On modular representations of $\text{GL}_2(L)$ for unramified $L$

C. Breuil, F. Herzig, Y. Hu, S. Morra and B. Schraen

I.H.É.S.
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3. Some ideas on the proof
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  and at exactly one infinite place
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  totally odd modular representation.

General aim:

Understand better certain smooth admissible representations of
$GL_2(F_v)$ over $\mathbb{F}$ associated to $\bar{\rho}$ ($F_v :=$completion of $F$ at $v$).
Local factor at $v$ associated to $\overline{r}$
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For $K \subset (D \otimes_F \mathbb{A}_F^\infty)^\times$ a compact open subgroup, let $X_K/F :=$ associated Shimura curve $= \text{smooth projective algebraic variety}/F$. 
Local factor at $v$ associated to $\bar{r}$

For $K \subset (D \otimes_F \mathbb{A}_F^\infty)^\times$ a compact open subgroup, let $X_K/F :=$ associated Shimura curve $=$ smooth projective algebraic variety $/F$.

We first consider the smooth representation of $(D \otimes_F \mathbb{A}_F^\infty)^\times$ over $F$:

$$\pi(\bar{r}) := \text{Hom}_{\text{Gal}(\bar{F}/F)} \left( \bar{r}, \lim_{\rightarrow K} H^1_{\text{ét}}(X_K \times_F \bar{F}, \mathbb{F}) \right) \neq 0.$$
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One doesn't know if $\pi(\overline{r})$ has a Flath decomposition as a restricted tensor product of smooth $D_w^\times$-representations over finite places $w$ of $F$ ($D_w := D \otimes_F F_w$).
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One doesn’t know if $\pi(\overline{r})$ has a Flath decomposition as a restricted tensor product of smooth $D_w^\times$-representations over finite places $w$ of $F$ ($D_w := D \otimes_F F_w$).

But one can still define from $\pi(\overline{r})$ in an “ad hoc” way a local factor $\pi_{\nu}(\overline{r})$ at $\nu$ under technical assumptions on $\overline{r}$. 

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On modular representations of $\text{GL}_2(L)$ for unramified $L$
From now on assume:

- $p > 5$ and $\bar{r}|_{\text{Gal}(\overline{F}/F(\sqrt{\psi}))}$ still absolutely irreducible
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Then one can define an “optimal” open compact subgroup $K_\nu$ of $((D \otimes_F \mathbb{A}_F^\infty, \nu) \times$, a certain smooth finite dim. representation $M_\nu$ of $K_\nu$ over $\mathbb{F}$ (a “type”), and set (B.-Diamond, Emerton-Gee-Savitt):

$$\pi_\nu(\overline{r}) := \text{Hom}_{K_\nu}(M_\nu, \pi(\overline{r}))[m] \neq 0$$

where $[m] := \text{kernel of Hecke operators at certain places} \neq \nu$. 

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Then one can define an “optimal” open compact subgroup $K^v$ of $(D \otimes_F \mathbb{A}_F^\infty)^\times$, a certain smooth finite dim. representation $M^v$ of $K^v$ over $\mathbb{F}$ (a “type”), and set (B.-Diamond, Emerton-Gee-Savitt):

$$\pi_v(\bar{r}) := \text{Hom}_{K^v}(M^v, \pi(\bar{r}))[m] \neq 0$$

where $[m] :=$ kernel of Hecke operators at certain places $\neq v$. 

$$\pi_v(\bar{r}) = \text{smooth admissible representation of } D^\times_v \cong \text{GL}_2(F_v) \text{ over } \mathbb{F} \text{ with central character } \psi := \omega \text{det}(\bar{r}_v) \ (\omega := \text{cyclo mod } p).$$
Some known results
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Some known results

**Theorem 1 (Emerton, building on Colmez, B., Kisin, Berger,...)**

Assume $F = \mathbb{Q}$ and $D = \text{GL}_2$, then $\pi_v(\bar{r})$ is known. In particular:

- $\text{GK}(\pi_v(\bar{r})) = 1$
- $\pi_v(\bar{r})$ is of finite length
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For $n \geq 1$ let $K_v(n) := 1 + p^n M_2(\mathcal{O}_{F_v}) \subset K_v := \mathcal{O}_{D_v}^\times \cong \text{GL}_2(\mathcal{O}_{F_v})$. 
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**Definition 1 (Gelfand-Kirillov dimension)**

Let $\pi_v$ be a smooth admissible representation of $K_v(1)$ over $\mathbb{F}$.

There exists a unique $\text{GK}(\pi_v) \in \{0, \ldots, \dim_{\mathbb{Z}_p}(K_v)\}$ such that there are $a \leq b$ in $\mathbb{R}_{>0}$ with $a \leq \frac{\dim_{\mathbb{F}}(\pi_v^{K_v(n)})}{p^n \text{GK}(\pi_v)} \leq b$ for all $n \geq 1$. 
Let:

- $f := [F_v : \mathbb{Q}_p]$, $q := p^f$, $K := K_v$, $K(1) := K_v(1)$
- $\Gamma := K/K(1) \cong \text{GL}_2(\mathbb{F}_q)$, $Z(1) := \text{center of } K(1)$
- $m_K := \text{maximal ideal of Iwasawa algebra } \mathbb{F}[[K(1)/Z(1)]]$. 

