Cartier isomorphism in mixed characteristic

Let $R/\mathbb{F}_p$. (based on joint work w/ Bhatt, arXiv:2202.04818)

Construction

$$C^{-1} : \Omega^*_{R/\mathbb{F}_p} \longrightarrow H^*(\Omega^\bullet_{R/\mathbb{F}_p})$$

$$r \longmapsto [r^p]$$

$$ds \longmapsto [s^{p-1}ds] \text{ "}=\text{" } \frac{d(s^p)}{p},$$

$r, s \in R$

Thm $R/\mathbb{F}_p$ is regular noetherian, then C^{-1} is an iso.

Pf: Check directly for $\mathbb{F}_p[x_1, \dots, x_n]$.

Étale localize ⇒ check for smooth $\mathbb{F}_p$-algebras.

R is ind-smooth by Popescu's thm. ✓

Goal: Formulate & prove analog in mixed char.

<u>Rmk</u>: (Lurie): Can prove above thm w/o ind-smoothness, w/ a more elaborate argument. (9.5.6 of Bhatt-Lurie-M)

Q: Which other $\mathbb{F}_p$-alg satisfy Cartier?

<u>Def</u> (Kelly-Morrow, Kerz-Strunk-Tamme): $R/\mathbb{F}_p$ is <u>Cartier smooth</u> if

1) $L_{R/\mathbb{F}_p} \in \mathcal{D}(R)$ is in degree 0

& is flat R-module

2) C^{-1} is an iso.

Ex) 1) Regular $\mathbb{F}_p$-alg

2) Perfect $\mathbb{F}_p$-algs (or smooth over perfect $\mathbb{F}_p$-algs)

3) Algebras relatively perfect over a polynomial $\mathbb{F}_p$-alg ("p-basis").

4) (Gabber-Ramero, Gabber) Valuation $\mathbb{F}_p$-algebras

<u>Rmk</u> (Bhatt-Gabber)

It's possible for $R/\mathbb{F}_p$ to satisfy 1) but not 2).

Ex: $\exists R/\mathbb{F}_p$ s.t. R semiperfect, imperfect, w/ $L_{R/\mathbb{F}_p} = 0$.

not Cartier smooth!

Why is notion useful?

1) dRW complex behaves well for Cartier smooth $\mathbb{F}_p$-algs

2)

<u>Thm</u> (Kelly-Morrow, Kerz-Strunk-Tamme) $X = \mathrm{Spec}(R)$, $R/\mathbb{F}_p$ Cartier smooth, then

$$\mathbb{Z}/p^n(i)_X = W_n \Omega^i_{\log, X}[-i]$$

same formula as if X smooth $/\mathbb{F}_p$.

Goal: mixed char. analog.

Reformulate:

$R/\mathbb{F}_p \rightsquigarrow L\Omega_{R/\mathbb{F}_p}$

$\rightarrow$ derived de Rham complex (Illusie, Bhatt).

$\rightarrow$ animation of usual dR complex for polynomial $\mathbb{F}_p$-algs

Equipped w/

1) Increasing conj. filtration (exhaustive)

$\mathrm{Fil}_{\leq \cdot} L\Omega_{R/\mathbb{F}_p}$

(animation of Postnikov filtration)

2) Decending Hodge filtration

$L\Omega^{\geq \cdot}_{R/\mathbb{F}_p}$

(animation of Hodge filt)

— not always convergent.

In both cases,

$$\mathrm{gr}^i = \Lambda^i L_{R/\mathbb{F}_p}[-i].$$

Prop $R/\mathbb{F}_p$ is Cartier smooth

$\Longleftrightarrow$

1) $\forall i \geq 0$,

$$\mathrm{Fil}^{\mathrm{conj}}_{\leq i} L\Omega_{R/\mathbb{F}_p} \to L\Omega_{R/\mathbb{F}_p}$$

has fiber in $\mathcal{D}(R)$ w/ Tor-amplitude in degrees $\geq i+2$.

2) $L\Omega_{R/\mathbb{F}_p}$ is complete for Hodge filtration.

(given 1), 2) is the statement that $L\Omega_{R/\mathbb{F}_p}$ = actual dR complex).

These have analogs in mixed char.

Let $R \in \mathrm{QSyn}$.

We have

$$N^{\geq \bullet} \Delta_R \xrightarrow{\varphi} I^{\bullet} \Delta_R$$

a map of filtered complexes.

