $\mathfrak{g}$:

- the action of $G\times G$ on G by right and left multiplication: $(g_1, g_2)\cdot g = g_2 g g_1^{-1}$;
- the affine action of $G \ltimes \mathfrak{g}$ on $\mathfrak{g}$ through the adjoint action: $(g, Z)\cdot Y = \mathrm{Ad}(g)Y + Z$.

3.1. Uniform contraction in $\mathbb{H}^p$ and proper actions on $\mathrm{O}(p, 1)$ and $\mathfrak{o}(p, 1)$.

We start with the case $q = 0$.

Definition 3.1. Let $G = \mathrm{O}(p, 1)$ for $p \geq 1$. Let Γ be a discrete group and $\rho : \Gamma \to G$ a representation.

(1) A representation $\rho' : \Gamma \to G$ is *uniformly contracting* with respect to ρ if there is a (ρ, ρ')-equivariant map $f : \mathbb{H}^p \to \mathbb{H}^p$ which is C-Lipschitz for some $C < 1$, i.e. for all $x, y \in \mathbb{H}^p$,

$$d_{\mathbb{H}^p}(f(x), f(y)) \leq C\, d(x, y).$$

(2) A ρ-cocycle $u : \Gamma \to \mathfrak{g}$ is *uniformly contracting* if there is a (ρ, u)-equivariant vector field X on $\mathbb{H}^p$ which is c-lipschitz (lowercase ‘l’) for some $c < 0$, i.e. for all $x, y \in \mathbb{H}^p$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} d_{\mathbb{H}^p}\big(\exp_x(tX(x)), \exp_y(tX(y))\big) \leq c\, d_{\mathbb{H}^p}(x, y). \tag{3.1}$$

In (2), by (ρ, u)-*equivariant* we mean that

$$X(\rho(\gamma)\cdot x) = \rho(\gamma)_* X(x) + u(\gamma)(\rho(\gamma)\cdot x) \tag{3.2}$$

for all $\gamma \in \Gamma$ and $x \in \mathbb{H}^p$, where we see $u(\gamma)$ as a Killing vector field on $\mathbb{H}^p$ as in Section 2.3. The (ρ, u)-equivariance of X entails that both sides of (3.1) are invariant under replacing (x, y) with $(\rho(\gamma)\cdot x, \rho(\gamma)\cdot y)$.

Example 3.2. Let I be an open interval containing 0. If $(\rho_t)_{t\in I}$ and $u_0 := \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \rho_t\rho_0^{-1}$ are as in (1.2), then for any smooth family of (ρ_0, ρ_t)-equivariant maps $f_t : \mathbb{H}^p \to \mathbb{H}^p$ with $f_0 = \mathrm{Id}_{\mathbb{H}^p}$, the derivative $X(x) := \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} f_t(x)$ is (ρ_0, u_0)-equivariant. If moreover there exists $c < 0$ with $\mathrm{Lip}(f_t) \leq 1 + ct$ for all small $t > 0$, then X is c-lipschitz. This is how we will produce all the equivariant contracting vector fields that appear in this paper.

With this terminology, here are some sufficient conditions for properness.

Theorem 3.3. *Let $G = \mathrm{O}(p, 1)$ for $p \geq 1$. Let Γ be a discrete group and $\rho : \Gamma \to G$ a representation with finite kernel and discrete image.*

(1) *(see* [Sal, K2, GuK]*) If a representation $\rho' : \Gamma \to G$ is uniformly contracting with respect to ρ, then the action of Γ on G by right and left multiplication via (ρ, ρ') is properly discontinuous, and the corresponding quotient $(\rho, \rho')(\Gamma)\backslash G$ is an $(\mathrm{O}(p) \times \mathrm{O}(1))$-bundle over $\rho(\Gamma)\backslash\mathbb{H}^p$.*

(2) *(see* [DGK1]*) If a ρ-cocycle $u : \Gamma \to \mathfrak{g}$ is uniformly contracting, then the affine action of Γ on $\mathfrak{g} \simeq \mathbb{R}^{p,(p^2-p)/2}$ via (ρ, u) is properly discontinuous and the corresponding quotient $(\rho, u)(\Gamma)\backslash\mathfrak{g}$ is an $\mathfrak{o}(p)$-bundle over $\rho(\Gamma)\backslash\mathbb{H}^p$.*

Theorem 3.3.(1) was proved in [GuK, Prop. 7.2]. It first appeared in [Sal, K2] for $p = 2$ without the statement that $(\rho, \rho')(\Gamma)\backslash G$ fibers over $\rho(\Gamma)\backslash\mathbb{H}^2$. Theorem 3.3.(2) was proved in [DGK1, Prop. 6.3] for $p = 2$; the proof given there works without changes for any $p \geq 2$, as we shall see in Section 4.2.

Remark 3.4. When ρ is geometrically finite, the converse to Theorem 3.3.(1) holds up to switching ρ and ρ': this was proved in [K2] for $p = 2$ and convex cocompact ρ, and in [GuK] in general.

For $p = 2$ and convex cocompact ρ, the converse to Theorem 3.3.(2) holds up to replacing u by $-u$, by [DGK1, Th. 1.1]; a similar statement for geometrically finite ρ will be proved in [DGK6]. We believe that this fails for $p = 3$, as $\mathfrak{o}(3, 1) \simeq \mathfrak{psl}(2, \mathbb{C})$ has a complex structure and properness (unlike uniform contraction) is unaffected when we multiply a cocycle by a nonzero complex number.

Using Theorem 3.3, we can already construct proper actions on $\mathrm{O}(p, 1)$ and $\mathfrak{o}(p, 1)$ for certain right-angled Coxeter groups: see Section 5.

3.2. **Uniform spacelike contraction in $\mathbb{H}^{p,q}$ and proper actions on $\mathrm{O}(p, q+1)$ and $\mathfrak{o}(p, q+1)$.** In order to prove Theorem 1.1 and construct proper affine actions for *any* right-angled Coxeter group, we consider the group $G = \mathrm{O}(p, q+1)$ for any $p, q \in \mathbb{N}$ with $p+q \geq 1$, acting on the pseudo-Riemannian hyperbolic space $\mathbb{H}^{p,q}$ of Section 2.2. Recall Notation 2.2 for $d_{\mathbb{H}^{p,q}}$. We shall use the following terminology extending Definition 3.1.

Definition 3.5. Let $G = \mathrm{O}(p, q+1)$ for $p, q \in \mathbb{N}$ with $p+q \geq 1$. Let Γ be a discrete group and $\rho : \Gamma \to G$ a representation with finite kernel and discrete image, preserving a nonempty properly convex open subset Ω of $\mathbb{H}^{p,q}$.

(1) A representation $\rho' : \Gamma \to G$ is *uniformly contracting in spacelike directions* with respect to (ρ, Ω) if there exist a nonempty $\rho(\Gamma)$-invariant subset $\mathcal{O}$ of Ω (e.g. Ω itself, or a single $\rho(\Gamma)$-orbit) and a

(ρ,ρ')-equivariant map $f:\mathcal{O}\to\mathbb{H}^{p,q}$ such that f is *C-Lipschitz in spacelike directions* for some $C<1$, i.e. for all $x,y\in\mathcal{O}$ on a spacelike line,
$$d_{\mathbb{H}^{p,q}}(f(x),f(y))\leq C\,d(x,y).$$

(2) A ρ-cocycle $u:\Gamma\to\mathfrak{g}$ is *uniformly contracting in spacelike directions* with respect to Ω if there are a nonempty $\rho(\Gamma)$-invariant subset $\mathcal{O}$ of Ω, a (ρ,u)-equivariant vector field $X:\mathcal{O}\to T\mathbb{H}^{p,q}$ on $\mathcal{O}$ such that X is *c-lipschitz in spacelike directions* for some $c<0$, i.e. for all $x,y\in\mathbb{H}^{p,q}$ on a spacelike line,
$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} d_{\mathbb{H}^{p,q}}\left(\exp_x(tX(x)),\exp_y(tX(y))\right)\leq c\,d_{\mathbb{H}^{p,q}}(x,y).$$

In (2), by *(ρ,u)-equivariant* we mean that X satisfies (3.2) for all $\gamma\in\Gamma$ and $x\in\mathcal{O}$, where we see $u(\gamma)$ as a Killing vector field on $\mathbb{H}^{p,q}$ as in Section 2.3.

Given a smooth family of maps $f_t:\Omega\to\mathbb{H}^{p,q}$ (for $t\geq 0$) such that $f_0=\mathrm{Id}_\Omega$ and f_t is $(1+ct)$-Lipschitz in spacelike directions, note that, as in Example 3.2, the derivative vector field $X(x):=\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}f_t(x)$ is c-lipschitz in spacelike directions.

The following result, proved in Section 6 below, generalizes Theorem 3.3. Recall that a representation into $\mathrm{GL}(\mathbb{R}^{p,q+1})$ is called *strongly irreducible* if its image does not preserve any nonempty finite union of nonzero proper linear subspaces of $\mathbb{R}^{p,q+1}$. An element $g\in\mathrm{End}(\mathbb{R}^{p,q+1})$ is called *proximal in* $\mathbb{P}(\mathbb{R}^{p,q+1})$ if it has a unique eigenvalue of maximal modulus, counting multiplicities; equivalently, g admits a unique attracting fixed point in $\mathbb{P}(\mathbb{R}^{p,q+1})$.

Theorem 3.6. *Let $G=\mathrm{O}(p,q+1)$ for $p,q\in\mathbb{N}$ with $p+q>1$. Let Γ be a discrete group and $\rho:\Gamma\to G$ a representation with finite kernel and discrete image, preserving a nonempty properly convex open subset Ω of $\mathbb{H}^{p,q}$.*

(1) Let $\rho':\Gamma\to G$ be a strongly irreducible representation such that $\rho'(\Gamma)$ contains a proximal element. If ρ' is uniformly contracting in spacelike directions with respect to ρ and Ω, then the action of Γ on G by right and left multiplication via (ρ,ρ') is properly discontinuous.

(2) Let $u:\Gamma\to\mathfrak{g}$ be a ρ-cocycle. If u is uniformly contracting in spacelike directions with respect to Ω, then the affine action of Γ on $\mathfrak{g}\simeq\mathbb{R}^{(p+q+1)(p+q)/2}$ via (ρ,u) is properly discontinuous. This action preserves the Killing form $\kappa_{p,q+1}$ on $\mathfrak{g}$, of signature given by (2.4).

We deduce Theorems 1.5 and 1.7 (hence Theorem 1.1) from Theorem 3.6 by constructing representations, cocycles, and equivariant maps and vector fields, all explicit, which are uniformly contracting in spacelike directions. This is done by deforming the canonical representation of a Coxeter group on k generators in $\mathrm{GL}(k,\mathbb{R})$: see Section 7.

4. Uniform contraction in $\mathbb{H}^p$ implies properness

In this section we give a proof of Theorem 3.3.

4.1. **Properness for actions on $\mathrm{O}(p,1)$ by right and left multiplication.** For the reader's convenience, we recall the proof of Theorem 3.3.(1) from [GuK, § 7.4].