Note that $Z(1)$ acts trivially on $\pi_v(r)$ as $\psi|_{Z(1)} = 1$. For arbitrary $F$, $D$ and $r$ (as before), one has the following:

**Theorem 2 (Emerton-Gee-Savitt, Le, Hu-Wang, Le-Morra-Schraen, building on B.-Paškūnas and Buzzard-Diamond-Jarvis)**

The finite-dimensional $\Gamma$-representation $\pi_v(r)_{K(1)} = \pi_v(r)[m_K]$ is explicitly known, in particular is local and multiplicity free. If $D_v \neq \text{GL}_2(\mathbb{Q}_p)$ none of the statements in Theorem 1 are known.
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Note that \( Z(1) \) acts trivially on \( \pi_v(\overline{\rho}) \) as \( \psi|_{Z(1)} = 1 \).

For arbitrary \( F, D \) and \( \overline{\rho} \) (as before), one has the following:

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Fix an embedding $\mathbb{F}_{q^2} \hookrightarrow \mathbb{F}$ and let $\omega_f, \omega_{2f} :=$ associated Serre’s fundamental characters of level $f, 2f$ of inertia subgroup $I_v$. 

\[ \rho \text{ reducible: } \rho \mid_{I_v} \sim = (\omega (r_0 + 1) + \cdots + p_f - 1 (r_f - 1 + 1) ) \otimes \omega^* \]

for some $r_i$ with $8 \leq r_i \leq p_f - 11$ ($\Rightarrow p \geq 19$).

\[ \rho \text{ irreducible: } \rho \mid_{I_v} \sim = (\omega (r_0 + 1) + \cdots + p_f - 1 (r_f - 1 + 1) ) \otimes \omega^* \]

for $9 \leq r_0 \leq p_f - 10$ and $8 \leq r_i \leq p_f - 11$ if $i > 0$.

This strong genericity assumption on $\rho$ is not optimized!
Hypothesis on $\overline{r}_v$

Fix an embedding $\mathbb{F}_{q^2} \hookrightarrow \mathbb{F}$ and let $\omega_f, \omega_{2f} := \text{associated Serre's fundamental characters of level } f, 2f$ of inertia subgroup $I_v$.

We set $\overline{\rho} := \overline{r}_v$ and assume $\overline{\rho}$ is semi-simple such that:
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This strong genericity assumption on $\overline{\rho}$ is not optimized!
Main result

Theorem 3

With the previous assumptions on $F$, $D$, $r$ and $\rho$, we have:

$$GK(\pi v(r)) = f.$$ 

Remarks

The assumptions on $\rho$ should (conjecturally) be unnecessary, i.e. one should have $GK(\pi v(r)) = f$ for $F$, $D$, $r$ as before.

Gee-Newton proved (without the assumptions on $\rho$) that $GK(\pi v(r)) \geq f$, so our main result is $GK(\pi v(r)) \leq f$.

Even under the assumptions on $\rho$, we do not know if $\pi v(r)$ is of finite length or if $\pi v(r)$ is local.

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Quick review of past results

Statement of the main theorem

Some ideas on the proof
First intermediate theorem

We first prove the following extension of Theorem 2 (much harder):

**Theorem 4**

The smooth finite-dimensional $K$-representation $\pi_v(r)[m_2K]$ is explicitly known, in particular is local and multiplicity free.

Let:

$I := \{g \in K, g \equiv (\ast \ast 0 \ast) \mod p\} = \text{Iwahori}$

$I(1) := \{g \in K, g \equiv (1 \ast 0 1) \mod p\} = \text{pro-Iwahori}$

$m_I := \text{maximal ideal of Iwasawa algebra } F[[I(1)/Z(1)]]$.

**Corollary 1**

The smooth finite-dimensional $I$-representation $\pi_v(r)[m_3I]$ is multiplicity free.
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**Corollary 1**

The smooth finite-dimensional $I$-representation $\pi_v(\overline{r})[m_I^3]$ is multiplicity free.
Second intermediate theorem

Let $\pi_v$ be a smooth admissible representation of $I/Z(1)$ over $F$ such that $\pi_v |_{I^3}$ is multiplicity free. Then $GK(\pi_v) \leq f$.

It then directly follows from Corollary 1 and Theorem 5:

**Corollary 2**

We have $GK(\pi_v(r)) \leq f$.

Using Gee-Newton for the reverse inequality, one gets Theorem 4.
Theorem 5

Let $\pi_v$ be a smooth admissible representation of $I/Z(1)$ over $\mathbb{F}$ such that $\pi_v[m_I^3]$ is multiplicity free. Then $GK(\pi_v) \leq f$. 

