Parallel transport for Higgs bundles over p-adic curves

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Classical results /$\mathbb{C}$

X smooth projective variety over $\mathbb{C}$

Theorem (Hodge decomposition)

$$\mathrm{H}^n(X^{\mathrm{an}}, \underline{\mathbb{C}}) \simeq \bigoplus_{i+j=n} \mathrm{H}^i(X, \Omega^j_{X/\mathbb{C}})$$

Non-abelian Hodge theory provides a generalisation for cohomologies with coefficients.

Narasimhan–Seshadri correspondence

- $\dim X = 1$

$$\left\{ \begin{array}{c} \text{irreducible unitary} \\ \mathbb{C}\text{-representations of } \pi_1^{\mathrm{top}}(X) \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{stable vector bundles} \\ \text{of degree 0 over } X \end{array} \right\}$$

$$V \mapsto \widetilde{X} \times V / \pi_1^{\mathrm{top}}(X)$$

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- $\widetilde{X}$ = universal covering of X
- Action of $g \in \pi_1^{\mathrm{top}}(X)$ is given by $g(x, v) = (g(x), g(v))$.

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- $\widetilde{X}$ = universal covering of X
- Action of $g \in \pi_1^{\mathrm{top}}(X)$ is given by $g(x, v) = (g(x), g(v))$.
- A vector bundle E over X is *stable* (resp. *semi-stable*) if for any sub vector bundle F of E, we have

$$\mu(F) < \ (\text{resp. } \leq)\ \mu(E), \quad \text{where } \mu(E) = \frac{\deg(E)}{\mathrm{rank}(E)}$$

Hitchen–Simpson correspondence

- A Higgs bundle $(E, \theta)/X$: E a vector bundle over X and $\theta : E \to E \otimes_{\mathscr{O}_X} \Omega^1_{X/\mathbb{C}}$ an $\mathscr{O}_X$-linear morphism such that $\theta \wedge \theta = 0$.

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- For a $\mathbb{C}$-representation V and (E,θ) associated Higgs bundle,

$$\mathrm{H}^n(X^{an}, V) \simeq \mathbb{H}^n(E \otimes_{\mathscr{O}_X} \Omega^\bullet_{X/\mathbb{C}})$$

P-adic theory: Hodge–Tate decomposition

- K a finite extension of $\mathbb{Q}_p$, $\overline{K}$ an algebraic closure of K, $G_K = \mathrm{Gal}(\overline{K}/K)$, $\mathscr{O}_K, \mathscr{O}_{\overline{K}}$ rings of integers, $\mathscr{O}_{\mathbf{C}} = \widehat{\mathscr{O}_{\overline{K}}}$ and $\mathbf{C} = \mathscr{O}_{\mathbf{C}}[\frac{1}{p}]$.

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- X a smooth proper variety over K.

Theorem (Faltings, Niziol, Tsuji/ Scholze)

$\exists$ a canonical G_K-equivariant $\mathbf{C}$-linear isomorphism

$$\mathrm{H}^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbf{C} \xrightarrow{\sim} \bigoplus_{i+j=n} \mathrm{H}^i(X, \Omega^j_{X/K}) \otimes_K \mathbf{C}(-j), \tag{2.1}$$

where the action of G_K on $\mathrm{H}^i(X, \Omega^j_{X/K})$ is trivial and on $\mathbf{C}(-j)$ is given by powers of p-adic cyclotomic character

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- Non-abelian theories:
- P-adic Simpson correspondence: Faltings, Abbes–Gros and Tsuji.
- P-adic Riemann–Hilbert correspondence: Liu–Zhu, Diao–Lan–Liu–Zhu.

P-adic Simpson correspondence

- $\mathscr{O}_{\mathbf{C}}^{\flat} = \varprojlim_{x \mapsto x^p} \mathscr{O}_{\mathbf{C}}/p\mathscr{O}_{\mathbf{C}}$, $A_{\inf} = \mathrm{W}(\mathscr{O}_{\mathbf{C}}^{\flat}) \xrightarrow{\theta} \mathscr{O}_{\mathbf{C}}$.
- $\mathrm{Ker}(\theta) = (\xi)$, $A_{\inf,n} = A_{\inf}/\xi^n$, $\mathscr{O}_{\mathbf{C}} = A_{\inf,1}$.

