THE p-ADIC SIMPSON CORRESPONDENCE: FUNCTORIALITY BY PROPER DIRECT IMAGE AND HODGE-TATE LOCAL SYSTEMS

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Abstract. Faltings initiated in 2005 a p-adic analogue of the (complex) Simpson correspondence whose construction has been taken up by various authors and whose properties have been developed according to several approaches. I will present in these lectures the approach I developed with Michel Gros, inspired by the Cartier transform of Ogus and Vologodsky, which is an analogue in characteristic p of Simpson correspondence. The p-adic Simpson correspondence can be considered as a categorification of the Hodge-Tate decomposition. I will present its construction for small generalized representations using a suitable period ring, establish its functoriality by proper direct image and discuss the link with Hodge-Tate local systems.

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1. Introduction

The leitmotif of these lectures is that *the p-adic Simpson correspondence is a categorification of the Hodge-Tate decomposition*, like the Cartier transform of Ogus-Vologodsky [24] is a categorification of the decomposition of the de Rham complex in characteristic p of Deligne-Illusie [8].

1.1. Let K be a complete discrete valuation field of characteristic 0, with *algebraically closed* residue field of characteristic $p>0$, $\mathscr{O}_K$ the valuation ring of K, $\overline{K}$ an algebraic closure of K, $\mathscr{O}_{\overline{K}}$ the integral closure of $\mathscr{O}_K$ in $\overline{K}$. We denote by G_K the Galois group of $\overline{K}$ over K, by $\mathscr{O}_C$ the p-adic completion of $\mathscr{O}_{\overline{K}}$, by C its field of fractions.

We set $S=\mathrm{Spec}(\mathscr{O}_K)$ and $\overline{S}=\mathrm{Spec}(\mathscr{O}_{\overline{K}})$ and we denote by s (resp. η, resp. $\overline{\eta}$) the closed point of S (resp. generic point of S, resp. generic point of $\overline{S}$).

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For any proper and smooth variety X over K, there is a canonical functorial and G_K-equivariant spectral sequence, the *Hodge-Tate spectral sequence*,

$$\mathrm{E}_2^{i,j} = \mathrm{H}^i(X_C, \Omega^j_{X_C/C})(-j) \Rightarrow \mathrm{H}^{i+j}_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C. \tag{1.1.1}$$

It degenerates at E_2 and splits by Tate's result on Galois cohomology : $\mathrm{H}^i(G_K, C(j)) = 0$ for $i = 0, 1$ and all $j \neq 0$. Hence, it gives, for every integer $n \geq 0$, a canonical functorial G_K-equivariant decomposition

$$\mathrm{H}^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C \xrightarrow{\sim} \oplus_{i=0}^n \mathrm{H}^i(X_C, \Omega^{n-i}_{X_C/C})(i-n). \tag{1.1.2}$$

This decomposition was conjectured by Tate [27] and proved independently by Faltings [9], Niziol [22, 23] and Tsuji [28]. It was later generalized by Scholze [26] to rigid-analytic varieties, who was the first to formulate the absolute spectral sequence, although it is implicit in the work Faltings.

The Hodge-Tate spectral sequence has been extended to morphisms, independently by different methods by A. Caraiani and P. Scholze ([6] 2.2.4) and by M. Gros and myself ([1] 6.7.5). A new proof should appear soon by I. Gaisin and T. Koshikawa.

1.2. I will discuss in these lectures a categorical analogue of the Hodge-Tate spectral sequence known as the *p-adic Simpson correspondence*. Initiated by Faltings [11] and developed following different approches by T. Tsuji and by M. Gros and myself [3], this correspondence provides an equivalence of categories between certain *p-adic étale local systems* and certain *Higgs bundles*. Moreover, for a pair of associated objects, we have a canonical spectral sequence whose initial term is the cohomology of the Higgs bundle and whose abutment is the cohomology of the local system, generalizing the Hodge-Tate spectral sequence.

The notion of a *Higgs module* was first introduced by Hitchin, Donaldson and Simpson (for complex varieties). A Higgs module on X/K is a pair (M, θ) formed of an $\mathscr{O}_X$-module M and an $\mathscr{O}_X$-linear map $\theta \colon M \to M \otimes_{\mathscr{O}_X} \Omega^1_{X/K}$ such that $\theta \wedge \theta \colon M \to M \otimes_{\mathscr{O}_X} \Omega^2_{X/K}$ vanishes. We can then form the *Dolbeault complex*

$$0 \longrightarrow M \xrightarrow{\theta} M \otimes_{\mathscr{O}_X} \Omega^1_{X/K} \xrightarrow{\theta \wedge \mathrm{id}} M \otimes_{\mathscr{O}_X} \Omega^2_{X/K} \xrightarrow{\theta \wedge \mathrm{id}} \dots \tag{1.2.1}$$

1.3. Let X be a proper smooth S-scheme and let L be a locally constant constructible sheaf of $\mathbb{Z}/p^n\mathbb{Z}$-modules on $X_{\overline{\eta},\text{ét}}$ for an integer $n \geq 0$. To study the cohomology of L, Faltings introduced a ringed topos $(\widetilde{E}, \overline{\mathscr{B}})$ equipped with two morphisms of topos (the definitions will be given later)

$$X_{\overline{\eta},\text{ét}} \xrightarrow{\psi} \widetilde{E} \xrightarrow{\sigma} X_{\text{ét}} . \tag{1.3.1}$$

For any integer $j \geq 1$, we have $\mathrm{R}^j\psi_*(L) = 0$. In particular, for any $i \geq 0$, we have a canonical isomorphism

$$\mathrm{H}^i(X_{\overline{\eta},\text{ét}}, L) \xrightarrow{\sim} \mathrm{H}^i(\widetilde{E}, \psi_*(L)). \tag{1.3.2}$$

Using Artin-Schreier theory, Faltings proved a refinement of this result, namely that the canonical morphism

$$\mathrm{H}^i(X_{\overline{\eta},\text{ét}}, L) \otimes_{\mathbb{Z}_p} \mathscr{O}_C \to \mathrm{H}^i(\widetilde{E}, \psi_*(L) \otimes_{\mathbb{Z}_p} \overline{\mathscr{B}}) \tag{1.3.3}$$

is an *almost isomorphism*, *i.e.*, its kernel and cokernel are annihilated by $\mathfrak{m}_C$.

Setting $\mathscr{M} = \psi_*(L) \otimes_{\mathbb{Z}_p} \overline{\mathscr{B}}$, we can compute $\mathrm{H}^i(\widetilde{E}, \mathscr{M})$ by the Cartan-Leray specral sequence

$$\mathrm{E}_2^{i,j} = \mathrm{H}^i(X_{\text{ét}}, \mathrm{R}^j\sigma_*(\mathscr{M})) \Rightarrow \mathrm{H}^{i+j}(\widetilde{E}, \mathscr{M}). \tag{1.3.4}$$

Roughly speaking, for certain local systems L, the complex $\mathrm{R}\sigma_*(\mathscr{M})$ is the Dolbeault complex of a Higgs bundle on X/S canonically associated to L by the *p-adic Simpson correspondence.* I will explain for which local systems this property holds true and I will construct the correspondence explicitly.

It turns out that the theory works well only after passing to the limit on n and inverting p. However, there is one case where we can compute $\mathrm{R}\sigma_*(\mathscr{M})$ modulo p^n (up to bounded torsion), namely for $L = \mathbb{Z}/p^n\mathbb{Z}$ in which case the Higgs bundle is trivial. The canonical morphism $\mathbb{Z}/p^n\mathbb{Z} \to \psi_*(\mathbb{Z}/p^n\mathbb{Z})$ is an isomorphism, so $\mathscr{M} = \overline{\mathscr{B}}/p^n\overline{\mathscr{B}}$. Let $\overline{X}_n$ and $\overline{S}_n$ the reduction of $\overline{X} = X \times_S \overline{S}$ and $\overline{S}$ modulo p^n. By globalizing Faltings' computation of Galois cohomology, we prove that there exists a canonical homomorphism of graded $\mathscr{O}_{\overline{X}_n}$-algebras of $X_{s,\text{ét}}$

$$\wedge\,(\xi^{-1}\Omega^1_{\overline{X}_n/\overline{S}_n}) \to \oplus_{i\geq 0}\mathrm{R}^i\sigma_*(\overline{\mathscr{B}}/p^n\overline{\mathscr{B}}), \tag{1.3.5}$$

whose kernel and cokernel are annihilated by a fixed power of p, where ξ is a variant of Tate twist. By passing to projective limits and inverting p, the Cartan-Leray spectral sequence (1.3.4) induces the Hodge-Tate spectral sequence (1.1.1).

1.4. In general, we can build a correspondence between certain $\overline{\mathscr{B}}$-modules of $\widetilde{E}$ and certain Higgs bundles on X/S, through the classical scheme of Fontaine correspondences involving a period ring $\mathscr{C}$ of $\widetilde{E}$. A first natural candidate for $\mathscr{C}$ could be Hyodo's ring B_{HT}. This can be done in the affine case and in Scholze's approach [21]. But in our context, to sheafify a $\mathbb{Q}_p$-algebra in $\widetilde{E}$, we need a good integral model. Inspired by Faltings' original approach and the work of Ogus-Vologodsky on the Cartier transform in characteristic p [24], we construct such a model, that we call the *Higgs-Tate* algebra, using deformations. It turns out that a "weak" p-adic completion of $\mathscr{C}$ has nice cohomological properties that lead to an equivalence between the categories of admissible objects, namely *Dolbeault $\overline{\mathscr{B}}$-modules* and *solvable Higgs bundles.* The first condition is weaker than Hodge-Tate.

Faltings sketched a strategy to enlarge the correspondence beyond admissible objects for curves by descent: any p-adic representation of the fundamental group is admissible over a finite étale Galois cover. We will not discuss this aspect on which there has been a recent progress by D. Xu [30].

The goal of these lectures is to sketch the outlines of the theory following the approach I developed with M. Gros:

- the period ring $\mathscr{C}$;
- the equivalence of categories between admissible objects;
- the compatibility of the correspondence with the natural cohomologies, which leads to a generalization of the Hodge-Tate spectral sequence;
- the functoriality of the correspondence by proper direct image, which leads to a generalization of the relative Hodge-Tate spectral sequence;
- the relation to Hodge-Tate local systems.