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Proof of second intermediate theorem

Let \( \pi \lor v : = \text{Hom}_F(\pi v, F) \), then \( \pi \lor v / mI = (\pi I(1)v) \lor = \oplus \alpha \chi_\alpha \) for some characters \( \chi_\alpha : I/I(1) \to F \times \).

Let \( \text{Proj}_I\chi_\alpha : = \chi_\alpha \otimes F F[I/I(1)] = \text{projective envelope of } \chi_\alpha \text{ in the category of compact } F[I/I(1)]\text{-modules}. \)

As \( \chi_\alpha \) does not appear in \( mI \pi \lor v \) (by assumption), one proves there exist \( I\text{-equivariant maps } h_\alpha : (\text{Proj}_I\chi_\alpha) \oplus 2f \to \text{Proj}_I\chi_\alpha \) s.t.:

\[ \text{image}(h_\alpha) \subseteq m^2I \text{Proj}_I\chi_\alpha. \]

The map \( (\text{Proj}_I\chi_\alpha / mI) \oplus 2f \to m^2I \text{Proj}_I\chi_\alpha / m^3I \) is injective. \( \pi \lor v \) is a quotient of \( \oplus \alpha \text{coker}(h_\alpha) \).

Thm. 5 then follows from \( \text{GK}(\pi v) \leq \max \alpha \text{GK}(\text{coker}(h_\alpha) \lor) \).

Proposition 1 We have \( \text{GK}(\text{coker}(h_\alpha) \lor) \leq f(\text{calculation in } \text{gr}_I F F[I/I(1)]). \)
Proof of second intermediate theorem

Let $\pi_v^\vee := \text{Hom}_F(\pi_v, F)$, then $\pi_v^\vee / \mathfrak{m}_I = (\pi_v^{l(1)})^\vee = \bigoplus \alpha \chi_\alpha$ for some characters $\chi_\alpha : \mathcal{I}/\mathcal{I}^{l(1)} \to F^\times$.
Proof of second intermediate theorem

Let $\pi_v^\vee := \text{Hom}_F(\pi_v, F)$, then $\pi_v^\vee / m_I = (\pi_v^\vee)^{(1)} = \bigoplus \alpha \chi_\alpha$ for some characters $\chi_\alpha : I/I(1) \to \mathbb{F}^\times$.

Let $\text{Proj}_I \chi_\alpha := \chi_\alpha \otimes_F F[[I(1)/Z(1)]] = \text{projective envelope of } \chi_\alpha$ in the category of compact $F[[I/Z(1)]]$-modules.
Proof of second intermediate theorem

Let $\pi^\vee := \text{Hom}_F(\pi, F)$, then $\pi^\vee / m_l = (\pi^l(1))^\vee = \bigoplus \chi_\alpha$ for some characters $\chi_\alpha : l/l(1) \to F^\times$.

Let $\text{Proj}_l \chi_\alpha := \chi_\alpha \otimes_F F[[l(1)/Z(1)]] = \text{projective envelope of } \chi_\alpha$ in the category of compact $F[[l/Z(1)]]$-modules.

As $\chi_\alpha$ does not appear in $m_l \pi^\vee / m_l^3 \pi^\vee$ (by assumption), one proves there exist $l$-equivariant maps $h_\alpha : (\text{Proj}_l \chi_\alpha)^{\oplus 2f} \to \text{Proj}_l \chi_\alpha$ s.t.:
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As $\chi_\alpha$ does not appear in $m_I\pi_v^\vee / m_I^3\pi_v^\vee$ (by assumption), one proves there exist $I$-equivariant maps $h_\alpha : (\text{Proj}_I\chi_\alpha)^{\oplus 2^f} \to \text{Proj}_I\chi_\alpha$ s.t.:

- $\text{image}(h_\alpha) \subseteq m_I^2\text{Proj}_I\chi_\alpha$
Proof of second intermediate theorem

Let $\pi_v^\vee := \text{Hom}_F(\pi_v, F)$, then $\pi_v^\vee / m_I = (\pi_v^l(1))^\vee = \bigoplus \chi_\alpha$ for some characters $\chi_\alpha : I/I(1) \to F^\times$.

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As $\chi_\alpha$ does not appear in $m_I \pi_v^\vee / m_I^3 \pi_v^\vee$ (by assumption), one proves there exist $l$-equivariant maps $h_\alpha : (\text{Proj}_I \chi_\alpha)^{\oplus 2} \to \text{Proj}_I \chi_\alpha$ s.t.:

- $\text{image}(h_\alpha) \subseteq m_I^2 \text{Proj}_I \chi_\alpha$
- The map $(\text{Proj}_I \chi_\alpha / m_I)^{\oplus 2} \to m_I^2 \text{Proj}_I \chi_\alpha / m_I^3$ is injective
Proof of second intermediate theorem

Let $\pi_v^\vee := \text{Hom}_F(\pi_v, F)$, then $\pi_v^\vee / m_l = (\pi_v^{l(1)})^\vee = \bigoplus \chi_\alpha$ for some characters $\chi_\alpha : I/I(1) \to F^\times$.