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- X smooth projective $/\mathscr{O}_K$ (works with semi-stable reduction).
- A family of smooth Cartesian liftings $\mathcal{X} = (\mathcal{X}_n)/(A_{\mathrm{inf},n})_{n\geq 1}$ of $X_{\mathscr{O}_{\mathbf{C}}}/\mathscr{O}_{\mathbf{C}}$. Cartesian: $\mathcal{X}_{n+1} \times_{A_{\mathrm{inf},n+1}} A_{\mathrm{inf},n} \simeq \mathcal{X}_n$.

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- Roughly speaking, $\mathcal{X}$ induces an equivalence

$$\mathbf{H}_{\mathcal{X}} : \mathbf{GRep}_{\mathrm{small}}(X) \simeq \mathrm{HB}_{\mathrm{small}}(X_{\mathbf{C}})$$

  between small generalized representations and small Higgs bundles. It is established by a "period ring" $\mathscr{C}^{\dagger}$.

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- In the curve case, $\mathbf{H}_{\mathcal{X}}$ can be extended to $\mathbf{GRep}(X)$. $\mathbf{GRep}(X)$ contains the category $\mathbf{Rep}_{\mathbf{C}}(\pi_1(X_{\overline{K}}))$ of continuous finite dimensional $\mathbf{C}$-representations of $\pi_1(X_{\overline{K}})$ as a full subcategory.
- In this talk, we characterize the Higgs bundles associated to $\mathbf{Rep}_{\mathbf{C}}(\pi_1(X_{\overline{K}}))$ for curves.

P-adic Simpson correspondence: objects

- $\mathrm{HB}(X_{\mathbf{C}})$ Higgs bundle: (M, θ), $M \in \mathrm{VB}(X_{\mathbf{C}})$, $\theta : M \to \xi^{-1} M \otimes \Omega^1_{X_{\mathbf{C}}}$.
- $\mathrm{HB}_{\mathrm{small}}(X_{\mathbf{C}})$: (M, θ) is *small*, if $\exists$ a model $M^\circ \in \mathrm{Coh}(X_{\mathscr{O}_{\mathbf{C}}})$ of M such that $\theta(M^\circ) \subset p^\alpha M^\circ \otimes \Omega^1_{X/\mathscr{O}_K}$ for $\alpha \in \mathbb{Q}_{>\frac{1}{p-1}}$.

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- $\mathbf{GRep}(X)$ generalized representations: certain modules in Faltings ringed topos $(\widetilde{E}_X, \overline{\mathscr{B}}_X)$ of X;
  Locally, they are representations of $\pi_1(X_{\overline{K}})$ over finite projective modules over a certain p-adic ring.
- $\mathbf{Rep}_{\mathbf{C}}(\pi_1(X_{\overline{K}})) \to \mathbf{GRep}(X)$ full subcategory.

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- $\mathbf{Rep}_{\mathbf{C}}(\pi_1(X_{\overline{K}})) \to \mathbf{GRep}(X)$ full subcategory.
- $\mathbf{GRep}_{\mathrm{small}}(X)$: It is *small*, if locally, a representation admits a basis, whose $\pi_1(X_{\overline{K}})$-action is trivial modulo p^β for $\beta \in \mathbb{Q}_{>\frac{2}{p-1}}$.
- (Tsuji) Small generalized representations $=$ Dolbeault modules.
  Dolbeault modules are those generalized representations satisfying the admissible condition defined by $\mathscr{C}^\dagger$.

P-adic Simpson correspondence: picture

$$\begin{tikzcd}
\mathbf{GRep}_{\mathrm{small}}(X) \arrow[rr, "\mathbf{H}_{\mathcal{X}}", "\sim"'] \arrow[d] & & \mathrm{HB}_{\mathrm{small}}(X_{\mathbf{C}}) \arrow[d] \\
\mathbf{GRep}(X) \arrow[rr, dashed, "\mathbf{H}_{\mathcal{X},\mathrm{Exp}}", "\dim X_{\mathbf{C}}=1"'] & & \mathrm{HB}(X_{\mathbf{C}}) \\
\mathbf{Rep}(\pi_1(X_{\overline{K}}), \mathbf{C}) \arrow[u] \arrow[urr, "\mathbb{H}_{\mathcal{X},\mathrm{Exp}}"'] & &
\end{tikzcd}$$