Suppose $\rho' : \Gamma \to \mathrm{O}(p,1)$ is uniformly contracting with respect to ρ, and let $f : \mathbb{H}^p \to \mathbb{H}^p$ be a (ρ,ρ')-equivariant map which is C-Lipschitz for some $C < 1$. The group $\mathrm{O}(p) \times \mathrm{O}(1)$ is the stabilizer in $\mathrm{O}(p,1)$ of some point of $\mathbb{H}^p$. Therefore, for any $x \in \mathbb{H}^p$,

$$\mathcal{L}_x := \{g \in \mathrm{O}(p,1) \mid g \cdot x = f(x)\}$$

is a right-and-left translate of $\mathrm{O}(p) \times \mathrm{O}(1)$. An element $g \in \mathrm{O}(p,1)$ belongs to $\mathcal{L}_x$ if and only if x is a fixed point of $g^{-1} \circ f$; since $\mathrm{Lip}(g^{-1} \circ f) = \mathrm{Lip}(f) < 1$, such a fixed point exists and is unique for each given g, meaning that g belongs to exactly one set $\mathcal{L}_x$. We denote this x by $\Pi(g)$. The fibration $\Pi : \mathrm{O}(p,1) \to \mathbb{H}^p$ is continuous: if $h \in G$ is close enough to g so that $d(\Pi(g), h^{-1} \circ f \circ \Pi(g)) \leq (1 - \mathrm{Lip}(f))\,\varepsilon$, then $h^{-1} \circ f$ takes the ε-ball centered at $\Pi(g)$ to itself, hence $\Pi(h)$ is within ε from $\Pi(g)$. Moreover, $\Pi : \mathrm{O}(p,1) \to \mathbb{H}^p$ is by construction $((\rho,\rho'),\rho)$-equivariant:

$$\rho'(\gamma)\,\mathcal{L}_x\,\rho(\gamma)^{-1} = \mathcal{L}_{\rho(\gamma)\cdot x}$$

for all $\gamma \in \Gamma$ and $x \in \mathbb{H}^p$. Since the action of Γ on $\mathbb{H}^p$ via ρ is properly discontinuous, by $((\rho,\rho'),\rho)$-equivariance, the action of Γ on $\mathrm{O}(p,1)$ by right and left multiplication via (ρ,ρ') is also properly discontinuous. The fibration Π descends to a topological fibration of the quotient $(\rho,\rho')(\Gamma)\backslash\mathrm{O}(p,1)$, with base $\rho(\Gamma)\backslash\mathbb{H}^p$ and fiber $\mathrm{O}(p) \times \mathrm{O}(1)$.

Remark 4.1. The fibers $\mathcal{L}_x$, for $x \in \mathbb{H}^p$, are totally geodesic subspaces of $\mathrm{O}(p,1)$ which are negative for the natural pseudo-Riemannian structure of signature $(p',q') = (p,(p^2-p)/2)$ on $\mathrm{O}(p,1)$ induced by the Killing form $\kappa_{p,1}$ (see Section 2.3) and maximal for this property.

4.2. **Properness for affine actions on $\mathfrak{o}(p,1)$.** We now recall the proof of Theorem 3.3.(2) which was given in [DGK1, § 6.2] for $p = 2$. This proof actually works for any $p \geq 2$, and runs parallel to that of Section 4.1.

Suppose the ρ-cocycle $u : \Gamma \to \mathfrak{o}(p,1)$ is uniformly contracting, and let $X : \mathbb{H}^p \to T\mathbb{H}^p$ be a (ρ,u)-equivariant vector field on $\mathbb{H}^p$ which is c-lipschitz for some $c < 0$. The Lie algebra $\mathfrak{o}(p)$ is the set of Killing vector fields on $\mathbb{H}^p$ that vanish at some specific point of $\mathbb{H}^p$. Therefore, for any $x \in \mathbb{H}^p$,

$$\ell_x := \{Y \in \mathfrak{o}(p,1) \mid Y(x) = X(x)\}$$

is the image of $\mathfrak{o}(p)$ by an affine transformation of $\mathfrak{o}(p,1)$.

A Killing vector field $Y \in \mathfrak{o}(p,1)$ belongs to ℓ_x if and only if $X - Y$ vanishes at x. The vector field $X - Y$ is c-lipschitz because X is c-lipschitz and Y is a Killing field. Since $c < 0$, the vector field $X - Y$ points inwards on the boundary of any large enough ball, hence has a zero in $\mathbb{H}^p$ by Brouwer's theorem. This zero is unique because $c < 0$, and we denote it by $\pi(Y) \in \mathbb{H}^p$.

The fibration $\pi : \mathfrak{o}(p,1) \to \mathbb{H}^p$ is continuous: if $Y' \in \mathfrak{o}(p,1)$ is close enough to Y in the sense that $\|(Y - Y')(x)\| < |c|\,\delta$ for $x = \pi(Y)$, then the

c-lipschitz vector field $X - Y' = (X - Y) + (Y - Y')$ points inward along the sphere of radius δ centered at x (for the Killing field $Y - Y'$ has constant component along any given line through x), hence the unique zero $\pi(Y')$ of $X - Y'$ is within δ of $x = \pi(Y)$.

Moreover, $\pi : \mathfrak{o}(p, 1) \to \mathbb{H}^p$ is by construction $((\rho, u), \rho)$-equivariant:

$$\pi\big(\operatorname{Ad}(\rho(\gamma))Y + u(\gamma)\big) = \rho(\gamma) \cdot \pi(Y) \tag{4.1}$$

for all $\gamma \in \Gamma$ and $Y \in \mathfrak{o}(p, 1)$. Indeed, $\pi(Y) = x$ means $Y(x) = X(x)$, hence by (3.2)

$$\big(\operatorname{Ad}(\rho(\gamma))Y + u(\gamma)\big)\big(\rho(\gamma)x\big) = \rho(\gamma)_* X(x) + u(\gamma)(\rho(\gamma)x) = X(\rho(\gamma)x),$$

proving (4.1).

Since the action of Γ on $\mathbb{H}^p$ via ρ is properly discontinuous, by $((\rho, u), \rho)$-equivariance the affine action of Γ on $\mathfrak{o}(p, 1)$ via (ρ, u) is properly discontinuous. The fibration π descends to a topological fibration of the quotient $(\rho, u)(\Gamma)\backslash\mathfrak{o}(p, 1)$, with base $\rho(\Gamma)\backslash\mathbb{H}^p$ and fiber $\mathfrak{o}(p)$.

Remark 4.2. The fibers ℓ_x, for $x \in \mathbb{H}^p$, are affine subspaces of $\mathfrak{o}(p, 1)$ which are negative for the Killing form $\kappa_{p,1}$ on $\mathfrak{o}(p, 1)$ (see Section 2.3) and maximal for this property.

5. Examples of proper actions on $\mathrm{O}(p, 1)$ and $\mathfrak{o}(p, 1)$ for small p

In this section we prove Propositions 1.6 and 1.8.

5.1. **Uniformly contracting maps obtained by colorings.**

Our proof is based on the following construction.

Proposition 5.1. *Let Γ be a discrete subgroup of $\mathrm{O}(p, 1)$ generated by the reflections $\{\gamma_i\}_{1 \leq i \leq k}$ in the faces $\{F_i\}_{1 \leq i \leq k}$ of a convex compact right-angled polyhedron of $\mathbb{H}^p$. Let $v_i = (w_i, 1) \in \mathbb{R}^{p,1}$ be normal vectors to the F_i. Suppose there exists a "coloring" $\sigma : \{1, \dots, k\} \to \{0, \dots, c\}$ such that $\sigma(i) \neq \sigma(j)$ when F_i intersects F_j. Let $u_0, \dots, u_c$ be the vertices of a regular simplex inscribed in the unit sphere of $\mathbb{R}^c$. For any $t \in \mathbb{R}$, we set*

$$v_i^t := \big(\cosh(t)\, w_i, \sqrt{c}\, \sinh(t)\, u_{\sigma(i)}, 1\big) \subset \mathbb{R}^{p+c,1}.$$

Then for any $t \in \mathbb{R}$, the representation $\rho_t : \Gamma \to \mathrm{O}(p + c, 1)$ taking γ_i to the orthogonal reflection in $(v_i^t)^\perp \subset \mathbb{R}^{p+c,1}$ is well defined, and for small enough $|t|$ it is faithful and discrete. Moreover, for any $0 < t < t'$ with t small enough, there exists a $(\rho_t, \rho_{t'})$-equivariant, $\frac{\cosh(t)}{\cosh(t')}$-Lipschitz map $f_{t'}^t : \mathbb{H}^{p+c} \to \mathbb{H}^{p+c}$.

Proof. Let $t \in \mathbb{R}$. To prove that ρ_t is well defined, we only need to check that $\langle v_i^t, v_j^t \rangle_{p+c,1} = 0$ when F_i intersects F_j. Since $\langle v_i, v_j \rangle_{p,1} = 0$ we have $\langle w_i, w_j \rangle_{p,0} = 1$, and $\langle u_{\sigma(i)}, u_{\sigma(j)} \rangle_{c,0} = -1/c$. Therefore

$$\begin{aligned} \langle v_i^t, v_j^t \rangle_{p+c,1} &= \cosh^2(t) \langle w_i, w_j \rangle_{p,0} + c\, \sinh^2(t) \langle u_{\sigma(i)}, u_{\sigma(j)} \rangle_{c,0} - 1 \\ &= \cosh^2(t) - \sinh^2(t) - 1 = 0. \end{aligned}$$

For small enough $|t|$ the representation ρ_t is faithful and discrete because it is a small deformation of the convex cocompact Fuchsian representation ρ_0 (valued in $\mathrm{O}(p,1) \subset \mathrm{O}(p+c,1)$).

We now assume that $t > 0$ is such that ρ_t is faithful and discrete, and fix $t' > t$. Let $P_t \subset \mathbb{P}(\mathbb{R}^{p+c,1})$ be the polytope bounded by the $\mathbb{P}(v_i^t)^\perp$ for $1 \leq i \leq k$, so that $P_t \cap \mathbb{H}^{p+c}$ is the fundamental polyhedron of $\rho_t(\Gamma)$. Define similarly $P_{t'}$. In the affine chart $\{x_{p+c+1} = 1\}$ of $\mathbb{P}(\mathbb{R}^{p+c,1})$, the linear transformation $\frac{\cosh(t')}{\cosh(t)}\mathrm{Id}_{\mathbb{R}^p} \oplus \frac{\sinh(t')}{\sinh(t)}\mathrm{Id}_{\mathbb{R}^c}$ takes the v_i^t to the $v_i^{t'}$. It follows that dually, $f := \frac{\cosh(t)}{\cosh(t')}\mathrm{Id}_{\mathbb{R}^p} \oplus \frac{\sinh(t)}{\sinh(t')}\mathrm{Id}_{\mathbb{R}^c}$ takes P_t to $P_{t'}$. The restriction of f to $P_t \cap \mathbb{H}^{p+c}$ can be $(\rho_t, \rho_{t'})$-equivariantly, continuously extended by reflections in the faces of P_t and $P_{t'}$, yielding a $(\rho_t, \rho_{t'})$-equivariant map $f_{t'}^t : \mathbb{H}^{p+c} \to \mathbb{H}^{p+c}$. By the triangle inequality, it only remains to show that $f|_{\mathbb{H}^{p+c} \cap P_t}$ is $\frac{\cosh(t)}{\cosh(t')}$-Lipschitz. Since the ellipsoid $f(\mathbb{H}^{p+c})$ is contained in the ball of radius $\frac{\cosh(t)}{\cosh(t')}$ (itself contained in the unit ball $\mathbb{H}^{p+c}$ of the chart), the result is an immediate consequence of the following Lemma 5.2, which quantifies Remark 2.1. $\square$