Let $\text{Proj}_l \chi_\alpha := \chi_\alpha \otimes_F F[[I(1)/Z(1)]] = $ projective envelope of $\chi_\alpha$ in the category of compact $F[[I/Z(1)]]$-modules.

As $\chi_\alpha$ does not appear in $m_l \pi_v^\vee / m_l^3 \pi_v^\vee$ (by assumption), one proves there exist $l$-equivariant maps $h_\alpha : (\text{Proj}_l \chi_\alpha)^{\oplus 2f} \to \text{Proj}_l \chi_\alpha$ s.t.:

- $\text{image}(h_\alpha) \subseteq m_l^2 \text{Proj}_l \chi_\alpha$
- the map $(\text{Proj}_l \chi_\alpha / m_l)^{\oplus 2f} \to m_l^2 \text{Proj}_l \chi_\alpha / m_l^3$ is injective
- $\pi_v^\vee$ is a quotient of $\bigoplus \alpha \text{coker}(h_\alpha)$.
Proof of second intermediate theorem

Let $\pi^\vee_v := \text{Hom}_F(\pi_v, F)$, then $\pi^\vee_v / m_l = (\pi^l_v)^\vee = \bigoplus \alpha \chi_\alpha$ for some characters $\chi_\alpha : l/l(1) \to F^\times$.

Let $\text{Proj}_l \chi_\alpha := \chi_\alpha \otimes_F F[[l(1)/Z(1)]] = \text{projective envelope of } \chi_\alpha$ in the category of compact $F[[l/Z(1)]]$-modules.

As $\chi_\alpha$ does not appear in $m_l \pi^\vee_v / m_l^3 \pi^\vee_v$ (by assumption), one proves there exist $l$-equivariant maps $h_\alpha : (\text{Proj}_l \chi_\alpha)^{\oplus 2f} \to \text{Proj}_l \chi_\alpha$ s.t.:

- $\text{image}(h_\alpha) \subseteq m_l^2 \text{Proj}_l \chi_\alpha$
- the map $(\text{Proj}_l \chi_\alpha / m_l)^{\oplus 2f} \to m_l^2 \text{Proj}_l \chi_\alpha / m_l^3$ is injective
- $\pi^\vee_v$ is a quotient of $\bigoplus \alpha \text{coker}(h_\alpha)$.

Thm. 5 then follows from $\text{GK}(\pi_v) \leq \max_\alpha \text{GK}(\text{coker}(h_\alpha)^\vee)$ and:

\[ \text{Proposition 1} \]
We have $\text{GK}(\text{coker}(h_\alpha)^\vee) \leq f(\text{calculation in } \text{gr} F[[l/Z(1)]])$. 

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On modular representations of $\text{GL}_2(L)$ for unramified $L$. 

\[ \text{C. Breuil, F. Herzig, Y. Hu, S. Morra and B. Schraen} \]

\[ \text{On modular representations of } \text{GL}_2(L) \text{ for unramified } L \]
Proof of second intermediate theorem

Let \( \pi_v^\vee := \text{Hom}_F(\pi_v, F) \), then \( \pi_v^\vee / \mathfrak{m}_I = (\pi_v^{I(1)})^\vee = \bigoplus \chi_\alpha \) for some characters \( \chi_\alpha : I/I(1) \to F^\times \).

Let \( \text{Proj}_I \chi_\alpha := \chi_\alpha \otimes_F F[[I(1)/Z(1)]] \) = projective envelope of \( \chi_\alpha \) in the category of compact \( F[[I/Z(1)]] \)-modules.

As \( \chi_\alpha \) does not appear in \( \mathfrak{m}_I \pi_v^\vee / \mathfrak{m}_I^3 \pi_v^\vee \) (by assumption), one proves there exist \( I \)-equivariant maps \( h_\alpha : (\text{Proj}_I \chi_\alpha)^{\oplus 2f} \to \text{Proj}_I \chi_\alpha \) s.t.:

- \( \text{image}(h_\alpha) \subseteq \mathfrak{m}_I^2 \text{Proj}_I \chi_\alpha \)
- the map \( (\text{Proj}_I \chi_\alpha / \mathfrak{m}_I)^{\oplus 2f} \to \mathfrak{m}_I^2 \text{Proj}_I \chi_\alpha / \mathfrak{m}_I^3 \) is injective
- \( \pi_v^\vee \) is a quotient of \( \bigoplus \alpha \text{coker}(h_\alpha) \).