Remark 1.5. We treat in [1, 2, 3] schemes with toric singularities using logarithmic geometry, but for simplicity, I will stick in these lectures to the smooth case.

2. Faltings topos. Faltings main comparison theorem

2.1. Let X be a smooth S-scheme, E the category of morphisms $(V \to U)$ above the canonical morphism $X_{\overline{\eta}} \to X$, that is, commutative diagrams

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ X_{\overline{\eta}} & \longrightarrow & X \end{array} \tag{2.1.1}$$

such that U is étale over X and the canonical morphism $V \to U_{\overline{\eta}}$ is *finite étale*. It is useful to consider the category E as fibred by the functor

$$\pi\colon E \to \mathbf{\acute{E}t}_{/X}, \quad (V \to U) \mapsto U, \tag{2.1.2}$$

over the étale site of X.

The fiber of π above an object U of $\mathbf{\acute{E}t}_{/X}$ is canonically equivalent to the category $\mathbf{\acute{E}t}_{\mathrm{f}/U_{\overline{\eta}}}$ of finite étale morphisms over $U_{\overline{\eta}}$. We equip it with the étale topology and denote by $U_{\overline{\eta},\mathrm{f\acute{e}t}}$ the associated topos. If $U_{\overline{\eta}}$ is connected and if $\overline{y}$ is a geometric point of $U_{\overline{\eta}}$, then the topos $U_{\overline{\eta},\mathrm{f\acute{e}t}}$ is equivalent to the classifying topos of the profinite group $\pi_1(U_{\overline{\eta}}, \overline{y})$, *i.e.*, the category of discrete sets equipped with a continuous left action of $\pi_1(U_{\overline{\eta}}, \overline{y})$.

2.2. We equip E with the *covanishing* topology, that is the topology generated by coverings $\{(V_i \to U_i) \to (V \to U)\}_{i \in I}$ of the following two types :

(v) $U_i = U$ for all $i \in I$ and $(V_i \to V)_{i \in I}$ is a covering;
(c) $(U_i \to U)_{i \in I}$ is a covering and $V_i = V \times_U U_i$ for all $i \in I$.

We denote by $\widetilde{E}$ the topos of sheaves of sets on E.

To give a sheaf F on E is equivalent to give a collection of sheaves $\{U \in \mathbf{\acute{E}t}_{/X} \mapsto F_U \in U_{\overline{\eta},\mathrm{f\acute{e}t}}\}$ satisfying a cocycle condition (for the composition of morphisms) and a gluing condition (for coverings of $\mathbf{\acute{E}t}_{/X}$).

2.3. Any specialization map $\overline{y} \rightsquigarrow \overline{x}$ from a geometric point $\overline{y}$ of $X_{\overline{\eta}}$ to a geometric point $\overline{x}$ of X, determines a point of $\widetilde{E}$ that we denote by $\rho(\overline{y} \rightsquigarrow \overline{x})$. *The collection of these points of* $\widetilde{E}$ *is conservative.*

2.4. There are two morphisms of topos

$$X_{\overline{\eta},\mathrm{\acute{e}t}} \xrightarrow{\ \psi\ } \widetilde{E} \xrightarrow{\ \sigma\ } X_{\mathrm{\acute{e}t}}\ , \tag{2.4.1}$$

defined by

$$U \in \mathrm{Ob}(\mathbf{\acute{E}t}_{/X}) \quad \mapsto \quad \sigma^*(U) = (U_{\overline{\eta}} \to U)^a, \tag{2.4.2}$$

$$(U \mapsto V) \in \mathrm{Ob}(E) \quad \mapsto \quad \psi^*(V \to U) = V. \tag{2.4.3}$$

2.5. The higher direct images of σ sheafify Galois cohomology ([3] VI.10.40): if $F = \{U \mapsto F_U\}$ is an abelian group of $\widetilde{E}$, for each integer $i \geq 0$, $\mathrm{R}^i\sigma_*(F)$ is canonically isomorphic to the sheaf associated to the presheaf

$$U \mapsto \mathrm{H}^i(U_{\overline{\eta},\mathrm{f\acute{e}t}}, F_U). \tag{2.5.1}$$

Proposition 2.6 (Faltings [9], Achinger [4])**.** *For any locally constant constructible torsion abelian sheaf* F *of* $X_{\overline{\eta},\mathrm{\acute{e}t}}$*, we have* $\mathrm{R}^i\psi_*(F) = 0$ *for any* $i \geq 1$.

This statement is a consequence of the fact that for any geometric point $\overline{x}$ of X over s, denoting $\underline{X}$ the strict localization of X at $\overline{x}$, $\underline{X}_{\overline{\eta}}$ is a $K(\pi,1)$ scheme, *i.e.*, if $\overline{y}$ is a geometric point of $\underline{X}_{\overline{\eta}}$, for any locally constant constructible torsion abelian sheaf F on $\underline{X}_{\overline{\eta}}$ and any $i\geq 0$, we have an isomorphism

(2.6.1) $$\mathrm{H}^i(\underline{X}_{\overline{\eta}},F)\xrightarrow{\sim}\mathrm{H}^i(\pi_1(\underline{X}_{\overline{\eta}},\overline{y}),F_{\overline{y}}).$$

This property was proved by Faltings ([9] Lemma 2.3 page 281), generalizing results of Artin ([5] XI). It was generalized further by Achinger in the log-smooth case [4].

2.7. For any object $(V\to U)$ of E, we denote by $\overline{U}^V$ the integral closure of $\overline{U}$ in V and we set

(2.7.1) $$\overline{\mathscr{B}}(V\to U)=\Gamma(\overline{U}^V,\mathscr{O}_{\overline{U}^V}).$$

The presheaf on E defined above is in fact a sheaf. Like any sheaf, we can write $\overline{\mathscr{B}}=\{U\mapsto\overline{\mathscr{B}}_U\}$.

Let $U=\mathrm{Spec}(R)$ be an étale X-scheme, $\overline{y}$ a geometric point of $U_{\overline{\eta}}$. The stalk $\overline{\mathscr{B}}_{U,\overline{y}}$ can be described as follows. We denote by $(V_i)_{i\in I}$ the universal cover of $U_{\overline{\eta}}$ at $\overline{y}$. For each $i\in I$, let $U_i=\mathrm{Spec}(R_i)$ be the normalization of $\overline{U}=U\times_S\overline{S}$ in V_i.

(2.7.2) $$\begin{array}{ccc} V_i & \longrightarrow & U_i \\ \downarrow & & \downarrow \\ U_{\overline{\eta}} & \longrightarrow & \overline{U} \end{array}$$

Then, the stalk $\overline{\mathscr{B}}_{U,\overline{y}}$ is isomorphic to the following $\mathscr{O}_{\overline{K}}$-representation of $\pi_1(U_{\overline{\eta}},\overline{y})$:

(2.7.3) $$\overline{R}=\varinjlim_{i\in I}R_i.$$

Using Artin-Schreier theory, Faltings proved the following refinement of 2.6:

Theorem 2.8 (Faltings [10], A.-Gros [1] 4.8.13)**.** *For any locally constant constructible sheaf of $(\mathbb{Z}/p^n\mathbb{Z})$-modules F of $X_{\overline{\eta},\text{ét}}$, the canonical morphism*

(2.8.1) $$\mathrm{H}^i(X_{\overline{\eta},\text{ét}},F)\otimes_{\mathbb{Z}_p}\mathscr{O}_C\to\mathrm{H}^i(\widetilde{E},\psi_*(F)\otimes_{\mathbb{Z}_p}\overline{\mathscr{B}})$$

is an almost isomorphism*, i.e., its kernel and cokernel are annihilated by $\mathfrak{m}_C$.*

Faltings derived all comparaison theorems between p-adic étale cohomology and other p-adic cohomologies from this main p-adic comparaison theorem. We revisit in [1] the proof of this important result providing more details.

3. The torsor of deformations. The Higgs-Tate algebra

3.1. Recall ([3] II.9, [28] 1.1) that Fontaine associated functorially to any $\mathbb{Z}_{(p)}$-algebra A such that $A/pA\neq 0$, the ring

(3.1.1) $$A^\flat=\varprojlim_{\mathbb{N}}A/pA,$$

where the transition morphisms are the absolute Frobenius of A/pA; and the ring homomorphism

(3.1.2) $$\theta\colon \mathrm{W}(A^\flat)\to\widehat{A},$$

from the Witt vectors of $A^\flat$ to the p-adic completion of A, defined for $x=(x_0,x_1,\dots)\in\mathrm{W}(A^\flat)$ by

(3.1.3) $$\theta(x)=\lim_{m\to+\infty}(\widetilde{x}_{0m}^{p^m}+p\widetilde{x}_{1m}^{p^{m-1}}+\cdots+p^m\widetilde{x}_{mm})$$

where for each $n \geq 0$, we write $x_n = (x_{nm})_{m \geq 0} \in A^\flat$ and for $x \in A/pA$, $\widetilde{x}$ denotes a lifting in A.

The ring $A^\flat$ is perfect of characteristic p, and the homomorphism θ is surjective if the absolute Frobenius of A/pA is surjective.

