Syntomic complexes

(after Bhatt–Morrow–Scholze, Bhatt–Lurie)

Let X be a scheme.

There are objects

$$R\Gamma_{syn}(X, \mathbb{Z}_p(i)) = \mathbb{Z}_p(i)(X)$$

$$\in \mathcal{D}(\mathbb{Z}_p),$$

"ith syntomic complex of X" $\quad (i \geq 0)$

<u>Heuristic</u>: $R\Gamma_{syn}(X, \mathbb{Z}_p(i))$ give p-adic étale motivic complexes.

E.g. given a vector bundle on X,

one has Chern classes

$$c_i \in H^{2i}_{syn}(X, \mathbb{Z}_p(i)).$$

<u>Variant</u>: Can globalize the construction (w/ mod p^n coeffs)

$\rightsquigarrow$ produces $\mathbb{Z}/p^n(i)_X \in D(X_{\acute{e}t}, \mathbb{Z}/p^n)$.

$$\rightsquigarrow \mathbb{Z}/p^n(i)(X) = R\Gamma_{\acute{e}t}(X, \mathbb{Z}/p^n(i)_X).$$

Ex) If $X/\mathbb{Z}[1/p]$,

$$\mathbb{Z}_p(i)(X) = R\Gamma_{pro\acute{e}t}(X, \mathbb{Z}_p(i))$$

$$\mathbb{Z}/p^n(i)_X = \mu_{p^n}^{\otimes i}$$

(in general, for any X, $\mathbb{Z}/p^n(i)_X\big|_{X[1/p]} = \mu_{p^n}^{\otimes i}$).

Ex) $X/\mathbb{F}_p$ is regular

$$\mathbb{Z}/p^n(i)_X = W_n\Omega^i_{\log,X}[-i]$$

(Milne, Illusie, Gros).

Ex) $$\mathbb{F}_p(i)_X = \ker\left(\Omega^i_X \xrightarrow{1-C^{-1}} \frac{\Omega^i_X}{d\Omega^{i-1}_X}\right)[-i].$$

Ex) $$\mathbb{Z}_p(i)(X) = \mathrm{fib}(1-F : W\Omega^i_X \to W\Omega^i_X).$$
$$[-i]$$

K-theoretic def of syntomic complexes (Nizioł).

Construction: X quasi-projective scheme,

$K_0(X)$ = Groth gp of vector bundles on X

$$= \left\{ \begin{array}{l} [V] \quad V \text{ vector bundle} \\ {[V] = [V'] + [V'']} \quad \text{if} \\ \quad \exists \text{ ses } 0 \to V' \to V \to V'' \to 0 \end{array} \right.$$

Construction of higher K-groups

$K_i(X)$, $i \geq 0$ $(K_i(X) = \pi_i K(X))$

"animation" of construction of K_0.

Construction: $K_i(- ; \mathbb{Z}/p^n)$, $i \geq 0$

define presheaves on cat of

schemes.

Consider étale sheafification $K_i^{ét}(-; \mathbb{Z}/p^n)$.

<u>Thm</u> (Gabber–Suslin)

On $\mathbb{Z}[1/p]$-schemes,

1) $K_{2i+1}^{ét}(-; \mathbb{Z}/p^n) = 0$

2) $K_{2i}^{ét}(-; \mathbb{Z}/p^n) = \mu_{p^n}^{\otimes i}$.

Not true for arbitrary schemes, (b/c $K_1(-; \mathbb{Z}/p^n)$)

Work w/ <u>syntomic topology</u>: generated by faithfully flat, lci morphisms.

Consider $K_i^{syn}(-; \mathbb{Z}/p^n)$ - syntomic sheafification of K-groups

<u>Thm</u> (Bhatt - Scholze, Česnavičius - Scholze)
On lci noetherian schemes,
$K_{2i+1}^{syn}(-; \mathbb{Z}/p^n) = 0$, $i \geq 0$.

Consider (on lci noeth schemes)
$K_{2i}^{syn}(-; \mathbb{Z}/p^n)$ & take
the derived pushforward to
$X_{ét}$, X lci.

gives $\rightsquigarrow$ $\mathbb{Z}/p^n(i)_X$.