Thm. 5 then follows from \( \text{GK}(\pi_v) \leq \max \alpha \text{GK}(\text{coker}(h_\alpha)^{\vee}) \) and:

**Proposition 1**

We have \( \text{GK}(\text{coker}(h_\alpha)^{\vee}) \leq f \) (calculation in \( \text{gr}_{\mathfrak{m}_I} F[[I(1)/Z(1)]] \)).
Proof of first intermediate theorem
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Let:

- $\sigma$ a Serre weight (\(= \) irreducible representation of $\Gamma$ over $\mathbb{F}$)
Proof of first intermediate theorem

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Proof of first intermediate theorem

Let:

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Enough to prove: $\dim_{\mathbb{F}} \text{Hom}_K(\text{Proj}_K \sigma / m_K^2, \pi_v(\bar{r})) \leq 1$. 
Proof of first intermediate theorem

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Can assume $\text{Hom}_K(\sigma, \pi_v(\bar{r})) \neq 0$ (i.e. $\sigma =$ Serre weight of $\bar{\rho}$).
Proof of first intermediate theorem

Let:

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Enough to prove: \( \dim_{\mathbb{F}} \text{Hom}_K(\text{Proj}_K \sigma / \mathfrak{m}_K^2, \pi_v(\overline{r})) \leq 1 \).

Can assume \( \text{Hom}_K(\sigma, \pi_v(\overline{r})) \neq 0 \) (i.e. \( \sigma = \) Serre weight of \( \overline{\rho} \)).

Main tool: patching functor \( M_\infty \) of Emerton-Gee-Savitt (building on Taylor-Wiles, Kisin) = exact functor from continuous repres. of \( K \) over finite type \( W(\mathbb{F}) \)-modules + central character lifting \( \psi \) to finite type \( R_\infty \)-modules satisfying several properties (cf. E.-G.-S.).

\( R_\infty = \) patched deformation ring = power series ring over \( W(\mathbb{F}) \).
Proof of first intermediate theorem

Let:

\[ m_\infty := \text{maximal ideal of } R_\infty \]
Proof of first intermediate theorem

Let:

- $m_\infty :=$ maximal ideal of $R_\infty$
- $V :=$ any finite dimensional representation of $K$ over $\mathbb{F}$
Proof of first intermediate theorem

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from the construction of $M_\infty$ one gets:

$$\text{Hom}_F(M_\infty(V)/m_\infty, \mathbb{F}) \cong \text{Hom}_K(V, \pi_V(\bar{r})).$$
Proof of first intermediate theorem

Let:

- \( m_\infty := \text{maximal ideal of } R_\infty \)
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from the construction of \( M_\infty \) one gets:

\[
\text{Hom}_F(M_\infty(V)/m_\infty, F) \cong \text{Hom}_K(V, \pi_V(\bar{r})).
\]

Hence Theorem 4 (multiplicity free part) follows from:

**Theorem 6**

The \( R_\infty \)-module \( M_\infty(\text{Proj}_K\sigma/m_K^2) \) is cyclic.

Equivalently \( M_\infty(\text{Proj}_K\sigma/m_K^2) \cong \text{quotient of } R_\infty. \)
Proof of first intermediate theorem

We now prove Theorem 6. First: need to describe $\text{Proj}_K \sigma / \mathfrak{m}_K^2$. 
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- $\text{Proj}_\Gamma \sigma = \text{Proj}_K \sigma / m_K :=$ projective envelope of $\sigma$ in category of $\Gamma$-representations over $\mathbb{F}$
- $V_2^\tau := (\text{Sym}^2(F^2) \otimes_F \text{det}^{-1})^\tau =$ algebraic representation of $\Gamma$ via $\tau : F_q \hookrightarrow F$ (arbitrary embedding),
Proof of first intermediate theorem

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  - via $\tau : \mathbb{F}_q \hookrightarrow \mathbb{F}$ (arbitrary embedding),

then $\text{Proj}_K \sigma / m_K^2$ is a non-split extension:

$$\text{Proj}_K \sigma / m_K^2 \cong (\bigoplus_\tau (V_2^\tau \otimes_{\mathbb{F}} \text{Proj}_\Gamma \sigma)) \rightarrow \text{Proj}_\Gamma \sigma.$$

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then $\text{Proj}_K \sigma / \mathfrak{m}_K^2$ is a non-split extension:

$$\text{Proj}_K \sigma / \mathfrak{m}_K^2 \cong (\bigoplus_{\tau} (V_2^\tau \otimes_{\mathbb{F}} \text{Proj}_\Gamma \sigma)) \cong \text{Proj}_\Gamma \sigma .$$

Moreover $V_2^\tau \otimes_{\mathbb{F}} \text{Proj}_\Gamma \sigma \cong \text{Proj}_\Gamma \sigma + 2\tau \oplus \text{Proj}_\Gamma \sigma \oplus \text{Proj}_\Gamma \sigma - 2\tau$. 
We now prove Theorem 6. First: need to describe $\text{Proj}_K \sigma / m_K^2$.

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then $\text{Proj}_K \sigma / m_K^2$ is a non-split extension:

$$\text{Proj}_K \sigma / m_K^2 \cong \left( \oplus_\tau (V_2^\tau \otimes_{\mathbb{F}} \text{Proj}_\Gamma \sigma) \right) \hookrightarrow \text{Proj}_\Gamma \sigma.$$

Moreover $V_2^\tau \otimes_{\mathbb{F}} \text{Proj}_\Gamma \sigma \cong \text{Proj}_\Gamma \sigma_{+2\tau} \oplus \text{Proj}_\Gamma \sigma \oplus \text{Proj}_\Gamma \sigma_{-2\tau}$.

