Derivative of L-functions for unitary groups, III.

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We consider a cuspidal automorphic representation $\pi = \otimes_v \pi_v$ of $G_r(\mathbb{A}_F)$ satisfying:

(R1) If $v \mid \infty$, then π_v is a holomorphic discrete series of weights $(\frac{1-n}{2}, \frac{3-n}{2}, \ldots, \frac{n-3}{2}, \frac{n-1}{2})$.

(R2) If $v \nmid \infty$ and is nonsplit in E, then π_v is $K_{r,v}$-spherical, that is, $\pi_v^{K_{r,v}} \neq \{0\}$.

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Recall that V is **standard indefinite** if it has signature $(n-1, 1)$ at the default embedding $F \subseteq \mathbb{R}$ and signature $(n, 0)$ at other real places. For a standard indefinite hermitian space V over E of rank n, we have a system of Shimura varieties $\{X_L\}$ indexed by neat open compact subgroups $L \subseteq H(\mathbb{A}_F^\infty)$, which are smooth, quasi-projective, of dimension $n-1$ over E, together with the complex uniformization:

$$X_L(\mathbb{C}) = H(F)\backslash \mathbb{P}(V_\mathbb{C})^- \times H(\mathbb{A}_F^\infty)/L,$$

where $\mathbb{P}(V_\mathbb{C})^- \subseteq \mathbb{P}(V_\mathbb{C})$ is the complex open domain of negative definite lines.

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$$\{x \in V \mid \mathrm{Tr}_{E/\mathbb{Q}}(x, \Lambda)_V \in \mathbb{Z}\} = \Lambda.$$

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For every finite set Σ of nonarchimedean places of F, we have the abstract Hecke algebra

$$\mathbb{T}^\Sigma := \mathbb{Z}[L_0^\Sigma \backslash H(\mathbb{A}_F^{\infty,\Sigma})/L_0^\Sigma],$$

which is commutative and canonically isomorphic to $\mathbb{Z}[K_r^\Sigma \backslash G_r(\mathbb{A}_F^{\infty,\Sigma})/K_r^\Sigma]$.

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For every finite set Σ of nonarchimedean places of F containing Σ_π, we have the Satake homomorphism

$$\chi_\pi^\Sigma := \mathbb{T}^\Sigma \otimes \mathbb{C} \to \mathbb{C},$$

which is the eigen-character of the action of $\mathbb{T}^\Sigma$ on $\pi^{K_r^\Sigma}$. Put $\mathfrak{m}_\pi^\Sigma := \ker \chi_\pi^\Sigma$.

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We propose the following assumption that limits our latter theorems.

(E) The field E properly contains an imaginary quadratic subfield E_0 in which 2 splits and satisfying $(d_{E_0}, d_F) = 1$.

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Theorem (Chao Li–L.)

Assume (E). If $\mathrm{ord}_{s=1/2} L(s,\pi) = 1$, *then for every finite set* Σ *of nonarchimedean places of* F *containing* Σ_π *and satisfying* $|\Sigma \cap \Sigma^{\mathrm{spl}}| \geqslant 2$, *we have*

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- If we apply the Beilinson–Bloch conjecture for unitary Shimura varieties yesterday to our particular V and $\tilde{\pi}^\infty=\mathrm{Hom}_{G_r(\mathbb{A}_F^\infty)}(\omega_r^\infty\otimes\pi^\infty,1)$, then $\varinjlim_{L=L_\Sigma L_0^\Sigma}\mathrm{CH}^r(X_L)^0_{\mathbb{C}}[(\tilde{\pi}^\infty)^L]$ should have dimension 1.

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- Indeed, the theory of local theta lifting tells us that when $\mathrm{ord}_{s=1/2} L(s,\pi) = \mathrm{ord}_{s=1/2} L(s,\mathrm{BC}(\pi)) = 1$, $\Pi_{\tilde{\pi}^\infty}$ is exactly the isobaric factor of $\mathrm{BC}(\pi)$ such that $\mathrm{ord}_{s=1/2} L(s,\Pi_{\tilde{\pi}^\infty}) = 1$.