3.2. The ring $\mathscr{O}_{\overline{K}^\flat} = (\mathscr{O}_{\overline{K}})^\flat$ is a complete non discrete valuation ring of height 1. We denote by $\overline{K}^\flat$ its fractions field. We choose a sequence $(p_n)_{n\geq 0}$ of elements of $\mathscr{O}_{\overline{K}}$ such that $p_0 = p$ and $p_{n+1}^p = p_n$ for all $n \geq 0$. We denote by ϖ the element of $\mathscr{O}_{\overline{K}^\flat}$ defined by (p_n) and we set

(3.2.1) $$\xi = [\varpi] - p \in \mathrm{W}(\mathscr{O}_{\overline{K}^\flat}).$$

It's a generator of the kernel of θ. We set

(3.2.2) $$\mathscr{A}_2(\mathscr{O}_{\overline{K}}) = \mathrm{W}(\mathscr{O}_{\overline{K}^\flat}) / \ker(\theta)^2.$$

Then, we have an exact sequence

(3.2.3) $$0 \longrightarrow \mathscr{O}_C \xrightarrow{\cdot \xi} \mathscr{A}_2(\mathscr{O}_{\overline{K}}) \xrightarrow{\theta} \mathscr{O}_C \longrightarrow 0.$$

We have a canonical homomorphism $\mathbb{Z}_p(1) \to \mathscr{O}_{\overline{K}^\flat}^\times$. For any $\zeta \in \mathbb{Z}_p(1)$, we have $\theta([\zeta] - 1) = 0$. We deduce a group homomorphism

(3.2.4) $$\mathbb{Z}_p(1) \to \mathscr{A}_2(\mathscr{O}_{\overline{K}}), \quad \zeta \mapsto \log([\zeta]) = [\zeta] - 1,$$

whose image generates the ideal $p^{\frac{1}{p-1}}\xi\mathscr{O}_C$ de $\mathscr{A}_2(\mathscr{O}_{\overline{K}})$. It induces an $\mathscr{O}_C$-linear isomorphism ([3] II.9.18)

(3.2.5) $$\mathscr{O}_C(1) \xrightarrow{\sim} p^{\frac{1}{p-1}}\xi\mathscr{O}_C.$$

3.3. Let $X = \mathrm{Spec}(R)$ be an affine smooth S-scheme which is *small* in the sense of Faltings (*i.e.* it admits an étale S-morphism $X \to \mathbb{G}^d_{m,S} = \mathrm{Spec}(\mathscr{O}_K[T_1^{\pm 1}, \dots, T_d^{\pm 1}])$, for an integer $d \geq 0$), such that $X_s \neq \emptyset$. We fix a geometric point $\overline{y}$ of $X_{\overline{\eta}}$ and we apply Fontaine's construction to the algebra $\overline{R} = \overline{\mathscr{B}}_{X,\overline{y}}$ (2.7.3). Then, setting

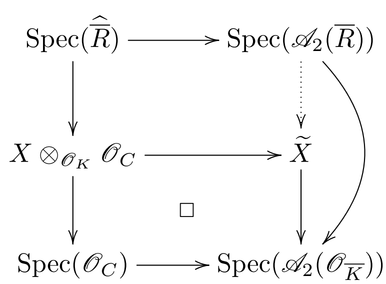
(3.3.1) $$\mathscr{A}_2(\overline{R}) = \mathrm{W}(\overline{R}^\flat) / \ker(\theta)^2,$$

we have an exact sequence

(3.3.2) $$0 \longrightarrow \widehat{\overline{R}} \xrightarrow{\cdot \xi} \mathscr{A}_2(\overline{R}) \xrightarrow{\theta} \widehat{\overline{R}} \longrightarrow 0.$$

3.4. By functoriality of θ, the big rectangle below is commutative

(3.4.1)



We fix a smooth $\mathscr{A}_2(\mathscr{O}_{\overline{K}})$-deformation $\widetilde{X}$ of $X \otimes_{\mathscr{O}_K} \mathscr{O}_C$. We consider the Zariski sheaf $\mathscr{L}$ on $\mathrm{Spec}(\widehat{\overline{R}})$ whose sections are given by the dotted arrows that complete the diagram (3.4.1) so as to leave it commutative. It's a torsor under the $\widehat{\overline{R}}$-module $\mathrm{Hom}_R(\Omega^1_{R/\mathscr{O}_K}, \xi\widehat{\overline{R}})$. Such a torsor is very easy to describe. Let $\mathscr{F}$ be the $\widehat{\overline{R}}$-module of affine functions on $\mathscr{L}$ ([3] II.4.9). The latter fits into a canonical exact sequence

$$0 \to \widehat{\overline{R}} \to \mathscr{F} \to \xi^{-1}\Omega^1_{R/\mathscr{O}_K} \otimes_R \widehat{\overline{R}} \to 0. \tag{3.4.2}$$

Consider the $\widehat{\overline{R}}$-algebra

$$\mathscr{C} = \varinjlim_{n\geq 0} \mathrm{Sym}^n_{\widehat{\overline{R}}}(\mathscr{F}), \tag{3.4.3}$$

where the transition morphisms are defined by mapping $x_1 \otimes \cdots \otimes x_n$ to $1 \otimes x_1 \otimes \cdots \otimes x_n$. Then, $\mathscr{L}$ is represented by $\mathrm{Spec}(\mathscr{C})$ ([3] II.4.10).

The natural action of $\pi_1(X_{\overline{\eta}}, \overline{y})$ on $\overline{R}$ induces an action on $\mathscr{A}_2(\overline{R})$, and hence an $\widehat{\overline{R}}$-semi-linear action on $\mathscr{F}$, such that the morphisms of (3.4.3) are $\pi_1(X_{\overline{\eta}}, \overline{y})$ equivariant ([3] II.10.4). We deduce an action of $\pi_1(X_{\overline{\eta}}, \overline{y})$ on $\mathscr{C}$ by ring homomorphisms. These actions are continuous for the p-adic topology ([3] II.12.4).

The ring $\mathscr{C}$ is in fact a model of Hyodo's ring ([3] II.15.6) which defines the notion of *Hodge-Tate* representations. A "weak" p-adic completion $\mathscr{C}^\dagger$ of $\mathscr{C}$ defines a weaker admissibility condition, namely *Dolbeault* representations, leading the p-adic Simpson correspondence. The construction of this correspondence in the affine case is developed in ([3] II.12). I will not explain it here since it cannot be glued in Faltings topos. Instead, I will explain the global variant defined by sheafifying $\mathscr{C}$ in Faltings topos.

4. Dolbeault modules

4.1. Let X be a smooth S-scheme. We assume that there exists a smooth $\mathscr{A}_2(\mathscr{O}_{\overline{K}})$-deformation $\widetilde{X}$ of $X \otimes_{\mathscr{O}_K} \mathscr{O}_C$ that we fix:

$$\begin{array}{ccc} X \otimes_{\mathscr{O}_K} \mathscr{O}_C & \longrightarrow & \widetilde{X} \\ \downarrow & \square & \downarrow \\ \mathrm{Spec}(\mathscr{O}_C) & \longrightarrow & \mathrm{Spec}(\mathscr{A}_2(\mathscr{O}_{\overline{K}})) \end{array} \tag{4.1.1}$$

Recall that we have the ring $\overline{\mathscr{B}} = \{U \mapsto \overline{\mathscr{B}}_U\}$ of $\widetilde{E}$ (2.7). For every $n \geq 0$, we set $\overline{\mathscr{B}}_n = \overline{\mathscr{B}}/p^n\overline{\mathscr{B}}$ and for every $U \in \mathrm{Ob}(\mathbf{\acute{E}t}_{/X})$, $\overline{\mathscr{B}}_{U,n} = \overline{\mathscr{B}}_U/p^n\overline{\mathscr{B}}_U$, which is a ring of $U_{\overline{\eta},\text{fét}}$.

For any small affine étale X-scheme U, there exists a canonical exact sequence of $\overline{\mathscr{B}}_{U,n}$-modules of $U_{\overline{\eta},\text{fét}}$

$$0 \to \overline{\mathscr{B}}_{U,n} \to \mathscr{F}_{U,n} \to \xi^{-1}\Omega^1_{X/S}(U) \otimes_{\mathscr{O}_X(U)} \overline{\mathscr{B}}_{U,n} \to 0, \tag{4.1.2}$$

such that for any geometric point $\overline{y}$ of $U_{\overline{\eta}}$, we have a canonical isomorphism of $\overline{\mathscr{B}}_{U,\overline{y}}$-representations of $\pi_1(U_{\overline{\eta}}, \overline{y})$

$$(\mathscr{F}_{U,n})_{\overline{y}} \xrightarrow{\sim} \mathscr{F}^{\overline{y}}_U/p^n\mathscr{F}^{\overline{y}}_U, \tag{4.1.3}$$

where $\mathscr{F}^{\overline{y}}_U$ is the Higgs-Tate $\overline{\mathscr{B}}_{U,\overline{y}}$-extension (3.4.2) defined relatively to the restriction of $\widetilde{X}$ over U.

For every rational number $r \geq 0$, let

$$0 \to \overline{\mathscr{B}}_{U,n} \to \mathscr{F}^{(r)}_{U,n} \to \xi^{-1}\Omega^1_{X/S}(U) \otimes_{\mathscr{O}_X(U)} \overline{\mathscr{B}}_{U,n} \to 0 \tag{4.1.4}$$

be the extension of $\overline{\mathscr{B}}_{U,n}$-modules of $U_{\overline{\eta},\mathrm{f\acute{e}t}}$ obtained from $\mathscr{F}_{U,n}$ by pull-back by the multiplication by p^r on $\xi^{-1}\Omega^1_{X/S}(U)\otimes_{\mathscr{O}_X(U)}\overline{\mathscr{B}}_{U,n}$, and let

$$\mathscr{C}^{(r)}_{U,n}=\varinjlim_{m\geq 0}\mathrm{Sym}^m_{\overline{\mathscr{B}}_{U,n}}(\mathscr{F}^{(r)}_{U,n}) \tag{4.1.5}$$

be the associated $\overline{\mathscr{B}}_{U,n}$-algebra of $U_{\overline{\eta},\mathrm{f\acute{e}t}}$.

The formation of $\mathscr{F}^{(r)}_{U,n}$ being functorial in U, the correspondences

$$\{U\mapsto\mathscr{F}^{(r)}_{U,n}\}\quad\text{and}\quad\{U\mapsto\mathscr{C}^{(r)}_{U,n}\} \tag{4.1.6}$$

define presheaves on the subcategory E^{sm} of E of objects $(V\to U)$ such that U is affine and small. The latter is topologically generating of E. Therefore, by taking associated sheaves, we get a $\overline{\mathscr{B}}_n$-module $\mathscr{F}^{(r)}_n$ and a $\overline{\mathscr{B}}_n$-algebra $\mathscr{C}^{(r)}_n$ of $\widetilde{E}$.

4.2. To take into account the p-adic topology, we consider the formal p-adic completion of the ringed topos $(\widetilde{E},\overline{\mathscr{B}})$, similar to the formal p-adic completion $\mathfrak{X}$ of $\overline{X}=X\times_S\overline{S}$. First, we define the *special fiber* $\widetilde{E}_s$ of $\widetilde{E}$, a topos that fits into a commutative diagram

$$\begin{array}{ccc}\widetilde{E}_s & \xrightarrow{\sigma_s} & X_{s,\mathrm{\acute{e}t}}\\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\iota_{\mathrm{\acute{e}t}}}\\ \widetilde{E} & \xrightarrow{\sigma} & X_{\mathrm{\acute{e}t}}\end{array} \tag{4.2.1}$$

where $\iota\colon X_s\to X$ is the canonical injection (cf. [3] III.9.8). Concretely, $\widetilde{E}_s$ is the full subcategory of $\widetilde{E}$ of objects F such that $F|\sigma^*(X_\eta)$ is the final object of $\widetilde{E}_{/\sigma^*(X_\eta)}$, and $\delta_*\colon\widetilde{E}_s\to\widetilde{E}$ is the canonical injection functor.