(will give algebraic defn below)

Fontaine-Messing, Kato defn.

Let X be a p-adic formal scheme, smooth over $\mathrm{Spf}(\mathbb{Z}_p)$.

$\rightsquigarrow R\Gamma_{dR}(X)$ = p-adic algebraic dR complex of X

Equipped w/

1) Hodge filtration (naive filt on dR complex)

$$R\Gamma_{dR}^{\geq i}(X)$$

2) $\varphi : R\Gamma_{dR}(X) \to R\Gamma_{dR}(X)$

(from crystalline theory).

Def (Fontaine-Messing, Kato)

$$\mathbb{Z}_p(i)^{\mathrm{crys}}(X) = \mathrm{fib}\left(\varphi - p^i : R\Gamma_{dR}^{\geq i}(X) \to R\Gamma_{dR}(X)\right),$$

(for $i \geq 0$).

Motivation: $V \to X$ a vector bundle

$c_i(V) \in H^{2i}_{dR}(X)$ live in filtration

$\geq i$ & are $\varphi = p^i$.

Can generalize this construction to flat, lci p-adic formal schemes.

<u>Construction:</u>

X as above.

There is a natural map <u>up to isogeny</u>

$$\mathbb{Z}_p(i)^{\mathrm{crys}}(X) \longrightarrow \mathbb{Z}_p(i)(X[1/p])$$

coh of rigid generic fiber

<u>Thm</u> (Kurihara, Kato, Tsuji, Colmez–Nizioł)

X formally smooth, or semistable, over $\mathcal{O}_K$ or $\mathcal{O}_{\overline{K}}$, for K a p-adic field, then above map is an isogeny in degrees $< i$.

Cor (Colmez-Niziol)

$X/\mathcal{O}_K$ formally smooth, then if $j \le i-1$, there is a natural iso

$$H^{j-1}_{dR}(X)_K \cong H^j_{\text{proét}}(X[1/p], \mathbb{Q}_p(i)).$$

Rmk (Kato)

If $i \le p-2$, then

$$\varphi : R\Gamma^{\geq i}_{dR}(X) \longrightarrow R\Gamma_{dR}(X)$$

is divisible by p^i, & can consider

$\mathrm{fib}\left(1 - \varphi/p^i : R\Gamma_{dR}^{\geq i}(X) \to R\Gamma_{dR}(X)\right)$.

Gives correct integral theory for $i \leq p-2$.

Goal: define integral theory in general

Quasisyntomic site

<u>Def</u>: (BMS2)

A ring R is <u>quasisyntomic</u> if

1) R is p-complete w/ bounded p-power torsion

2) $L_{R/\mathbb{Z}} \otimes^L_R (R/p) \in \mathcal{D}(R/p)$

has Tor-amplitude in $[-1, 0]$.

Let QSyn = cat of quasisyntomic rings.

QSyn^{op} has a structure of a site.

Given $A \to B$ in QSyn, say that it's a cover if

1) $A/p \to B \otimes^L_A A/p$ is faithfully flat (in particular, target in degree 0).

2) $L_{B/A} \otimes^L_B (B/p) \in \mathcal{D}(B/p)$

has Tor-amplitude in $[-1, 0]$.

<u>Rmk</u>: If R is p-complete noetherian,

R is quasisyntomic $\iff$ lci.

Ex) If $R \in QSyn$, if $t \in R$,

then $$\frac{R[x^{1/p^\infty}]^\wedge}{(x-t)}$$ is

a quasisyntomic cover of R.

Basis for QSyn:

R is quasiregular semiperfectoid if R ∈ QSyn & R is a quotient of a perfectoid.

Let QRSPerfd = cat of qrsp rings.

Ex) Perfectoid/regular sequence.
(assume bounded p-power torsion).

Strategy: define everything by descent from QRSPerfd on QSyn

Prismatic cohomology

Construction Let $R \in \mathrm{QRSPerfd}$.