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- Thus, our theorem aligns with and provide evidence toward the Beilinson–Bloch conjecture for higher dimensional unitary Shimura varieties.
- In fact, the theorem we proved is stronger. First, we only need a weaker version of Assumption (E); in particular, E does not have to contain an imaginary quadratic field. Second, we allow a finite set Σ' of primes of F inert in E at which π can be "slightly ramified" (which we call almost unramified); in response, we have $\epsilon(\frac{1}{2},\pi)=(-1)^{r[F:\mathbb{Q}]+|\Sigma'|}$.

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By the modularity of A and the very recent breakthrough on the automorphy of symmetric powers of holomorphic modular forms obtained by Newton–Thorne, there exists a unique cuspidal automorphic representation $\Pi(\mathrm{Sym}^{n-1} A)$ of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$ satisfying

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$$\prod_{j=0}^{n-1} \left(T - \alpha_{p,1}^{j} \alpha_{p,2}^{n-1-j}\right) \in \mathbb{Q}[T],$$

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By the endoscopic classification for quasi-split unitary groups, there exists a cuspidal automorphic representation $\pi(\mathrm{Sym}^{n-1} A_E)$ of $G_r(\mathbb{A}_F)$ satisfying (R1–3) and that for every $v \nmid \infty d_A$, the base change of $\pi(\mathrm{Sym}^{n-1} A_E)_v$ to $\mathrm{GL}_n(E_v)$ is isomorphic to $\Pi(\mathrm{Sym}^{n-1} A_E)_v$.

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Definition

For every $\phi^\infty \in \mathscr{S}(V^r \otimes_F \mathbb{A}_F^\infty)^L$ and $\varphi \in \pi$ that has parallel weight r (the lowest weight), we define the **arithmetic theta lift** to be

$$\Theta_{\phi^\infty}(\varphi)_L := \int_{G_r(F)\backslash G_r(\mathbb{A}_F)} \overline{\varphi(g)}\mathscr{Z}(\phi^\infty)_L(g)\mathrm{d}g \in \mathrm{CH}^r(X_L)_{\mathbb{C}}.$$

Arithmetic theta lifting

Now we state a quantitative version of the previous theorem by giving a height formula for an arithmetic analogue of the classical theta lifting, which we call **arithmetic theta lifting**. However, the construction of arithmetic theta lifting relies on the Modularity Hypothesis, which is unknown when $r > 1$. Thus, our height formula will be conditional (on the construction of the object). From now on, we assume **Assumption (E)** and the **Modularity Hypothesis** for V with $m = r$, that is, for every $\phi^\infty \in \mathscr{S}(V^r \otimes_F \mathbb{A}_F^\infty)^L$, there exists a (necessarily unique) holomorphic element $\mathscr{Z}(\phi^\infty)_L \in \mathcal{A}_r(G_r) \otimes \mathrm{CH}^r(X_L)$ such that for every $g^\infty \in G_r(\mathbb{A}_F^\infty)$, the Siegel–Fourier expansion of $g^\infty \mathscr{Z}(\phi^\infty)_L$ coincides with

$$\sum_{T \in \mathrm{Herm}_r(F)} Z_T(\omega_r^\infty(g^\infty)\phi^\infty)_L \cdot q^T.$$

Definition

For every $\phi^\infty \in \mathscr{S}(V^r \otimes_F \mathbb{A}_F^\infty)^L$ and $\varphi \in \pi$ that has parallel weight r (the lowest weight), we define the **arithmetic theta lift** to be

$$\Theta_{\phi^\infty}(\varphi)_L \coloneqq \int_{G_r(F)\backslash G_r(\mathbb{A}_F)} \overline{\varphi(g)}\mathscr{Z}(\phi^\infty)_L(g)\mathrm{d}g \in \mathrm{CH}^r(X_L)_{\mathbb{C}}.$$

In fact, we show that $\Theta_{\phi^\infty}(\varphi)_L$ belongs to $\mathrm{CH}^r(X_L)_{\mathbb{C}}^{\langle \ell \rangle}$ for every sufficiently large prime ℓ.

Arithmetic inner product formula

Theorem (Arithmetic inner product formula, Chao Li–L.)

