Derivative of L-functions for unitary groups, I. & II.

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The Gross–Zagier formula

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- a (modular) elliptic curve $E/\mathbb{Q}$ of conductor N, corresponding to a normalized new cusp form $f_E = \sum_{n\geqslant 1} a_n(E)q^n \in \mathcal{S}_2(\Gamma_0(N))$,

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Denote by $X_0(N)$ the compactified modular curve of level $\Gamma_0(N)$ over $\mathbb{Q}$. Recall that away from cusps, $X_0(N)(\mathbb{C})$ is the set of isomorphism classes of cyclic isogenies $[E \to E']$ of complex elliptic curves of degree N.

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Choose an ideal $\mathfrak{n}$ of O_K satisfying $O_K/\mathfrak{n} = \mathbb{Z}/N\mathbb{Z}$. For a modular parameterization $\varphi\colon X_0(N) \to E$, we define the Heegner point

$$P_K^{\varphi} := \sum_{\mathfrak{a}\in \mathrm{Cl}(K)} \varphi([\mathbb{C}/\mathfrak{a} \to \mathbb{C}/\mathfrak{n}^{-1}\mathfrak{a}]) \in E(\mathbb{C}).$$

By the theory of CM elliptic curves, one sees that the above sum is independent of $\mathfrak{n}$ and the choice of representatives of $\mathrm{Cl}(K)$; moreover, P_K^{φ} belongs to $E(K)$.

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Theorem (Gross–Zagier, 1986)

$$L'(1, E/K) = \frac{32\pi^2 \|f_E\|^2_{\mathrm{Pet}}}{|O_K^{\times}|^2\sqrt{|d_K|}} \frac{\mathrm{h}_{\mathrm{NT}}(P_K^{\varphi})}{\deg \varphi}.$$

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Let J_C be the Jacobian variety of C over K, so that there is a canonical homomorphism $\alpha\colon \mathrm{Div}(C)^0 \to \mathrm{J}_C(K)$ of abelian groups. (Twice of) the theta divisor on J_C gives rise to a height function $\mathrm{h}\colon \mathrm{J}_C(K) \to \mathbb{R}$. For every $D \in \mathrm{Div}(C)^0$, we define

$$\mathrm{h}_{\mathrm{NT}}(D) := \lim_{n\to\infty} \frac{\mathrm{h}(\alpha(nD))}{n^2}$$

in which the limit exists. We have

- h_{NT} descends to a function on $\mathrm{CH}^1(C)^0 = \mathrm{Div}(C)^0/\sim_{\mathrm{rat}}$;
- h_{NT} is a positive definite quadratic function on $\mathrm{CH}^1(C)^0$.

In what follows, we denote by $\langle\ ,\ \rangle_{\mathrm{NT}}\colon \mathrm{CH}^1(C)^0 \times \mathrm{CH}^1(C)^0 \to \mathbb{R}$ the associated quadratic form.

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In what follows, we denote by $\langle\ ,\ \rangle_{\mathrm{NT}}\colon \mathrm{CH}^1(C)^0 \times \mathrm{CH}^1(C)^0 \to \mathbb{R}$ the associated quadratic form. The quadratic form $\langle\ ,\ \rangle_{\mathrm{NT}}$ admits a decomposition into local heights over all places of K, which we review. For $? \in \{\ , u\}$ where u is a place of K, put $C_? \coloneqq C \otimes_K K_?$ and denote by $(\mathrm{Div}(C_?)^0 \times \mathrm{Div}(C_?)^0)^*$ the subgroup of $\mathrm{Div}(C_?)^0 \times \mathrm{Div}(C_?)^0$ consisting of pairs of degree zero divisors with disjoint support.

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For every place u of K, there is a unique function (called **Néron symbol**)

$$\langle\ \ \rangle_u\colon (\mathrm{Div}(C_u)^0 \times \mathrm{Div}(C_u)^0)^* \to \mathbb{R}$$

that is bi-additive, symmetric, continues, and satisfies

$$\langle a, b\rangle_u = -\sum m_x \log|f(x)|_u$$

when $a = \sum m_x x$ and $b = \mathrm{div}(f)$.

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Since the natural map $(\mathrm{Div}(C)^0 \times \mathrm{Div}(C)^0)^* \to \mathrm{CH}^1(C)^0 \times \mathrm{CH}^1(C)^0$ is surjective, this gives a decomposition formula for the Néron–Tate height pairing.