For every integer $n\geq 0$, $\overline{\mathscr{B}}_n$ is an object of $\widetilde{E}_s$. We denote by $\overline{X}_n$ and $\overline{S}_n$ the reductions of $\overline{X}$ and $\overline{S}$ modulo p^n. Then, the morphism σ_s is underlying a canonical morphism of ringed topos

$$\sigma_n\colon(\widetilde{E}_s,\overline{\mathscr{B}}_n)\to(X_{s,\mathrm{\acute{e}t}},\mathscr{O}_{\overline{X}_n}), \tag{4.2.2}$$

where we identified the étale topos of X_s and $\overline{X}_n$, since k is algebraically closed.

For any rational number $r\geq 0$, we have a canonical locally split exact sequence

$$0\to\overline{\mathscr{B}}_n\to\mathscr{F}^{(r)}_n\to\sigma_n^*(\xi^{-1}\Omega^1_{\overline{X}_n/\overline{S}_n})\to 0 \tag{4.2.3}$$

and a canonical isomorphism of $\overline{\mathscr{B}}_n$-algebras

$$\mathscr{C}^{(r)}_n\xrightarrow{\sim}\varinjlim_{m\geq 0}\mathrm{Sym}^m_{\overline{\mathscr{B}}_n}(\mathscr{F}^{(r)}_n). \tag{4.2.4}$$

The formal p-adic completion of $(\widetilde{E},\overline{\mathscr{B}})$ is the ringed topos $(\widetilde{E}_s^{\mathbb{N}^\circ},\breve{\overline{\mathscr{B}}})$, where $\widetilde{E}_s^{\mathbb{N}^\circ}$ is the topos of projective systems of objects of $\widetilde{E}_s$ indexed by the ordered set $\mathbb{N}$ ([3] III.7) and $\breve{\overline{\mathscr{B}}}=(\overline{\mathscr{B}}_n)_{n\geq 0}$. The morphisms σ_n induce a morphism of topos

$$\widehat{\sigma}\colon(\widetilde{E}_s^{\mathbb{N}^\circ},\breve{\overline{\mathscr{B}}})\to(X_{s,\mathrm{zar}},\mathscr{O}_{\mathfrak{X}}). \tag{4.2.5}$$

We work in the category $\mathbf{Mod}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}})$ of $\breve{\overline{\mathscr{B}}}$*-modules up to isogeny*, *i.e.*, the category having for objects $\breve{\overline{\mathscr{B}}}$-modules, and for any $\breve{\overline{\mathscr{B}}}$-modules $\mathscr{F}$ and $\mathscr{G}$,

$$\mathrm{Hom}_{\mathbf{Mod}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}})}(\mathscr{F},\mathscr{G})=\mathrm{Hom}_{\mathbf{Mod}(\breve{\overline{\mathscr{B}}})}(\mathscr{F},\mathscr{G})\otimes_{\mathbb{Z}}\mathbb{Q}. \tag{4.2.6}$$

We denote the localization functor $\mathbf{Mod}(\breve{\overline{\mathscr{B}}}) \to \mathbf{Mod}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}})$ by $\mathscr{F} \mapsto \mathscr{F}_{\mathbb{Q}}$. We call $\breve{\overline{\mathscr{B}}}_{\mathbb{Q}}$*-modules* the objects of $\mathbf{Mod}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}})$.

We set $\breve{\mathscr{F}}^{(r)} = (\mathscr{F}_n^{(r)})_{n\geq 0}$ which is a $\breve{\overline{\mathscr{B}}}$-module and $\breve{\mathscr{C}}^{(r)} = (\mathscr{C}_n^{(r)})_{n\geq 0}$ which is a $\breve{\overline{\mathscr{B}}}$-algebra, that we call the *Higgs-Tate algebra of thickness* r associated to $\widetilde{X}$. We have a canonical exact sequence of $\breve{\overline{\mathscr{B}}}$-modules

$$0 \to \breve{\overline{\mathscr{B}}} \to \breve{\mathscr{F}}^{(r)} \to \widehat{\sigma}^*(\xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}}) \to 0, \tag{4.2.7}$$

where $\mathscr{S} = \mathrm{Spf}(\mathscr{O}_C)$. The universal $\breve{\overline{\mathscr{B}}}$-derivation of $\breve{\mathscr{C}}^{(r)}$ can be identified with a derivation

$$d_{\breve{\mathscr{C}}^{(r)}} : \breve{\mathscr{C}}^{(r)} \to \widehat{\sigma}^*(\xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}}) \otimes_{\breve{\overline{\mathscr{B}}}} \breve{\mathscr{C}}^{(r)}. \tag{4.2.8}$$

It's a Higgs $\breve{\overline{\mathscr{B}}}$-field.

For any rational numbers $r \geq r' \geq 0$, we have a canonical homomorphism of $\breve{\overline{\mathscr{B}}}$-algebras $\breve{\mathscr{C}}^{(r)} \to \breve{\mathscr{C}}^{(r')}$. Observe that the restriction of the derivation $p^{r'} d_{\breve{\mathscr{C}}^{(r')}}$ is $p^r d_{\breve{\mathscr{C}}^{(r)}}$. Hence, we have a morphism of complexes

$$\mathbb{K}^\bullet(\breve{\mathscr{C}}^{(r)}) \to \mathbb{K}^\bullet(\breve{\mathscr{C}}^{(r')}), \tag{4.2.9}$$

where $\mathbb{K}^\bullet(\breve{\mathscr{C}}^{(r)})$ is the Dolbeault complex of the Higgs $\breve{\overline{\mathscr{B}}}$-module $(\breve{\mathscr{C}}^{(r)}, p^r d_{\breve{\mathscr{C}}^{(r)}})$.

Proposition 4.3 (A.-Gros [3] III.11.18)**.** *The canonical homomorphism*

$$\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}] \to \varinjlim_{r\in\mathbb{Q}_{>0}} \widehat{\sigma}_*(\breve{\mathscr{C}}^{(r)})[\frac{1}{p}] \tag{4.3.1}$$

is an isomorphism, and for any $q \geq 1$*,*

$$\varinjlim_{r\in\mathbb{Q}_{>0}} \mathrm{R}^q\widehat{\sigma}_*(\breve{\mathscr{C}}^{(r)})[\frac{1}{p}] = 0. \tag{4.3.2}$$

This result is a sheafification of the computation of the Galois cohomology of the Higgs-Tate algebra over a small affine scheme ([3] II.12.5). The Galois cohomology computation relies on Faltings' almost purity result and the sheafification requires to prove a version modulo p^n, up to some bounded defect ([3] II.12.7).

Proposition 4.4 (A.-Gros [3] III.11.24)**.** *The canonical morphism of* $\mathbf{Mod}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}})$

$$\breve{\overline{\mathscr{B}}}_{\mathbb{Q}} \to \varinjlim_{r\in\mathbb{Q}_{>0}} \mathrm{H}^0(\mathbb{K}^\bullet_{\mathbb{Q}}(\breve{\mathscr{C}}^{(r)})) \tag{4.4.1}$$

is an isomorphism, and for any $q \geq 1$*,*

$$\varinjlim_{r\in\mathbb{Q}_{>0}} \mathrm{H}^q(\mathbb{K}^\bullet_{\mathbb{Q}}(\breve{\mathscr{C}}^{(r)})) = 0. \tag{4.4.2}$$

This result is a sheafification of the computation of the de Rham cohomology of the Higgs-Tate algebra over a small affine scheme ([3] II.12.3).

Observe that filtered inductive limits are not a priori representable in $\mathbf{Mod}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}})$. We can naturally embed this category into the abelian category $\mathbf{Ind\text{-}Mod}(\breve{\overline{\mathscr{B}}})$ of ind-$\breve{\overline{\mathscr{B}}}$-modules where filtered inductive limits are representable and which has better properties (cf. [17] and [2]).

In the same way, we can naturally embed the category of coherent $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$-modules into the category $\mathbf{Ind\text{-}Mod}(\mathscr{O}_{\mathfrak{X}})$ of ind-$\mathscr{O}_{\mathfrak{X}}$-modules.

The morphism $\widehat{\sigma}$ induces two adjoint functors

$$\mathbf{Ind\text{-}Mod}(\breve{\overline{\mathscr{B}}}) \underset{\mathrm{I}\widehat{\sigma}^*}{\overset{\mathrm{I}\widehat{\sigma}_*}{\rightleftarrows}} \mathbf{Ind\text{-}Mod}(\mathscr{O}_{\mathfrak{X}}) . \tag{4.4.3}$$

that extend the adjoint functors $\widehat{\sigma}^*$ and $\widehat{\sigma}_*$.

Definition 4.5 (A.-Gros [2])**.** Let $\mathscr{M}$ be an ind-$\breve{\overline{\mathscr{B}}}$-module, $\mathscr{N}$ a Higgs $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$-bundle on $\mathfrak{X}/\mathscr{S}$, *i.e.*, a locally projective $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$-module of finite type $\mathscr{N}$ equipped with an $\mathscr{O}_{\mathfrak{X}}$-linear morphism $\theta\colon \mathscr{N} \to \xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{N}$ such that $\theta \wedge \theta = 0$.

(i) We say that $\mathscr{M}$ and $\mathscr{N}$ are r-*associated* (for $r \in \mathbb{Q}_{>0}$) if there exists an isomorphism of ind-$\breve{\mathscr{C}}^{(r)}$-modules

$$\mathscr{M} \otimes_{\breve{\overline{\mathscr{B}}}} \breve{\mathscr{C}}^{(r)} \xrightarrow{\sim} \mathrm{I}\widehat{\sigma}^*(\mathscr{N}) \otimes_{\breve{\overline{\mathscr{B}}}} \breve{\mathscr{C}}^{(r)}, \tag{4.5.1}$$

compatible with the total Higgs $\breve{\overline{\mathscr{B}}}$-fields with coefficients in $\widehat{\sigma}^*(\xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}})$, where $\mathscr{M}$ is equipped with the zero Higgs field and $\breve{\mathscr{C}}^{(r)}$ with $p^r d_{\breve{\mathscr{C}}^{(r)}}$.