1) Δ_R = prismatic cohomology

Here (Δ_R, φ, I) is a prism

w/ a map

$$R \to \Delta_R / I.$$

(in fact, (Δ_R, φ, I) is the universal prism w/ such a map)

2) Nygaard filtration:

$$N^{\geq n} ⧍_R = \varphi^{-1}(I^n ⧍_R).$$

3) There is a natural invertible $⧍_R$-module $⧍_R\{1\}$ ("Breuil-Kisin twist").

w/ a φ-linear map

$$\varphi_1 : ⧍_R\{1\} \longrightarrow I^{-1} ⧍_R\{1\}$$

whose φ-linearization is an iso.

4) $\overline{⧍}_R = ⧍_R / I$

called the Hodge-Tate coh of R (receives map $R \to \overline{\Delta}_R$),

$$\Delta_R\{i\} \otimes_{\Delta_R} \overline{\Delta}_R \simeq I/I^2$$

5) φ induces a filtered map

$$\varphi : N^{\geq \bullet} \Delta_R \longrightarrow I^{\bullet} \Delta_R$$

and on associated graded terms, get a map

$$\bigoplus_{i \geq 0} \underset{\text{\rotatebox{90}{$\cong$}}}{N^i \Delta_R} \longrightarrow \bigoplus_{i \geq 0} \underset{\substack{\| \\ I^i/I^{i+1}}}{\overline{\Delta}_R\{i\}},$$

$\mathrm{gr}^i N^{\geq 0} \Delta_R$

Can descend to all of QSyn. b/c the above define sheaves on QRSPerfd w/ no higher cohomology.

<u>Rmk</u>: Suffices to consider only p-torsionfree R in all of the above. (in that case Δ_R is transversal prism, $\overline{\Delta}_R$ is p-torsionfree, Δ_R is p-torsionfree).

Ex) R perfectoid,

then $\Delta_R = A_{\inf}(R)$ &

the map

$$R \xrightarrow{\sim} \overline{\Delta}_R \text{ is an iso}$$

(inverse comes from $\theta : A_{\inf}(R) \to R$).

<u>Def</u> (Syntomic cohomology).

Let $R \in$ QRSPerfd.

Consider

$$\Delta_R\{i\} \xrightarrow{\varphi_i} I^{-i} \Delta_R\{i\}$$

$$\varphi_i(N^{\geq i} \Delta_R\{i\}) \subseteq \Delta_R\{i\}.$$

- -

Def: For $R \in \mathbb{Q}\mathrm{RSPerfd}$, define

$$\mathbb{Z}_p(i)(R) = \mathrm{fib}\left(\varphi_i - \mathrm{can} : N^{\geq i}\Delta_R\{i\} \to \Delta_R\{i\}\right).$$

$$N^{\geq i}\Delta_R\{i\} = N^{\geq i}\Delta_R \cdot \Delta_R\{i\}$$

By descent, can define $\mathbb{Z}_p(i)(R)$, $R \in \mathbb{Q}\mathrm{Syn}$,

by animation, extends to all

$\cup$

p-complete animated rings.

(In these talks, I will focus on QSyn).

<u>Construction</u>

Let $R \in \mathrm{QRSPerfd}$.

There is a natural map

$$\log_{\Delta} : T_p(R^\times) \longrightarrow \Delta_R\{1\}$$

whose image is exactly those $x \in \Delta_R\{1\}$ fixed by φ_1.

Gives an iso
$$T_p(R^\times) \simeq H^0_{syn}(\mathbb{Z}_p(1)(R)).$$

see. 2 of APC

for any prism (A, I)

$$(1+I)_{rk\,1} \longrightarrow A\{1\}.$$

<u>Thm</u> (Bhatt–Lurie, Moron)

For any p-adic formal scheme X, there is a natural multiplicative map

$$\bigoplus_{i\geq 0} \mathbb{Z}_p(i)(X) \longrightarrow \bigoplus_{i\in\mathbb{Z}} \mathbb{Z}_p(i)(X[1/p])$$

étale coh of
generic fiber.

Pf sketch:

- consider the case $X = \mathrm{Spf}(\mathcal{O}_C)$

  C alg closed NA field of complete mixed char.