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We review the definition of $\langle\ ,\ \rangle_u$:

- Suppose that $u < \infty$. If C_u admits a smooth projective model $\mathcal{C}_u$ over O_{K_u}, then

  $$\langle a, b\rangle_u = \log q_u \cdot (\overline{a}, \overline{b})_{\mathcal{C}_u},$$

  where $(\overline{a}, \overline{b})_{\mathcal{C}_u}$ denotes the intersection number of the Zariski closures of a and b in $\mathcal{C}_u$ and q_u denotes the residue cardinality of K_u.

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  More generally, C_u always admits a regular projective model $\mathcal{C}_u$ and we shall take $\overline{a}$ and $\overline{b}$ to be **flat extensions** of a and b, respectively. Here, an extension is flat if it has zero intersection number with every component of the special fiber of $\mathcal{C}_u$.

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- Suppose that $u \mid \infty$. We have

  $$\langle a, b\rangle_u = \frac{[K_u : \mathbb{R}]}{2} \sum m_x G_b(x)$$

  if $a = \sum m_x x$, where G_b is a **Green function** for b, that is, a smooth function on $C_u(\mathbb{C}) \setminus |b|$ such that $\mathrm{dd^c} G_b + \delta_b = 0$ as currents (recall: $\mathrm{d^c} = (4\pi i)^{-1}(\partial - \overline{\partial})$).

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- We denote by $\mathrm{Z}^m(X)^0$ the kernel of the de Rham cycle class map
$$\mathrm{cl}_{X,\mathrm{dR}} \colon \mathrm{Z}^m(X) \to \mathrm{H}^{2m}_{\mathrm{dR}}(X/K)(m),$$
and by $\mathrm{CH}^m(X)^0$ the image of $\mathrm{Z}^m(X)^0$ in $\mathrm{CH}^m(X)$.

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- When K is a non-archimedean local field, we denote by $\mathrm{Z}^m(X)^{\langle\ell\rangle}$ the kernels of the (absolute) ℓ-adic cycle class map
$$\mathrm{cl}_{X,\ell}\colon \mathrm{Z}^m(X) \to \mathrm{H}^{2m}(X, \mathbb{Q}_\ell(m)).$$
By the comparison theorem between de Rham and ℓ-adic cohomology, we have $\mathrm{Z}^m(X)^{\langle\ell\rangle} \subseteq \mathrm{Z}^m(X)^0$. In fact, the Monodromy–Weight conjecture for X implies that when ℓ is invertible on O_K, $\mathrm{Z}^m(X)^{\langle\ell\rangle} = \mathrm{Z}^m(X)^0$.

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- When K is a number field, we define $\mathrm{Z}^m(X)^{\langle\ell\rangle}$ via the following Cartesian diagram
$$\begin{tikzcd} \mathrm{Z}^m(X)^{\langle\ell\rangle} \arrow[r] \arrow[d] & \prod_{u\nmid\infty\ell} \mathrm{Z}^m(X_u)^{\langle\ell\rangle} \arrow[d] \\ \mathrm{Z}^m(X) \arrow[r] & \prod_{u\nmid\infty\ell} \mathrm{Z}^m(X_u) \end{tikzcd}$$
where the product is taken over all non-archimedean places u of K not above ℓ. We denote by $\mathsf{CH}^m(X)^{\langle\ell\rangle}$ the image of $\mathrm{Z}^m(X)^{\langle\ell\rangle}$ in $\mathsf{CH}^m(X)$.

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Now we let K be a number field and consider

- a smooth projective scheme X over K of pure dimension $n-1$ (for some $n \geqslant 2$),
- a prime number ℓ such that X_u has good reduction for every place u of K above ℓ,
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For $? \in \{\ , u\}$ where u is a place of K and $\# \in \{\langle\ell\rangle, 0\}$, denote by $(\mathrm{Z}^{d_1}(X_?)^{\#} \times \mathrm{Z}^{d_2}(X_?)^{\#})^{*}$ the subgroup of $\mathrm{Z}^{d_1}(X_?)^{\#} \times \mathrm{Z}^{d_2}(X_?)^{\#}$ consisting of pairs of cycles with disjoint support.