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- $\log : 1+\mathfrak{m}_{\mathbf{C}} \to \mathbf{C}$, $x \mapsto \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n}$.
  $\exp : B = \{x \in \mathbf{C} \| x| < p^{-\frac{1}{p-1}}\} \to 1+\mathfrak{m}_{\mathbf{C}}$, $x \mapsto \sum_{n\geq 0}\frac{x^n}{n!}$
  is an inverse of log on B.
- As $1+\mathfrak{m}_{\mathbf{C}}$ is divisible, exp extends to a section of log:

$$\mathrm{Exp} : (\mathbf{C}, +) \to (1+\mathfrak{m}_{\mathbf{C}}, \times).$$

In the following, we assume $\dim X_{\mathbf{C}} = 1$

$$
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\mathbf{GRep}(X) \arrow[r, dashed, "\mathbf{H}_{\mathcal{X},\mathrm{Exp}}"] & \mathrm{HB}(X_{\mathbf{C}}) & \\
\mathbf{Rep}_{\mathbf{C}}(\pi_1(X_{\overline{K}})) \arrow[u] \arrow[ur, "{\color{red}\mathbb{H}_{\mathcal{X},\mathrm{Exp}}}"'] & \mathrm{VB}^{\mathrm{DW}}(X_{\mathbf{C}}) \arrow[l, "\mathbb{V}^{\mathrm{DW}}"] \arrow[u] \arrow[uu, bend right] & E \arrow[uu]
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$$

- Conjecture: image $\mathbb{H}_{\mathcal{X},\mathrm{Exp}}(\mathbf{Rep}_{\mathbf{C}}(\pi_1(X_{\overline{K}}))) =$ semi-stable Higgs bundles of degree zero $/X_{\mathbf{C}}$.
- Faltings sketched a proof of $\subset$.

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- Faltings sketched a proof of $\subset$.
- Describe essential image of $\mathbb{H}_{\mathcal{X},\mathrm{Exp}}$ with the help of Deninger–Werner's functor, which fits into above diagram:

$$\mathbb{V}^{\mathrm{DW}} : \mathrm{VB}^{\mathrm{DW}}(X_{\mathbf{C}}) \to \mathbf{Rep}_{\mathbf{C}}(\pi_1(X_{\overline{K}})).$$

A p-adic analogue of Narasimhan–Seshadri correspondence.

Deninger–Werner's theory

Definition

Let k be the residue field of K and Z a smooth proper $\overline{k}$-curve. A vector bundle E on Z is *strongly semi-stable*, if $F_Z^{n,*}(E)$ is semi-stable for every integer $n \geq 0$.

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Example: Suppose genus of Z is ≥ 2. Then $F_{Z,*}(\mathscr{O}_Z)$ is a stable bundle of rank p, but $F_Z^*(F_{Z,*}(\mathscr{O}_Z))$ is not semi-stable.

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Definition (Deninger-Werner)

Let X be a projective $\mathscr{O}_K$-curve with semi-stable reduction.

(i) $\mathrm{VB}^{\mathrm{DW}}(X_{\mathscr{O}_{\mathbf{C}}})$: A vector bundle $\mathscr{F}$ is DW if for every irreducible component Z_i of $X_{\overline{k}}$ and $\widetilde{Z}_i =$ normalisation of Z_i, the pullback $\mathscr{F}_{\overline{k}}|\widetilde{Z}_i$ is strongly semi-stable of degree zero.

(ii) $\mathrm{VB}^{\mathrm{DW}}(X_{\mathbf{C}})$: Image of $\mathrm{VB}^{\mathrm{DW}}(X_{\mathscr{O}_{\mathbf{C}}}) \to \mathrm{VB}(X_{\mathbf{C}})$.

Theorem (Deninger–Werner)

TFAE: (i) $\mathscr{F}/X_{\mathscr{O}_{\mathbb{C}}}$ *is DW.*
(ii) $\forall n \geq 1$, *after taking a finite extension of* K, $\exists$ *a projective* $\mathscr{O}_K$*-curve* Y *with semi-stable reduction and a proper map* $f : Y \to X$ *such that*

- $f_{\overline{K}} : Y_{\overline{K}} \to X_{\overline{K}}$ *is a Galois étale cover.*
- *The action of* $\mathrm{Aut}(Y_{\overline{K}}/X_{\overline{K}})$ *extends to an action over* Y/X.
- $\overline{f}_n^*(\mathscr{F}_n)$ *is a trivial bundle,* $\overline{f}_n = f \otimes_{\mathscr{O}_K} \mathscr{O}_{\overline{K}}/p^n\mathscr{O}_{\overline{K}}$.