Lemma 5.2. *Fix a Euclidean chart $\mathbb{R}^n$ of $\mathbb{P}^n(\mathbb{R})$. If B_r denotes the ball of radius r in $\mathbb{R}^n$ centered at 0, then the Hilbert metrics satisfy $d_{B_r}(x,y) \geq d_{B_1}(x,y)/r$ for any $r \in (0,1)$ and $x, y \in B_r$.*

Proof. Consider a line ℓ of the Euclidean chart $\mathbb{R}^n$ through points $x, y \in B_1$, with $\ell \cap B_1 = \{a, b\}$ and a, x, y, b lying in this order on ℓ. We can parametrize ℓ at unit velocity by $(x_t)_{t\in\mathbb{R}}$ so that $(a,x,y,b) = (x_{-\alpha}, x_0, x_\delta, x_\beta)$ for some $\delta, \alpha, \beta > 0$. We have

$$d_{B_1}(x,y) = \frac{1}{2}\log\left(\frac{\delta+\alpha}{\delta-\beta}\Big/\frac{0+\alpha}{0-\beta}\right) \underset{\delta\to 0}{\sim} \delta\,\frac{\alpha^{-1}+\beta^{-1}}{2}.$$

The factor $\nu_{\ell,x}^{B_1} := (\alpha^{-1}+\beta^{-1})/2$ expresses the Finsler norm associated to the Hilbert metric d_{B_1} near x, in the direction of ℓ, in terms of the ambient Euclidean norm. If we replace B_1 with a scaled ball B_{1-t} for some $t > 0$, then the new endpoints of $\ell \cap B_{1-t}$ lie at linear coordinates $-\alpha_t$ and β_t such that $\frac{\mathrm{d}}{\mathrm{d}\tau}\big|_{\tau=t}\alpha_\tau \leq -1$ and $\frac{\mathrm{d}}{\mathrm{d}\tau}\big|_{\tau=t}\beta_\tau \leq -1$. Therefore

$$\frac{\frac{\mathrm{d}}{\mathrm{d}\tau}\big|_{\tau=t}\nu_{\ell,x}^{B_{1-\tau}}}{\nu_{\ell,x}^{B_{1-t}}} \geq \frac{\alpha_t^{-2}+\beta_t^{-2}}{\alpha_t^{-1}+\beta_t^{-1}} = \frac{\alpha_t/\beta_t+\beta_t/\alpha_t}{\alpha_t+\beta_t} \geq \frac{1}{1-t},$$

where we use $\alpha_t + \beta_t \leq 2 - 2t$ for the last inequality. Integrating this logarithmic derivative over $t \in [0, 1-r]$, we find $\nu_{\ell,x}^{B_r} \geq \nu_{\ell,x}^{B_1}/r$. This is valid for all ℓ and x, hence $d_{B_r} \geq d_{B_1}/r$. $\square$

5.2. **Proof of Propositions 1.6 and 1.8.** Let Γ be the discrete subgroup of $\mathrm{O}(2,1)$ generated by the reflections in the faces of a convex right-angled k-gon in $\mathbb{H}^p = \mathbb{H}^2$, for $k \geq 6$ even. Color the sides of the k-gon, alternatingly, with labels 0 and 1. Applying Proposition 5.1 with $c = 1$ yields, for small enough $0 < t < t'$, faithful and discrete representations $\rho_t, \rho_{t'} : \Gamma \to \mathbb{H}^3$ and $(\rho_t, \rho_{t'})$-equivariant, $\frac{\cosh(t)}{\cosh(t')}$-Lipschitz maps $f^t_{t'} : \mathbb{H}^3 \to \mathbb{H}^3$ (Figure 2 shows a fundamental polyhedron). In particular, $\rho_{t'}$ is uniformly contracting with respect to ρ_t (Definition 3.1), and by Example 3.2 the ρ_t-cocycle $u_t := \frac{\mathrm{d}}{\mathrm{d}\tau}\big|_{\tau=t} \rho_\tau \rho_t^{-1}$ is uniformly contracting since $\frac{\mathrm{d}}{\mathrm{d}\tau}\big|_{\tau=t} \frac{\cosh(t)}{\cosh(\tau)} = -\tanh(t) < 0$. Applying Theorem 3.3, we obtain Propositions 1.8.(a) and 1.6.(a).

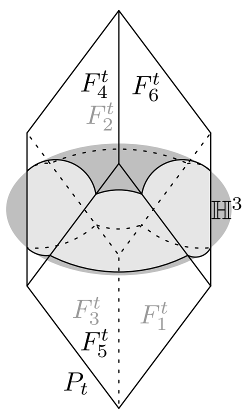

FIGURE 2. A fundamental domain $P_t \cap \mathbb{H}^3$ for the action of $\rho_t(\Gamma)$ on $\mathbb{H}^3$, bounded by k planes $F^t_i = (v^t_i)^\perp$ (here $k = 6$ so Γ is a right-angled hexagon group). The hexahedron P_t becomes vertically more elongated as $t \to 0$. The faces F^t_1, F^t_2, F^t_3 are at the back.

Similarly, in order to prove Propositions 1.8.(b) and 1.6.(b), it is enough to color the faces of the regular (Euclidean) 120-cell with $c+1 = 5$ colors so that adjacent faces receive different colors. This is a well-known construction which we briefly recall below.

The 120-cell can be described as follows. Let $\varphi = \frac{\sqrt{5}+1}{2} = 1.618\ldots$ be the golden ratio. Let $w_1 \ldots, w_{120} \in \mathbb{R}^4$ be the unit vectors obtained from the rows of the matrix

$$\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 1 \\ 0 & \varphi^{-1} & 1 & \varphi \end{pmatrix}$$

by sign changes and even permutations of the four coordinates. We endow $\mathbb{R}^4$ with its standard scalar product $\langle \cdot, \cdot \rangle_{4,0}$. The affine hyperplanes $(w_i + w_i^\perp)_{1 \le i \le 120}$ cut out a regular 120-cell in $\mathbb{R}^4$. Cells of the 120-cell are regular dodecahedra, four of which meet at each vertex, and two cells share a (pentagonal) face if and only if the dual vectors w_i, w_j are *neighbors*, which means by definition that $\langle w_i, w_j \rangle_{4,0} = \varphi/2$. Each w_i has 12 neighbors.

We now explain how to color the 120 vectors w_i (i.e. the corresponding cells) with 5 colors, so that no two neighbors have the same color. Seen as unit quaternions, the w_i form a group which maps surjectively, via the covering $\psi : \mathbb{S}^3 \to \mathrm{SO}(3)$, to the icosahedron group $\mathfrak{A}_5$ (even permutations on 5 symbols $\{0, 1, 2, 3, 4\}$) with kernel $\{1, -1\}$. We color each w_i with the value $\sigma(i) \in \{0, 1, 2, 3, 4\}$ that the associated permutation takes at the symbol 0. Any neighbors w_i, w_j always have different colors: indeed the corresponding permutations differ by a 5-cycle, since

$\mathrm{Re}(w_i^{-1}w_j) = \langle w_i, w_j\rangle_{4,0} = \varphi/2 = \cos(\pi/5)$ shows that $\psi(w_i^{-1}w_j)$ has order 5 in $\mathfrak{A}_5$. Propositions 1.8.(b) and 1.6.(b) are proved.

Remarks 5.3.
(1) The maps $f_{t'}^t$ produced by Proposition 5.1 above are not smooth, but continuous and piecewise projective. Similarly, the equivariant vector fields are not smooth, but they can be made smooth (while remaining uniformly contracting), e.g. by the equivariant convolution procedure described in [DGK1, § 5.5].
(2) If Γ is one of the reflection groups in Propositions 1.6 and 1.8, then Γ admits a finite-index, torsion-free subgroup Γ_1, which is either a surface group (case (a)) or a 4-manifold group (case (b)).
(3) As noticed in the proof of Theorem 3.3.(2), the map $\pi : \mathfrak{o}(p+c,1) \to \mathbb{H}^{p+c}$ endows the affine manifold $\Gamma_1\backslash\mathfrak{o}(p+c,1)$ constructed in the proof above with the structure of an $\mathfrak{o}(p+c)$-bundle over $\Gamma_1\backslash\mathbb{H}^{p+c}$. This bundle structure is smooth if the equivariant vector field is.

5.3. **A variant of Lemma 5.2.** Later, in order to prove Theorem 3.6, we will need the following variant of Lemma 5.2, in which we endow the projective space $\mathbb{P}^n(\mathbb{R})$ with its standard spherical metric: for all $v, w \in \mathbb{R}^{n+1} \smallsetminus \{0\}$,
$$d_{\mathbb{P}^n(\mathbb{R})}([v],[w]) = \measuredangle(v,w) \in \left[0, \frac{\pi}{2}\right].$$

Lemma 5.4. *Let $(H_t)_{t\geq 0}$ be a family of smooth connected open subsets of $\mathbb{P}^n(\mathbb{R})$. Suppose that ∂H_t moves inwards with normal velocity ≥ 1 everywhere at $t=0$. Let $[a,b] \subset \overline{H_0}$ be a segment transverse to ∂H at both endpoints, containing points x, y in its interior. Let a_t, b_t be the endpoints of $[a,b] \cap H_t$, for small t. Then*
$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} d_{[a_t,b_t]}(x,y) \geq 2\, d_{[a_0,b_0]}(x,y).$$

Proof. We may assume that a, x, y, b are lined up in this order. Let ℓ be that line, parametrized at unit speed for the spherical metric $d_{\mathbb{P}^n(\mathbb{R})}$ so that $(a_t, x, y, b_t) = (x_{-\alpha_t}, x_0, x_\delta, x_{\beta_t})$ for some $\delta > 0$, and some $\alpha_t, \beta_t \in (0,\pi)$. By construction, $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \alpha_t \leq -1$ and $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \beta_t \leq -1$, and $\alpha_0 + \beta_0 < \pi$. Then
$$d_{[a_t,b_t]}(x,y) = \frac{1}{2}\log\left(\frac{\tan\delta + \tan\alpha_t}{\tan\delta - \tan\beta_t}\Big/\frac{0+\tan\alpha_t}{0-\tan\beta_t}\right) \underset{\delta\to 0}{\sim} \delta\,\frac{\cot\alpha_t + \cot\beta_t}{2}\ .$$
The factor $\nu_{\ell,x}^t := (\cot\alpha_t + \cot\beta_t)/2$ expresses the Hilbert metric $d_{[a_t,b_t]}$ near x in terms of the ambient spherical metric. Its logarithmic derivative at $t=0$ satisfies
$$\frac{\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \nu_{\ell,x}^t}{\nu_{\ell,x}^0} = \frac{-\left(\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\alpha_t\right)\sin^{-2}\alpha_0 - \left(\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\beta_t\right)\sin^{-2}\beta_0}{\cot\alpha_0 + \cot\beta_0}$$
$$\geq \frac{\sin^{-2}\alpha_0 + \sin^{-2}\beta_0}{\cot\alpha_0 + \cot\beta_0} = \frac{\frac{\sin\alpha_0}{\sin\beta_0} + \frac{\sin\beta_0}{\sin\alpha_0}}{\sin(\alpha_0+\beta_0)} \geq 2.$$
This is valid for all $x \in \ell$: integrating along ℓ, the result follows. $\square$

6. Uniform spacelike contraction implies properness

In this section we prove Theorem 3.6. We fix $p, q \in \mathbb{N}$ with $p+q \geq 1$ and set $G = \mathrm{O}(p, q+1)$.