- Both sides are polynomial algs on a class in degree 1

$$(1, \zeta_p, \zeta_{p^2}, \ldots) \to H^0_{syn}(\mathbb{Z}_p(1)(\mathcal{O}_C))$$

$$\xrightarrow{\sim} H^0(\mathbb{Z}_p(1)(C))$$

- Extend to $\mathrm{Spf}(\prod \mathcal{O}_C)$

- Reduce to this case by arc-descent of RHS.
- Explained in APC paper. = Bhatt–Lurie (absolute prismatic coh).

Consequently, can define syntomic complexes of a scheme, let $X = \mathrm{Spec}(A)$.

Have a pullback

$$\begin{tikzcd}
\mathbb{Z}_p(i)(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathbb{Z}_p(i)(A[1/p]) \arrow[d] \\
\mathbb{Z}_p(i)(\hat{A}_p) \arrow[r] & \mathbb{Z}_p(i)(\hat{A}_p[1/p])
\end{tikzcd}$$

derived p-completions

Gives a def of syntomic coh of arbitrary schemes.

Ex) $\mathbb{Z}_p(0)(X) = R\Gamma_{\text{proét}}(X, \mathbb{Z}_p)$

Ex) $\mathbb{Z}_p(1)(X) = R\Gamma_{\text{ét}}(X, \mathbb{G}_m)[-1]^{\wedge}_p$

(via $\log_\Delta$ for formal schemes).

Ex) $\mathbb{Z}_p(i)$ agrees w/ Fontaine-Messing / Kato defn for $i \le p-2$, or for all i up to isogeny (Antieau-M-Morrow-Nikolaus).

Ex) $\mathbb{Z}_p(i)$ satisfy a projective bundle formula, & there is a theory of Chern classes.

Goal: describe the syntomic complexes of regular schemes.

Ex) (BMS2) If $X/\mathbb{F}_p$ regular,

$$\mathbb{Z}/p^n(i)_X = W_n\Omega^i_{\log, X}[-i].$$

Suppose X is a regular p-torsionfree scheme, want to describe $\mathbb{Z}/p^n(i)_X \in D(X_{et}, \mathbb{Z}/p^n)$.

First, $j: X[1/p] \longrightarrow X$ is

open inclusion, $j^*(\mathbb{Z}/p^n(i)_X) = \mu_{p^n}^{\otimes i}$,

so have comparison

$$\mathbb{Z}/p^n(i)_X \longrightarrow Rj_*(\mu_{p^n}^{\otimes i}).$$

Fact: (Antieau-M-Morrow-Nikolaus):

LHS $\in \mathcal{D}^{\leq i}$. (for all X).

Above comparison lifts to

$$\mathbb{Z}/p^n(i)_X \longrightarrow \tau^{\leq i} Rj_*(\mu_{p^n}^{\otimes i}).$$

$(\ast)$

Theorem (Bhatt–M): If X regular, p-torsionfree
The comparison map $(*)$ is
an iso in degrees $< i$, and in
degree i is an injection.
The image in H^i is precisely
the image of the symbol map

$$(\mathcal{O}_X^\times)^{\otimes i} \longrightarrow R^i j_*(\mu_{p^n}^{\otimes i}).$$

(here $\mathcal{O}_X^\times \to R^1 j_*(\mu_p)$)

(Reason: $\bigoplus_{i \geq 0} H^i(\mathbb{Z}/p^n(i)_X)$ is ← Lüders–Morrow
generated in degree 1, via

$\mathcal{O}_X^\times \to \mathcal{H}^1(\mathbb{Z}/p^n(1)_X)$.)

- X smooth/semistable DVR, generalizing Kurihara, Kato, Tsuji, Colmez-Nizioł.
- Construction of RHS (modified $\tau^{\leq i} Rj_* \mu_{p^n}^{\otimes i}$) appears in work of Schneider, Geisser, Sato (semistable case)

Questions:

1) Do the syntomic complexes of regular schemes arise as p-adic étale sheafifications of

motivic complexes?
(cf. Sato).

2) Can one compute syntomic complexes for singular schemes?
Work of Antieau-Krause-Nikolaus for syntomic complexes of $\mathbb{Z}/p^n$.

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2 main ingredients for proof:
1) Étale comparison theorem
2) F-smoothness (analog of Cartier iso over $\mathbb{F}_p$).

