Ramification of ℓ-adic sheaves on varieties over curves

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Table of contents

Nearby cycles and vanishing cycles

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X_{\overline{\eta}} \arrow[r, "\overline{j}"] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \arrow[dr, phantom, "\square"] & X_s \arrow[l, "i"'] \arrow[d] \\
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Definition

$$R\Psi(\mathscr{K}, f) := i^* R\overline{j}_* \overline{j}^* \mathscr{K} \in \mathrm{ob}\, D^+(X_s, \mathbb{Z}/\ell^n\mathbb{Z})$$

is called the nearby cycle of $\mathscr{K}$.

Definition

We have a distinguished triangle:

$$i^*\mathscr{K} \xrightarrow{\text{adj}} i^*R\bar{j}_*\bar{j}^*\mathscr{K} \to R\Phi(\mathscr{K}, f) \xrightarrow{+1}$$

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$R^*\Psi(\mathscr{K}, f)$, $R^*\Phi(\mathscr{K}, f)$ (resp. $H^*(X_{\bar{\eta}}, \mathscr{K}|_{X_{\bar{\eta}}})$) are étale sheaves on X_s (resp. $\mathbb{Z}/\ell^n\mathbb{Z}$-modules) with continuous G-actions.

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When $f : X \to S$ is proper,

$$E_2^{ab} = H^a(X_s, R^b\Psi(\mathscr{K}, f)) \Rightarrow H^{a+b}(X_{\bar{\eta}}, \mathscr{K}|_{X_{\bar{\eta}}})$$

Ramification properties

- Grothendieck, Rapoport-Zink and Illusie:

  $f : X \to S$ is proper semi-stable and $\mathscr{K}$ is a smooth étale sheaf on the generic fiber tamely ramified along the special fiber $\Longrightarrow$ $H^*(X_{\overline{\eta}}, \mathscr{K}|_{X_{\overline{\eta}}})$ and $R^*\Psi(\mathscr{K}, f)$ are tame.

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- Milnor formula (Deligne):

  $f : X \to S$ is smooth except a closed point $x \in X$.

$$\sum_i (-1)^i \mathrm{dimtot}(R^i\Phi_{\overline{x}}(\Lambda, f)) = \text{Milnor number}$$

- Bloch conductor formula (Bloch, Abbes, Kato-Saito, Saito, Töen-Vezzosi,. . .)

  X is regular and $f : X \to S$ is proper, flat and generically smooth.

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Question

Can we study the ramification of each $H^*(X_{\overline{\eta}}, \mathscr{K}|_{X_{\overline{\eta}}})$ and $R^*\Psi(\mathscr{K}, f)$ if we allow wildly ramifications of $\mathscr{K}$?

Leal's conjecture

R strictly henselian discrete valuation field;
$F = R/\mathfrak{m}$ is algebraically closed and $\mathrm{char}(F) = p > 0$;
$S = \mathrm{Spec}(R)$, s, η, $\overline{\eta}$; $\quad G = \mathrm{Gal}(\overline{\eta}/\eta)$.

$$\begin{array}{ccccc} U & \xrightarrow{j} & X & \xleftarrow{i} & Z \\ & & \downarrow{\scriptstyle f} & & \\ & & S & & \end{array}$$

(X, Z) semi-stable pair: $f : X \to S$ is semi-stable, $Z = Z_f \bigcup X_s$ is a divisor with simple normal crossings of X, where each irreducible component of Z_f is flat over S. $U = X - Z$.

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$\mathscr{F}$ smooth étale sheaf of Λ-modules on U.

Conjecture (Leal 2016)

Assume that $f : X \to S$ is proper and that $\mathscr{F}$ is tamely ramified along irreducible components of Z_f. Then, the ramification of each $H_c^m(U_{\bar\eta}, \mathscr{F}|_{U_{\bar\eta}})$ is bounded by $r_{\log}$, i.e., the action of $G^{r_{\log}+}$ on each $H_c^m(U_{\bar\eta}, \mathscr{F}|_{U_{\bar\eta}})$ is trivial.
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What is the Abbes-Saito logarithmic slope?