Let $Q_\tau :=$ unique quotient of $\text{Proj}_K \sigma / m_K^2$ which is a non-split extension $\left( \text{Proj}_\Gamma \sigma_{+2\tau} \oplus \text{Proj}_\Gamma \sigma_{-2\tau} \right) \hookrightarrow \text{Proj}_\Gamma \sigma$. 

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Proof of first intermediate theorem

To proceed, we lift the $K$-representation $\text{Proj}_K \sigma/m_K^2$ to $W(\mathbb{F})$. 
Proof of first intermediate theorem

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Let:

- $\widetilde{\text{Proj}}_{\Gamma} \sigma := \text{unique representation of } \Gamma \text{ lifting } \text{Proj}_{\Gamma} \sigma \text{ over } W(F)$
Proof of first intermediate theorem

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On modular representations of $\text{GL}_2(L)$ for unramified $L$
Proof of first intermediate theorem

To proceed, we lift the $K$-representation $\text{Proj}_K\sigma/m_K^2$ to $W(\mathbb{F})$.

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- $\widetilde{V}_2^\tau := (\text{Sym}^2(W(\mathbb{F})^2) \otimes W(\mathbb{F}) \det^{-1})^\tau$.

One can prove:

**Proposition 3**

(i) There is an invariant $W(\mathbb{F})$-lattice $L_2^\tau$ in $(\widetilde{V}_2^\tau \otimes W(\mathbb{F}) \widetilde{\text{Proj}}_\Gamma\sigma)[\frac{1}{p}]$ such that $L_2^\tau/p \cong Q_\tau$. 
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**Proposition 3**

(i) There is an invariant $W(\mathbb{F})$-lattice $L_2^\tau$ in $(\tilde{V}_2^\tau \otimes_{W(\mathbb{F})} \tilde{\text{Proj}}_{\Gamma} \sigma)[\frac{1}{p}]$ such that $L_2^\tau / p \cong Q_\tau$.

(ii) Let $L := \ker \left( \tilde{\text{Proj}}_{\Gamma} \sigma \oplus (\oplus_\tau L_2^\tau) \rightarrow (\text{Proj}_{\Gamma} \sigma)^f \right)$, then $L / p \cong \text{Proj}_K \sigma / m_K^2$. 
Proof of first intermediate theorem

To proceed, we lift the $K$-representation $\text{Proj}_K\sigma/m^2_K$ to $W(\mathbb{F})$.

Let:
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**Proposition 3**

(i) There is an invariant $W(\mathbb{F})$-lattice $L_2^\tau$ in $(\widetilde{V}_2^\tau \otimes W(\mathbb{F}) \widetilde{\text{Proj}}_\Gamma \sigma)[\frac{1}{p}]$ such that $L_2^\tau/p \cong Q_\tau$.

(ii) Let $L := \ker \left( \widetilde{\text{Proj}}_\Gamma \sigma \oplus (\bigoplus_\tau L_2^\tau) \longrightarrow (\text{Proj}_\Gamma \sigma) \oplus f \right)$, then $L/p \cong \text{Proj}_K \sigma/m^2_K$.

It is enough to prove that $M_\infty(L)$ is cyclic.
Proof of first intermediate theorem

We know $M_\infty(\widetilde{\text{Proj}}_\Gamma \sigma)$ is cyclic (Hu-Wang, Le-Morra-Schraen).
We know $M_{\infty}(\widetilde{\text{Proj}}_{\Gamma}\sigma)$ is cyclic (Hu-Wang, Le-Morra-Schraen).

**Proposition 2**

The $R_{\infty}$-module $M_{\infty}(L_2^T/p)$, and hence $M_{\infty}(L_2^T)$, are cyclic.
Quick review of past results

Statement of the main theorem

Some ideas on the proof

Proof of first intermediate theorem

We know $M_{\infty} (\widetilde{\text{Proj}}_{\Gamma} \sigma)$ is cyclic (Hu-Wang, Le-Morra-Schraen).

**Proposition 2**

The $R_{\infty}$-module $M_{\infty} (L_{2}^T/p)$, and hence $M_{\infty} (L_{2}^T)$, are cyclic.

The proof is by dévissage, using:

- $M_{\infty} (\sigma') \neq 0 \iff \sigma' \hookrightarrow \pi_v (\overline{r})[m_K] \iff \sigma'$ Serre weight of $\overline{\rho}$
- $M_{\infty} (\text{Proj}_{\Gamma} \sigma')$ cyclic (Hu-Wang, Le-Morra-Schraen)
- $M'' \subsetneq M' \subseteq M$ finite type $R_{\infty}$-modules with $M'$ cyclic, then $M$ cyclic $\iff M / M''$ cyclic (E.-G.-S.).
Proof of first intermediate theorem

We know $M_{\infty}(\widetilde{\text{Proj}} \Gamma \sigma)$ is cyclic (Hu-Wang, Le-Morra-Schraen).