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We define maps

$$\begin{aligned}
\langle\ ,\ \rangle_u &\colon (\mathrm{Z}^{d_1}(X_u)^0 \times \mathrm{Z}^{d_2}(X_u)^0)^* \to \mathbb{R}, \\
\langle\ ,\ \rangle_u &\colon (\mathrm{Z}^{d_1}(X_u)^{\langle\ell\rangle} \times \mathrm{Z}^{d_2}(X_u)^{\langle\ell\rangle})^* \to \mathbb{R} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell, \\
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when $u \mid \infty$, $u \nmid \ell\infty$ and $u \mid \ell$, respectively. Take an element (c_1, c_2) in the source of these maps and denote by Z_i the support of c_i so that $Z_1 \cap Z_2 = \emptyset$.

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Case 1: $u \mid \infty$. We define

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\langle c_1, c_2 \rangle_u := \frac{[K_u : \mathbb{R}]}{2} \int_{X_u(\mathbb{C})} \delta_{c_1} \wedge g_{c_2} \in \mathbb{R},
$$

where δ_{c_1} denotes the Dirac current of c_1 and g_{c_2} is a regular harmonic Green current for c_2, that is, a smooth (d_2-1, d_2-1)-form on $X_u(\mathbb{C}) \setminus Z_2$ such that $\mathrm{dd}^c g_{c_2} + \delta_{c_2} = 0$ as currents.

Beilinson's height pairing

Case 2: $u \nmid \infty\ell$. Let $\alpha_i \in \mathrm{H}^{2d_i}_{Z_i}(X_u, \mathbb{Q}_\ell(d_i))$ be the refined cycle class of c_i. As α_i goes to zero in $\mathrm{H}^{2d_i}(X_u, \mathbb{Q}_\ell(d_i))$ by definition, there exists $\gamma_i \in \mathrm{H}^{2d_i-1}(U_i, \mathbb{Q}_\ell(d_i))$ that goes to α_i under the coboundary map $\mathrm{H}^{2d_i-1}(U_i, \mathbb{Q}_\ell(d_i)) \to \mathrm{H}^{2d_i}_{Z_i}(X_u, \mathbb{Q}_\ell(d_i))$, where $U_i := X_u \setminus Z_i$.

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$$\langle c_1, c_2\rangle_u := \log q_u \otimes \langle c_1, c_2\rangle'_u,$$

where $\langle c_1, c_2\rangle'_u$ is the image of $\gamma_1 \cup \gamma_2$ under the composite map

$$\mathrm{H}^{2n-2}(U_1 \cap U_2, \mathbb{Q}_\ell(n)) \to \mathrm{H}^{2n-1}(X_u, \mathbb{Q}_\ell(n)) \xrightarrow{\mathrm{Tr}_{X_u/K_u}} \mathrm{H}^1(\mathrm{Spec}\, K_u, \mathbb{Q}_\ell(1)) = \mathbb{Q}_\ell$$

in which the first arrow is the coboundary map in the Mayer–Vietoris exact sequence for the covering $X_u = U_1 \cup U_2$. Here, the identification $\mathrm{H}^1(\mathrm{Spec}\, K_u, \mathbb{Q}_\ell(1)) = \mathbb{Q}_\ell$ is the composition

$$\mathrm{H}^1(\mathrm{Spec}\, K_u, \mathbb{Q}_\ell(1)) \to \mathrm{H}^2_{\mathrm{Spec}\, \kappa_u}(\mathrm{Spec}\, O_{K_u}, \mathbb{Q}_\ell(1)) \xrightarrow{\sim} \mathrm{H}^0(\mathrm{Spec}\, \kappa_u, \mathbb{Q}_\ell) = \mathbb{Q}_\ell$$

where κ_u denotes the residue field of K_u; this is *negative* to the one given by the Kummer isomorphism for Galois cohomology.

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where κ_u denotes the residue field of K_u; this is *negative* to the one given by the Kummer isomorphism for Galois cohomology.
It is conjectured that $\langle c_1, c_2 \rangle'_u$ belongs to $\mathbb{Q}$ and is independent of ℓ.

Case 2: $u \nmid \infty\ell$. Let $\alpha_i \in \mathrm{H}^{2d_i}_{Z_i}(X_u, \mathbb{Q}_\ell(d_i))$ be the refined cycle class of c_i. As α_i goes to zero in $\mathrm{H}^{2d_i}(X_u, \mathbb{Q}_\ell(d_i))$ by definition, there exists $\gamma_i \in \mathrm{H}^{2d_i-1}(U_i, \mathbb{Q}_\ell(d_i))$ that goes to α_i under the coboundary map $\mathrm{H}^{2d_i-1}(U_i, \mathbb{Q}_\ell(d_i)) \to \mathrm{H}^{2d_i}_{Z_i}(X_u, \mathbb{Q}_\ell(d_i))$, where $U_i := X_u \setminus Z_i$. Then