6.1. **A preliminary lemma.** We first make the following observations. For $x \in \mathbb{H}^{p,q}$, we denote by $T_x^{+1}\mathbb{H}^{p,q}$ the set of unit spacelike tangent vectors at x; it is isometric to the quadric $\{v \in \mathbb{R}^{p,q} \mid \langle v, v\rangle_{p,q} = +1\}$.

Lemma 6.1. *Let Γ be a discrete group and $\rho : \Gamma \to \mathrm{O}(p, q+1)$ a representation with finite kernel and discrete image, preserving a nonempty properly convex open subset Ω of $\mathbb{H}^{p,q} \subset \mathbb{P}(\mathbb{R}^{p,q+1})$. For any compact subset $\mathcal{D}$ of Ω,*

(1) all accumulation points of the $\rho(\Gamma)$-orbit of $\mathcal{D}$ are contained in $\partial\mathbb{H}^{p,q}$;
(2) there exists a bounded family of compact sets $\mathcal{K}_x \subset T_x^{+1}\mathbb{H}^{p,q}$, for x ranging over $\mathcal{D}$, such that for all but finitely many $\gamma \in \Gamma$,
$$\rho(\gamma)\cdot\mathcal{D} \subset \bigcap_{x\in\mathcal{D}} \exp_x(\mathbb{R}^+\mathcal{K}_x);$$
(3) in particular, if $(\gamma_n) \in \Gamma^{\mathbb{N}}$ goes to infinity (i.e. leaves every finite subset of Γ), then for any sequences $(x_n), (x_n') \in \mathcal{D}^{\mathbb{N}}$ we have $d_{\mathbb{H}^{p,q}}(x_n, \rho(\gamma_n)\cdot x_n') \to +\infty$.

Proof. (1) Suppose by contradiction that there are sequences $(x_n) \in \mathcal{D}^{\mathbb{N}}$ and $(\gamma_n) \in \Gamma^{\mathbb{N}}$ such that the γ_n are pairwise distinct and $y_n := \rho(\gamma_n)\cdot x_n$ converges to some $y \in \mathbb{H}^{p,q}$. We can lift the $x_n \in \mathbb{H}^{p,q}$ to vectors $v_n \in \mathbb{R}^{p,q+1}$ with $\langle v_n, v_n\rangle_{p,q+1} = -1$; both the v_n and the $\rho(\gamma_n)\cdot v_n$ stay in a compact subset of $\mathbb{R}^{p,q+1}$ and $\rho(\gamma_n)\cdot v_n$ converges to a unit timelike vector v. On the other hand, since ρ has finite kernel and discrete image, there exists a basis vector w of $\mathbb{R}^{p,q+1}$ such that $(\rho(\gamma_n)\cdot w)_{n\in\mathbb{N}}$ leaves every compact subset of $\mathbb{R}^{p,q+1}$. Up to passing to a subsequence, we may assume that the direction of $\rho(\gamma_n)\cdot w$ converges to some null direction ℓ. There exists $\varepsilon > 0$ such that all segments $[v_n - \varepsilon w, v_n + \varepsilon w] \subset \mathbb{R}^{p,q+1} \smallsetminus \{0\}$ project to segments σ_n contained in Ω. The images $\rho(\gamma_n)\cdot\sigma_n$, which are again contained in Ω, converge to the full projective line spanned by v and ℓ. This contradicts the proper convexity of Ω. Thus the $\rho(\Gamma)$-orbit of $\mathcal{D}$ does not have any accumulation point in $\mathbb{H}^{p,q}$.

(2) Let $y \in \partial\mathbb{H}^{p,q}$ be an accumulation point of the orbit $\rho(\Gamma)\cdot\mathcal{D}$, and consider $x \in \mathcal{D}$. Then y cannot be seen from x in a timelike direction since timelike geodesics do not meet $\partial\mathbb{H}^{p,q}$. It cannot be seen in a lightlike direction either: otherwise, the tangent plane to $\partial\mathbb{H}^{p,q}$ at y contains the interval $[x, y) \subset \Omega$, but any small perturbation $[x', y)$ still lies in Ω — however x' can be chosen so that this perturbation crosses $\partial\mathbb{H}^{p,q}$, which would contradict $\Omega \subset \mathbb{H}^{p,q}$. Therefore $y \in \partial\mathbb{H}^{p,q}$ is seen from x in a spacelike direction. We conclude using the compactness of the accumulation set.

(3) The third statement is an easy consequence of (1) and (2). $\square$

6.2. **Properness for affine actions on $\mathfrak{g} = \mathfrak{o}(p, q+1)$.** In order to prove Theorem 3.6.(2), the strategy is to use a "coarse" analogue of the proof of Theorem 3.3.(2) from Section 4.2: given a (ρ, u)-equivariant vector field X with contracting properties, we build an equivariant coarse projection π from $\mathfrak{g}$ to some set on which Γ is known to act properly discontinuously. There are several possibilities for this set; one of the them is the set $\mathcal{F}(\Gamma)$ of finite subsets of Γ, endowed with the discrete topology and with the action of Γ by left multiplication. Theorem 3.6.(2) is an immediate consequence of the following.

Proposition 6.2. *Let Γ be a discrete group, $\rho : \Gamma \to G = \mathrm{O}(p, q+1)$ a representation with finite kernel and discrete image, preserving a properly convex open subset Ω of $\mathbb{H}^{p,q}$, and $u : \Gamma \to \mathfrak{g}$ a ρ-cocycle. Choose $x \in \Omega$ and an equivariant family of norms $\|\cdot\|_\gamma$ on $T_{\rho(\gamma)\cdot x}\mathbb{H}^{p,q}$, for $\gamma \in \Gamma$. Let X be a (ρ, u)-equivariant vector field on $\rho(\Gamma)\cdot x$ which is c-lipschitz in spacelike directions, for some $c < 0$. Then the map*

$$\begin{array}{rccc} \pi : & \mathfrak{g} & \longrightarrow & \mathcal{F}(\Gamma) \\ & Y & \longmapsto & \big\{\gamma \in \Gamma \mid \|(X - Y)(\rho(\gamma)\cdot x)\|_\gamma \text{ is minimal}\big\} \end{array}$$

(where we view $\mathfrak{g}$ as the set of Killing vector fields on $\mathbb{H}^{p,q}$, see Section 2.3) is well defined and takes any compact set to a compact set. Moreover, π is equivariant with respect to the affine action of Γ on $\mathfrak{g}$ via (ρ, u) and the action of Γ on $\mathcal{F}(\Gamma)$ by left multiplication. In particular, the affine action of Γ on $\mathfrak{g}$ via (ρ, u) is properly discontinuous.

Proof. We first observe that for any $a, b \in \mathbb{H}^{p,q}$ on a spacelike line and for any $Z(a) \in T_a\mathbb{H}^{p,q}$ and $Z(b) \in T_b\mathbb{H}^{p,q}$,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} d_{\mathbb{H}^{p,q}}\big(\exp_a(tZ(a)),\ \exp_b(tZ(b))\big) \\ &\qquad = -\mathsf{g}^{p,q}_a\big(Z(a), v_a^b\big) - \mathsf{g}^{p,q}_b\big(Z(b), v_b^a\big), \end{aligned} \tag{6.1}$$

where $\mathsf{g}^{p,q}$ is the pseudo-Riemannian metric on $\mathbb{H}^{p,q}$ as in Section 2.2, and $v_a^b \in T_a^{+1}\mathbb{H}^{p,q}$ is the unit vector at a pointing to b, and similarly for v_b^a.

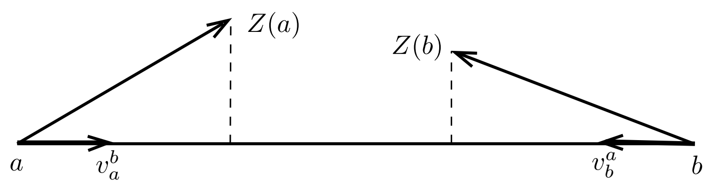

FIGURE 3. Illustration of formula (6.1)

By Lemma 6.1.(2), there is a compact set $\mathcal{K}_x \subset T_x^{+1}\mathbb{H}^{p,q}$ of spacelike directions in which the point x sees the points $\rho(\gamma)\cdot x$ for all $\gamma \in \Gamma$ outside some finite set F. By compactness, there exists $R > 0$ such that

$$\mathsf{g}^{p,q}_x(w, v) \leq R\,\|w\|_e$$

for all $w \in T_x\mathbb{H}^{p,q}$ and $v \in \mathcal{K}_x$, where e denotes the identity element of Γ. Consider $\gamma \in \Gamma \smallsetminus (F \cup F^{-1})$. By equivariance, $\rho(\gamma)\cdot x$ sees x in a spacelike direction in $\mathcal{K}_{\rho(\gamma)\cdot x} := \rho(\gamma)_*\mathcal{K}_x$, and $\mathsf{g}^{p,q}_{\rho(\gamma)\cdot x}(w,v) \leq R\,\|w\|_\gamma$ for all $w \in T_{\rho(\gamma)\cdot x}\mathbb{H}^{p,q}$ and $v \in \mathcal{K}_{\rho(\gamma)\cdot x}$. Applying (6.1) with $a = x$ and $b = \rho(\gamma)\cdot x$, we obtain that for any vector field Z defined at both x and $\rho(\gamma)\cdot x$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} d_{\mathbb{H}^{p,q}}\big(\exp_x(tZ(x)),\ \exp_{\rho(\gamma)\cdot x}(tZ(\rho(\gamma)\cdot x))\big)&\\ \geq -R\,\|Z(x)\|_e - R\,\|Z(\rho(\gamma)\cdot x)\|_\gamma.&\end{aligned}$$

Now consider $Y \in \mathfrak{g}$ and take the vector field $Z = X - Y$ on $\rho(\Gamma)\cdot x$, which is c-lipschitz in spacelike directions. Since $(x, \rho(\gamma)\cdot x)$ are in spacelike position, by (3.1) we obtain

$$R\,\|(X-Y)(\rho(\gamma)\cdot x)\|_\gamma \geq |c|\, d_{\mathbb{H}^{p,q}}(x, \rho(\gamma)\cdot x) - R\,\|(X-Y)(x)\|_e$$

for all $\gamma \in \Gamma \smallsetminus (F \cup F^{-1})$. The term $R\,\|(X-Y)(x)\|_e$ is independent of γ and remains bounded as Y varies in a compact set, while the term $|c|\, d_{\mathbb{H}^{p,q}}(x, \rho(\gamma)\cdot x)$ is independent of Y and goes to $+\infty$ as γ goes to infinity in Γ, by Lemma 6.1.(3). This shows that π is well defined and takes compact sets to compact sets.

The Γ-equivariance of π follows from that of the collection of norms $\|\cdot\|_\gamma$ and from the identity

$$\begin{aligned}&\big(X - (\rho,u)(\gamma)Y\big)(\rho(\gamma)\cdot x)\\ =\ &\rho(\gamma)_*X(x) + u(\gamma)(\rho(\gamma)\cdot x) - (\mathrm{Ad}(\rho(\gamma))Y + u(\gamma))(\rho(\gamma)\cdot x)\\ =\ &\rho(\gamma)_*((X-Y)(x))\end{aligned}$$

for all $\gamma \in \Gamma$, where Γ acts affinely on $\mathfrak{g}$ via (ρ, u).