On gr^i, obtain a map

$$\varphi_i : N^i \Delta_R \longrightarrow I^i / I^{i+1} = \overline{\Delta}_R\{i\}.$$

<u>Rmk</u>: Recovers, for $R/\mathbb{F}_p$,

$$\mathrm{Fil}^{\mathrm{conj}}_{\leq i} L\Omega_{R/\mathbb{F}_p} \longrightarrow L\Omega_{R/\mathbb{F}_p}.$$

Def $R \in \mathrm{QSyn}$ is F-smooth

1) For each $i \geq 0$, $\mathrm{fib}(\varphi_i : N^i \Delta_R \to \overline{\Delta}_R\{i\})$ has p-complete Tor-amplitude in degrees $\geq i+2$.

2) For each i, $N^{\geq 0} \Delta_R\{i\}$ is complete.

Ex) 1) For $R/\mathbb{F}_p$, equivalent to Cartier smoothness

2) If R perfectoid, then R is F-smooth (φ_i are isos).

Ex) If R smooth / perfectoid, then

R is F-smooth

(follows from HT comparison)
in relative theory

Prop (Bouis): If R_0 is perfectoid, then R/R_0 is F-smooth $\Longleftrightarrow$

1) $\widehat{L}_{R/R_0}$ is p-completely flat

2) $\widehat{L\Omega}_{R/R_0}$ is (p-completely) Hodge complete.

Ex) $\mathbb{Z}_p$ is F-smooth (cf. APC).

$$H^*\left(\bigoplus N^i \Delta_{\mathbb{Z}_p/p}\right) = E(\alpha) \otimes P(\theta)$$

$|\alpha| = (1, p)$

$|\theta| = (0, p)$

$$H^*\left(\bigoplus \overline{\Delta}_{\mathbb{Z}_p}\{i\}/p\right) = \mathcal{E}(\alpha) \otimes P(\Theta^{\pm 1})$$

(Here $\Theta \in H^0(N^p \Delta_{\mathbb{Z}_p}/p)$)

Observation:

If R is F-smooth,

$$\Theta : N^i \Delta_R/p \longrightarrow N^{i+p}\Delta_R/p$$

in $\mathcal{D}(R)$

has fib w/ Tor-amplitude in degrees $\geq i+1$. ("Θ is periodic in some range").

Prop The class of F-smooth rings is

1) Étale local

2) Closed under passage to smooth algebras.

3) Closed under filtered colimits.

<u>Observation</u>: If R is F-smooth,

then

$$N^{\geq i} \overline{\Delta}\{*\}/p \quad \text{is}$$

uniformly pro-zero. ← in any cohomological range.
(in R)

<u>Thm A</u>: (Bhatt-M, Bouis):
If R is p-~~torsion~~free and F-smooth,

then

$$\mathbb{F}_p(i)(R) \longrightarrow \mathbb{F}_p(i)(R[1/p])$$

is iso on H^j, $j<i$

is inj on H^i.

(Cf BMS2 for smooth$/\mathcal{O}_C$, C alg closed complete NA).

Thm B (Bhatt-M):

If R is p-complete noetherian, then R is F-smooth $\Leftrightarrow$ R is regular.

Prop $R \in QSyn$, $x \in R$ nonzerodivisor, then if $R[1/x]$, R/x are F-smooth, $\Rightarrow$ R is F-smooth.

Rmk: Implies Thm B($\Leftarrow$). (by noeth. induction, localization, & reduction to field of char. p).

Reproves Cartier iso for regular $\mathbb{F}_p$-algebras (Luric).

To prove, use

Prop For each i, there are fiber sequences

$$\overline{\Delta}_R\{i\}/x \to \overline{\Delta}_{R/x}\{i\} \to \overline{\Delta}_{R/x}\{i-1\}$$

(also on Nygaard completions)

&

$$(N^i \Delta_R)/x \to N^i \Delta_{R/x} \to N^{i-1} \Delta_{R/x}$$

Implies previous prop.

Rmk Should be related to work of Saito, Hochster-Jeffries.

__Cor__ : Let R be a regular local ring w/ perfect residue field. Suppose R/p is F-finite.

Then $\overline{\Delta}_R\{c\} \in \mathcal{D}^{\leq \dim(R)}$.

__Questions__

1) Are valuation rings F-smooth?
(Bouis, over perfectoids)

2) Given a prism (A, I) with A p-torsionfree and A/p Cartier smooth, is A/I F-smooth?

Thm A proof:

Strategy: If R F-smooth, p-torsionfree

$v_1 : \mathbb{F}_p(i)(R) \to \mathbb{F}_p(i+p-1)(R)$

is iso in range. (By étale comp, enough)

Use description of $\mathbb{F}_p(i)$ as equalizer, and identification of v_1 by θ, & θ is periodicity operator.