(ii) We say that $\mathscr{M}$ and $\mathscr{N}$ are *associated* if they are r-associated for a rational number $r > 0$.

In fact, (4.5.1) is an isomorphism of *ind-$\breve{\mathscr{C}}^{(r)}$-modules with p^r-connection relatively to the extension $\breve{\mathscr{C}}^{(r)}/\breve{\overline{\mathscr{B}}}$.* In particular, for any rational numbers $r \geq r' > 0$, if $\mathscr{M}$ and $\mathscr{N}$ are r-associated, they are r'-associated.

Definition 4.6 (A.-Gros [2])**.**

(i) We say that an ind-$\breve{\overline{\mathscr{B}}}$-module is *Dolbeault* if it's associated to a Higgs $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$-bundle on $\mathfrak{X}/\mathscr{S}$.

(ii) We say that a Higgs $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$-bundle on $\mathfrak{X}/\mathscr{S}$ is *solvable* if it's associated to an ind-$\breve{\overline{\mathscr{B}}}$-module.

The notion of being Dolbeault applies to $\breve{\overline{\mathscr{B}}}_{\mathbb{Q}}$-module. We call the associated Higgs bundles *rationally solvable.*

In ([3] III.12.11), we considered only $\breve{\overline{\mathscr{B}}}_{\mathbb{Q}}$-modules and we requested moreover that they are adic of finite type. We renamed them in [2] *strongly Dolbeault* $\breve{\overline{\mathscr{B}}}_{\mathbb{Q}}$-modules and renamed the associated Higgs bundles *strongly solvable.*

Theorem 4.7 (A.-Gros [2])**.** *There are explicit equivalences of categories quasi-inverse to each other*

$$\mathbf{Ind\text{-}Mod}^{\mathrm{Dolb}}(\breve{\overline{\mathscr{B}}}) \underset{\mathscr{V}}{\overset{\mathscr{H}}{\rightleftarrows}} \mathbf{HIG}^{\mathrm{sol}}(\mathscr{O}_{\mathfrak{X}}[\tfrac{1}{p}], \xi^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}}) . \tag{4.7.1}$$

These functors are in fact explicitly defined ([2], [3] III.12.13 and III.12.23). Let $\vec{\sigma}_*$ be the composed functor

$$\mathbf{Ind\text{-}Mod}(\breve{\overline{\mathscr{B}}}) \xrightarrow{\mathrm{I}\widehat{\sigma}_*} \mathbf{Ind\text{-}Mod}(\mathscr{O}_{\mathfrak{X}}) \xrightarrow{\kappa_{\mathscr{O}_{\mathfrak{X}}}} \mathbf{Mod}(\mathscr{O}_{\mathfrak{X}}) , \tag{4.7.2}$$

where $\mathrm{I}\widehat{\sigma}_*$ is defined in (4.4.3) and

$$\kappa_{\mathscr{O}_{\mathfrak{X}}}(``\underset{\longrightarrow}{\lim}"\alpha) = \underset{\underset{J}{\longrightarrow}}{\lim}\ \alpha. \tag{4.7.3}$$

Then, the functor $\mathscr{H}$ can in fact be defined for any ind-$\breve{\overline{\mathscr{B}}}$-module $\mathscr{M}$ by

$$\mathscr{H}(\mathscr{M}) = \varinjlim_{r\in\mathbb{Q}_{>0}} \vec{\boldsymbol{\sigma}}_*(\mathscr{M}\otimes_{\breve{\overline{\mathscr{B}}}}\breve{\mathscr{C}}^{(r)}, p^r\mathrm{id}\otimes d_{\breve{\mathscr{C}}^{(r)}}). \tag{4.7.4}$$

We have a similar definition of $\mathscr{V}$.

Theorem 4.8 (A.-Gros [2])**.** *For any Dolbeault ind-$\breve{\overline{\mathscr{B}}}$-module $\mathscr{M}$ and any integer $q\geq 0$, there is a canonical functorial isomorphism of $\mathbf{D}^+(\mathbf{Mod}(\mathscr{O}_{\mathfrak{X}}))$*

$$\mathrm{R}\vec{\boldsymbol{\sigma}}_*(\mathscr{M}) \xrightarrow{\sim} \mathbb{K}^\bullet(\mathscr{H}(\mathscr{M})), \tag{4.8.1}$$

where $\mathbb{K}^\bullet(\mathscr{H}(\mathscr{M}))$ is the Dolbeault complex of $\mathscr{H}(\mathscr{M})$.

Corollary 4.9. *Let $M=(M_n)_{n\geq 0}$ be a $\breve{\mathbb{Z}}_p$-local system of $X^{\mathbb{N}^\circ}_{\overline{\eta},\text{ét}}$, $\mathscr{M}=\breve{\psi}_*(M)\otimes_{\breve{\mathbb{Z}}_p}\breve{\overline{\mathscr{B}}}$, where $\breve{\mathbb{Z}}_p=(\mathbb{Z}/p^n\mathbb{Z})_{n\geq 0}$ and $\breve{\psi}$ is induced by ψ (2.4.1). Assume that X is proper over S and that the $\breve{\overline{\mathscr{B}}}_{\mathbb{Q}}$-module $\mathscr{M}_{\mathbb{Q}}$ is Dolbeault. Then, there exists a canonical spectral sequence*

$$\mathrm{E}_2^{i,j}=\mathrm{H}^i(X_s,\mathrm{H}^j(\mathbb{K}^\bullet))\Rightarrow \mathrm{H}^{i+j}(X^{\mathbb{N}^\circ}_{\overline{\eta},\text{ét}}, M)\otimes_{\mathbb{Z}_p} C, \tag{4.9.1}$$

where $\mathbb{K}^\bullet$ be the Dolbeault complex of $\mathscr{H}(\mathscr{M}_{\mathbb{Q}})$.

It follows from 2.8 and 4.8.

Remark 4.10. In 4.9, if we take $M=\breve{\mathbb{Z}}_p$, then $\mathscr{M}=\breve{\overline{\mathscr{B}}}$, $\mathscr{H}(\breve{\overline{\mathscr{B}}}_{\mathbb{Q}})$ is equal to $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$ equipped with the zero Higgs field ([3] III.12.14), and the spectral sequence (4.9.1) is the Hodge-Tate spectral sequence (1.1.1).

5. Functoriality of the p-adic Simpson correspondence by proper direct image

5.1. Let $g\colon X'\to X$ be a smooth morphism of smooth S-schemes. We equip with a prime $'$ the objects associated to X'/S. By functoriality of Faltings topos, g induces a canonical morphism Θ between Faltings topos that fits into a commutative diagram

$$\begin{array}{ccccc} X'_{\overline{\eta},\text{ét}} & \xrightarrow{\psi'} & \widetilde{E}' & \xrightarrow{\sigma'} & X'_{\text{ét}} \\ \downarrow{\scriptstyle g_{\overline{\eta}}} & & \downarrow{\scriptstyle \Theta} & & \downarrow{\scriptstyle g} \\ X_{\overline{\eta},\text{ét}} & \xrightarrow{\psi} & \widetilde{E} & \xrightarrow{\sigma} & X_{\text{ét}} \end{array} \tag{5.1.1}$$

We have also a canonical ring homomorphism

$$\overline{\mathscr{B}}\to\Theta_*(\overline{\mathscr{B}}'). \tag{5.1.2}$$

Theorem 5.2 (Faltings [10], A.-Gros [1] 5.7.3)**.** *Assume that $g\colon X'\to X$ is projective, and let F' be a locally constant constructible sheaf of $(\mathbb{Z}/p^n\mathbb{Z})$-modules of $X'_{\overline{\eta},\text{ét}}$ $(n\geq 1)$. Then, for any integer $i\geq 0$, the canonical morphism*

$$\psi_*(\mathrm{R}^i g_{\overline{\eta}*}(F'))\otimes_{\mathbb{Z}_p}\overline{\mathscr{B}}\to \mathrm{R}^i\Theta_*(\psi'_*(F')\otimes_{\mathbb{Z}_p}\overline{\mathscr{B}}') \tag{5.2.1}$$

is an almost isomorphism.

Observe that the sheaves $\mathrm{R}^i g_{\overline{\eta}*}(F)$ are locally constant constructible on $X_{\overline{\eta}}$ by the smooth and the proper base change theorems.

Faltings formulated this *relative version* of his main p-adic comparison theorem in [10] and he very roughly sketched a proof in the appendix. Some arguments have to be modified and the actual proof in [1] requires much more work. It is based on a fine study of the local structure of certain almost-étale φ-modules which is interesting in itself ([1] 5.5.22).

The condition *g projective* is used here to obtain an almost finiteness result for almost coherent modules. We rely on the finiteness results of SGA6 instead of those of Kiehl. It should be possible to replace it by *g proper*.

5.3. Diagram (5.1.1) induces a commutative diagram of morphisms of ringed topos

(5.3.1)
$$\begin{array}{ccc} (\widetilde{E}'^{\mathbb{N}^\circ}_s, \breve{\overline{\mathscr{B}}}') & \xrightarrow{\breve{\uptheta}} & (\widetilde{E}^{\mathbb{N}^\circ}_s, \breve{\overline{\mathscr{B}}}) \\ {\scriptstyle \widehat{\upsigma}'}\downarrow & & \downarrow{\scriptstyle \widehat{\upsigma}} \\ (X'_{s,\mathrm{zar}}, \mathscr{O}_{\mathfrak{X}'}) & \xrightarrow{\mathfrak{g}} & (X_{s,\mathrm{zar}}, \mathscr{O}_{\mathfrak{X}}) \end{array}$$

where the horizontal arrows are induced by Θ and g, and the vertical arrows are induced by σ' and σ as in (4.2.5).