Since the action of Γ on $\mathcal{F}(\Gamma)$ by left multiplication is properly discontinuous, it follows from equivariance that the affine action of Γ on $\mathfrak{g}$ via (ρ, u) is properly discontinuous. ☐

Remark 6.3. Let Γ be a discrete group, $\rho : \Gamma \to G = \mathrm{O}(p, q+1)$ a representation with finite kernel and discrete image, preserving a properly convex open subset Ω of $\mathbb{H}^{p,q}$, and $u : \Gamma \to \mathfrak{g}$ a ρ-cocycle. Suppose Γ acts cocompactly on a $\rho(\Gamma)$-invariant subset $\mathcal{O}$ of Ω, and that there exist a (ρ, u)-equivariant vector field X on $\mathcal{O}$ and constants $c < 0$ and $c' \geq 0$ such that for any $x, y \in \mathcal{O}$ on a spacelike line,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} d_{\mathbb{H}^{p,q}}\big(\exp_x(tX(x)), \exp_y(tX(y))\big) \leq c\, d_{\mathbb{H}^{p,q}}(x,y) + c'.$$

Similarly to Proposition 6.2, we can then use Lemma 6.1 to construct a $((\rho,u),\rho)$-equivariant map

$$\pi : \mathfrak{g} \longrightarrow \{\text{compact subsets of } \mathcal{O}\}$$

sending any compact set to a compact set. This map is defined by choosing a ρ-equivariant family of norms $\|\cdot\|_y$ on $T_y\mathbb{H}^{p,q}$ for $y \in \mathcal{O}$, and sending any $Y \in \mathfrak{g}$ to the set of $y \in \mathcal{O}$ that minimize $\|(X-Y)(y)\|_y$. Proposition 6.2 is the case where $\mathcal{O}$ is a single orbit.

6.3. **Properness for actions on $G = \mathrm{O}(p, q+1)$ by right and left multiplication.** We now prove Theorem 3.6.(1), again using a coarse projection. We still denote by $\mathcal{F}(\Gamma)$ the set of finite subsets of Γ, endowed with the discrete topology. We use the notation $\|\cdot\|$ both for the standard Euclidean norm on $\mathbb{R}^{p,q+1}$ and for the corresponding operator norm on $\mathrm{End}(\mathbb{R}^{p,q+1})$.

Proposition 6.4. *Let Γ be a discrete group and $\rho : \Gamma \to G = \mathrm{O}(p, q+1)$ a representation with finite kernel and discrete image, preserving a nonempty properly convex open subset Ω of $\mathbb{H}^{p,q}$. Let $\rho' : \Gamma \to G$ be a strongly irreducible representation such that $\rho'(\Gamma)$ contains a proximal element. If ρ' is uniformly contracting in spacelike directions with respect to ρ, then the map*

$$\begin{array}{rccc} \Pi : & G & \longrightarrow & \mathcal{F}(\Gamma) \\ & g & \longmapsto & \big\{\gamma \in \Gamma \mid \|\rho'(\gamma)^{-1} g \rho(\gamma)\| \text{ is minimal}\big\} \end{array}$$

is well defined and takes any compact set to a compact set. Moreover, Π is equivariant with respect to the action of Γ on G by right and left multiplication via (ρ, ρ') and the action of Γ on $\mathcal{F}(\Gamma)$ by left multiplication. In particular, the action of Γ on G by right and left multiplication via (ρ, ρ') is properly discontinuous.

In order to prove Proposition 6.4, we need some preliminary results. For any $1 \leq i \leq p+q+1$ and any $g \in \mathrm{End}(\mathbb{R}^{p,q+1})$, we denote by $\lambda_i(g)$ (resp. $\mu_i(g)$) the logarithm of the modulus of the i-th largest eigenvalue (resp. singular value) of g. We have $\mu_1(g) = \log \|g\|$. An element $g \in G$ is proximal if and only if $\lambda_1(g) > \lambda_2(g)$. Note that $\lambda_i(g) = -\lambda_{p+q+2-i}(g)$ for all i, which implies in particular that any proximal element $g \in G = \mathrm{O}(p, q+1)$ has, not only an attracting fixed point, but also a repelling fixed point in $\mathbb{P}(\mathbb{R}^{p,q+1})$; these points belong to $\partial\mathbb{H}^{p,q}$.

Our first preliminary result is the following.

Lemma 6.5. *Let $g \in G = \mathrm{O}(p, q+1)$ and let $y \in \mathbb{H}^{p,q}$.*

(1) We have
$$\limsup_{n\to+\infty} \frac{1}{n}\, d_{\mathbb{H}^{p,q}}(y, g^n y) \leq \lambda_1(g).$$

(2) If g is proximal in $\mathbb{P}(\mathbb{R}^{p,q+1})$, with attracting and repelling fixed points $\xi_g^\pm \in \partial\mathbb{H}^{p,q}$, and if $y \notin (\xi_g^+)^\perp \cup (\xi_g^-)^\perp$, then
$$\frac{1}{n}\, d_{\mathbb{H}^{p,q}}(y, g^n y) \underset{n\to+\infty}{\longrightarrow} \lambda_1(g).$$

Proof. (1) By writing the Jordan decomposition of g as the commuting product of a hyperbolic, a unipotent, and an elliptic element, we see that $\|g^n\|$ is bounded above by a polynomial times $\mathrm{e}^{n\lambda_1(g)}$, hence so is $\langle v, g^n v\rangle_{p,q+1}$ where $[v] = y$. We conclude using (2.2).

(2) Again, by (2.2), it suffices to study the growth of $\langle v, g^n v\rangle_{p,q+1}$ where $[v] = y$. The projective hyperplane $(\xi_g^\pm)^\perp$ is the projectivization of the sum of the generalized eigenspaces of g for eigenvalues other than $\mathrm{e}^{\mp\lambda_1(g)}$.

Therefore the assumption on y means that v, when decomposed over the generalized eigenspaces of g, has nonzero components v^+, v^- along ξ_g^+ and ξ_g^-. These components are *not* orthogonal. In the pairing $\langle v, g^n v\rangle_{p,q+1}$, the term $\langle v^-, g^n v^+\rangle_{p,q+1} = e^{n\lambda_1(g)}\langle v^-, v^+\rangle_{p,q+1}$ therefore dominates all the others, and grows like $e^{n\lambda_1(g)}$ as $n \to +\infty$. We conclude using (2.2). $\square$

We shall also use the following fact, obtained by combining a result of Abels–Margulis–Soifer [AMS1, Th. 5.17] with a small compactness argument of Benoist [Be, Lem. 2.4].

Fact 6.6 ([AMS1, Be])**.** *Let Γ be a discrete group and $\rho' : \Gamma \to G = \mathrm{O}(p, q+1)$ a strongly irreducible representation such that $\rho'(\Gamma)$ contains a proximal element. Then there exist a finite set $F \subset \Gamma$ and a constant $C_{\rho'} > 0$ such that for any $\gamma \in \Gamma$, we may find $f \in F$ such that $\rho'(\gamma f)$ is proximal in $\mathbb{P}(\mathbb{R}^{p,q+1})$ and satisfies $\lambda_1(\rho'(\gamma f)) \geq \mu_1(\rho'(\gamma)) - C_{\rho'}$.* $\square$

Lemma 6.7. *Let Γ be a discrete group and $\rho : \Gamma \to G = \mathrm{O}(p, q+1)$ a representation with finite kernel and discrete image, preserving a nonempty properly convex open subset Ω of $\mathbb{H}^{p,q}$. Let $\rho' : \Gamma \to G$ be a strongly irreducible representation such that $\rho'(\Gamma)$ contains a proximal element. If ρ' is uniformly contracting in spacelike directions with respect to ρ, then there exist $C < 1$ and $C' \geq 0$ such that*

(1) $\lambda_1(\rho'(\gamma)) \leq C\,\lambda_1(\rho(\gamma))$ for all $\gamma \in \Gamma$ with $\rho'(\gamma)$ proximal in $\mathbb{P}(\mathbb{R}^{p,q+1})$;
(2) $\mu_1(\rho'(\gamma)) \leq C\,\mu_1(\rho(\gamma)) + C'$ for all $\gamma \in \Gamma$.

Proof of Lemma 6.7. (1) Let $\mathcal{O}$ be a $\rho(\Gamma)$-invariant subset of Ω and $f : \mathcal{O} \to \mathbb{H}^{p,q}$ a (ρ, ρ')-equivariant map which is C-Lipschitz in spacelike directions for some $C < 1$. Consider $\gamma \in \Gamma$ such that $\rho'(\gamma)$ is proximal in $\mathbb{P}(\mathbb{R}^{p,q+1})$; in particular, $\lambda_1(\rho'(\gamma)) > 0$. Let $H^{\pm} \subset \mathbb{R}^{p,q+1}$ be the sum of the generalized eigenspaces of $\rho'(\gamma)$ for eigenvalues of modulus $\neq e^{\mp\lambda_1(\rho'(\gamma))}$. Suppose by contradiction that $f(\mathcal{O}) \subset H^+ \cup H^-$. Since $f(\mathcal{O})$ is $\rho'(\Gamma)$-invariant, so is the Zariski closure Z of $f(\mathcal{O})$. Any irreducible component Z_i of Z is contained either in H^+ or in H^-, hence spans a proper subspace of $\mathbb{R}^{p,q+1}$. The union of these subspaces is preserved by $\rho'(\Gamma)$, contradicting strong irreducibility. Therefore there exists $x \in \mathcal{O}$ such that $f(x) \notin H^+ \cup H^-$, and Lemma 6.5.(2) gives

$$\lim_{n\to+\infty} \frac{1}{n}\, d_{\mathbb{H}^{p,q}}(f(x), \rho'(\gamma)^n \cdot f(x)) = \lambda_1(\rho'(\gamma)).$$

On the other hand, by Lemma 6.1, for any large enough $n \in \mathbb{N}$ the points x and $\rho(\gamma^n)\cdot x$ are on a spacelike line. Hence, by assumption on f, the number above is at most

$$\limsup_{n\to+\infty} \frac{1}{n}\, C\, d_{\mathbb{H}^{p,q}}(x, \rho(\gamma)^n \cdot x) \leq C\lambda_1(\rho(\gamma)),$$

where we use Lemma 6.5.(1) for the last inequality.