This allows us to associate representation to DW vector bundles.

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This allows us to associate representation to DW vector bundles.

- $\exists$ an universal isomorphism $\mathscr{O}_K \xrightarrow{\sim} \lambda_*(\mathscr{O}_Y)$, λ canonical map.
- If $r = \mathrm{rank}\,\mathscr{F}$, we obtain a representation (parallel transport)

$$\mathrm{Gal}(Y_{\overline{K}}/X_{\overline{K}}) \circlearrowright \Gamma(Y_{\overline{k}}, \overline{f}_n^*(\mathscr{F}_n)) \xrightarrow{\sim} (\mathscr{O}_{\overline{K}}/p^n\mathscr{O}_{\overline{K}})^{\oplus r}.$$

- $\mathbb{V}_n^{\mathrm{DW}} : \mathrm{VB}^{\mathrm{DW}}(X_{\mathscr{O}_\mathbb{C}}) \to \mathbf{Rep}_{\mathscr{O}_{\overline{K}}/p^n\mathscr{O}_{\overline{K}}}(\pi_1(X_{\overline{K}})), \quad \mathbb{V}^{\mathrm{DW}} = \varprojlim \mathbb{V}_n^{\mathrm{DW}}[\frac{1}{p}]$

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- $\mathbb{V}_n^{\mathrm{DW}} : \mathrm{VB}^{\mathrm{DW}}(X_{\mathscr{O}_{\mathbb{C}}}) \to \mathbf{Rep}_{\mathscr{O}_{\overline{K}}/p^n \mathscr{O}_{\overline{K}}}(\pi_1(X_{\overline{K}}))$, $\quad \mathbb{V}^{\mathrm{DW}} = \varprojlim \mathbb{V}_n^{\mathrm{DW}}[\frac{1}{p}]$
- Compatibility between $\mathbb{V}^{\mathrm{DW}}$ and p-adic Simpson correspondence (X.)
- Higher dimensional case: Deninger–Werner, Würthen.

Definition

Let X be a smooth projective curve $/\mathscr{O}_K$.
(i) $\mathrm{HB}^{\mathrm{DW}}(X_{\mathscr{O}_{\mathbf{C}}})$: $M \in \mathrm{VB}^{\mathrm{DW}}(X_{\mathscr{O}_{\mathbf{C}}})$ and a *small* Higgs field θ on M.
(ii) $\mathrm{HB}^{\mathrm{DW}}(X_{\mathbf{C}})$: Image of $\mathrm{HB}^{\mathrm{DW}}(X_{\mathscr{O}_{\mathbf{C}}}) \to \mathrm{HB}(X_{\mathbf{C}})$.

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- To extend above definition for a general Higgs field, we need *twisted pullback functor* for Higgs bundles (compatible with pullback functoriality of p-adic Simpson correspondence).
- Let Y be a smooth projective $\mathscr{O}_K$-curve, $\mathcal{Y} = (\mathcal{Y}_N)/(A_{\mathrm{inf},N})$ a smooth Cartesian lifting of $Y_{\mathscr{O}_{\mathbf{C}}}/\mathscr{O}_{\mathbf{C}}$.
- A proper map $f : Y \to X$ such that f_K is finite induces:

$$f^{\circ}_{\mathcal{Y},\mathcal{X},\mathrm{Exp}} (= f^{\circ}) : \mathrm{HB}(X_{\mathbf{C}}) \to \mathrm{HB}(Y_{\mathbf{C}}).$$

Definition

Let X be a smooth projective curve $/\mathscr{O}_K$.
(i) $\mathrm{HB}^{\mathrm{DW}}(X_{\mathscr{O}_{\mathbf{C}}})$: $M \in \mathrm{VB}^{\mathrm{DW}}(X_{\mathscr{O}_{\mathbf{C}}})$ and a *small* Higgs field θ on M.
(ii) $\mathrm{HB}^{\mathrm{DW}}(X_{\mathbf{C}})$: Image of $\mathrm{HB}^{\mathrm{DW}}(X_{\mathscr{O}_{\mathbf{C}}}) \to \mathrm{HB}(X_{\mathbf{C}})$.