Abbes-Saito's ramification theory

K henselian discrete valuation field,

$\mathscr{O}_K$ integer ring of K,

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Theorem (Abbes-Saito, 2002–)

There are two filtrations $\{G_K^r\}_{r\in\mathbb{Q}_{\geq 1}}$ and $\{G_{K,\log}^r\}_{r\in\mathbb{Q}_{\geq 1}}$ by closed normal subgroups, which are called ramification filtration and logarithmic ramification filtration of G_K, satisfying following conditions:

Theorem (Abbes-Saito, 2002 –)

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- $G^{1+}_K = G^{0+}_{K,\log} = P_K$;
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- K'/K *a finite tamely ramified extension of ramification degree* e. *Then* $G_{K,\log}^r = G_{K',\log}^{er}$ *for any* $r \in \mathbb{Q}_{>0}$.

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Λ a finite field of characteristic ℓ, $(\ell, p) = 1$,

M a Λ-vector space of finite rank with a continuous G_K-action, Slope decompositions:

$$M = \bigoplus_{r\in\mathbb{Q}_{\geq 1}} M^{(r)} \quad \text{and} \quad M = \bigoplus_{r\in\mathbb{Q}_{\geq 0}} M^{(r)}_{\log}$$

$M^{(1)} = M^{P_K} = M^{(0)}_{\log}$,

for any $r \in \mathbb{Q}_{>0}$ ($t \in \mathbb{Q}_{>0}$),

$(M^{(r)})^{G^r_K} = \{0\}$, $(M^{(r)})^{G^{r+}_K} = M^{(r)}$;
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Definition

The largest rational number $r = r(M)$ (resp. $r' = r_{\log}(M)$) such that $M^{(r)} \neq 0$ (resp. $M^{(r')}_{\log} \neq 0$) is called the largest Abbes-Saito slope (resp. largest Abbes-Saito logarithmic slope) of M.

Proposition (A)

Let k be a perfect field of characteristic $p > 0$, Y a smooth k-scheme and E an irreducible smooth divisor of Y, $h : V = Y - E \to Y$ the canonical injection and $\mathscr{G}$ a smooth sheaf of Λ-modules on V.

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- [Saito, 2017; H.-Yang 2017] Let C be a smooth k-curve in Y meet E transversely at $x \in E$. Then

$$r(\mathscr{G}, E) \geq r_{\mathrm{up}}(\mathscr{G}|_C, x) + 1$$

It is an equality when the canonical immersion $i : C \to Y$ is $SS(h_!\mathscr{G})$-transversal and the ramification of $\mathscr{G}$ along E is non-degenerate.

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- [Barrientos, 2016; H., 2017] Let $\mathcal{C}$ be the category of smooth k-curves in Y meet E at a single point. Then

  $$r_{\log}(\mathscr{G}, E) = \sup_{C \in \mathrm{ob}\, \mathcal{C}} \frac{r_{\mathrm{up}}(\mathscr{G}|_C, C \cap E)}{(C, E)_{C \cap E}}$$

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$X_s = \sum_i Z_i$, $\xi_i \in Z_i$ generic point, $\eta_i \in \mathrm{Spec}(\widehat{\mathscr{O}}_{X,\xi_i})$ generic point,

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$\mathscr{F}|_{\eta_i}$ = finite dimensional Λ-modules with a continuous $\mathrm{Gal}(\overline{\eta}_i/\eta_i)$-action.

$r_{i,\log}$ = largest Abbes-Saito logarithmic slope of $\mathscr{F}|_{\eta_i}$;

$r_{\log} = \max_i\{r_{i,\log}\}$ in Leal's conjecture.

Main theorem

Theorem (H.-Teyssier, 2018)

Assume that S is a strict localization at a closed point of a smooth curve over a perfect field and that $f : X \to S$ is smooth. Then, for any $m \in \mathbb{Z}$, the action of $G^{+r_{\log}}$ on $R^m\Psi(j_!\mathscr{F}, f)$ is trivial.

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When $f : X \to S$ is proper,

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Corollary

Leal's conjecture is true when S is a strict localization at a closed point of a smooth curve over a perfect field and that $f : X \to S$ is smooth and proper.

Sketch of the proof

a) Main theorem is equivalent to:
For any sheaf of Λ-modules $\mathscr{N}$ on $\eta \in S$ of pure upper numbering slope $\alpha > r_{\log}$, we have

$$R\Psi^{\mathrm{t}}(j_!(\mathscr{F} \otimes f^*\mathscr{N}), f) = 0,$$

where $R\Psi^{\mathrm{t}}$ denotes the tame nearby cycle.