Assume that there exists a commutative diagram, that we fix,

(5.3.2)
$$\begin{array}{ccc} X' \otimes_{\mathscr{O}_K} \mathscr{O}_C & \longrightarrow & \widetilde{X}' \\ {\scriptstyle g\otimes \mathrm{id}}\downarrow & \Box & \downarrow{\scriptstyle \widetilde{g}} \\ X \otimes_{\mathscr{O}_K} \mathscr{O}_C & \longrightarrow & \widetilde{X} \\ \downarrow & \Box & \downarrow \\ \mathrm{Spec}(\mathscr{O}_C) & \longrightarrow & \mathrm{Spec}(\mathscr{A}_2(\mathscr{O}_{\overline{K}})) \end{array}$$

where $\widetilde{X}$ and $\widetilde{X}'$ are smooth over $\mathscr{A}_2(\mathscr{O}_{\overline{K}})$.

For any rational number $r \geq 0$, the lifting $\widetilde{g}$ induces a homomorphism of the Higgs-Tate algebras

(5.3.3)
$$\breve{\uptheta}^*(\breve{\mathscr{C}}^{(r)}) \to \breve{\mathscr{C}}'^{(r)}.$$

The construction is rather subtle (cf. [2]).

Theorem 5.4 (A.-Gros [2])**.** *Assume $g\colon X' \to X$ proper. Let $\mathscr{M}$ be a Dolbeault ind-$\breve{\overline{\mathscr{B}}}'$-module (4.6),*

(5.4.1)
$$\mathscr{H}'(\mathscr{M}) \to \xi^{-1}\Omega^1_{\mathfrak{X}'/\mathscr{S}} \otimes_{\mathscr{O}_{\mathfrak{X}'}} \mathscr{H}'(\mathscr{M})$$

the associated Higgs bundle (4.7.1),

(5.4.2)
$$\underline{\mathscr{H}}'(\mathscr{M}) \to \xi^{-1}\Omega^1_{\mathfrak{X}'/\mathfrak{X}} \otimes_{\mathscr{O}_{\mathfrak{X}'}} \underline{\mathscr{H}}'(\mathscr{M})$$

the induced relative Higgs bundle, $\underline{\mathbb{K}}^\bullet$ the Dolbeault complex of $\underline{\mathscr{H}}'(\mathscr{M})$. Then, for any integer $q \geq 0$, there exists a rational number $r > 0$ and a $\breve{\mathscr{C}}^{(r)}$-isomorphism

(5.4.3)
$$\mathrm{R}^q\mathrm{I}\breve{\uptheta}_*(\mathscr{M}) \otimes_{\breve{\overline{\mathscr{B}}}} \breve{\mathscr{C}}^{(r)} \xrightarrow{\sim} \mathrm{I}\widehat{\upsigma}^*(\mathrm{R}^q\mathfrak{g}_*(\underline{\mathbb{K}}^\bullet)) \otimes_{\breve{\overline{\mathscr{B}}}} \breve{\mathscr{C}}^{(r)},$$

compatible with the total Higgs fields, where $\breve{\mathscr{C}}^{(r)}$ is equipped with the Higgs field $p^r d_{\breve{\mathscr{C}}^{(r)}}$, $\mathrm{R}^q\mathrm{I}\breve{\uptheta}_*(\mathscr{M})$ with the zero Higgs field and $\mathrm{R}^q\mathfrak{g}_*(\underline{\mathbb{K}}^\bullet)$ with the Katz-Oda field (cf. [20] and [2]).

Observe that the $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$-module $\mathrm{R}^q\mathfrak{g}_*(\underline{\mathbb{K}}^\bullet)$ is coherent, and that the functor

$$\mathrm{I}\widehat{\sigma}^*\colon \mathbf{Mod}^{\mathrm{coh}}(\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]) \to \mathbf{Ind\text{-}Mod}(\breve{\overline{\mathscr{B}}}) \tag{5.4.4}$$

is exact [2].

Corollary 5.5. *Under the assumptions of 5.4, if the $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$-module $\mathrm{R}^q\mathfrak{g}_*(\underline{\mathbb{K}}^\bullet)$ is locally projective of finite type, then the ind-$\breve{\overline{\mathscr{B}}}$-module $\mathrm{R}^q\mathrm{I}\breve{\theta}_*(\mathscr{M})$ is* Dolbeault*, and we have an isomorphism*

$$\mathscr{H}(\mathrm{R}^q\mathrm{I}\breve{\theta}_*(\mathscr{M})) \xrightarrow{\sim} \mathrm{R}^q\mathfrak{g}_*(\underline{\mathbb{K}}^\bullet), \tag{5.5.1}$$

where $\mathrm{R}^q\mathfrak{g}_*(\underline{\mathbb{K}}^\bullet)$ is equipped with the Katz-Oda field.

Corollary 5.6. *Assume $g\colon X' \to X$ projective, and let $\mathscr{M}^n = \breve{\psi}_*(\mathrm{R}^n\breve{g}_{\overline{\eta}*}(\breve{\mathbb{Z}}_p)) \otimes_{\breve{\mathbb{Z}}_p} \breve{\overline{\mathscr{B}}}$ for an integer $n \geq 0$. Then, the $\breve{\overline{\mathscr{B}}}_{\mathbb{Q}}$-module $\mathscr{M}^n_{\mathbb{Q}}$ is Dolbeault and we have an isomorphism*

$$\mathscr{H}(\mathscr{M}^n_{\mathbb{Q}}) \xrightarrow{\sim} \oplus_{0\leq i\leq n}\mathrm{R}^i g_*(\xi^{i-n}\Omega^{n-i}_{X'/X}) \otimes_{\mathscr{O}_X} \mathscr{O}_{\mathfrak{X}}[\frac{1}{p}], \tag{5.6.1}$$

where the Higgs field on the right hand side is nilpotent, induced by the Kodaira-Spencer maps of g

$$\mathrm{R}g_*(\xi^{-j}\Omega^j_{X'/X}) \to \xi^{-1}\Omega^1_{X/S} \otimes_{\mathscr{O}_X} \mathrm{R}g_*(\xi^{1-j}\Omega^{j-1}_{X'/X})[+1]. \tag{5.6.2}$$

It follows from 5.2 and 5.5. Indeed, $\breve{\psi}'_*(\breve{\mathbb{Z}}_p) = \breve{\mathbb{Z}}_p$ and $\underline{\mathscr{H}}'(\breve{\overline{\mathscr{B}}}'_{\mathbb{Q}})$ is equal to $\mathscr{O}_{\mathfrak{X}'}[\frac{1}{p}]$ equipped with the zero Higgs field ([3] III.12.14). Therefore, with the notation of 5.4, for any $q \geq 0$, the $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$-module $\mathrm{R}^q\mathfrak{g}_*(\underline{\mathbb{K}}^\bullet)$ is locally free of finite type by ([7] 5.5), which implies the first statement. The second statement follows easily from the definition of the Katz-Oda field ([19] 1.2).

Corollary 5.7. *Let $M = (M_n)_{n\geq 0}$ be a $\breve{\mathbb{Z}}_p$-local system of $X'^{\mathbb{N}^\circ}_{\overline{\eta},\text{ét}}$, $\mathscr{M} = \breve{\psi}'_*(M) \otimes_{\breve{\mathbb{Z}}_p} \breve{\overline{\mathscr{B}}}'$. Assume that $g\colon X' \to X$ is projective and that the $\breve{\overline{\mathscr{B}}}'_{\mathbb{Q}}$-module $\mathscr{M}_{\mathbb{Q}}$ is Dolbeault. Then, there exists a rational number $r > 0$ and a spectral sequence*

$$\mathrm{E}_2^{i,j} = \widehat{\sigma}^*_{\mathbb{Q}}(\mathrm{R}^i\mathfrak{g}_*(\mathrm{H}^j(\underline{\mathbb{K}}^\bullet))) \otimes_{\breve{\overline{\mathscr{B}}}_{\mathbb{Q}}} \breve{\mathscr{C}}^{(r)}_{\mathbb{Q}} \Rightarrow \breve{\psi}_*(\mathrm{R}^{i+j}\breve{g}_{\overline{\eta}*}(M)) \otimes_{\breve{\mathbb{Z}}_p} \breve{\mathscr{C}}^{(r)}_{\mathbb{Q}}, \tag{5.7.1}$$

where $\underline{\mathbb{K}}^\bullet$ its Dolbeault complex of the relative Higgs $\mathscr{O}_{\mathfrak{X}'}[\frac{1}{p}]$-bundle $\underline{\mathscr{H}}'(\mathscr{M}_{\mathbb{Q}})$ (5.4.2).

It follows from 5.2 and 5.4.

If we take $M = \breve{\mathbb{Z}}_p$, the spectral sequence (5.7.1) can be deduced from the following relative Hodge-Tate spectral sequence, by extension of scalars from $\breve{\overline{\mathscr{B}}}$ to $\breve{\mathscr{C}}^{(r)}$.

Theorem 5.8 (A.-Gros [1] 6.7.5)**.** *Assume $g\colon X' \to X$ projective. Then, we have a canonical spectral sequence of $\breve{\overline{\mathscr{B}}}_{\mathbb{Q}}$-modules,* the relative Hodge-Tate spectral sequence*,*

$$\mathrm{E}_2^{i,j} = \sigma^*(\mathrm{R}^i g_*(\Omega^j_{X'/X})) \otimes_{\sigma^*(\mathscr{O}_X)} \breve{\overline{\mathscr{B}}}_{\mathbb{Q}}(-j) \Rightarrow \breve{\psi}_*(\mathrm{R}^{i+j}\breve{g}_{\overline{\eta}*}(\breve{\mathbb{Z}}_p)) \otimes_{\breve{\mathbb{Z}}_p} \breve{\overline{\mathscr{B}}}_{\mathbb{Q}}. \tag{5.8.1}$$

This spectral sequence does not require any deformation (5.3.2). It's G_K-equivariant for the natural G_K-equivariant structures on the various topos and objects involved. Hence it degenerates at E_2.

5.9. The proof of 5.4 can be divided into three steps. First, we compute the relative Galois and Higgs cohomologies of the Higgs-Tate algebra by adapting Faltings' computation in the absolute case. To sheafify these computations, we consider the following fiber product of topos

(5.9.1)
$$\begin{array}{ccc} \widetilde{E}' & & \\ \downarrow{\scriptstyle \tau} & \searrow^{\sigma'} & \\ \widetilde{E}\times_{X_{\text{ét}}} X'_{\text{ét}} & \longrightarrow & X'_{\text{ét}} \\ \downarrow & \Box & \downarrow{\scriptstyle g} \\ \widetilde{E} & \xrightarrow{\ \sigma\ } & X_{\text{ét}} \end{array}$$

The local relative Galois cohomology computation can be globalized into a computation of the sheaves $\mathrm{R}^i\tau_*(\mathscr{C}_n'^{(r)})$. The last step is a base change statement relatively to the Cartesian square.