Étale comparison theorem

Gives a way to compare syntomic coh of a scheme & étale coh of generic fiber.

Recall:

$$\mathbb{Z}_p(1)(X) = R\Gamma_{\acute{e}t}(X, \mathbb{G}_m)[-1]^\wedge_p.$$

$$= \varprojlim R\Gamma_{fl}(X, \mu_{p^n}).$$

$X = \mathbb{Z}[\zeta_{p^\infty}]$, obtain a class

$$\varepsilon \in H^0(\mathbb{Z}_p(1)(\mathbb{Z}[\zeta_{p^\infty}])).$$

Theorem (Bhatt–Scholze, Bhatt–Lurie)

If $X/\mathbb{Z}[\zeta_{p^\infty}]$ qcqs, then

$$\bigoplus_{i\geq 0} \mathbb{Z}_p(i)(X) \to \bigoplus_{i\in\mathbb{Z}} \mathbb{Z}_p(i)(X[1/p])$$

exhibits the target as the localization of the source at ε (in p-complete sense).

Can extend to an arbitrary qcqs scheme.

Construction:

There is a class $v_1 \in H^0(\mathbb{F}_p(p-1)(\mathbb{Z}))$ which maps to ε^{p-1} in $\mathbb{F}_p(p-1)(\mathbb{Z}[\zeta_{p^\infty}])$.

Can construct v_1 by flat descent from ε^{p-1}.

Cor: X any qcqs scheme, then

$$\bigoplus_{i \geq 0} \mathbb{F}_p(i)(X) \to \bigoplus_{i \in \mathbb{Z}} \mathbb{F}_p(i)(X[1/p])$$

exhibits the target as the source with v_1 inverted.

Rest of today: describe
$$v_1 \in H^0(\mathbb{F}_p(p-1)(\mathbb{Z}_p))$$
(via prismatic coh of $\mathbb{Z}_p$).

<u>Enough</u>: for every transversal (A/I p-torsionfree) prism (A, I), the image of v_1 in
$$\frac{A}{p}\{p-1\}$$
s.t.

this image is a fixed pt
for
φ_{p-1}.

What is v_1 in $\frac{A}{p}\{p-1\}$?

Choose a generator
$y_1 \in A\{1\}$

which exists locally.
(One checks then independence of choices).

$$\varphi_1 : A\{1\} \longrightarrow I^{-1} A\{1\}.$$

& $\dfrac{y_1}{\varphi_1(y_1)}$ is a generator for ideal I.

Def:

$$V_1 = \frac{y_1}{\varphi_1(y_1)}, \quad y_1^{\otimes p-1} \in \frac{A}{p}\{p-1\}.$$

Observation: 1) v_1 independent of choice of y_1 b/c φ_1 is Frob-semilinear.

2) v_1 is a fixed pt for

$$\varphi_{p-1} = \varphi_1^{\otimes p-1}.$$

Taking inverse limit over all transversal prisms, produces class in $H^0(\mathbb{F}_p(p-1)(\mathbb{Z}_p))$

One other way to describe v_1:

$v_1 \rightsquigarrow$ gives a class

in $H^0\left(\frac{\Delta_{\mathbb{Z}_p}}{p}\{p-1\}\right)$.

Here $\frac{\Delta_{\mathbb{Z}_p}}{p}\{p-1\}$ carries the

the Nygaard filtration, &

<u>Thm</u>:

$H^*\left(\bigoplus_{i\geq 0} N^i \Delta_{\mathbb{Z}_p}/p\right)$

$\simeq \mathcal{E}(\alpha) \otimes P(\theta)$ (over $\mathbb{F}_p$)

where $\alpha = (1, p)$ (coh degree, internal degree).
$\theta = (0, p)$

Follows from the description of Hodge-Tate stack of $\mathbb{Z}_p$ (in APC) as $B\mathbb{G}_m^{\#}$

<u>Fact:</u> $v_1 \in \frac{\Delta_{\mathbb{Z}_p}\{p-1\}}{p}$ lifts

to a class

$$\tilde{\Theta} \in \frac{N^{\geq p} \Delta_{\mathbb{Z}_p}\{p-1\}}{p}$$

and is detected by Θ in

gr^p,