$$\langle c_1, c_2\rangle_u := \log q_u \otimes \langle c_1, c_2\rangle'_u,$$

where $\langle c_1, c_2\rangle'_u$ is the image of $\gamma_1 \cup \gamma_2$ under the composite map

$$\mathrm{H}^{2n-2}(U_1 \cap U_2, \mathbb{Q}_\ell(n)) \to \mathrm{H}^{2n-1}(X_u, \mathbb{Q}_\ell(n)) \xrightarrow{\mathrm{Tr}_{X_u/K_u}} \mathrm{H}^1(\mathrm{Spec}\, K_u, \mathbb{Q}_\ell(1)) = \mathbb{Q}_\ell$$

in which the first arrow is the coboundary map in the Mayer–Vietoris exact sequence for the covering $X_u = U_1 \cup U_2$. Here, the identification $\mathrm{H}^1(\mathrm{Spec}\, K_u, \mathbb{Q}_\ell(1)) = \mathbb{Q}_\ell$ is the composition

$$\mathrm{H}^1(\mathrm{Spec}\, K_u, \mathbb{Q}_\ell(1)) \to \mathrm{H}^2_{\mathrm{Spec}\,\kappa_u}(\mathrm{Spec}\, O_{K_u}, \mathbb{Q}_\ell(1)) \xrightarrow{\sim} \mathrm{H}^0(\mathrm{Spec}\,\kappa_u, \mathbb{Q}_\ell) = \mathbb{Q}_\ell$$

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It is conjectured that $\langle c_1, c_2\rangle'_u$ belongs to $\mathbb{Q}$ and is independent of ℓ.

Case 3: $u \mid \ell$. Choose a smooth projective model $\mathcal{X}_u$ of X_u over O_{K_u}. Then

$$\langle c_1, c_2\rangle_u := \log q_u \cdot (\mathcal{C}_1, \mathcal{C}_2)_{\mathcal{X}_u},$$

where $\mathcal{C}_i$ denotes the Zariski closure of c_i in $\mathcal{X}_u$. Later, we will justify this definition and in particular show that it is independent of the choice of the model.

Beilinson's height pairing

Finally, we define

$$\langle\ ,\ \rangle_{\mathrm{B}} \coloneqq \sum_{u} \langle\ ,\ \rangle_u \colon (\mathrm{Z}^{d_1}(X)^{\langle\ell\rangle} \times \mathrm{Z}^{d_2}(X)^{\langle\ell\rangle})^* \to \mathbb{R} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell.$$

It is not hard to show that $\langle\ ,\ \rangle_{\mathrm{B}}$ is symmetric and descends to a map from $\mathrm{CH}^{d_1}(X)^{\langle\ell\rangle} \times \mathrm{CH}^{d_2}(X)^{\langle\ell\rangle}$ (which we now assume). Moreover, for every correspondence $t \in \mathrm{CH}^{n-1}(X \times X)$, $\langle t^*\ ,\ \rangle_{\mathrm{B}} = \langle\ , t_*\ \rangle_{\mathrm{B}}$.

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The above conjecture is known when $n = 2$, that is, X is a curve.

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In what follows, we will often use the complex sesquilinear (linear in the first variable and conjugate linear in the second variable) extension of $\langle\ ,\ \rangle_{\mathrm{B}}$ or $\langle\ ,\ \rangle_{u}$.

Nonarchimedean local index

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To ease notation, we suppress u on this and the next two pages. Denote by $Y := \mathcal{X} \otimes_{O_K} \kappa$ the special fiber of $\mathcal{X}$. For two closed subschemes $\mathcal{Z}_1$ and $\mathcal{Z}_2$ of $\mathcal{X}$ of codimension d_1 and d_2, respectively, such that $\mathcal{Z}_1 \cap \mathcal{Z}_2 \subseteq Y$, we have the intersection number

$$(\mathcal{Z}_1, \mathcal{Z}_2)_{\mathcal{X}} := \chi\left(Y, \mathcal{O}_{\mathcal{Z}_1} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{Z}_2}\right).$$

By sesquilinear extension, we have the intersection number $(\mathcal{C}_1, \mathcal{C}_2)_{\mathcal{X}} \in \mathbb{C}$ for every pair $(\mathcal{C}_1, \mathcal{C}_2) \in \mathrm{Z}^{d_1}(\mathcal{X})_{\mathbb{C}} \times \mathrm{Z}^{d_2}(\mathcal{X})_{\mathbb{C}}$ satisfying $|\mathcal{C}_1| \cap |\mathcal{C}_2| \subseteq Y$.