- To extend above definition for a general Higgs field, we need *twisted pullback functor* for Higgs bundles (compatible with pullback functoriality of p-adic Simpson correspondence).
- Let Y be a smooth projective $\mathscr{O}_K$-curve, $\mathcal{Y} = (\mathcal{Y}_N)/(A_{\mathrm{inf},N})$ a smooth Cartesian lifting of $Y_{\mathscr{O}_{\mathbf{C}}}/\mathscr{O}_{\mathbf{C}}$.
- A proper map $f : Y \to X$ such that f_K is finite induces:

$$f^{\circ}_{\mathcal{Y},\mathcal{X},\mathrm{Exp}}(= f^{\circ}) : \mathrm{HB}(X_{\mathbf{C}}) \to \mathrm{HB}(Y_{\mathbf{C}}).$$

- When Higgs field $= 0$, $f^{\circ}(M, 0) = (f^*_{\mathbf{C}}(M), 0)$.
- When Higgs field $\neq 0$, f° is different to f^* by a "twist".

Definition

$\mathrm{HB}^{\mathrm{pDW}}_{\mathcal{X},\mathrm{Exp}}(X_{\mathbf{C}})(= \mathrm{HB}^{\mathrm{pDW}}(X_{\mathbf{C}}))$: $(M, \theta) \in \mathrm{HB}(X_{\mathbf{C}})$ such that after $\exists$ $f : Y \to X$ as above and $f^{\circ}(M, \theta) \in \mathrm{HB}^{\mathrm{DW}}(Y_{\mathbf{C}})$.

Theorem (X.)

(i) $\mathbb{H}_{\mathcal{X},\mathrm{Exp}}$ *sends* $\mathbf{Rep}_{\mathbf{C}}(\pi_1(X_{\overline{K}}))$ *to* $\mathrm{HB}^{\mathrm{pDW}}_{\mathcal{X},\mathrm{Exp}}(X_{\mathbf{C}})$.
(ii) $\exists$ *a quasi-inverse of* $\mathbb{H}_{\mathcal{X},\mathrm{Exp}}$:

$$\mathbb{V}_{\mathcal{X},\mathrm{Exp}}(=\mathbb{V}) : \mathrm{HB}^{\mathrm{pDW}}_{\mathcal{X},\mathrm{Exp}}(X_{\mathbf{C}}) \xrightarrow{\sim} \mathbf{Rep}_{\mathbf{C}}(\pi_1(X_{\overline{K}})).$$

(iii) $(E,\theta) \in \mathrm{HB}^{\mathrm{pDW}}_{\mathcal{X},\mathrm{Exp}}(X_{\mathbf{C}})$ *and* $V = \mathbb{V}(E,\theta)$, *we have*

$$\mathrm{H}^*_{\text{ét}}(X_{\overline{K}}, V) \simeq \mathbb{H}^*(X_{\mathbf{C}}, E \xrightarrow{\theta} E \otimes \Omega^1_{X_{\mathbf{C}}}).$$

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Proposition

(i) $\mathrm{HB}^{\mathrm{pDW}}_{\mathcal{X},\mathrm{Exp}}(X_{\mathbf{C}}) \subset$ *semi-stable Higgs bundles of degree zero* $/X_{\mathbf{C}}$.
(ii) *Every Higgs line bundle of degree zero* $\in \mathrm{HB}^{\mathrm{pDW}}_{\mathcal{X},\mathrm{Exp}}(X_{\mathbf{C}})$.
(iii) $\mathrm{HB}^{\mathrm{pDW}}_{\mathcal{X},\mathrm{Exp}}(X_{\mathbf{C}})$ *is abelian and closed under extensions.*

- It is expected that assertion (i) is an equivalence.

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- Higgs line bundles: Heuer, Abeloid: Heuer–Mann–Werner.

Tsuji's approach to p-adic Simpson correspondence

- $\mathbb{V}$ is an extension of $\mathbb{V}^{\mathrm{DW}}$ to Higgs bundles.
  It is based on Tsuji's approach to p-adic Simpson correspondence.
- In this approach, small Higgs bundles are interpreted as crystals on a site $(\mathfrak{X}/A_{\mathrm{inf}})$, where $\mathfrak{X} =$ p-adic completion of $X_{\mathscr{O}_{\mathbb{C}}}$.

