· jt work in progress w J. Newton

Thm A
(Wiles, Taylor-Wiles
Breuil-Conrad-Diamond-Taylor)

Let $E/\mathbb{Q}$ be an elliptic curve

Then E is modular,

i.e. $\exists \quad f \in S_2(\Gamma_0(N), \mathbb{C})$
Hecke eigenform s.t.

$$a_p(f) = a_p(E)$$

eigenvalue of T_p on f

$p+1-\#E(\mathbb{F}_p)$

$\forall p \nmid N$

Outline:

① Prove modularity lifting thm :

if $\bar{\rho}_{E,p} : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{F}_p)$

modular & has large image

$\Rightarrow$ $\rho_{E,p} : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{Q}_p)$

is modular

Key: Taylor-Wiles patching
- Kisin (MLT in Barsotti-Tate case)

Calegari-Geraghty

② Residual modularity

$p = 3$: $\bar{\rho}_{E,3}$ is modular

by Langlands-Tunnell

Allen-Khare

Thorne ($GL_2(\mathbb{F}_3)$ is solvable)

(subtlety: weight 1 $\rightsquigarrow$ weight 2)

3   3-5 modularity

+ 3-7 modularity switch

switch: find $E'/\mathbb{Q}$ s.t. $\bar\rho_{E,5} \simeq \bar\rho_{E',5}$

& s.t. $\bar\rho_{E',3}$ has large image

$X_E(5) \simeq \mathbb{P}^1$ $X_E(7)$ twist of Klein quartic

produce rat'l points by Hilbert irreducibility

4   understand exceptions,

exceptions are classified by

e.g. if $\bar\rho_{E,3}$ & $\bar\rho_{E,5}$ are reducible

curves
of genus
> 1

$\rightsquigarrow$ E corresponds to point in $\underbrace{X_0(15)}(\mathbb{Q})$
ell curve w MW rk 0
8 rat'l points

not able
to do
3-7
modularity
switch

<u>Thm B</u> (Freitas - Le Hung - Siksek)

Let F be a real quadratic field, E/F elliptic curve. Then E is modular (∃ Hilbert modular form of parallel weight 2 w. system of eigenvalues $\{a_p(E)\}$

f function on $\Gamma \backslash \mathcal{H} \times \mathcal{H}$

$\Gamma \subset SL_2(\mathcal{O}_F)$ congruence subgp

$\rightsquigarrow$ contributes to

$$H^2(\Gamma \backslash \mathcal{H} \times \mathcal{H}, \mathbb{C})$$

<u>Pf</u>: Similar method to Thm A

F imaginary quadratic field

look for systems of Hecke eigenvalues in

$$H^*(X_\Gamma, \mathbb{C})$$

1 or 2

$\Gamma \subset SL_2(\mathcal{O}_F)$
congruence

subgp

$$X_\Gamma = \Gamma \backslash \mathcal{H}^3$$

Bianchi manifold

Construction of Galois reps

more subtle : GL_n / F ← CM field

Harris-Lan-Taylor-Thorne

Scholze: also handles torsion

<u>Thm C</u> (C-Newton, in progress)

Let F im. quad field,

E/F ell curve, non-CM, s.t.

<u>Hypothesis</u> (M) for $p = 3$ or 5

the action of $\mathrm{Gal}(\overline{F}/F(\zeta_p))$

on $E[p]$ is
abs irreducible.
Then E is modular.

<u>Remarks:</u>

1). Potential modularity
known: Boxer-Calegari-Gee-Pilloni
ten author

2). If F fixed, 100%
of elliptic curves satisfy (H)
& we hope to also
understand some exceptions.
If F is s.t. $X_0(15) \times_{\mathbb{Q}} F$

has MW rk 0, we hope
to prove modularity of all
elliptic curves over F
e.g. $F = \mathbb{Q}(i), \mathbb{Q}(\sqrt{-2}), \mathbb{Q}(\sqrt{-3})$

3). Residual modularity
results of Allen-Khare-Thorne
crucial for us.

D. Whitmore: modularity
of positive proportion
of ell. curves over im. quad
fields

- Key new ingredient: step 1
(crystalline, non-ordinary
modularity lifting)

More precisely, key ingredient is local-global compatibility for Galois representations constructed by Scholze.

$$\left.\begin{array}{l} G = \mathrm{Res}_{F/\mathbb{Q}}\, GL_n \\ F \quad \text{CM field} \\ ([F:\mathbb{Q}] \neq 2) \\ K \subset GL_n(\mathbb{A}_{F,f}) \end{array}\right\} \begin{array}{l} X_K \\ \text{locally} \\ \text{symm.} \\ \text{space} \end{array}$$

λ highest wt for G

$\rightsquigarrow$ local system $\mathcal{V}_\lambda$ on X_K of $\mathcal{O}$-modules ($\mathcal{O} \subset E/\mathbb{Q}_p$ fin.)

$\mathbb{T} \curvearrowright H^*(X_K, \mathcal{O}_\lambda)$

$\mathbb{T}(K,\lambda) := \operatorname{Im}(\mathbb{T} \to \operatorname{End}_{\mathcal{O}}(H^*(X_K, \mathcal{O}_\lambda)))$

$\cup$

$\mathfrak{m}$ max'l ideal

$\rightsquigarrow$ Scholze

$\bar{\rho}_{\mathfrak{m}} : \operatorname{Gal}(\bar{F}/F) \to GL_n(\mathbb{F})$

abs irreducible $\qquad \underbrace{\mathbb{F}/\mathbb{F}_p}_{\text{finite}}$

$\rightsquigarrow$

$\rho_{\mathfrak{m}} : \operatorname{Gal}(\bar{F}/F) \to GL_n(\mathbb{T}(K,\lambda)^{\wedge}_{\mathfrak{m}} / I)$

$\left(\begin{array}{c} I \subset \mathbb{T}(K,\lambda)^{\wedge}_{\mathfrak{m}} \\ \text{nilpotent ideal} \ldots \end{array} \right)$

Let $v|p$ be a prime of F

$K = K^v K_v$ where

$K_v = GL_n(\mathcal{O}_{F_v})$

<u>Expect</u> (Gee-Newton):

$$\begin{tikzcd}
R^{\square}_{\bar{\rho}_m|G_{F_v}} \arrow[r] \arrow[rr, bend left] \arrow[drr] & R^{univ}_{\bar{\rho}_m} \arrow[r] & \mathbb{T}(K,\lambda)^{\wedge}_{m}/I \\
 & & R^{\square, cris}_{\bar{\rho}|G_{F_v}}(\lambda_v) \arrow[u, dashed]
\end{tikzcd}$$

existence of ρ_m

unrestricted (framed) local deformation ring for $\bar{\rho}_m|G_{F_v}$

- Kisin

E'-valued pts are crystalline w HT wts det by λ_v

E'/E finite

- T. Liu: mod p^n pts are those that admit lifts to char 0 that are crystalline & w. HT wts det by λ_v.

- We construct char 0 lift of $\rho_m|_{G_{F_v}}$ that has the right properties.

Key ingredient:

theory of P-ordinary parts

$P \subset \tilde{G}$ $\quad \tilde{G} = \mathcal{U}(n,n)/F^+$

P has Levi quotient G.

$\rightsquigarrow$ 2n-dim'l

$$\rho_{\tilde{\mathfrak{m}}}|_{G_{F_v}} \simeq \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

(n×n blocks)