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Definition

We say that an extension $\mathcal{C} \in \mathrm{Z}^d(\mathcal{X})_{\mathbb{C}}$ of $c \in \mathrm{Z}^d(X)^{\langle\ell\rangle}_{\mathbb{C}}$ is ℓ-**flat** if the cycle class of $\mathcal{C}$ in $\mathrm{H}^{2d}(\mathcal{X}, \mathbb{Q}_\ell(d)) \otimes_{\mathbb{Q}} \mathbb{C}$ vanishes.

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Proposition

Given $(c_1, c_2) \in (\mathrm{Z}^{d_1}(X)^{\langle\ell\rangle}_{\mathbb{C}} \times \mathrm{Z}^{d_2}(X)^{\langle\ell\rangle}_{\mathbb{C}})^*$ and a pair $(\mathcal{C}_1, \mathcal{C}_2) \in \mathrm{Z}^{d_1}(\mathcal{X})^{\langle\ell\rangle}_{\mathbb{C}} \times \mathrm{Z}^{d_2}(\mathcal{X})^{\langle\ell\rangle}_{\mathbb{C}}$ of extensions of (c_1, c_2) in which at least one is ℓ-flat, we have

$$\langle c_1, c_2\rangle'_u = (\mathcal{C}_1, \mathcal{C}_2)_{\mathcal{X}}.$$

In particular, when $\mathcal{X}$ is smooth over O_K, we can take $\mathcal{C}_i$ to be the Zariski closure of c_i in $\mathcal{X}$, hence $\langle c_1, c_2\rangle'_u$ belongs to $\mathbb{C}$ and is independent of ℓ. (This justifies the definition of the local height when $u \mid \ell$.)

Étale correspondences and flat extensions

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Keep the setup from the previous page. We propose a method of finding flat extensions using étale correspondences, with the application to Shimura varieties and Hecke correspondences in mind. For simplicity, we assume $n = 2r$ even and $d_1 = d_2 = r$.

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of $\mathcal{X}$ is **étale** if both p and q are finite étale. A **complex étale correspondence** of $\mathcal{X}$ is a complex linear combination of étale correspondences of $\mathcal{X}$.

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Proposition

Let t be an ℓ-tempered complex étale correspondence of $\mathcal{X}$. Then for every pair $(c_1, c_2) \in \mathrm{Z}^r(X)_{\mathbb{C}} \times \mathrm{Z}^r(X)_{\mathbb{C}}$ satisfying $\mathrm{supp}(t^*c_1) \cap \mathrm{supp}(t^*c_2) = \emptyset$, we have $(t^*c_1, t^*c_2) \in (\mathrm{Z}^r(X)_{\mathbb{C}}^{\langle\ell\rangle} \times \mathrm{Z}^r(X)_{\mathbb{C}}^{\langle\ell\rangle})^*$ and

$$\langle t^*c_1, t^*c_2\rangle'_u = (t^*\mathcal{C}_1, t^*\mathcal{C}_2)_{\mathcal{X}},$$

where $\mathcal{C}_i \in \mathrm{Z}^r(\mathcal{X})_{\mathbb{C}}$ is an arbitrary extension of c_i in $\mathcal{X}$ for $i = 1, 2$. In particular, we have $\langle t^*c_1, t^*c_2\rangle'_u \in \mathbb{C}$.

Beilinson–Bloch conjecture

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Let K be a number field and X a projective smooth scheme over K of (odd) dimension $n-1$. We have the L-function $L(s, \mathrm{H}^{n-1}(X_{\overline{K}}, \mathbb{Q}_\ell(r)))$ for the middle degree ℓ-adic cohomology of X for every rational prime ℓ, which is conjectured to be meromorphic, independent of ℓ, and satisfy a functional equation with center $s = 0$.

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The unrefined Beilinson–Bloch conjecture predicts that

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We have an equivariant version of the Beilinson–Bloch conjecture as follows. Suppose that X admits an action of an algebra T via étale correspondences. Then T acts on both $\mathrm{CH}^r(X)^0$ and $\mathrm{H}^{n-1}(X_{\overline{K}}, \mathbb{Q}_\ell(r))$. Let ϱ be a nonzero irreducible finite-dimensional complex representation of T. Then for every ℓ and every embedding $\mathbb{Q}_\ell \hookrightarrow \mathbb{C}$, we have the L-function

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Then it is expected that

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holds, which can be regarded as the Beilinson–Bloch conjecture for the (conjectural Chow) motive $\mathrm{Hom}_{\mathrm{T}}(\varrho, \mathrm{h}^{n-1}(X)(r)_{\mathbb{C}})$, where $\mathrm{h}^{n-1}(X)$ is the (conjectural Chow) motive of X of degree $n-1$.