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b) Reduce to the case where $Z = X_s$ and it is irreducible.
(Abhyankar + Second part of Proposition (A)).

c) $S_n = \mathrm{Spce}(R[t]/(t^n - \pi)), \quad X_n = S_n \times_S X.$

$$\begin{tikzcd} U_n \arrow[r, "j_n"] \arrow[d, "\pi_n"'] \arrow[dr, phantom, "\square"] & X_n \arrow[d] & X_s \arrow[l, "i"'] \arrow[d, equal] \\ U \arrow[r] & X & X_s \arrow[l] \end{tikzcd}$$

$$R\Psi^{\mathrm{t}}(j_!(\mathscr{F} \otimes f^*\mathscr{N}), f) = \varinjlim_{(n,p)=1} i_n^* R j_{n*}(\pi_n^*(\mathscr{F} \otimes f^*\mathscr{N}))$$

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To show $R\Psi^{\mathrm{t}}(j_!(\mathscr{F} \otimes f^*\mathscr{N}), f) = 0$, we are left to show, for any $(n,p) = 1$,

$$i_n^* Rj_{n*}(\pi_n^*(\mathscr{F} \otimes f^*\mathscr{N})) = 0,$$

i.e., to prove the following proposition:

Proposition (B)

Let k be a perfect field of characteristic $p > 0$,

$$\begin{tikzcd} U \arrow[r, "j"] \arrow[d, "f_V"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \arrow[dr, phantom, "\square"] & D \arrow[l] \arrow[d] \\ V \arrow[r] & C & \{c\} \arrow[l] \end{tikzcd}$$

C a smooth k-curve, c a closed point of C, $V = C - \{c\}$ and $f : X \to C$ is smooth. Let $\mathscr{F}$ be a smooth sheaf on U and $\mathscr{N}$ a smooth sheaf on V of pure upper numbering slope $\alpha > r_{\log}(\mathscr{F}, D)$. Then we have

$$Rj_*(\mathscr{F} \otimes f_V^* \mathscr{N}) = j_!(\mathscr{F} \otimes f_V^* \mathscr{N}).$$

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- For any smooth k-curve T in X meet D transversely at a closed point $x\in X$, the $(\mathscr{F}\otimes f_V^*\mathscr{N})|_T$ has pure upper numbering slope α. (By the first part of Proposition (A)).

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- For any smooth k-curve T in X meet D transversely at a closed point $x \in X$, the $(\mathscr{F} \otimes f_V^* \mathscr{N})|_T$ has pure upper numbering slope α. (By the first part of Proposition (A)).
- It implies any morphism $h : X \to B$ to a smooth k-curve B, such that $h|_D : D \to B$ is smooth, is universally locally acyclic with respect to $j_!(\mathscr{F} \otimes f_V^* \mathscr{N})$. (Deligne-Laumon's semi-continuity of Swan conductors + Saito's talk 1.2.4).

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- It implies any morphism $h: X\to B$ to a smooth k-curve B, such that $h|_D: D\to B$ is smooth, is universally locally acyclic with respect to $j_!(\mathscr{F}\otimes f_V^*\mathscr{N})$. (Deligne-Laumon's semi-continuity of Swan conductors + Saito's talk 1.2.4).
- It shows that $SS^w(j_!(\mathscr{F}\otimes f_V^*\mathscr{N})) = [T^*_X X]\cup[T^*_D X]$. Since $SS^w = SS$, we obtain $(*)$.

- By $(*)$, for any closed point $x \in D$ and any k-curve T in X meet D transversely at x, the immersion $h : T \to X$ is $SS(Rj_*(\mathscr{F} \otimes f_V^* \mathscr{N}))$-transversal. By the equivalence

$$\text{singular support transversal} \quad \Longleftrightarrow \quad \text{sheaf transversal},$$

we get a base change formula

$$(Rj_*(\mathscr{F} \otimes f_V^* \mathscr{N}))|_T \cong Rj'_*((\mathscr{F} \otimes f_V^* \mathscr{N})|_{T-\{x\}})$$

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- Proposition (A) $\Longrightarrow$ $(\mathscr{F} \otimes f_V^* \mathscr{N})|_{T-\{x\}}$ is purely wildly ramified at x (with the same pure upper numbering slope as $\mathscr{N}$) $\Longrightarrow$

$$Rj'_*((\mathscr{F} \otimes f_V^* \mathscr{N})|_{T-\{x\}}) \cong j'_!((\mathscr{F} \otimes f_V^* \mathscr{N})|_{T-\{x\}}).$$

Proposition (B) and the main theorem is proved.