For $\phi_1^\infty = \otimes_v \phi_{1,v}^\infty, \phi_2^\infty = \otimes_v \phi_{2,v}^\infty \in \mathscr{S}(V^r \otimes \mathbb{A}_F^\infty)^L$ and $\varphi_1 = \otimes_v \varphi_{1,v}, \varphi_2 = \otimes \varphi_{2,v} \in \pi$ that have parallel weight r, we have

$$\mathrm{vol}(L) \cdot \langle \Theta_{\phi_1^\infty}(\varphi_1)_L, \Theta_{\phi_2^\infty}(\varphi_2)_L \rangle_{X_L} = \frac{L'(\frac{1}{2}, \pi)}{b_n(0)} C_r^{[F:\mathbb{Q}]} \prod_{v \nmid \infty} Z_v^\natural(\varphi_{1,v}^\infty, \varphi_{2,v}^\infty; \phi_{1,v}^\infty, \phi_{2,v}^\infty).$$

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If we take $\phi_1^\infty = \phi_2^\infty$ and $\varphi_1 = \varphi_2$ in the theorem, the signs on both sides are compatible under the Beilinson–Bloch conjecture and the generalized Riemann Hypothesis.

Height pairing between special cycles

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Fix a sufficiently large prime ℓ. We compare the "height pairing" between $Z_{T_1}(\phi_1^\infty)_L$ and $Z_{T_2}(\phi_2^\infty)_L$ and the derivative

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where $\mathcal{E}(s,\phi_1^\infty\otimes\overline{\phi_2^\infty})$ denotes the Siegel–Eisenstein series $E(s,g,\Phi_\infty^0\otimes(\phi_1^\infty\otimes\overline{\phi_2^\infty}))$ on $G_{2r}(\mathbb{A}_F)$, in which Φ_∞^0 is the standard hermitian Gaussian function on $(E^n\otimes_{\mathbb{Q}}\mathbb{R})^{2r}$.

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By the multiplicity one property of local theta lifting, it suffices to consider L of the form $L=(\prod_{v\in\Sigma}L_v)\times L_0^\Sigma$ for some finite set $\Sigma_\pi\subseteq\Sigma\subseteq\Sigma^{\mathrm{spl}}$ of cardinality at least 2.

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Put $\mathbb{T}:=\mathbb{T}^\Sigma\otimes\mathbb{C}$, $\chi:=\chi_{\pi^\vee}^\Sigma$ and $\mathfrak{m}:=\ker\chi$ for short.

Height pairing between special cycles

To solve (D1), we use Hecke operators. We show that there exists an element $t \in \mathbb{T} \setminus \mathfrak{m}$ (depending on L) such that $t^* Z_{T_i}(\phi_i^\infty)_L$ belongs to $\mathrm{Z}^r(X_L)_{\mathbb{C}}^{\langle \ell \rangle}$ for $i = 1, 2$, every $T_i \in \mathrm{Herm}_r^\circ(F)^+$ and every $\phi_i^\infty \in \mathscr{S}(V^r \otimes_F \mathbb{A}_F^\infty)^L$. Considering $t^* Z_{T_i}(\phi_i^\infty)_L$ will not lose information toward the AIPF since

$$\int \overline{\varphi_i} \cdot t^* \mathscr{Z}(\phi_i^\infty)_L = \int \overline{\varphi_i} \cdot \mathscr{Z}(t\phi_i^\infty)_L = \int \overline{t\varphi_i} \cdot \mathscr{Z}(\phi_i^\infty)_L = \chi(t) \int \overline{\varphi_i} \cdot \mathscr{Z}(\phi_i^\infty)_L.$$

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To solve (D2) and (D3), we use the same trick. For $v \nmid \infty$, we say that $(\phi_{1,v}^\infty, \phi_{2,v}^\infty)$ is a **regular pair** if the support of $\phi_{1,v}^\infty \otimes \overline{\phi_{2,v}^\infty}$ is contained in the subset $\{x \in V_v^{2r} \mid T(x) \in \mathrm{Herm}_{2r}^\circ(F_v)\}$.

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To solve (D2) and (D3), we use the same trick. For $v \nmid \infty$, we say that $(\phi_{1,v}^\infty, \phi_{2,v}^\infty)$ is a **regular pair** if the support of $\phi_{1,v}^\infty \otimes \overline{\phi_{2,v}^\infty}$ is contained in the subset $\{x \in V_v^{2r} \mid T(x) \in \mathrm{Herm}_{2r}^\circ(F_v)\}$.