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One can specify the equivariant Beilinson–Bloch conjecture to certain unitary Shimura varieties.

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$$X_L(\mathbb{C}) = H(F)\backslash \mathbb{P}(V_\mathbb{C})^- \times H(\mathbb{A}_F^\infty)/L,$$

where $\mathbb{P}(V_\mathbb{C})^- \subseteq \mathbb{P}(V_\mathbb{C})$ is the complex open domain of negative definite lines.

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Let π be a tempered cuspidal automorphic representation of $G_r(\mathbb{A}_F)$, where G_r denotes the quasi-split unitary group over E/F of rank $n = 2r$. Let V be a standard indefinite hermitian space over E of rank n, with $H := \mathrm{U}(V)$. For every irreducible admissible representation $\tilde\pi^\infty$ of $H(\mathbb{A}_F^\infty)$ satisfying

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holds. Here, $\Pi_{\tilde{\pi}^\infty}$ is the cuspidal factor of $\mathrm{BC}(\pi)$ determined by $\tilde{\pi}^\infty$ via Arthur's multiplicity formula; in particular, $\Pi_{\tilde{\pi}^\infty} = \mathrm{BC}(\pi)$ if $\mathrm{BC}(\pi)$ is already cuspidal (that is, π is stable).

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In (R2), since π_v is $K_{r,v}$-spherical, there exist unitary unramified characters $\chi_{v,1}, \dots \chi_{v,r} \colon E_v^\times \to \mathbb{C}^\times$, unique up to permutation and taking inverse, such that π_v is isomorphic to the (normalized) principal series of $(\chi_{v,1}, \dots \chi_{v,r})$.

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We introduce some notation for future use. For a positive integer m, denote by

- $\mathrm{Herm}_m\subseteq\mathrm{Res}_{E/F}\mathrm{Mat}_m$ the subscheme of hermitian matrices, $\mathrm{Herm}_m^\circ:=\mathrm{Herm}_m\cap\mathrm{Res}_{E/F}\mathrm{GL}_m$,
- $\mathrm{Herm}_m(F)^+$ and $\mathrm{Herm}_m^\circ(F)^+$ the subsets of $\mathrm{Herm}_m(F)$ of totally semi-positive and positive definite elements, respectively.

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$$\omega_{m,v}\left(\begin{pmatrix} a & \\ & \overline{a}^{\mathrm{t},-1}\end{pmatrix}\right)\phi(x)=|\det a|_{E_v}^r\cdot\phi(xa).$$

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$$\omega_{m,v}\left(\begin{pmatrix} 1_m & b\\ & 1_m\end{pmatrix}\right)\phi(x)=\psi_v(\mathrm{tr}\, bT(x))\cdot\phi(x),$$
where $T(x)=(x_i,x_j)_{1\leqslant i,j\leqslant m}\in\mathrm{Herm}_m(F_v)$ is the moment matrix of x.
- For $\phi\in\mathscr{S}(V_v^m)$, we have
$$\omega_{m,v}\left(\begin{pmatrix} & 1_m\\ -1_m & \end{pmatrix}\right)\phi(x)=\gamma_{V_v}^m\cdot\widehat{\phi}(x),$$
where $\gamma_{V_v}\in\{\pm1\}$ is the Weil constant of V_v.
- For $h\in\mathrm{U}(V_v)(F_v)$ and $\phi\in\mathscr{S}(V_v^m)$, we have
$$\omega_{m,v}(h)\phi(x)=\phi(h^{-1}x).$$

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For every $\phi \in \mathscr{S}(V^r \otimes \mathbb{A}_F)$, the theta function

$$\theta_\phi(g,h) := \sum_{x \in V^r} \omega_r(g,h)\phi(x) = \sum_{x \in V^r} \omega_r(g)\phi(h^{-1}x)$$

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For every $\varphi \in \pi$ (it is now known that π has a unique realization in the space of cusp forms of $G_r(\mathbb{A}_F)$), we have the **(global) theta lift** $\theta_\phi(\varphi)$, defined by the formula

