Prismatic Dieudonné theory

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The results presented today are a joint work with Johannes Anschütz.

In all the talk, we fix a prime number p.

The goal is to establish classification results for p-divisible groups. Our main tool is the theory of prisms and prismatic cohomology, recently developed by Bhatt and Scholze.

Plan of the talk

Prisms and prismatic cohomology (after Bhatt-Scholze)

In this section, all rings are assumed to be $\mathbf{Z}_{(p)}$-algebras.

Definition

A δ*-ring* is a commutative ring A, together with a map of sets $\delta : A \to A$, such that

- $\delta(0) = 0$, $\delta(1) = 0$.
- For all $x, y \in A$,

$$\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y).$$

- For all $x, y \in A$,

$$\delta(x+y) = \delta(x) + \delta(y) + \frac{x^p + y^p - (x+y)^p}{p}.$$

For any δ-ring (A, δ), denote by φ the map defined by

$$\varphi : A \to A, \quad x \mapsto x^p + p\delta(x).$$

The identities satisfied by δ are made to make φ a ring endomorphism lifting Frobenius modulo p.
Conversely, a p-torsion free ring equipped with a lift of Frobenius gives rise to a δ-ring.

Remark

If A is a ring, specifying a δ-structure on A is the same as specifying a ring morphism $A \to W_2(A)$ which is a section of the natural projection on the first component $W_2(A) \to A$.
This implies in particular that the category of δ-rings has all limits and colimits (which are computed at the level of underlying rings).
It follows formally that the forgetful functor from δ-rings to rings has both a left and a right adjoint. The right adjoint is the Witt vectors functor.

Remark

There is no non-zero δ-ring in which $p^n = 0$ for some $n \geq 0$.

Definition

A pair (A, I) formed by a δ-ring A and an ideal $I \subset A$ is a *prism* if I defines a Cartier divisor on $\mathrm{Spec}(A)$, if A is (derived) (p, I)-complete and if I is pro-Zariski locally generated by a distinguished element, i.e. an element d such that $\delta(d)$ is a unit.

Example

For any p-complete p-torsion free δ-ring A, the pair $(A, (p))$ is a prism.

Example

Say that a prism is *perfect* if the Frobenius φ on the underlying δ-ring is an isomorphism. Then

$$\{\text{perfect prisms}\} \cong \{\text{integral perfectoid rings}\}$$

via the functors :

$$R \mapsto (A_{\inf}(R) := W(R^\flat), \ker(\theta))$$

and

$$(A, I) \mapsto A/I.$$

Therefore, prisms are some kind of *"deperfection" of perfectoid rings* (the choice of I being like the choice of an untilt).

The crucial definition for us is the following.

Definition

Let R be a p-complete ring. The *(absolute) prismatic site* $(R)_{\Delta}$ *of* R is the opposite of the category of bounded prisms (B, J) together with a map $R \to B/J$, endowed with the Grothendieck topology for which covers are morphisms of prisms $(B, J) \to (B', J')$, such that the underlying ring map $B \to B'$ is (p, J)-completely faithfully flat.

Proposition

The functor $\mathcal{O}_{\Delta}$ (resp. $\overline{\mathcal{O}}_{\Delta}$) on the prismatic site valued in (p, I)-complete δ-rings (resp. in p-complete R-algebras), sending $(B, J) \in (R)_{\Delta}$ to B (resp. B/J), is a sheaf. The sheaf $\mathcal{O}_{\Delta}$ (resp. $\overline{\mathcal{O}}_{\Delta}$) is called the prismatic structure sheaf (resp. the reduced prismatic structure sheaf).

From the proposition, one easily deduces that the presheaf

$$I_{\Delta} : (B, J) \mapsto J$$

is also a sheaf on $(R)_{\Delta}$.

If (A, I) is a fixed bounded prism, and R is a p-complete A/I-algebra, one easily defines a relative variant $(R/A)_{\Delta}$ of the absolute prismatic site :

Definition

The prismatic site $(R/A)_{\Delta}$ is the opposite of the category of prisms (B, J) with a map $(A, I) \to (B, J)$, and a map $R \to B/J$ of A/I-algebras, with topology defined as before.

Definition

The complex of A-modules

$$\Delta_{R/A} = R\Gamma((R/A)_{\Delta}, \mathcal{O}_{\Delta})$$

is called the *prismatic cohomology of R relatively to A*, when R is p-completely smooth over A/I. (In general, the prismatic cohomology should rather be defined by left Kan extension from the smooth case.)

Bhatt and Scholze prove several comparison results between prismatic cohomology and other p-adic cohomology theories. Here, I only want to quote two of them.

Let (A, I) be a bounded prism and R be a p-completely smooth A/I-algebra.