(2) Let F and $C_{\rho'}$ be given by Fact 6.6, and let

$$C' := C_{\rho'} + C \max_{f\in F} \mu_1(\rho(f)) > 0.$$

For any $\gamma \in \Gamma$, we can find $f \in F$ such that $\rho'(\gamma f)$ is proximal in $\mathbb{P}(\mathbb{R}^{p,q+1})$ and $\mu_1(\rho'(\gamma)) \leq \lambda_1(\rho'(\gamma f)) + C_{\rho'}$. By (1), we have $\lambda_1(\rho'(\gamma f)) \leq C\,\lambda_1(\rho(\gamma f))$. For any $g \in G$ we have $\mu_1(g) = \log\|g\|$, hence $\mu_1(g) \geq \lambda_1(g)$ and $\mu_1(gg') \leq \mu_1(g) + \mu_1(g')$ for all $g, g' \in G$. We deduce

$$\mu_1(\rho'(\gamma)) \leq C\,\lambda_1(\rho(\gamma f)) + C_{\rho'} \leq C\,\mu_1(\rho(\gamma f)) + C_{\rho'} \leq C\,\mu_1(\rho(\gamma)) + C'. \qquad \square$$

Proof of Proposition 6.4. Let $K = \mathrm{O}(p) \times \mathrm{O}(q+1)$, let $\mathbb{X}_{p,q+1} = G/K$ be the Riemannian symmetric space of $G = \mathrm{O}(p, q+1)$, with basepoint $x_0 \in \mathbb{X}_{p,q+1}$ fixed by K. The function $d_1 : \mathbb{X}_{p,q+1} \times \mathbb{X}_{p,q+1} \to \mathbb{R}^+$ given by

$$d_1(g \cdot x_0, g' \cdot x_0) = \mu_1(g'^{-1}g) = \log\|g'^{-1}g\|$$

for $g, g' \in G$ is a G-invariant Finsler metric on $\mathbb{X}_{p,q+1}$. Lemma 6.7.(2) states the existence of $C < 1$ and $C' > 0$ such that for all $\gamma \in \Gamma$,

$$d_1(x_0, \rho'(\gamma) \cdot x_0) \leq C\,d_1(x_0, \rho(\gamma) \cdot x_0) + C'.$$

For any $g \in G$ and $\gamma \in \Gamma$ we have

$$\begin{aligned}
& \|\rho'(\gamma)^{-1} g \rho(\gamma)\| \\
= \;& d_1(\rho(\gamma) \cdot x_0, g^{-1}\rho'(\gamma) \cdot x_0) \\
\geq \;& d_1(x_0, \rho(\gamma) \cdot x_0) - d_1(x_0, g^{-1} \cdot x_0) - d_1(g^{-1} \cdot x_0, g^{-1}\rho'(\gamma) \cdot x_0) \\
= \;& d_1(x_0, \rho(\gamma) \cdot x_0) - d_1(x_0, g^{-1} \cdot x_0) - d_1(x_0, \rho'(\gamma) \cdot x_0) \\
\geq \;& (1 - C)\,d_1(x_0, \rho(\gamma) \cdot x_0) - \big(d_1(x_0, g^{-1} \cdot x_0) + C'\big).
\end{aligned}$$

Since ρ has finite kernel and discrete image, for any $R > 0$ there are only finitely many elements $\gamma \in \Gamma$ such that $d_1(x_0, \rho(\gamma) \cdot x_0) \leq R$. We deduce that Π is well defined. Moreover, since the function μ_1 is continuous, we see that Π sends any compact set to a compact set. The Γ-equivariance is clear.

Since the action of Γ on $\mathcal{F}(\Gamma)$ by left multiplication is properly discontinuous, it follows from equivariance that the action of Γ on G by right and left multiplication via (ρ, ρ') is properly discontinuous. $\square$

Remark 6.8. Let Γ be a discrete group and $\rho, \rho' : \Gamma \to G = \mathrm{O}(p, q+1)$ two representations, such that ρ has finite kernel and discrete image. Suppose Γ acts cocompactly on a $\rho(\Gamma)$-invariant subset $\mathbb{O}$ of $\mathbb{X}_{p,q+1}$, and that there are a (ρ, ρ')-equivariant map $f_{\mathbb{X}} : \mathbb{O} \to \mathbb{X}_{p,q+1}$, a G-invariant metric d on $\mathbb{X}_{p,q+1}$, and constants $C < 1$ and $C' \geq 0$ such that for all $x, y \in \mathbb{O}$,

$$d\big(f_{\mathbb{X}}(x), f_{\mathbb{X}}(y)\big) \leq C\,d(x, y) + C'.$$

Extending Proposition 6.4, we can then construct a $((\rho, \rho'), \rho)$-equivariant map

$$\Pi : G \longrightarrow \{\text{compact subsets of } \mathbb{O}\}$$

sending any compact set to a compact set. This map is defined by sending any $g \in G$ to the set of $y \in \mathbb{O}$ minimizing $d(y, g^{-1} \cdot f_{\mathbb{X}}(y))$. Proposition 6.4 corresponds to $\mathbb{O}$ a single orbit and to $d = d_1$.

7. Uniform spacelike contraction for right-angled Coxeter groups

In this section we prove Theorems 1.5 and 1.7 using the sufficient conditions for properness provided by Theorem 3.6.

In the whole section we fix a right-angled Coxeter group

$$\Gamma = \langle \gamma_1, \dots, \gamma_k \mid (\gamma_i\gamma_j)^{m_{i,j}} = 1 \quad \forall i, j \rangle,$$

where $m_{i,i} = 1$ and $m_{i,j} \in \{2, \infty\}$ for all $i \neq j$. Recall that Γ is said to be *irreducible* if the generating set $S = \{\gamma_1, \dots, \gamma_k\}$ cannot be written as a nontrivial disjoint union $S = S' \sqcup S''$ such that the groups generated by S' and by S'' commute. In the whole section we assume Γ to be irreducible, and infinite (i.e. $k \geq 2$).

7.1. **The canonical representation and its deformations.** The matrix $M_1 = (-\cos(\pi/m_{i,j}))_{1\leq i,j\leq k}$ (with the convention $\pi/\infty = 0$) is called the *Gram matrix* for Γ. It defines a (possibly degenerate) symmetric bilinear form $\langle\cdot,\cdot\rangle_1$ on $\mathbb{R}^k$. Let $(e_1, \dots, e_k)$ be the standard basis of $\mathbb{R}^k$. The *canonical representation* $\rho_1 : \Gamma \to \mathrm{GL}(k, \mathbb{R})$ is defined by

$$\rho_1(\gamma_i) \ : \ x \mapsto x - 2\langle x, e_i\rangle_1 \, e_i$$

for all generators γ_i. Tits proved that ρ_1 is injective and discrete (see [H, Cor 5.4] or [Bo, § V.4]) and acts as a reflection group on a convex open subset of $\mathbb{P}(\mathbb{R}^k)$. Since the Coxeter group Γ is irreducible, the canonical representation is irreducible: see [Da, Cor. 6.12.8].

For $t > 1$, the matrix $M_t = (M_t(i,j))_{1\leq i,j\leq k}$ where

$$M_t(i,j) = \begin{cases} 1 & \text{if } m_{i,j} = 1, \text{ i.e. } i = j, \\ 0 & \text{if } m_{i,j} = 2, \\ -t & \text{if } m_{i,j} = \infty \end{cases}$$

still defines a symmetric bilinear form $\langle\cdot,\cdot\rangle_t$ on $\mathbb{R}^k$. Note that $\det(M_t)$ is a nonzero polynomial in t (take $t = 0$), hence it is nonzero outside of some finite set F of exceptional values of t. Let $I \subset (1, +\infty) \smallsetminus F$ be an open interval. For any $t \in I$, the form $\langle\cdot,\cdot\rangle_t$ is nondegenerate of constant signature $(p, q+1)$ for some $p, q \in \mathbb{N}$. We define the representation $\rho_t : \Gamma \to \mathrm{Aut}(\langle\cdot,\cdot\rangle_t) \simeq \mathrm{O}(p, q+1)$ by

$$\rho_t(\gamma_i) : v \mapsto v - 2\langle v, e_i\rangle_t \, e_i$$

for all i. It is immediate to check that for any $i \neq j$ with $m_{i,j} = \infty$, the element $\rho_t(\gamma_i\gamma_j)$ is proximal in $\mathbb{P}(\mathbb{R}^k)$. In particular, $\rho_t(\Gamma)$ is infinite and so $p \geq 1$. The convex cone

$$\widetilde{\Delta}_t = \{v \in \mathbb{R}^k \mid \langle v, e_i\rangle_t \leq 0 \ \ \forall i\}$$

descends to a convex polytope Δ_t in an affine chart of $\mathbb{P}(\mathbb{R}^k)$. By [V, Th. 2 & 5], the representation ρ_t is discrete, faithful, the set $\rho_t(\Gamma) \cdot \Delta_t$ is convex in $\mathbb{P}(\mathbb{R}^k)$, and the action of Γ on the open set

$$\mathcal{U}_t := \mathrm{Int}\,(\rho_t(\Gamma)\cdot\Delta_t)$$

is properly discontinuous. By [V, Prop. 19], the representation ρ_t is irreducible, hence $\mathcal{U}_t$ is properly convex, i.e. contains no projective subspace. In fact, if $k \geq 3$, then ρ_t is strongly irreducible (see [DGK5, Prop. 3.8.(3)]).

Remark 7.1. The polytope Δ_t is a simplex spanned by the $[e'_i(t)]$, where $e'_1(t), \dots, e'_k(t)$ are the column vectors of the matrix $-M_t^{-1}$, i.e. $\langle e'_i(t), e_j\rangle_t = -\delta_{ij}$ for all $1 \leq i,j \leq k$.

7.2. **Construction of Ω_t.** The $\rho_t(\Gamma)$-invariant open set $\mathcal{U}_t \subset \mathbb{P}(\mathbb{R}^k)$ above is properly convex, but not necessarily contained in

$$\mathbb{H}^{p,q}_t := \{[v] \in \mathbb{P}(\mathbb{R}^k) \mid \langle v, v\rangle_t < 0\}.$$

With the eventual goal of applying Theorem 3.6, we must find a $\rho_t(\Gamma)$-invariant properly convex open set Ω_t which is contained in $\mathbb{H}^{p,q}_t$.

Using the nondegenerate symmetric bilinear form $\langle\cdot,\cdot\rangle_t$, we can view the dual convex set $\mathcal{U}_t^*$ of $\mathcal{U}_t$ as a properly convex open subset of $\mathbb{P}(\mathbb{R}^k)$ as in (2.1). The intersection

$$\Omega_t := \mathcal{U}_t \cap \mathcal{U}_t^*$$

is still open and properly convex.

Lemma 7.2. *For any $t \in I$, the set Ω_t is nonempty and contained in $\mathbb{H}^{p,q}_t$.*

Proof. We first check that Ω_t is nonempty. For any i,j with $m_{i,j} = \infty$, the element $\rho_t(\gamma_i\gamma_j)$ is proximal in $\mathbb{P}(\mathbb{R}^{p,q+1})$, and its attracting fixed point must belong to $\overline{\mathcal{U}_t} \cap \overline{\mathcal{U}_t^*}$. Thus $\overline{\mathcal{U}_t} \cap \overline{\mathcal{U}_t^*}$ is nonempty, closed, and $\rho_t(\Gamma)$-invariant. It must have nonempty interior since ρ_t is irreducible. Hence Ω_t is nonempty.

Let $e'_1(t), \dots, e'_k(t)$ be as in Remark 7.1. Lift $\Omega_t \subset \mathcal{U}_t$ to convex open cones $\widetilde{\Omega}_t \subset \widetilde{\mathcal{U}}_t$ of $\mathbb{R}^k$: by definition, points of $\widetilde{\Omega}_t$ pair negatively with all points of $\widetilde{\mathcal{U}}_t$. Taking limits, for all $x \in \widetilde{\Omega}_t$ we have $\langle x, e'_i(t)\rangle_t \leq 0$. It follows that $\Omega_t \cap \Delta_t$ is contained in the truncated simplex

$$\Sigma_t := \Delta_t \cap \mathbb{P}\{v \,|\, \langle v, e'_i(t)\rangle_t \leq 0\} = \Delta_t \cap \mathbb{P}\left(\textstyle\sum_{i=1}^k \mathbb{R}^+ e_i\right)$$

$$= \mathbb{P}\left\{v = \textstyle\sum_{i=1}^k s_i e_i \;\middle|\; s_i \geq 0 \text{ and } \langle v, e_i\rangle_t \leq 0 \text{ for all } i\right\} \tag{7.1}$$

and $\Omega_t \subset \rho_t(\Gamma)\Sigma_t$. The proof concludes by showing that $\Sigma_t \subset \mathbb{H}^{p,q}_t$.