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Theorem (Rallis inner product formula)

For $\phi_1 = \otimes_v \phi_{1,v}, \phi_2 = \otimes_v \phi_{2,v} \in \mathscr{S}(V^r \otimes \mathbb{A}_F)$ and $\varphi_1 = \otimes_v \varphi_{1,v}, \varphi_2 = \otimes \varphi_{2,v} \in \pi$, we have

$$\langle \theta_{\phi_1}(\varphi_1), \theta_{\phi_2}(\varphi_2)\rangle_H = \frac{L(\frac{1}{2},\pi)}{b_n(0)} \prod_v Z_v^\natural(\varphi_{1,v},\varphi_{2,v};\phi_{1,v},\phi_{2,v}).$$

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- $\langle\ ,\ \rangle_H$ denotes the Peterson inner product of automorphic forms on $H(\mathbb{A}_F)$ with respect to the Tamagawa measure.
- $b_n(s) = \prod_v b_{n,v}(s)$, where $b_{n,v}(s) = \prod_{i=1}^{n} L(2s+i, \eta_v^{n-i})$ with $\eta\colon \mathbb{A}_F^\times \to \mathbb{C}^\times$ the automorphic character associated with the extension E/F.

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- $Z_v^\natural$ is the normalized doubling zeta integral:

$$Z_v^\natural(\varphi_{1,v}, \varphi_{2,v}; \phi_{1,v}, \phi_{2,v}) = \left(\frac{L(\frac{1}{2}, \pi_v)}{b_{n,v}(0)}\right)^{-1} \int_{G_r(F_v)} \overline{\langle g\varphi_{1,v}, \varphi_{2,v}\rangle_{\pi_v}} \cdot \langle g\phi_{1,v}, \phi_{2,v}\rangle_{\omega_{r,v}} \mathrm{d}g_v,$$

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Remark

The two functionals $\langle\theta_{\phi_1}(\varphi_1),\theta_{\phi_2}(\varphi_2)\rangle_H$ and $\prod_v Z_v^\natural(\varphi_{1,v},\varphi_{2,v};\phi_{1,v},\phi_{2,v})$ define two elements in

$$\bigotimes_v \mathrm{Hom}_{G_r(F_v)\times G_r(F_v)}\left(\mathscr{S}(V_v^{2r})_{H(F_v)},\pi_v\boxtimes\pi_v^\vee\right).$$

It is known that the above space has dimension 1 of which $\prod_v Z_v^\natural$ is a basis. Thus, Rallis inner product formula is nothing but the proportion of the two invariant functionals.

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Step 1. We regard $G_r \times G_r$ as a subgroup of G_{2r} via the embedding

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Step 2. Use the Siegel–Weil formula $\langle \theta_{\phi_1}(g_1, -), \theta_{\phi_2}(g_2, -) \rangle_H = E(0, (g_1, g_2), \phi_1 \otimes \overline{\phi_2})$. Here, for every $\Phi \in \mathscr{S}(V^{2r} \otimes \mathbb{A}_F)$, we have the Siegel–Eisenstein series

$$E(s, g, \Phi) := \sum_{\gamma \in P_{2r}(F) \backslash G_{2r}(F)} \omega_{2r}(\gamma g) \Phi(0) \cdot \mathrm{H}(\gamma g)^s$$

on $G_{2r}(\mathbb{A}_F)$, where $P_{2r} \subseteq G_{2r}$ denotes the upper-triangle Siegel parabolic subgroup and H denotes the "height" function with respect to P_{2r}.

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Step 3. Using cuspidality, we have

$$\begin{aligned}\langle \theta_{\phi_1}(\varphi_1), \theta_{\phi_2}(\varphi_2)\rangle_H &= \iint_{[G_r(F)\backslash G_r(\mathbb{A}_F)]^2} \overline{\varphi_1(g_1)}\varphi_2(g_2) E(s, (g_1, g_2), \phi_1 \otimes \overline{\phi_2}) \mathrm{d}g_1 \mathrm{d}g_2 \\ &= \frac{L(s+\frac{1}{2}, \pi)}{b_n(s)} \prod_v Z_v^\natural(s; \varphi_{1,v}, \varphi_{2,v}; \phi_{1,v}, \phi_{2,v}).\end{aligned}$$

Special cycles and generating series

Let V be a standard indefinite hermitian space over E of rank $n = 2r$, with $H := \mathrm{U}(V)$. Take a neat open compact subgroup $L \subseteq H(\mathbb{A}_F^\infty)$. We first recall the construction of Kudla's special cycle $Z(x)_L$ for every element $x \in V^m \otimes_F \mathbb{A}_F^\infty$ with $1 \leqslant m \leqslant n-1$.