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- $\mathbb{V}$ is an extension of $\mathbb{V}^{\mathrm{DW}}$ to Higgs bundles.
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- In this approach, small Higgs bundles are interpreted as crystals on a site $(\mathfrak{X}/A_{\mathrm{inf}})$, where $\mathfrak{X} =$ p-adic completion of $X_{\mathscr{O}_{\mathrm{C}}}$.
- An object $(\mathcal{T}, z)$ of $(\mathfrak{X}/A_{\mathrm{inf}})$ consists of a sequence of morphisms $\mathcal{T} = (\dots \mathcal{T}_N \to \mathcal{T}_{N+1} \dots)_{N \geq 1}$ of p-adic formal schemes $\mathcal{T}_N$ over $\mathrm{Spf}(A_{\mathrm{inf}}/\xi^N)$ such that
  - $\mathcal{T}_N \to \mathcal{T}_{N+1}$ is a closed immersion;
  - for $N \geq 2$, the morphism of the modulo p reduction of $\mathcal{T}_N \to \mathcal{T}_{N+1}$ is a nilpotent immersion;
  - The morphism $p : \mathscr{O}_{\mathcal{T}_N} \to \mathscr{O}_{\mathcal{T}_N}$ is injective for all $N \geq 1$;
  - $\mathrm{Ker}(\mathscr{O}_{\mathcal{T}_N} \to \mathscr{O}_{\mathcal{T}_n}) = \xi^n \mathscr{O}_{\mathcal{T}_N}$ for $1 \leq n \leq N$;
  - $\mathrm{Ker}(\xi : \mathscr{O}_{\mathcal{T}_N} \to \mathscr{O}_{\mathcal{T}_N})$ is $\xi^{N-1}\mathscr{O}_{\mathcal{T}_N}$;

  and a morphism $z : \mathcal{T}_1 \to \mathfrak{X}$.
- A morphism $(\mathcal{T}', z') \to (\mathcal{T}, z)$ is a family of compatible morphisms $f_N : \mathcal{T}'_N \to \mathcal{T}_N$ of formal schemes over A_{inf} such that $z \circ f_1 = z'$.

Higgs crystals

- Topology: $\mathrm{Cov}(\mathcal{T}, z) = \{(u_\alpha : (\mathcal{T}_\alpha, z_\alpha) \to (\mathcal{T}, z))_{\alpha \in A}\}$ such that
  (i) u_α is étale and Cartesian for all $\alpha \in A$
  (ii) $\bigcup_{\alpha \in A} u_{\alpha,1}(\mathcal{T}_{\alpha,1}) = \mathcal{T}_1$.
- Functor $(\mathcal{T}, z) \mapsto \Gamma(\mathcal{T}_1, \mathscr{O}_{\mathcal{T}_1})$ defines a sheaf of rings $\overline{\mathscr{O}}_{\mathfrak{X}/A_{\mathrm{inf}}}$.
- Consider $\overline{\mathscr{O}}_{\mathfrak{X}/A_{\mathrm{inf}},n} = \overline{\mathscr{O}}_{\mathfrak{X}/A_{\mathrm{inf}}}/p^n\overline{\mathscr{O}}_{\mathfrak{X}/A_{\mathrm{inf}}}$, $\overline{\mathscr{O}}_{\mathfrak{X}/A_{\mathrm{inf}}}[\frac{1}{p}]$.

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- $\mathrm{HC}_{\mathbb{Q}}(\mathfrak{X}/A_{\mathrm{inf}})$: Crystals of $\overline{\mathscr{O}}_{\mathfrak{X}/A_{\mathrm{inf}},\mathbb{Q}}[\frac{1}{p}]$-modules on $(\mathfrak{X}/A_{\mathrm{inf}})$, defined as in crystalline site.
- A smooth Cartesian (formal) lifting $\mathcal{X} = (\mathcal{X}_N)$ of $\mathfrak{X}$ over A_{inf} defines an object of $(\mathfrak{X}/A_{\mathrm{inf}})$. The evaluation at $\mathcal{X}$ induces:

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- $\mathrm{HB}_{\mathrm{small}}(X_{\mathbf{C}})$ is contained in the image of $\iota_{\mathcal{X}}$.
- $\mathrm{HC}_{\mathbb{Q},\mathrm{fin}}(\mathfrak{X}/A_{\mathrm{inf}})$ its essential image.

From Higgs crystals to generalized representations

- Suppose $X = \mathrm{Spec}(R) \to \mathbb{G}_m^d$ étale. Let $\{V_i \to X_{\overline{K}}\}$ be the universal cover, R_i the integral closure of R in V_i and $\overline{R} = \varinjlim_{i \in I} R_i$.