It is easy to see that $\Sigma_t \subset \overline{\mathbb{H}^{p,q}_t}$, just check that any $v = \sum_{i=1}^k s_i e_i \in (\mathbb{R}^+)^k$ pairs to a nonpositive value with itself: we have $\langle v, v\rangle_t = \sum_{i=1}^k s_i \langle v, e_i\rangle_t \leq 0$ since $s_i \geq 0$ and $\langle v, e_i\rangle_t \leq 0$ by definition of Δ_t. To see that $\Sigma_t \subset \mathbb{H}^{p,q}_t$, suppose $v = \sum_{a=1}^{\ell} s_{i_a} e_{i_a}$ with $s_{i_a} > 0$ and the i_a distinct. Assume $\langle v, v\rangle_t = 0$ and aim for a contradiction: we have

$$0 = \langle v, v\rangle_t = \textstyle\sum_{a=1}^{\ell} s_{i_a} \langle e_{i_a}, v\rangle_t,$$

hence $\langle e_{i_a}, v\rangle_t = \sum_{b=1}^{\ell} s_{i_b}\langle e_{i_a}, e_{i_b}\rangle_t = 0$ for all $1 \leq a \leq \ell$. Thus

$$s_{i_a} = t \sum_{\substack{b\in\{1,\dots,\widehat{a},\dots,\ell\}\\ m_{i_a,i_b}=\infty}} s_{i_b}$$

where the sum on the right is nonempty. Choosing a such that s_{i_a} is the smallest nonzero coefficient and using $t > 1$, this yields a contradiction. $\square$

Remark 7.3. Similar convex domains in $\mathbb{H}^{p,q}$ were studied by Dyer [Dy] and Dyer–Hohlweg–Ripoll [DHR] in the context of Kac–Moody algebras. Although the language is somewhat different, it follows from their work that indeed Ω_t is the interior of the orbit $\rho_t(\Gamma)\Sigma_t$ of the truncated simplex. More specifically, $\rho_t(\Gamma)\Sigma_t$ is the smallest convex set containing the closure of the attracting fixed points of $\rho_t(\Gamma)$.

Remark 7.4. The region $\rho_t(\Gamma)\Sigma_t$ is a union of closed sets. In the case where Γ is word hyperbolic, $\rho_t(\Gamma)\Sigma_t$ is a closed subset of $\mathbb{H}^{p,q}$ and its only accumulation points lie on $\partial\mathbb{H}^{p,q}$. Indeed, the condition that no point of Δ_t with infinite stabilizer survives in Σ_t can be shown to be equivalent to Moussong's criterion [Mo] for hyperbolicity of Γ. The action of Γ via ρ_t is proper and cocompact, and indeed the subgroup $\rho_t(\Gamma)$ satisfies a notion of convex cocompactness in $\mathbb{H}^{p,q}$ recently introduced in [DGK3] (see also [DGK4, DGK5]).

7.3. **Building contracting maps and cocycles.** For $t < t'$ in I, let us build a $(\rho_{t'}, \rho_t)$-equivariant map $\Omega_{t'} \to \mathbb{H}^{p,q}_t$ that is uniformly contracting in spacelike directions. (Note that this is the opposite direction from the $(\rho_t, \rho_{t'})$-equivariant maps of Section 5.)

Let $\langle\cdot,\cdot\rangle_t$, $\langle\cdot,\cdot\rangle_0$, and $\langle\!\langle\cdot,\cdot\rangle\!\rangle_t$ be the symmetric bilinear forms on $\mathbb{R}^k$ defined by the matrices M_t, Id, and M_t^{-1} respectively, and let $\perp$, $\perp_0$, and $\perp_{\langle\!\langle}$ refer to the corresponding notions of orthogonality. We have $\langle x, y\rangle_t = \langle\!\langle M_t x, M_t y\rangle\!\rangle_t$ for all $x, y \in \mathbb{R}^k$. The map M_t thus takes the pair $(\Sigma_t, \langle\cdot,\cdot\rangle_t)$ to $(\Sigma_t^{\langle\!\langle}, \langle\!\langle\cdot,\cdot\rangle\!\rangle_t)$ where

$$\Sigma_t^{\langle\!\langle} := \mathbb{P}\left\{ v \in \textstyle\sum_{i=1}^k \mathbb{R}^+ M_t e_i \;\middle|\; \langle\!\langle v, M_t e_i\rangle\!\rangle_t \leq 0 \right\}.$$

The reflection walls of $\Sigma_t^{\langle\!\langle}$ are the $(M_t e_i)^{\perp_{\langle\!\langle}} = e_i^{\perp_0}$: these are independent of t. Therefore $\Phi_t^{t'} := M_t^{-1} M_{t'}$ takes the reflecting walls of $\Sigma_{t'}$ to those of Σ_t; in the commutative diagram

$$\begin{array}{ccc}
(\mathbb{R}^k, \langle\cdot,\cdot\rangle_{t'}) & \xrightarrow{\Phi_t^{t'}} & (\mathbb{R}^k, \langle\cdot,\cdot\rangle_t) \\
{\scriptstyle M_{t'}}\big\downarrow & & \big\downarrow{\scriptstyle M_t} \\
(\mathbb{R}^k, \langle\!\langle\cdot,\cdot\rangle\!\rangle_{t'}) & \xrightarrow{\mathrm{Id}_{\mathbb{R}^k}} & (\mathbb{R}^k, \langle\!\langle\cdot,\cdot\rangle\!\rangle_t)
\end{array}$$

vertical arrows (but not horizontal ones) are isometric.

Claim 7.5. *There exists $c < 0$ such that for $t < t'$ close enough, the top arrow $\Phi_t^{t'}$ is $\mathrm{e}^{c\,|t-t'|}$-Lipschitz in spacelike directions on $\Sigma_{t'}$.*

Proof. By isometry, we can deal instead with the bottom arrow $\mathrm{Id}_{\mathbb{R}^k} : (\mathbb{R}^k, \langle\!\langle \cdot, \cdot \rangle\!\rangle_{t'}) \to (\mathbb{R}^k, \langle\!\langle \cdot, \cdot \rangle\!\rangle_t)$. It is enough to prove that the ideal boundary $\mathrm{Null}(M_\tau^{-1})$ of $M_\tau(\mathbb{H}^{p,q}_\tau)$ shrinks uniformly inwards as $\tau \in [t, t']$ increases, at some normal velocity $\geq \frac{|c|}{2} > 0$ for the spherical metric on $\mathbb{P}(\mathbb{R}^k)$, where $c < 0$. Indeed, Lemma 5.4 then gives

$$\frac{\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=\tau}\, d_{\mathbb{H}^{p,q}_s}(x,y)}{d_{\mathbb{H}^{p,q}_\tau}(x,y)} \geq |c|$$

for fixed x, y in spacelike position; the result follows by integrating over $\tau \in [t, t']$.

By compactness, this shrinking of the null cone can be written simply: $\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=\tau} \langle x, M_s^{-1} x\rangle_0 > 0$ for all $x \in \mathrm{Null}(M_\tau^{-1})$. Since

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=\tau} M_s^{-1} = -M_\tau^{-1} \left(\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=\tau} M_s \right) M_\tau^{-1} = -\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=\tau} \big(M_\tau^{-1} M_s M_\tau^{-1}\big),$$

under the change of variable $y = M_\tau^{-1} x$ the shrinking criterion becomes: $\langle y, \frac{\mathrm{d}M_s}{\mathrm{d}s}\big|_{s=\tau} y\rangle_0 < 0$ for all $y \in \mathrm{Null}(M_\tau)$. But $M_s = \mathrm{Id} - sM$ and $\tau > 0$, so the latter criterion is clearly satisfied: $y \in \mathrm{Null}(M_\tau)$ means $\langle y, My\rangle_0 = \frac{1}{\tau}\langle y, y\rangle_0$, hence implies $\langle y, \frac{\mathrm{d}M_s}{\mathrm{d}s}\big|_{s=\tau} y\rangle_0 = -\langle y, My\rangle_0 < 0$. ☐

We can extend $\Phi_t^{t'}$ continuously equivariantly by reflections in the walls of $\Sigma_{t'}$ and Σ_t. We also pre- and post-compose with isometries $\iota_{t'} : \mathbb{H}^{p,q} \to \mathbb{H}^{p,q}_{t'}$ and $\iota_t^{-1} : \mathbb{H}^{p,q}_t \to \mathbb{H}^{p,q}$, chosen smoothly in terms of t, and denote the resulting map by

$$f_t^{t'} := \iota_t^{-1} \circ \Phi_t^{t'} \circ \iota_{t'} : \mathbb{H}^{p,q} \longrightarrow \mathbb{H}^{p,q}.$$

The map $f_t^{t'}$ is $(\widehat{\rho}_{t'}, \widehat{\rho}_t)$-equivariant where

$$\widehat{\rho}_t := \iota_t^{-1} \circ \rho_t : \Gamma \longrightarrow \mathrm{Isom}(\mathbb{H}^{p,q}) = \mathrm{PO}(p, q+1)$$

and similarly for $\widehat{\rho}_{t'}$. By Claim 7.5, for $t < t'$ close enough, the map $f_t^{t'}$ is $e^{c|t-t'|}$-Lipschitz in spacelike directions on $\iota_{t'}^{-1}(\Sigma_{t'})$, where $c < 0$. However, a priori $f_t^{t'}$ might not be contracting in spacelike directions on its whole domain: indeed, the triangle inequality fails in $\mathbb{H}^{p,q}$, and so if $x_0, \dots, x_m$ lie in this order on a spacelike line, with each segment $[x_i, x_{i+1}]$ contained in some $\widehat{\rho}_{t'}(\Gamma)$-translate of $\iota_{t'}^{-1}(\Sigma_{t'})$, we cannot combine the inequalities $d_{\mathbb{H}^{p,q}}(f_t^{t'}(x_i), f_t^{t'}(x_{i+1})) < d_{\mathbb{H}^{p,q}}(x_i, x_{i+1})$ from Claim 7.5 into $d_{\mathbb{H}^{p,q}}(f_t^{t'}(x_0), f_t^{t'}(x_m)) < d_{\mathbb{H}^{p,q}}(x_0, x_m)$.

Infinitesimally however, the triangle inequality is an equality to first order:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t'} d_{\mathbb{H}^{p,q}}\left(f_t^{t'}(x_0), f_t^{t'}(x_m)\right) = \sum_{i=1}^{m} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t'} d_{\mathbb{H}^{p,q}}\left(f_t^{t'}(x_{i-1}), f_t^{t'}(x_i)\right)$$

because the points x_i are lined up. It follows that the *vector field* $X_{t'} = -\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=t'} f_t^{t'}$ is c-lipschitz in spacelike directions. This vector field is $(\widehat{\rho}_{t'}, u_{t'})$-equivariant where $u_{t'} : \Gamma \to \mathfrak{o}(p, q+1)$ is the $\widehat{\rho}_{t'}$-cocycle given by

$$u_{t'}(\gamma) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \widehat{\rho}_{t'-s}(\gamma)\widehat{\rho}_{t'}(\gamma)^{-1}.$$

Thus $u_{t'}$ is uniformly contracting in spacelike directions with respect to $\iota_{t'}^{-1}(\Omega_{t'})$. By Theorem 3.6.(2), the affine action of Γ on $\mathfrak{o}(p, q+1)$ via $(\widehat{\rho}_{t'}, u_{t'})$ is proper, yielding Theorem 1.5, hence also Theorem 1.1.