Let k be a perfect field of characteristic $p > 0$,

$$\begin{tikzcd} U \arrow[r, "j"] \arrow[d, "f_V"'] \arrow[dr, phantom, "\Box"] & X \arrow[d, "f"] \arrow[dr, phantom, "\Box"] & D \arrow[l] \arrow[d] \\ V \arrow[r] & C & \{c\} \arrow[l] \end{tikzcd}$$

C a smooth k-curve, c a closed point of C, $V = C - \{c\}$ and f is smooth projective with geometrically connected fibers.
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Proposition (C)

Assume that $f : X \to S$ is a relative curve of genus g.

$$Z = \{x \in D\,;\, T^*_x X \subset SS(j_!\mathscr{F})\}.$$

Then, we have

$$\sharp Z \leq \max\{0, ((2g-2) + 2g \cdot r_{\log}(\mathscr{F}, D)) \cdot \mathrm{rk}_\Lambda \mathscr{F}\}.$$

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Main theorem + Saito's proper push-forward of $CC \Longrightarrow$ Proposition (C).

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Main theorem + Saito's proper push-forward of $CC \Longrightarrow$ Proposition (C).

For $g = 1$, we have a much stronger result due to Saito.

Ramification of ℓ-adic sheaves on abelian schemes

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Let $\mathscr{F}$ be a smooth sheaf of Λ-modules on U.

Theorem (Saito, 2019)

The ramification of $\mathscr{F}$ along D is non-degenerate. In particular, for any closed point x in D, we have

$$\dim(SS(j_!\mathscr{F}) \times_X x) = 1.$$

- Non-degenerate roughly means that the ramification of a sheaf along a divisor is "almost" the same as its Abbes-Saito ramification at divisor's generic points. In general, we can always find an open dense subset of D on which the ramification of $\mathscr{F}$ is non-degenerate.

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- When $\mathrm{rel.dim}(X/S) = 1$ and $g = 1$,
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Proposition (D)

For any closed point x of D and any smooth k-curve T in X meeting D at x such that the immersion $h : T \to X$ is $SS(j_!\mathscr{F})$-transversal, the Newton polygon of the (upper numbering) slope decomposition $\mathscr{F}|_T$ at x are constant.

Non-dengenerate & $h : T \to X$ is $SS(j_!\mathscr{F})$-transversal imply

$$\begin{aligned} & \text{NP of the slope decomposition } \mathscr{F}|_T \text{ at } x \\ = \; & \text{NP of the slope decomposition } \mathscr{F} \text{ at the generic point of } D \end{aligned}$$

Proof of Saito's theorem

Proposition (E)

Let K be a field and A an abelian variety over K. Let $\mathscr{G}$ be a smooth étale sheaf of Λ-modules on A (if $\mathrm{char}(K) = p > 0$, $p \in \Lambda^{\times}$). Then, there is an integer number $m(\mathscr{G})$ depending on $\mathscr{G}$ such that, for any n co-prime to $m(\mathscr{G})$ and any $a \in A[n](K)$, we have $\mathscr{G} \cong (+a)^*\mathscr{G}$.

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- Saito's theorem is étale local.
- Let K be the function field of the localization $C_{(c)}$ and $\mathscr{G} = \mathscr{F}|_{X_K}$. For any integer n coprime to $p \cdot m(\mathscr{G})$, $X_K[n^\infty](K)$ and $D[n^\infty](k)$ are dense in X_K and D, respectively.
- Proposition (E) $\Longrightarrow$ Non-degenerate locus of $\mathscr{F}$ in D is stable by translations in $D[n^\infty](k)$ $\Longrightarrow$ Saito's theorem.

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Proposition (F)

Let $\overline{\eta}$ be a geometric generic point of the strict localization $C_{(\bar{c})}$. Then, we have

$$\sum_i (-1)^i \mathrm{sw}(H^i(X_{\overline{\eta}}, \mathscr{F}|_{\overline{\eta}})) = 0$$

Conductor formula for ℓ-adic sheaves on abelian varieties in a geometric case.

Thank you!