  Set $\Delta = \pi_1(X_{\overline{K}})$, which acts continuously on $\widehat{\overline{R}}$.

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- Crystals on prismatic site: Morrow–Tsuji, Min–Wang.

Pullback functoriality: review of $f^\circ : \mathrm{HB}(X_{\mathbf{C}}) \to \mathrm{HB}(Y_{\mathbf{C}})$

- $f : Y \to X$ a map of smooth projective curve and f_K is finite étale.

  $$f_{\mathrm{HIG}} : (\widetilde{\mathfrak{Y}/A_{\mathrm{inf}}}) \to (\widetilde{\mathfrak{X}/A_{\mathrm{inf}}}), \quad f^*_{\mathrm{HIG}} : \mathrm{HC}_{\mathbb{Q},\mathrm{fin}}(\mathfrak{X}/A_{\mathrm{inf}}) \to \mathrm{HC}_{\mathbb{Q},\mathrm{fin}}(\mathfrak{Y}/A_{\mathrm{inf}})$$

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- $(M, \theta) \in \mathrm{HB}(X_{\mathbf{C}})$ of rank r, $\rightsquigarrow$ a point in the Hitchin base $c \in \oplus_{i=1}^{r} \Gamma(X_{\mathbf{C}}, (\Omega^1_{X_{\mathbf{C}}})^{\otimes i})$ and a spectral cover:

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  π_* induces an equivalence between line bundles on Z_θ and Higgs bundle whose Hitchin image is c.
- The obstruction of lifting $f_{\mathscr{O}_{\mathbf{C}}}$ to $\mathcal{Y}_2 \dashrightarrow \mathcal{X}_2/A_{\mathrm{inf},2}$ defines a class $o_f \in \mathrm{H}^1(\mathfrak{Y}, f^*(T_{\mathfrak{Y}}))$. Consider line bundle $\mathcal{L}^{\mathrm{Exp}}_{f,\theta}$ as the image:

$$\mathrm{H}^1(\mathfrak{Y}, f^*(T_{\mathfrak{Y}})) \longrightarrow \mathrm{H}^1(Z_{f^*(\theta)}, \mathscr{O}_{Z_{f^*(\theta)}}) \xrightarrow{\mathrm{Exp}_{Z_{f^*(\theta)}}} \mathrm{H}^1(Z_{f^*(\theta)}, \mathscr{O}^\times).$$

- $f^\circ(M, \theta) := f^*(M, \theta) \otimes_{\mathscr{O}_{Z_{f^*(\theta)}}} \mathcal{L}^{\mathrm{Exp}}_{f,\theta}$, viewed as line bundles $/Z_{f^*(\theta)}$.

Trivializable property of $\mathrm{HB}^{\mathrm{DW}}$

We extend the trivilizable property of $\mathrm{VB}^{\mathrm{DW}}$ to $\mathrm{HB}^{\mathrm{DW}}$.

Theorem

Let M be a vector bundle of rank r over $X_{\mathscr{O}_{\mathbf{C}}}$, θ a small Higgs field on M, and $\mathcal{M}$ the associated Higgs crystal. TFAE:

(i) (M, θ) *belongs to* $\mathrm{HB}^{\mathrm{DW}}(X_{\mathscr{O}_{\mathbf{C}}})$.

(ii) *For every integer $n \geq 1$, after taking a finite extension of K, there exists an $\mathscr{O}_K$-curve Y with semi-stable reduction and a proper map $f : Y \to X$ such that:*

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- Idea: we can find such covers $f : Y \to X$ trivializing bundles or Higgs fields modulo some power of p.

Construction of $\mathbb{V} : \mathrm{HB}^{\mathrm{pDW}}(X_{\mathbf{C}}) \to \mathbf{Rep}_{\mathbf{C}}(\pi_1(X_{\overline{K}}))$

- $(M, \theta) \in \mathrm{HB}^{\mathrm{DW}}(X_{\mathscr{O}_{\mathbf{C}}})$ of rank r
- $\mathcal{M}$ associated Higgs crystal and n an integer ≥ 1.
- $f : Y \to X$ a proper map of curves trivializing $\mathcal{M}_n$ as above.