Theorem (Hodge-Tate comparison)

There is a canonical R-module isomorphism

$$\Omega^i_{R/(A/I)}\{i\} \cong H^i(\Delta_{R/A} \otimes^{\mathbb{L}}_A A/I).$$

Here, the symbol $(-)\{i\}$ *is a Breuil-Kisin twist : if* M *is an* A/I*-module, one sets* $M\{i\} = M \otimes_{A/I} (I/I^2)^{\otimes i}$.

Remark

Once the definition of prismatic cohomology is suitably extended to all p-complete A/I-algebras, the Hodge-Tate comparison theorem generalizes as follows :

$$\Delta_{R/A} \otimes_A^{\mathbb{L}} A/I$$

is endowed with a natural increasing $\mathbb{N}$-indexed filtration, with

$$\mathrm{gr}_i \cong (\wedge^i L_{R/(A/I)}\{-i\}[-i])^{\wedge_p}.$$

Theorem (Crystalline comparison)

Assume that $I = (p)$. There is a canonical Frobenius-equivariant isomorphism

$$R\Gamma_{\mathrm{crys}}(R/A) \cong \varphi_A^* \Delta_{R/A}$$

of commutative algebras in $D(A)$.

Quasi-syntomic rings

Definition

A ring R is *quasi-syntomic* if R is p-complete with bounded p^∞-torsion and if the cotangent complex $L_{R/\mathbf{Z}_p}$ has p-complete Tor-amplitude in $[-1,0]$, i.e. if

$$L_{R/\mathbf{Z}_p} \otimes^{\mathbb{L}}_R N \in D^{[-1,0]}(R/p)$$

for any R/p-module N.

Similarly, a map $R \to R'$ of p-complete rings with bounded p^∞-torsion is a *quasi-syntomic morphism* (resp. a *quasi-syntomic cover*) if R' is p-completely flat over R (resp. p-completely faithfully flat) and $L_{R'/R} \in D(R')$ has p-complete Tor-amplitude in $[-1,0]$.

Remark

This definition, due to Bhatt-Morrow-Scholze, extends (in the p-complete world) the usual notion of syntomic ring and syntomic morphism (flat and local complete intersection) to the non-Noetherian, non finite-type setting.

Definition

The category of all quasi-syntomic rings is denoted by QSyn. We endow $\mathrm{QSyn}^{\mathrm{op}}$ with the structure of a site using quasi-syntomic covers. If $R \in \mathrm{QSyn}$, let R_{qsyn} be the sub-site formed by rings which are quasi-syntomic over R.

Example

Any p-complete l.c.i. Noetherian ring is in QSyn.

Example

There are also has big rings in QSyn. For example, any (integral) perfectoid ring is in QSyn. Indeed, if R is such a ring, the transitivity triangle for

$$\mathbf{Z}_p \to A_{\mathrm{inf}}(R) \to R$$

and the fact that $A_{\mathrm{inf}}(R)$ is relatively perfect over $\mathbf{Z}_p$ modulo p identify the p-completions of $L_{R/\mathbf{Z}_p}$ and $L_{R/A_{\mathrm{inf}}(R)}$. But

$$L_{R/A_{\mathrm{inf}}(R)} = \ker(\theta)/\ker(\theta)^2[1] = R[1],$$

as $\ker(\theta)$ is generated by a non-zero divisor.

Example

The p-completion of a smooth algebra over a perfectoid ring is also quasi-syntomic, as well as any bounded p^∞-torsion ring which can be presented as the quotient of an integral perfectoid ring by a finite regular sequence. For example,

$$\mathcal{O}_{\mathbb{C}_p}\langle T\rangle \ , \ \mathcal{O}_{\mathbb{C}_p}/p \ , \ \mathbb{F}_p[T^{1/p^\infty}]/(T-1)$$

are in QSyn.

Filtered prismatic Dieudonné crystals

Proposition

Let R be a quasi-syntomic ring. There is a natural morphism of topoi :

$$v : \mathrm{Shv}(R_{\Delta}) \to \mathrm{Shv}(R_{\mathrm{qsyn}}),$$

coming from the morphism of topoi attached to the functor from R_{Δ} to the category of p-complete rings over R, sending (A, I) to A/I, composed with restriction.