Lemma

(1) *Suppose that $(\phi_{1,v}^\infty, \phi_{2,v}^\infty)$ is a regular pair for some $v \in \Sigma$, then*

- $Z_{T_i}(\phi_i^\infty)_L = 0$ *if* $T_i \notin \mathrm{Herm}_r^\circ(F)^+$ *for* $i = 1, 2$;
- $Z_{T_1}(\phi_1^\infty)_L$ *and* $Z_{T_2}(\phi_2^\infty)_L$ *have disjoint support for every pair* $(T_1, T_2) \in \mathrm{Herm}_r^\circ(F)^+ \times \mathrm{Herm}_r^\circ(F)^+$.

Height pairing between special cycles

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(2) *For every $v \in \Sigma$, there exists a regular pair $(\phi_{1,v}^\infty, \phi_{2,v}^\infty)$ such that the functional*

$$(\varphi_{1,v}, \varphi_{2,v}) \in \pi_v^\vee \times \pi_v \mapsto Z_v^\natural(\varphi_{1,v}, \varphi_{2,v}; \phi_{1,v}^\infty, \phi_{2,v}^\infty)$$

is nontrivial.

Derivative of Eisenstein series

The above discussion provides us with a decomposition

$$\langle t^* Z_{T_1}(\phi_1^\infty)_L, t^* Z_{T_2}(\phi_2^\infty)_L\rangle_{X_L} = \sum_u \langle t^* Z_{T_1}(\phi_1^\infty)_L, t^* Z_{T_2}(\phi_2^\infty)_L\rangle_u$$

over all places u of E for every pair $(T_1, T_2) \in \mathrm{Herm}_r^\circ(F)^+ \times \mathrm{Herm}_r^\circ(F)^+$ and every pair $(\phi_1^\infty, \phi_2^\infty)$ that is regular at some place in Σ.

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We say that an element $T \in \mathrm{Herm}_{2r}(F)$ is **nearby to** V if it satisfies:

(1) T belongs to $\mathrm{Herm}_{2r}^\circ(F)$;

(2) if we denote by V_T the hermitian space over E defined by T, then there exists a unique place v_T of F such that V_T and V are isomorphic exactly away from $\{F \subseteq \mathbb{R}\} \triangle \{v_T\}$.

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Lemma

Suppose that $(\phi_1^\infty, \phi_2^\infty)$ is regular at at least two places in Σ. Then

$$\mathcal{E}'(0, t\phi_1^\infty \otimes \overline{t\phi_2^\infty}) = \sum_v \sum_{\substack{T\in \mathrm{Herm}_{2r}(F) \text{ nearby to } V \\ v_T = v}} W_T'(0, \Phi_v) W_T(0, \Phi^v).$$

Here, $\Phi := \Phi_\infty^0 \otimes (t\phi_1^\infty \otimes \overline{t\phi_2^\infty}) \in \mathscr{S}(V^{2r} \otimes_F \mathbb{A}_F)$ and W_T denotes the T-th Siegel–Whittaker function.

Comparison of local terms

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We will maintain the setup from the previous two pages.

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Proposition

For every place u of E of degree 1 over F, there exists an element $s_u \in \mathbb{T} \setminus \mathfrak{m}$, such that

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for every pair $(T_1, T_2) \in \mathrm{Herm}_r^\circ(F)^+ \times \mathrm{Herm}_r^\circ(F)^+$ and every pair $(\phi_1^\infty, \phi_2^\infty)$ that is regular at at least two places in Σ. Moreover, one can take $\mathrm{s}_u = 1$ if u is not above Σ.

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Proposition

For every nonarchimedean place u of E of degree 2 over F, we have

$$\mathrm{vol}(L) \cdot \langle t^* Z_{T_1}(\phi_1^\infty)_L, t^* Z_{T_2}(\phi_2^\infty)_L \rangle_u = \sum_{\substack{T = \begin{pmatrix} T_1 & * \\ * & T_2 \end{pmatrix} \in \mathrm{Herm}_{2r}(F) \text{ nearby to } V \\ u \mid v_T}} W_T'(0, \Phi_v) W_T(0, \Phi^v).$$

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We also have a comparison theorem for terms indexed by $u \mid \infty$, whose form is rather technical, which we omit.

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We explain the idea for the first proposition of comparison. Take a place u of E of degree 1 over F, with $v \in \Sigma^{\mathrm{spl}}$ the underlying place.

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The second proposition of comparison will boil down to a similar comparison for the intersection number of special cycles on a certain Rapoport–Zink space, known as the **Kudla–Rapoport (type) conjecture**, whose content and proof require another short course.