It turns out that there is a very natural site underlying the topos $\widetilde{E}\times_{X_{\text{ét}}} X'_{\text{ét}}$, which is a relative version of Faltings topos and whose definition was inspired by oriented products of topos (beyond the covanishing topos which inspired the usual Faltings topos).

6. Relative Faltings topos

6.1. Let $g\colon X'\to X$ be a morphism of S-schemes. We denote by G the category of morphisms $(W\to U\leftarrow V)$ above the canonical morphisms $X'\to X\leftarrow X_{\overline{\eta}}$, *i.e.*, commutative diagrams

(6.1.1)
$$\begin{array}{ccccc} W & \longrightarrow & U & \longleftarrow & V \\ \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & X & \longleftarrow & X_{\overline{\eta}} \end{array}$$

such that W is étale over X', U is étale over X and the canonical morphism $V\to U_{\overline{\eta}}$ is *finite étale.* We equip it with the topology generated by coverings

$$\{(W_i\to U_i\leftarrow V_i)\to(W\to U\leftarrow V)\}_{i\in I}$$

of the following three types :

(a) $U_i=U$, $V_i=V$ for all $i\in I$ and $(W_i\to W)_{i\in I}$ is a covering;
(b) $W_i=W$, $U_i=U$ for all $i\in I$ and $(V_i\to V)_{i\in I}$ is a covering;
(c) diagrams

(6.1.2)
$$\begin{array}{ccccc} W' & \longrightarrow & U' & \longleftarrow & V' \\ \| & & \downarrow & \Box & \downarrow \\ W & \longrightarrow & U & \longleftarrow & V \end{array}$$

where $U'\to U$ is any morphism and the right square is Cartesian.

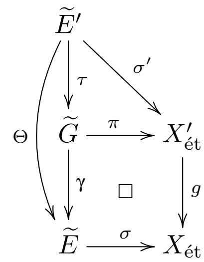
We denote by $\widetilde{G}$ the topos of sheaves of sets on G.

There is a canonical morphism of topos

(6.1.3)
$$\pi\colon \widetilde{G}\to X'_{\text{ét}},\quad W\in \mathrm{Ob}(\mathbf{\acute{E}t}_{/X'})\mapsto \pi^*(W)=(W\to X\leftarrow X_{\overline{\eta}})^a.$$

6.2. If $X' = X$, $\widetilde{G}$ is canonically equivalent to Faltings topos $\widetilde{E}$. By functoriality of the relative Faltings topos, we get a natural factorization of the canonical morphism $\Theta\colon \widetilde{E}' \to \widetilde{E}$ which fits into a commutative diagram

(6.2.1)



We prove that the lower left square is Cartesian.

We prove first a base change theorem relatively to this square for torsion abelian sheaves of $X'_{\text{ét}}$, inspired by a base change theorem for oriented products due to Gabber (A.-Gros [1] 6.5.5). It reduces to the proper base change theorem for the étale topos. Then, we prove the following result which plays a crucial role in the proofs of both 5.4 and 5.8:

Theorem 6.3 (A.-Gros [1] 6.5.31)**.** *Let $g\colon X' \to X$ be a* proper *and smooth morphism of smooth S-schemes. Then, there exists an integer $N \geq 0$ such that for any integers $n \geq 1$ and $q \geq 0$, and any quasi-coherent $\mathscr{O}_{X'_n}$-module $\mathscr{F}$, the kernel and cokernel of the base change morphism*

$$\sigma^*(\mathrm{R}^q g_*(\mathscr{F})) \to \mathrm{R}^q \gamma_*(\pi^*(\mathscr{F})), \tag{6.3.1}$$

are annihilated by p^N.

In this statement, π^* and σ^* denote the pull-backs for the morphisms of ringed topos

$$\pi\colon (\widetilde{G}, \tau_*(\overline{\mathscr{B}}')) \to (X'_{\text{ét}}, \mathscr{O}_{X'}), \tag{6.3.2}$$

$$\sigma\colon (\widetilde{E}, \overline{\mathscr{B}}) \to (X_{\text{ét}}, \mathscr{O}_X). \tag{6.3.3}$$

7. Hodge-Tate local systems

7.1. The constructions introduced so far depend on the existence and the choice of a deformation (4.1.1). We can get rid of this choice by developing a relative version over S. For this purpose, we define a *logarithmic relative version of the extension $\mathscr{A}_2(\mathscr{O}_{\overline{K}})$* (3.2.3) *over $\mathscr{O}_K$*. We fix a uniformizer π of $\mathscr{O}_K$ and a sequence $(\pi_n)_{n\geq 0}$ of elements of $\mathscr{O}_{\overline{K}}$ such that $\pi_0 = \pi$ and $\pi_{n+1}^p = \pi_n$ (for all $n \geq 0$) and let $\underline{\pi}$ be the associated element of $\mathscr{O}_{\overline{K}^\flat}$. We set

$$\mathrm{W}_{\mathscr{O}_K}(\mathscr{O}_{\overline{K}^\flat}) = \mathrm{W}(\mathscr{O}_{\overline{K}^\flat}) \otimes_{\mathrm{W}(k)} \mathscr{O}_K. \tag{7.1.1}$$

We denote by $\mathrm{W}^*_{\mathscr{O}_K}(\mathscr{O}_{\overline{K}^\flat})$ the sub-$\mathrm{W}_{\mathscr{O}_K}(\mathscr{O}_{\overline{K}^\flat})$-algebra of $\mathrm{W}_K(\mathscr{O}_{\overline{K}^\flat}) = \mathrm{W}(\mathscr{O}_{\overline{K}^\flat}) \otimes_{\mathrm{W}(k)} K$ generated by $[\underline{\pi}]/\pi$ and we set

$$\xi^*_\pi = \frac{[\underline{\pi}]}{\pi} - 1 \in \mathrm{W}^*_{\mathscr{O}_K}(\mathscr{O}_{\overline{K}^\flat}). \tag{7.1.2}$$

It's a generator of the kernel of the homomorphism $\theta^*_{\mathscr{O}_K}\colon \mathrm{W}^*_{\mathscr{O}_K}(\mathscr{O}_{\overline{K}^\flat}) \to \mathscr{O}_C$ induced by θ. Observe that this algebra depends on $(\pi_n)_{n\geq 0}$. We set

$$\mathscr{A}^*_2(\mathscr{O}_{\overline{K}}/\mathscr{O}_K) = \mathrm{W}^*_{\mathscr{O}_K}(\mathscr{O}_{\overline{K}^\flat})/(\xi^*_\pi)^2\mathrm{W}^*_{\mathscr{O}_K}(\mathscr{O}_{\overline{K}^\flat}). \tag{7.1.3}$$

Then, we have an exact sequence

$$0 \longrightarrow \mathscr{O}_C \xrightarrow{\cdot \xi^*_\pi} \mathscr{A}^*_2(\mathscr{O}_{\overline{K}}/\mathscr{O}_K) \xrightarrow{\theta^*_{\mathscr{O}_K}} \mathscr{O}_C \longrightarrow 0\ . \tag{7.1.4}$$

Denoting by K_0 the field of fractions of $\mathrm{W}(k)$ and by $\mathfrak{d}$ the different of K/K_0, the canonical homomorphism $\mathscr{A}_2(\mathscr{O}_{\overline{K}}) \to \mathscr{A}^*_2(\mathscr{O}_{\overline{K}}/\mathscr{O}_K)$ induces an $\mathscr{O}_C$-linear isomorphism

$$\xi\mathscr{O}_C \xrightarrow{\sim} \pi\mathfrak{d}\xi^*_\pi\mathscr{O}_C. \tag{7.1.5}$$

Let $\widetilde{S}$ be one of the schemes $\mathrm{Spec}(\mathscr{A}_2(\mathscr{O}_{\overline{K}}))$ or $\mathrm{Spec}(\mathscr{A}^*_2(\mathscr{O}_{\overline{K}}/\mathscr{O}_K))$, and let

$$i_S \colon \mathrm{Spec}(\mathscr{O}_C) \to \widetilde{S} \tag{7.1.6}$$

be the canonical closed immersion, defined by the ideal of square zero generated by $\widetilde{\xi} = \xi$ or ξ^*_π.

Given a smooth S-scheme X, we can develop the p-adic Simpson correspondence relatively to any smooth $\widetilde{S}$-deformation $\widetilde{X}$ of $X \otimes_{\mathscr{O}_K} \mathscr{O}_C$, as in sections 3 and 4. In the relative case (7.1.4), we can choose for $\widetilde{X}$ the trivial deformation $X \times_S \widetilde{S}$, since $\widetilde{S}$ is an S-scheme. The main defect of this case is the fact that the associated *smallness* depends on K as can be seen from (7.1.5). It's however good for Hodge-Tate local systems.

7.2. Let $X = \mathrm{Spec}(R)$ be a small affine smooth S-scheme such that $X_{\overline{\eta}}$ is connected and $X_s \neq \emptyset$, $\overline{y}$ a geometric generic point of $X_{\overline{\eta}}$, $\Delta = \pi_1(X_{\overline{\eta}}, \overline{y})$, $R_1 = R \otimes_{\mathscr{O}_K} \mathscr{O}_{\overline{K}}$ and $\overline{R} = \overline{\mathscr{B}}_{X,\overline{y}}$ (2.7.3). We fix a smooth $\widetilde{S}$-deformation $\widetilde{X}$ of $X \otimes_{\mathscr{O}_K} \mathscr{O}_C$.

$$\begin{array}{ccc} X \otimes_{\mathscr{O}_K} \mathscr{O}_C & \longrightarrow & \widetilde{X} \\ \downarrow & \square & \downarrow \\ \mathrm{Spec}(\mathscr{O}_C) & \longrightarrow & \widetilde{S} \end{array} \tag{7.2.1}$$

Let $\mathscr{C}$ be the associated Higgs-Tate $\widehat{\overline{R}}$-algebra (3.4.3), and for any rational number $r \geq 0$, $\mathscr{C}^{(r)}$ the Higgs-Tate $\widehat{\overline{R}}$-algebra of thickness r. We set

$$\mathscr{C}^\dagger = \varinjlim_{r \in \mathbb{Q}_{>0}} \widehat{\mathscr{C}}^{(r)}, \tag{7.2.2}$$

which is the period ring defining the local p-adic Simpson correspondence:

$$\mathbb{H} \colon \mathbf{Rep}_{\widehat{\overline{R}}}(\Delta) \to \mathbf{HIG}(\widehat{R_1}, \widetilde{\xi}^{-1}\Omega^1_{R/\mathscr{O}_K}), \quad M \mapsto \mathbb{H}(M) = (M \otimes_{\widehat{\overline{R}}} \mathscr{C}^\dagger)^\Delta, \tag{7.2.3}$$

the Higgs field of $\mathbb{H}(M)$ being induced by the $\widehat{\overline{R}}$-derivation of $\mathscr{C}^\dagger$ limit of the $p^r d_{\widehat{\mathscr{C}}^{(r)}}$'s.