Definition

Let

$$\mathcal{O}^{\mathrm{pris}} = v_* \mathcal{O}_{\Delta} \;\; ; \;\; \mathcal{I}^{\mathrm{pris}} = v_* \mathcal{I}_{\Delta}.$$

This sheaf comes with a natural surjection :

$$\mathcal{O}^{\mathrm{pris}} \to \mathcal{O},$$

whose kernel will be denoted by $\mathcal{N}^{\geq 1}\mathcal{O}^{\mathrm{pris}}$. The sheaf $\mathcal{O}^{\mathrm{pris}}$ is endowed with a Frobenius endomorphism φ and

$$\varphi(\mathcal{N}^{\geq 1}\mathcal{O}^{\mathrm{pris}}) \subset \mathcal{I}^{\mathrm{pris}}.$$

Definition

Let R be a quasi-syntomic ring. A *filtered prismatic Dieudonné crystal over* R is a collection $(\mathcal{M}, \mathrm{Fil}\mathcal{M}, \varphi_{\mathcal{M}})$ consisting of a finite locally free $\mathcal{O}^{\mathrm{pris}}$-module $\mathcal{M}$, a $\mathcal{O}^{\mathrm{pris}}$-submodule $\mathrm{Fil}\mathcal{M}$, and a φ-linear map $\varphi_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$, satisfying the following conditions :

1. $\varphi_{\mathcal{M}}(\mathrm{Fil}\mathcal{M}) \subset \mathcal{I}^{\mathrm{pris}}.\mathcal{M}$.
2. $\mathcal{N}^{\geq 1}\mathcal{O}^{\mathrm{pris}}.\mathcal{M} \subset \mathrm{Fil}\mathcal{M}$ and $\mathcal{M}/\mathrm{Fil}\mathcal{M}$ is a finite locally free $\mathcal{O}$-module.
3. $\varphi_{\mathcal{M}}(\mathrm{Fil}\mathcal{M})$ generates $\mathcal{I}^{\mathrm{pris}}.\mathcal{M}$ as an $\mathcal{O}^{\mathrm{pris}}$-module.

Definition

Let R be a quasi-syntomic ring. We denote by $\mathrm{DF}(R)$ the category of filtered prismatic Dieudonné crystals over R (with morphisms the $\mathcal{O}^{\mathrm{pris}}$-linear morphisms commuting with the Frobenius and respecting the filtration).

Main results

In all this section, R is a quasi-syntomic ring.

Definition

Let G be a p-divisible group over R. Set

$$\mathcal{M}_{\Delta}(G) = \mathcal{E}xt^1(G, \mathcal{O}^{\mathrm{pris}})$$

and

$$\mathrm{Fil}\mathcal{M}_{\Delta}(G) = \mathcal{E}xt^1(G, \mathcal{N}^{\geq 1}\mathcal{O}^{\mathrm{pris}}),$$

where the $\mathcal{E}xt$ are Ext-groups of abelian sheaves on $(R)_{\mathrm{qsyn}}$.

Theorem (Main Theorem 1)

Let G be a p-divisible group over R. The triple

$$\left(\mathcal{M}_{\Delta}(G), \mathrm{Fil}\mathcal{M}_{\Delta}(G), \varphi_{\mathcal{M}_{\Delta}(G)}\right)$$

where $\varphi_{\mathcal{M}_{\Delta}(G)}$ is the Frobenius induced by the Frobenius of $\mathcal{O}^{\mathrm{pris}}$, is a filtered prismatic Dieudonné crystal over R, denoted by $\underline{\mathcal{M}}_{\Delta}(G)$.

Remark

When $pR = 0$, the *crystalline comparison theorem* for prismatic cohomology allows us to prove that this construction coincides with the functor usually considered in crystalline Dieudonné theory, relying on Berthelot-Breen-Messing's constructions.

Theorem (Main Theorem 2)

The filtered prismatic Dieudonné functor

$$\underline{\mathcal{M}}_{\Delta} : G \mapsto \underline{\mathcal{M}}_{\Delta}(G)$$

induces an antiequivalence between the category $\mathrm{BT}(R)$ *of p-divisible groups over R and the category* $\mathrm{DF}(R)$ *of filtered prismatic Dieudonné crystals over R.*
Moreover, the prismatic Dieudonné functor :

$$\mathcal{M}_{\Delta} : G \mapsto \mathcal{M}_{\Delta}(G)$$

is fully faithful.

Remark

As a corollary of Theorem 2 and the comparison with the crystalline functor, one obtains that the filtered Dieudonné functor from crystalline Dieudonné theory is an equivalence for quasi-syntomic rings in characteristic p.
For excellent l.c.i. rings, fully faithfulness was proved by de Jong-Messing ; the equivalence was proved by Lau for F-finite l.c.i. rings (which are in particular excellent rings).

There is a large class of quasi-syntomic rings R for which the category $\mathrm{DF}(R)$ can be made more explicit.

Definition

A ring R is *quasi-regular semiperfectoid* if $R \in \mathrm{QSyn}$ and there exists a perfectoid ring S mapping surjectively to R.

Example

Any perfectoid ring, or any bounded p^∞-torsion quotient of a perfectoid ring by a finite regular sequence, is quasi-regular semiperfectoid.