Finally, integrating the contraction property of X_τ for $\tau \in [t, t']$ shows a posteriori that for *any* $t < t'$ in I the map $f_t^{t'}$ is $e^{c|t-t'|}$-Lipschitz in spacelike directions. Thus $\widehat{\rho}_t$ is uniformly contracting in spacelike directions with respect to $\widehat{\rho}_{t'}$ and $\iota_{t'}^{-1}(\Omega_{t'})$. If $k \geq 3$, then $\widehat{\rho}_t$ is strongly irreducible, and so by Theorem 3.6.(1) the action of Γ on $\mathrm{O}(p, q+1)$ by right and left multiplication via $(\widehat{\rho}_{t'}, \widehat{\rho}_t)$ is proper. If $k = 2$, then $\iota_{t'}^{-1}(\Omega_{t'}) = \mathbb{H}^1$, and so by Theorem 3.3.(1) the action of Γ on $\mathrm{O}(p, q+1)$ by right and left multiplication via $(\widehat{\rho}_{t'}, \widehat{\rho}_t)$ is proper. This yields Theorem 1.7.

References

[Ab] H. Abels, *Properly discontinuous groups of affine transformations: a survey*, Geom. Dedicata 87 (2001), p. 309–333.

[Ag] I. Agol, with an appendix by I. Agol, D. Groves, J. Manning, *The virtual Haken conjecture*, Doc. Math. 18 (2013), p. 1045–1087.

[AMS1] H. Abels, G. A. Margulis, G. A. Soifer, *Semigroups containing proximal linear maps*, Israel J. Math. 91 (1995), p. 1–30.

[AMS2] H. Abels, G. A. Margulis, G. A. Soifer, *Properly discontinuous groups of affine transformations with orthogonal linear part*, C. R. Acad. Sci. Paris 324 (1997), p. 253–258.

[AMS3] H. Abels, G. A. Margulis, G. A. Soifer, *On the Zariski closure of the linear part of a properly discontinuous group of affine transformations*, J. Differential Geom. 60 (2002), p. 315–344.

[AMS4] H. Abels, G. A. Margulis, G. A. Soifer, *The linear part of an affine group acting properly discontinuously and leaving a quadratic form invariant*, Geom. Dedicata 153 (2011), p. 1–46.

[AMS5] H. Abels, G. A. Margulis, G. A. Soifer, *The Auslander conjecture for dimension less than* 7, preprint, arXiv:1211.2525.

[Au] L. Auslander, *The structure of compact locally affine manifolds*, Topology 3 (1964), p. 131–139.

[Be] Y. Benoist, *Actions propres sur les espaces homogènes réductifs*, Ann. of Math. 144 (1996), p. 315–347.

[BW] N. Bergeron, D. T. Wise, *A boundary criterion for cubulation*, Amer. J. Math. 134 (2012), p. 843–859.

[BB] M. Bestvina, N. Brady, *Morse theory and finiteness properties of groups*, Invent. Math. 129 (1997), p. 123–139.

[Bi] L. Bieberbach, *Über die Bewegungsgruppen der Euklidischen Räume*, Math. Ann. 70 (1911), p. 297–336, and Math. Ann. 72 (1912), p. 400–412.

[Bo] N. Bourbaki, *Groupes et algèbres de Lie*, Chapitres IV, V, VI, Hermann, 1968.

[CDG] V. Charette, T. Drumm, W. M. Goldman, *Proper affine deformations of two-generator Fuchsian groups*, Transform. Groups 21 (2016), p. 953–1002.

[DGK1] J. Danciger, F. Guéritaud, F. Kassel, *Geometry and topology of complete Lorentz spacetimes of constant curvature*, Ann. Sci. Éc. Norm. Sup. 49 (2016), p. 1–56.

[DGK2] J. Danciger, F. Guéritaud, F. Kassel, *Margulis spacetimes via the arc complex*, Invent. Math. 204 (2016), p. 133–193.

[DGK3] J. Danciger, F. Guéritaud, F. Kassel, *Convex cocompactness in pseudo-Riemannian hyperbolic spaces*, Geom. Dedicata 192 (2018), p. 87–126, special issue *Geometries: A Celebration of Bill Goldman's 60th Birthday.*

[DGK4] J. Danciger, F. Guéritaud, F. Kassel, *Convex cocompact actions in real projective geometry*, preprint, arXiv:1704.08711.

[DGK5] J. Danciger, F. Guéritaud, F. Kassel, *Convex cocompactness for right-angled Coxeter groups*, preprint.

[DGK6] J. Danciger, F. Guéritaud, F. Kassel, *Margulis spacetimes with parabolic elements*, in preparation.

[Da] M. W. Davis, *The geometry and topology of Coxeter groups* London Mathematical Society Monographs Series 32, Princeton University Press, Princeton, NJ, 2008.

[DJ] M. W. Davis, T. Januszkiewicz, *Right-angled Artin groups are commensurable with right-angled Coxeter groups*, J. Pure Applied Alg. 153 (2000), p. 229–235.

[DT] B. Deroin, N. Tholozan, *Dominating surface group representations by Fuchsian ones*, Int. Math. Res. Not. 2016 (2016), p. 4145–4166.

[Dr] T. Drumm, *Fundamental polyhedra for Margulis space-times*, Topology 21 (1992), p. 677–683.

[Dy] M. J. Dyer, *Imaginary cone and reflection subgroups of Coxeter groups*, arXiv:1210.5206.

[DHR] M. Dyer, C. Hohlweg, V. Ripoll, *Imaginary cones and limit roots of infinite Coxeter groups*, Math. Z. 284 (2016), p. 715–780.

[FG] D. Fried, W. M. Goldman, *Three-dimensional affine crystallographic groups*, Adv. Math. 47 (1983), p. 1–49.

[Gh] É. Ghys, *Déformations des structures complexes sur les espaces homogènes de* $\mathrm{SL}_2(\mathbb{C})$, J. Reine Angew. Math. 468 (1995), p. 113–138.

[GM] O. Glorieux, D. Monclair, *Critical exponent and Hausdorff dimension for quasi-Fuchsian AdS manifolds*, preprint, arXiv:1606.05512.

[GoK] W. M. Goldman, Y. Kamishima, *The fundamental group of a compact flat Lorentz space form is virtually polycyclic*, J. Differential Geom. 19 (1983), p. 233–240.

[GLM] W. M. Goldman, F. Labourie, G. A. Margulis, *Proper affine actions and geodesic flows of hyperbolic surfaces*, Ann. of Math. 170 (2009), p. 1051–1083.

[Gu] F. Guéritaud, *On Lorentz spacetimes of constant curvature*, Contemp. Math. 639 (2015), p. 253–269.

[GuK] F. Guéritaud, F. Kassel, *Maximally stretched laminations on geometrically finite hyperbolic manifolds*, Geom. Topol. 21 (2017), p. 693–840.

[GKW] F. Guéritaud, F. Kassel, M. Wolff, *Compact anti-de Sitter 3-manifolds and folded hyperbolic structures on surfaces*, Pacific J. Math. 275 (2015), p. 325–359.

[HW1] F. Haglund, D. T. Wise, *Special cube complexes*, Geom. Funct. Anal. 17 (2008), p. 1551–1620.

[HW2] F. Haglund, D. T. Wise, *Coxeter groups are special*, Adv. Math. 224 (2010), p. 1890–1903.

[H] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics 29, Cambridge University Press, Cambridge, 1990.

[JS] T. Januszkiewicz, J. Świątkowski, *Hyperbolic Coxeter groups of large dimension*, Comment. Math. Helv. 78 (2003), p. 555–583.

[KM] J. Kahn, V. Markovic, *Immersing almost geodesic surfaces in a closed hyperbolic three manifold*, Ann. of Math. 175 (2012), p. 1127–1190.

[K1] F. Kassel, *Proper actions on corank-one reductive homogeneous spaces*, J. Lie Theory 18 (2008), p. 961–978.

[K2] F. Kassel, *Quotients compacts d'espaces homogènes réels ou p-adiques*, PhD thesis, Université Paris-Sud, 2009, available at `www.ihes.fr/~kassel/`.

[K] T. Kobayashi, *Deformation of compact Clifford–Klein forms of indefinite-Riemannian homogeneous manifolds*, Math. Ann. 310 (1998), p. 395–409.

[KR] R. S. Kulkarni, F. Raymond, *3-dimensional Lorentz space-forms and Seifert fiber spaces*, J. Differential Geom. 21 (1985), p. 231–268.

[LL] G. Lakeland, C. J. Leininger, *Strict contractions and exotic* $\mathrm{SO}_0(d,1)$ *quotients*, to appear in J. Lond. Math. Soc.

[L] Y. Liu, *Virtual cubulation of nonpositively curved graph manifolds*, J. Topol. 6 (2013), p. 793–822.

[LM] A. Lubotzky, A. Magid, *Varieties of representations of finitely generated groups*, Memoirs of the AMS, vol. 336, American Mathematical Society, 1985.

[M1] G. A. Margulis, *Free completely discontinuous groups of affine transformations* (in Russian), Dokl. Akad. Nauk SSSR 272 (1983), p. 785–788.

[M2] G. A. Margulis, *Complete affine locally flat manifolds with a free fundamental group*, J. Soviet Math. 1934 (1987), p. 129–139, translated from Zap. Naucha. Sem. Leningrad. Otdel. Mat. Inst. Steklov (LOMI) 134 (1984), p. 190–205.

[Mi] J. Milnor, *On fundamental groups of complete affinely flat manifolds*, Adv. Math. 25 (1977), p. 178–187.

[Mo] G. Moussong, *Hyperbolic Coxeter groups*, PhD thesis, Ohio State University, 1987.

[O] D. Osajda, *A construction of hyperbolic Coxeter groups*, Comment. Math. Helv. 88 (2013), p. 353–367.

[PW] P. Przytycki, D. T. Wise, *Mixed* 3*-manifolds are virtually special*, J. Amer. Math. Soc. 31 (2018), p. 319–347.

[Sag] M. Sageev, *Ends of group pairs and non-positively curved cube complexes*, Proc. London Math. Soc. 71 (1995), p. 585–617.

[Sal] F. Salein, *Variétés anti-de Sitter de dimension 3 exotiques*, Ann. Inst. Fourier 50 (2000), p. 257–284.

[S1] I. Smilga, *Proper affine actions on semisimple Lie algebras*, Ann. Inst. Fourier 66 (2016), p. 785–831.

[S2] I. Smilga, *Proper affine actions: a sufficient criterion*, arXiv:1612.08942.

[T] G. Tomanov, *The virtual solvability of the fundamental group of a generalized Lorentz space form*, J. Differential Geom. 32 (1990), p. 539–547.

[V] E. B. Vinberg, *Discrete linear groups generated by reflections*, Math. USSR Izv. 5 (1971), p. 1083–1119.

[W1] D. T. Wise, *The structure of groups with a quasiconvex hierarchy*, preprint (2011), see `http://www.math.mcgill.ca/wise/papers`.

[W2] D. T. Wise, *The cubical route to understanding groups*, in *Proceedings of the International Congress of Mathematicians, Seoul 2014*, volume II, p. 1075–1099, Kyung Moon, Seoul, Korea, 2014.

Department of Mathematics, The University of Texas at Austin, 1 University Station C1200, Austin, TX 78712, USA
E-mail address: jdanciger@math.utexas.edu

CNRS and Université Lille 1, Laboratoire Paul Painlevé, 59655 Villeneuve d'Ascq Cedex, France
E-mail address: francois.gueritaud@math.univ-lille1.fr

CNRS and Institut des Hautes Études Scientifiques, 35 route de Chartres, 91440 Bures-sur-Yvette, France
E-mail address: kassel@ihes.fr