Proposition 7.3 (A.-Gros [2])**.** *Let M be a projective $\widehat{\overline{R}}[\frac{1}{p}]$-module of finite type, equipped with an $\widehat{\overline{R}}[\frac{1}{p}]$-semi-linear action of Δ. Then, the following properties are equivalent:*

(i) *The $\widehat{\overline{R}}[\frac{1}{p}]$-representation M of Δ is Dolbeault (i.e. is continuous and admissible for $\mathscr{C}^\dagger$) and the associated Higgs $\widehat{R_1}[\frac{1}{p}]$-module $\mathbb{H}(M)$ (7.2.3) is nilpotent (i.e., there exists a finite decreasing filtration $(\mathbb{H}_i)_{0 \leq i \leq n}$ of $\mathbb{H}(M)$ by sub-$\widehat{R_1}[\frac{1}{p}]$-modules such that $\mathbb{H}_0 = \mathbb{H}(M)$, $\mathbb{H}_n = 0$ and for any $0 \leq i \leq n-1$, we have*

$$\theta(\mathbb{H}_i) \subset \widetilde{\xi}^{-1}\Omega^1_{R/\mathscr{O}_K} \otimes_R \mathbb{H}_{i+1}). \tag{7.3.1}$$

(ii) *There exists a projective* $\widehat{R_1}[\frac{1}{p}]$*-module of finite type* N*, a Higgs* $\widehat{R_1}[\frac{1}{p}]$*-field* θ *on* N *with coefficients in* $\widetilde{\xi}^{-1}\Omega^1_{R/\mathscr{O}_K}$ *and a* $\mathscr{C}$*-linear and* Δ*-equivariant isomorphism of Higgs* $\widehat{\overline{R}}[\frac{1}{p}]$*-modules*

$$N \otimes_{\widehat{R_1}} \mathscr{C} \xrightarrow{\sim} M \otimes_{\widehat{\overline{R}}} \mathscr{C}. \tag{7.3.2}$$

Moreover, under these conditions, we have an isomorphism of Higgs $\widehat{R_1}[\frac{1}{p}]$*-modules*

$$\mathbb{H}(M) \xrightarrow{\sim} (N, \theta). \tag{7.3.3}$$

Definition 7.4. We say that an $\widehat{\overline{R}}[\frac{1}{p}]$-representation M of Δ is *Hodge-Tate* if it satisfies the equivalent conditions of 7.3.

Remark 7.5. Tsuji [29] developed an arithmetic version of the local p-adic Simpson correspondence. He associates to a p-adic $\widehat{\overline{R}}[\frac{1}{p}]$-representation of $\Gamma = \pi_1(X_\eta, \overline{y})$ a Higgs field and an arithmetic Sen operator satisfying a compatibility relation that forces the Higgs field to be nilpotent (cf. also [15]). This explains the relation with the work of Liu and Zhu [21].

Definition 7.6 (A.-Gros [2])**.** Let X be a smooth S-scheme, $\widetilde{X}$ be a smooth $\widetilde{S}$-deformation of $X \otimes_{\mathscr{O}_K} \mathscr{O}_C$. We call *Hodge-Tate* $\breve{\overline{\mathscr{B}}}_{\mathbb{Q}}$*-module* any Dolbeault $\breve{\overline{\mathscr{B}}}_{\mathbb{Q}}$-module $\mathscr{M}$ (4.6) whose associated Higgs $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$-bundle $\mathscr{H}(\mathscr{M})$ (4.7.1) is nilpotent.

These notions do not depend on the choice of the $\widetilde{S}$-deformation $\widetilde{X}$, not even on the absolute or relative case [2].

Proposition 7.7 (A.-Gros [2])**.** *The functors* $\mathscr{H}$ *and* $\mathscr{V}$ (4.7.1) *induce equivalences of categories quasi-inverse to each other*

$$\mathbf{Mod}^{\mathrm{HT}}_{\mathbb{Q}}(\breve{\overline{\mathscr{B}}}) \underset{\mathscr{V}}{\overset{\mathscr{H}}{\rightleftarrows}} \mathbf{HIG}^{\mathrm{qsolnilp}}(\mathscr{O}_{\mathfrak{X}}[\tfrac{1}{p}], \widetilde{\xi}^{-1}\Omega^1_{\mathfrak{X}/\mathscr{S}}) \tag{7.7.1}$$

Proposition 7.8 (A.-Gros [2])**.** *Let* $g\colon X' \to X$ *be a projective smooth morphism of smooth* S*-schemes. Then, the* $\breve{\overline{\mathscr{B}}}_{\mathbb{Q}}$*-module* $\breve{\psi}_*(\mathrm{R}^n\breve{g}_{\overline{\eta}*}(\breve{\mathbb{Z}}_p)) \otimes_{\breve{\mathbb{Z}}_p} \breve{\overline{\mathscr{B}}}_{\mathbb{Q}}$ *is Hodge-Tate, for any integer* $n \geq 0$.

It follows from 5.6, by taking the trivial $\mathscr{A}_2^*(\mathscr{O}_{\overline{K}})$-deformation of g.

8. The local relative Hodge-Tate spectral sequences

We can deduce from the relative Hodge-Tate spectral sequence (5.8) two local versions that are easier to formulate.

Theorem 8.1 (A.-Gros [1] 6.9.6)**.** *Let* X *be a small affine smooth* S*-scheme,* $\overline{y}$ *a geometric point of* $X_{\overline{\eta}}$*,* $\Gamma = \pi_1(X_\eta, \overline{y})$*,* $\overline{R} = \overline{\mathscr{B}}_{X,\overline{y}}$*,* $g\colon X' \to X$ *a smooth projective morphism,* q *an integer* ≥ 0*. Then, there exists* $(\mathrm{fil}^q_r)_{0 \leq r \leq q+1}$*, a canonical exhaustive decreasing filtration of* $\mathrm{H}^q_{\text{ét}}(X'_{\overline{y}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \widehat{\overline{R}}[\frac{1}{p}]$ *by* $\widehat{\overline{R}}[\frac{1}{p}]$*-representations de* Γ*, such that* $\mathrm{fil}^q_{q+1} = 0$ *and for any integer* $0 \leq r \leq q$*, we have a canonical* Γ*-equivariant exact sequence*

$$0 \to \mathrm{fil}^q_{r+1} \to \mathrm{fil}^q_r \to \mathrm{H}^r(X', \Omega^{q-r}_{X'/X}) \otimes_R \widehat{\overline{R}}[\frac{1}{p}](r-q) \to 0. \tag{8.1.1}$$

It amounts to saying that there exists a canonical Γ-equivariant spectral sequence

$$\mathrm{E}_2^{i,j} = \mathrm{H}^i(X', \Omega^j_{X'/X}) \otimes_R \widehat{\overline{R}}[\frac{1}{p}](-j) \Rightarrow \mathrm{H}^{i+j}_{\text{ét}}(X'_{\overline{y}}, \mathbb{Q}_p) \otimes_{\mathbb{Z}_p} \widehat{\overline{R}}. \tag{8.1.2}$$

Indeed, it follows from Faltings' almost purity theorem that $\mathrm{H}^0(\Gamma, \widehat{\overline{R}}[\frac{1}{p}](j)) = 0$ for all $j \neq 0$. Therefore, the spectral sequence (8.1.2) degenerates at E_2. However in general, the group $\mathrm{H}^1(\Gamma, \widehat{\overline{R}}[\frac{1}{p}](1))$ does not vanish, and hence the abutment filtration does not split.

He ([14] 1.4) constructed the local Hodge-Tate filtration (8.1) in a more general setting. He deduced it from our global Hodge-Tate spectral sequence (5.8) and a cohomological descent result for Faltings topos that he established ([14] 1.14).

8.2. By localizing, we can get rid of the smallness conditions in 8.1. Let $f: X \to S$ and $g: X' \to X$ be smooth morphisms such that g is projective, $\overline{x}$ a geometric point of X above s, $\underline{X}$ the strict localisation of X at $\overline{x}$, $\underline{X}' = X' \times_X \underline{X}$, $\overline{y}$ a geometric point of $X_{\overline{\eta}}$, $\overline{y} \rightsquigarrow \overline{x}$ a specialization map. Consider the $\mathscr{O}_{\overline{K}}$-algebra

$$\underline{\overline{R}} = \varinjlim_{\overline{x} \to U} \overline{\mathscr{B}}_{U, \overline{y}}, \tag{8.2.1}$$

where the limit is taken over the category of affine étale neighborhoods U of $\overline{x}$ in X. It's naturally equipped with an action of $\underline{\Gamma} = \pi_1(\underline{X}_\eta, \overline{y})$.

Theorem 8.3 (A.-Gros [1] 6.8.7)**.** *There exists a canonical $\underline{\Gamma}$-equivariant spectral sequence*

$$\mathrm{E}_2^{i,j} = \mathrm{H}^i(\underline{X}', \Omega^j_{\underline{X}'/\underline{X}}) \otimes_{\mathscr{O}_{\underline{X}}} \widehat{\underline{\overline{R}}}[\frac{1}{p}](-j) \Rightarrow \mathrm{H}^{i+j}_{\text{ét}}(X'_{\overline{y}}, \mathbb{Q}_p) \otimes_{\mathbb{Z}_p} \widehat{\underline{\overline{R}}}. \tag{8.3.1}$$

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