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- When $T(x) \in \mathrm{Herm}_m^\circ(F)^+$, we may find elements $y \in V^m$ and $h \in H(\mathbb{A}_F^\infty)$ such that $hx = y \in V^m \otimes_F \mathbb{A}_F^\infty$. Denote by V_y the orthogonal complement of the subspace spanned by components of y in V, which is standard indefinite of rank $n-m$. Put $H_y := \mathrm{U}(V_y)$, which is naturally a subgroup of H. Define $Z(x)_L$ to be the image cycle of the composite morphism

$$(X_y)_{hLh^{-1} \cap H_y(\mathbb{A}_F^\infty)} \to X_{hLh^{-1}} \xrightarrow{\cdot h} X_L,$$

  where X_y denotes the system of Shimura varieties for V_y. It is straightforward to check that $Z(x)_L$ does not depend on the choice of y and h. Moreover, $Z(x)_L$ is a well-defined element in $\mathrm{Z}^m(X_L)$.

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- For $T(x) \in \mathrm{Herm}_m(F)^+$ in general, we have an element $Z(x)_L \in \mathrm{CH}^m(X_L)_{\mathbb{Q}}$ (not well-defined in $\mathrm{Z}^m(X_L)_{\mathbb{Q}}$). Since we will not use its precise definition, we omit.

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- When $T(x) \in \mathrm{Herm}_m^\circ(F)^+$, we may find elements $y \in V^m$ and $h \in H(\mathbb{A}_F^\infty)$ such that $hx = y \in V^m \otimes_F \mathbb{A}_F^\infty$. Denote by V_y the orthogonal complement of the subspace spanned by components of y in V, which is standard indefinite of rank $n - m$. Put $H_y \coloneqq \mathrm{U}(V_y)$, which is naturally a subgroup of H. Define $Z(x)_L$ to be the image cycle of the composite morphism

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For every $\phi^\infty \in \mathscr{S}(V^m \otimes_F \mathbb{A}_F^\infty)^L$ and $T \in \mathrm{Herm}_m(F)$, we put

$$Z_T(\phi^\infty)_L \coloneqq \sum_{x \in L \backslash V^m \otimes_F \mathbb{A}_F^\infty, T(x) = T} \phi^\infty(x) Z(x)_L.$$

As the above summation is finite, $Z_T(\phi^\infty)_L$ is a well-defined element in $\mathrm{CH}^m(X_L)_{\mathbb{C}}$. For $T \in \mathrm{Herm}_m^\circ(F)^+$, $Z_T(\phi^\infty)_L$ is even a well-defined element in $\mathrm{Z}^m(X_L)_{\mathbb{C}}$.

Special cycles and generating series

Denote by $\mathscr{A}_r(G_m)$ the space of automorphic forms on $G_m(\mathbb{A}_F)$ of "parallel weight r". For $\varphi \in \mathscr{A}_r(G_m)$, we have the Siegel–Fourier expansion

$$\varphi \sim \sum_{T \in \mathrm{Herm}_m(F)} \varphi_T \cdot q^T,$$

where

$$\varphi_T := \int_{\mathrm{Herm}_m(F) \backslash \mathrm{Herm}_m(\mathbb{A}_F)} \varphi\left(\begin{pmatrix} 1_m & b \\ & 1_m \end{pmatrix}\right) \psi(\mathrm{tr}\, bT)^{-1} \mathrm{d}b.$$

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We have the following conjecture due to Kudla.

Conjecture (Modularity Hypothesis)

For $\phi^\infty \in \mathscr{S}(V^m \otimes_F \mathbb{A}_F^\infty)^L$, there exists a (necessarily unique) holomorphic element $\mathscr{Z}(\phi^\infty)_L \in \mathscr{A}_r(G_m) \otimes \mathrm{CH}^m(X_L)$ such that for every $g^\infty \in G_m(\mathbb{A}_F^\infty)$, the Siegel–Fourier expansion of $g^\infty \mathscr{Z}(\phi^\infty)_L$ coincides with

$$\sum_{T \in \mathrm{Herm}_m(F)} Z_T(\omega_r^\infty(g^\infty)\phi^\infty)_L \cdot q^T.$$

This conjecture is formally known (that is, ignore the issue of convergence), and is only rigorously known when $m = 1$.