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- $f : Y \to X$ a proper map of curves trivializing $\mathcal{M}_n$ as above.
- Generalized representation $\mathbf{V}_X(\mathcal{M}_n)$ also satisfies a similar trivializable property in Faltings ringed topos $(\widetilde{E}_Y, \overline{\mathscr{B}}_Y)$ of Y:

$$\Phi^*(\mathbf{V}_X(\mathcal{M}_n)) \simeq \mathbf{V}_Y(f^*_{\mathrm{HIG}}(\mathcal{M}_n)) \simeq \overline{\mathscr{B}}_{Y,n}^{\oplus r},$$

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- Faltings' comparison theorem $\rightsquigarrow$ an almost isomorphism

$$\mathrm{Gal}(Y_{\overline{K}}/X_{\overline{K}}) \circlearrowright \Gamma(\widetilde{E}_Y, \Phi^*(\mathbf{V}_X(\mathcal{M}_n))) \simeq (\mathscr{O}_{\mathbf{C}}/p^n\mathscr{O}_{\mathbf{C}})^{\oplus r}.$$

- $\mathbb{V}_n : \mathrm{HB}^{\mathrm{DW}}(X_{\mathscr{O}_{\mathbf{C}}}) \to \mathbf{Rep}_{\mathscr{O}_{\mathbf{C}}/p^n\mathscr{O}_{\mathbf{C}}}(\pi_1(X_{\overline{K}}))$.

Construction of $\mathbb{V} : \mathrm{HB}^{\mathrm{pDW}}(X_{\mathbf{C}}) \to \mathbf{Rep}_{\mathbf{C}}(\pi_1(X_{\overline{K}}))$

- $(M, \theta) \in \mathrm{HB}^{\mathrm{DW}}(X_{\mathscr{O}_{\mathbf{C}}})$ of rank r
- $\mathcal{M}$ associated Higgs crystal and n an integer ≥ 1.
- $f : Y \to X$ a proper map of curves trivializing $\mathcal{M}_n$ as above.
- Generalized representation $\mathbf{V}_X(\mathcal{M}_n)$ also satisfies a similar trivializable property in Faltings ringed topos $(\widetilde{E}_Y, \overline{\mathscr{B}}_Y)$ of Y:

$$\Phi^*(\mathbf{V}_X(\mathcal{M}_n)) \simeq \mathbf{V}_Y(f_{\mathrm{HIG}}^*(\mathcal{M}_n)) \simeq \overline{\mathscr{B}}_{Y,n}^{\oplus r},$$

  where $\Phi : (\widetilde{E}_Y, \overline{\mathscr{B}}_Y) \to (\widetilde{E}_X, \overline{\mathscr{B}}_X)$ induced by f.
- Faltings' comparison theorem $\rightsquigarrow$ an almost isomorphism

$$\mathrm{Gal}(Y_{\overline{K}}/X_{\overline{K}}) \circlearrowright \Gamma(\widetilde{E}_Y, \Phi^*(\mathbf{V}_X(\mathcal{M}_n))) \simeq (\mathscr{O}_{\mathbf{C}}/p^n\mathscr{O}_{\mathbf{C}})^{\oplus r}.$$

- $\mathbb{V}_n : \mathrm{HB}^{\mathrm{DW}}(X_{\mathscr{O}_{\mathbf{C}}}) \to \mathbf{Rep}_{\mathscr{O}_{\mathbf{C}}/p^n\mathscr{O}_{\mathbf{C}}}(\pi_1(X_{\overline{K}}))$.
- Taking projective limit and descent, we obtain

$$\mathbb{V} : \mathrm{HB}^{\mathrm{pDW}}(X_{\mathbf{C}}) \to \mathbf{Rep}_{\mathbf{C}}(\pi_1(X_{\overline{K}})).$$

$\mathbb{V}$ and p-adic Simpson correspondence

- Show $\mathbb{V}$ is a quasi-inverse of $\mathbb{H}$, i.e. $\mathbb{V}$ is compatible with p-adic Simpson correspondence.

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$$\beta^*(\mathbb{V}(M,\theta)) \xrightarrow{\sim} \mathbf{V}_X(M,\theta).$$

- With above notations, such an isomorphism exists after pullback along $f : Y \to X$ and is equipped with descent data.

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- With above notations, such an isomorphism exists after pullback along $f : Y \to X$ and is equipped with descent data.
- We conclude the assertion by the cohomological descent in Faltings topos (T. He).

Thank You!