**Proposition 2**

The $R_{\infty}$-module $M_{\infty}(L_2^\tau/p)$, and hence $M_{\infty}(L_2^\tau)$, are cyclic.

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- $M'' \subsetneq M' \subseteq M$ finite type $R_{\infty}$-modules with $M'$ cyclic, then $M$ cyclic $\iff M/M''$ cyclic (E.-G.-S.).

Let $L_2^\tau := \ker (\widetilde{\text{Proj}} \Gamma \sigma \oplus L_2^\tau \rightarrow \text{Proj}_\Gamma \sigma) = \widetilde{\text{Proj}} \Gamma \sigma \times_{\text{Proj}_\Gamma \sigma} L_2^\tau$. 
Proof of first intermediate theorem

We know $M_\infty(\widetilde{\text{Proj}}_\Gamma \sigma)$ is cyclic (Hu-Wang, Le-Morra-Schraen).

**Proposition 2**

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Let $L_2^\tau := \ker (\widetilde{\text{Proj}}_\Gamma \sigma \oplus L_2^\tau \to \text{Proj}_\Gamma \sigma) = \widetilde{\text{Proj}}_\Gamma \sigma \times_{\text{Proj}_\Gamma \sigma} L_2^\tau$.

I explain why $M_\infty(L_2^\tau) = M_\infty(\widetilde{\text{Proj}}_\Gamma \sigma) \times_{M_\infty(\text{Proj}_\Gamma \sigma)} M_\infty(L_2^\tau)$ is cyclic. Proof for $L$ can be reduced to this case by induction.
Proof of first intermediate theorem

Let $R_v := R^\square(\bar{\rho}) :=$ framed deformations of $\bar{\rho}$ (no conditions, but need to fix determinant lifting $\omega^{-1}\psi|_{\text{Gal}(\bar{F}_v/F_v)}$, I forget this here).
Proof of first intermediate theorem

Let $R_v := R^\square(\overline{\rho}) :=$ framed deformations of $\overline{\rho}$ (no conditions, but need to fix determinant lifting $\omega^{-1}\psi|_{\text{Gal}(F_v/F_v)}$, I forget this here).

By previous cyclicities (using $R_\infty \cong R_v[[x_1, \ldots, x_h]]$):

- $M_\infty(\widehat{\text{Proj}}_{\Gamma} \sigma) \cong (R_v/J)[[x_1, \ldots, x_h]]$
Proof of first intermediate theorem

Let $R_v := R\Box(\overline{\rho}) :=$ framed deformations of $\overline{\rho}$ (no conditions, but need to fix determinant lifting $\omega^{-1}\psi|_{\text{Gal}(\overline{F}_v/F_v)}$, I forget this here).

By previous cyclicities (using $R_{\infty} \cong R_v[[x_1, \ldots, x_h]]$):

- $M_{\infty}(\tilde{\text{Proj}}_{\Gamma}\sigma) \cong (R_v/J)[[x_1, \ldots, x_h]]$
- $M_{\infty}(L_2^\tau) \cong (R_v/J_\tau)[[x_1, \ldots, x_h]]$
Proof of first intermediate theorem

Let $R_v := R\boxtimes(\bar{\rho}) :=$ framed deformations of $\bar{\rho}$ (no conditions, but need to fix determinant lifting $\omega^{-1}\psi|_{\text{Gal}(F_v/F_v)}$, I forget this here).

By previous cyclicities (using $R_\infty \cong R_v[[x_1, \ldots, x_h]]$):

- $M_\infty(\widehat{\text{Proj}}_\Gamma_\sigma) \cong (R_v/J)[[x_1, \ldots, x_h]]$
- $M_\infty(L_2^\tau) \cong (R_v/J_\tau)[[x_1, \ldots, x_h]]$
- $M_\infty(\text{Proj}_\Gamma_\sigma) \cong (R_v/(p, J))[x_1, \ldots, x_h]]$
Proof of first intermediate theorem

Let $R_v := R^\square(\overline{\rho}) := \text{framed deformations of } \overline{\rho} \text{ (no conditions, but need to fix determinant lifting } \omega^{-1}\psi|_{\text{Gal}(F_v/F_v)}, \text{ I forget this here).}$

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- $M_\infty(\widetilde{\text{Proj}}_\Gamma \sigma) \cong (R_v/J)[[x_1, \ldots, x_h]]$
- $M_\infty(L_2^\tau) \cong (R_v/J_\tau)[[x_1, \ldots, x_h]]$
- $M_\infty(\text{Proj}_\Gamma \sigma) \cong (R_v/(p, J))[x_1, \ldots, x_h]]$

where:

- $R_v/J$ parametrizes pot. cryst. lifts of $\overline{\rho}$ of any tame type whose reduction mod $p$ contains $\sigma$ and parallel HT weights $(1, 0)$
Proof of first intermediate theorem

Let $R_v := R_{\square}(\bar{\rho}) :=$ framed deformations of $\bar{\rho}$ (no conditions, but need to fix determinant lifting $\omega^{-1}\psi|_{\text{Gal}(\bar{F}/F)}$, I forget this here).