Let R be a quasi-regular semiperfectoid ring. The prismatic site $(R)_{\Delta}$ admits a final object (Δ_R, I). Moreover, one has a natural isomorphism

$$\theta \colon \Delta_R/\mathcal{N}^{\geq 1}\Delta_R \cong R,$$

where $\mathcal{N}^{\geq 1}\Delta_R = \varphi^{-1}(I)$.

Example

1. If R is a perfectoid ring,

$$(\Delta_R, I) = (A_{\mathrm{inf}}(R), \ker(\tilde{\theta})).$$

2. If R is quasi-regular semiperfectoid and $pR = 0$,

$$(\Delta_R, I) \cong (A_{\mathrm{crys}}(R), (p)).$$

Definition

A *filtered prismatic Dieudonné module over* R is a collection $(M, \mathrm{Fil}\ M, \varphi_M)$ consisting of a finite locally free Δ_R-module M, a Δ_R-submodule $\mathrm{Fil}\ M$, and a φ-linear map $\varphi_M : M \to M$, satisfying the following conditions :

1. $\varphi_M(\mathrm{Fil}\ M) \subset I.M$.
2. $\mathcal{N}^{\geq 1}\Delta_R.M \subset \mathrm{Fil}\ M$ and $M/\mathrm{Fil}\ M$ is a finite locally free R-module.
3. $\varphi_M(\mathrm{Fil}\ M)$ generates $I.M$ as a Δ_R-module.

Proposition

Let R be a quasi-regular semiperfectoid ring. The functor of evaluation on the initial prism (Δ_R, I) induces an equivalence between the category of filtered prismatic Dieudonné crystals over R and the category of filtered prismatic Dieudonné modules over R.

Remark

If moreover R is perfectoid, the forgetful functor (forgetting the filtration) from filtered prismatic Dieudonné modules over R to prismatic Dieudonné modules over R (usually called in this case *minuscule Breuil-Kisin-Fargues modules*) is an equivalence.

Remark

In particular, we see that Theorem 2 contains as a special case the results of Lau and Scholze-Weinstein on classification of p-divisible groups over perfectoid rings by minuscule Breuil-Kisin-Fargues modules.
But the proof of the theorem actually requires the special case of perfectoid valuation rings with algebraically closed fraction field as an *input*.

Remark

In general, the prismatic Dieudonné functor (without the filtration) is not essentially surjective. It is true for perfectoid rings, as we just saw. We prove it is also an equivalence for p-complete regular (Noetherian) rings.

Example

Let

$$R = \mathcal{O}_K,$$

where K is a discretely valued extension of $\mathbb{Q}_p$ with perfect residue field. Our theorem specialized to this case is equivalent to Breuil-Kisin's classification (as extended by Kim, Lau and Liu to all p) of p-divisible groups over $\mathcal{O}_K$, by evaluating the crystal on the prism $(\mathfrak{S}, (E))$.

The proof of Theorem 1 follows a strategy similar to the one used in Berthelot-Breen-Messing's book : via a theorem of Raynaud, one reduces to the case of p-divisible groups attached to abelian schemes.

One then needs a good understanding of the prismatic cohomology of abelian schemes, for which a key tool is provided by the *Hodge-Tate comparison theorem* for prismatic cohomology.

The proof of Theorem 2 uses quasi-syntomic descent. One knows that quasi-regular semiperfectoid rings form a basis of the quasi-syntomic topology, and thus one can reduce to these rings, for which everything is more concrete, as mentioned above.

This shows the interest of formulating and proving everything for all rings in QSyn, even if one is ultimately interested only in Noetherian rings.

Remark

Let R be a quasi-regular semiperfectoid ring. The map $\theta : \Delta_R \to R$ provides by adjunction a map of δ-rings

$$\Delta_R \to W(R)$$

which gives a functor from filtered prismatic Dieudonné modules over R to *displays over* R in the sense of Zink. As a corollary of Theorem 2, one recovers Zink's equivalence between *formal* p-divisible groups and nilpotent displays, over quasi-syntomic rings.

Remark

Theorem 2 gives a classification theorem for finite locally free group schemes over perfectoid rings.

- What about more general base rings ?
- Deformation theory (in the spirit of Grothendieck-Messing theory) for the prismatic Dieudonné functor ?

- Let R be a perfect ring. Define the sheaf $\mathcal{Q}$ on $(R)_{\Delta}$ as the quotient :

$$0 \to \mathcal{O}_{\Delta} \to \mathcal{O}_{\Delta}[1/p] \to \mathcal{Q} \to 0.$$

Then

$$M_{\Delta}(G) \cong \mathrm{Hom}_{(R)_{\mathrm{qsyn}}}(G, v_* \mathcal{Q}).$$

Can this be directly compared to Fontaine's original definition of the Dieudonné functor (using Witt covectors), without using the crystalline comparison theorem ?

Thank you !