By previous cyclicities (using $R_{\infty} \cong R_v[[x_1, \ldots, x_h]]$):

- $M_{\infty}(\widehat{\text{Proj}}_{\Gamma} \sigma) \cong (R_v/J)[[x_1, \ldots, x_h]]$
- $M_{\infty}(L_2^\tau) \cong (R_v/J_\tau)[[x_1, \ldots, x_h]]$
- $M_{\infty}(\text{Proj}_{\Gamma} \sigma) \cong (R_v/(p, J))[x_1, \ldots, x_h]]$

where:

- $R_v/J$ parametrizes pot. cryst. lifts of $\bar{\rho}$ of any tame type whose reduction mod $p$ contains $\sigma$ and parallel HT weights $(1, 0)$
- $R_v/J_\tau$ parametrizes pot. cryst. lifts of $\bar{\rho}$ of same tame types but HT weights $(1, 0)$ outside embedding $\tau$, $(2, -1)$ at $\tau$. 
Needed: fiber product \((R_v/J) \times_{R_v/(p,J)} (R_v/J_\tau)\) is a quotient of \(R_v\).
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Can explicitly compute \(J\) and \(J_\tau\) mod \(p^2\) and check:

**Lemma**

We have \(p \in J + J_\tau\).
Needed: fiber product \((R_v/J) \times_{R_v/\langle p, J \rangle} (R_v/J_\tau)\) is a quotient of \(R_v\). This holds if and only if \(J + J_\tau = \langle p, J \rangle\). Enough to prove \(p \in J + J_\tau\).

Can explicitly compute \(J\) and \(J_\tau\) mod \(p^2\) and check:

**Lemma**

We have \(p \in J + J_\tau\).

This finishes the proof of main result!
One application
Theorem 7 (Dotto-Le, building on C.-E.-G.-G.-P.-S.)

There is a “big” patched module $M_\infty$ finitely generated over $R_\infty[[\GL_2(O_{F_v})]] + \text{compatible action of } \GL_2(F_v)$ such that $M_\infty/m_\infty \cong \pi_v(\bar{r})^\vee$. 
**Theorem 7 (Dotto-Le, building on C.-E.-G.-G.-P.-S.)**

There is a “big” patched module $\mathcal{M}_\infty$ finitely generated over $R_\infty[[\text{GL}_2(\mathcal{O}_{F_v})]] + \text{compatible action of } \text{GL}_2(F_v)$ such that $\mathcal{M}_\infty/m_\infty \cong \pi_v(\bar{r})^\vee$.

**Corollary of our main result**

For any map $R_\infty \to \mathcal{O}_E$ of $W(F)$-algebras (where $[E : \mathbb{Q}_p] < \infty$), $(\mathcal{M}_\infty \otimes_{R_\infty} \mathcal{O}_E)^\vee[1/p] =$ non-zero admissible unitary continuous representation of $\text{GL}_2(F_v)$ over $E$ with a unit ball lifting $\pi_v(\bar{r})$. 
Quick review of past results
Statement of the main theorem
Some ideas on the proof

One application

Theorem 7 (Dotto-Le, building on C.-E.-G.-G.-P.-S.)

There is a “big” patched module $\mathbb{M}_\infty$ finitely generated over $R_\infty[[\text{GL}_2(\mathcal{O}_{F_v})]]$ + compatible action of $\text{GL}_2(F_v)$ such that $\mathbb{M}_\infty/m_\infty \cong \pi_v(\bar{r})^\vee$.

Corollary of our main result

For any map $R_\infty \to \mathcal{O}_E$ of $W(\overline{F})$-algebras (where $[E : \mathbb{Q}_p] < \infty$), $(\mathbb{M}_\infty \otimes_{R_\infty} \mathcal{O}_E)^\vee[1/p] = \text{non-zero admissible unitary continuous representation of } \text{GL}_2(F_v) \text{ over } E \text{ with a unit ball lifting } \pi_v(\bar{r})$.

Proof: The module $\mathbb{M}_\infty$ is CM over $R_\infty[[\text{GL}_2(\mathcal{O}_{F_v})]]$ (Gee-Newton) + $\text{GK}((\mathbb{M}_\infty/m_\infty)^\vee) = f$ (our main result) $\Rightarrow \mathbb{M}_\infty$ is flat over $R_\infty$ (“Miracle Flatness” in non-commutative setting, see Gee-Newton).
Remarks

- The case $\overline{\rho}$ non semi-simple should work as well (Hu-Wang).
Remarks

- The case $\bar{\rho}$ non semi-simple should work as well (Hu-Wang).
- Hope to prove for suitable level $K^\nu$:

$$
GK\left( \text{Hom}_{\text{Gal}(\overline{F}/F)}\left( \overline{r}, \lim_{\rightarrow K^\nu} H^1_{\text{ét}}(X_{K^\nu K^\nu} \times_F \overline{F}, \mathbb{F}) \right) \right) = f.
$$

Need to extend previous proof to cases without multiplicity 1.