On the Beilinson–Bloch–Kato conjecture for Rankin–Selberg motives

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If the (central critical) L-value $L(n, \mathrm{Sym}^{n-1} A_F \times \mathrm{Sym}^n A'_F)$ *does not vanish, then the Bloch–Kato Selmer group* $\mathrm{H}^1_f(F, \mathrm{Sym}^{n-1} \mathrm{H}^1_{\acute{e}t}(A_{\overline{F}}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \mathrm{Sym}^n \mathrm{H}^1_{\acute{e}t}(A'_{\overline{F}}, \mathbb{Q}_\ell)(n))$ *vanishes for all but finitely many rational primes* ℓ.

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We recall the definition of the Bloch–Kato Selmer group: For a Galois representation $\rho\colon \Gamma_F \to \mathrm{GL}(V)$ on a finite dimensional vector space V over a finite extension of $\mathbb{Q}_\ell$, we define $\mathrm{H}^1_f(F, V)$ to be the subspace of $\mathrm{H}^1(F, V)$ consisting of classes whose localization belongs to $\mathrm{H}^1_f(F_v, V)$ for every nonarchimedean place v of F. When $\ell \nmid v$, $\mathrm{H}^1_f(F_v, V) = \mathrm{H}^1_{\mathrm{unr}}(F_v, V)$. When $\ell \mid v$, $\mathrm{H}^1_f(F_v, V)$ is defined via ℓ-adic Hodge theory.

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- ✓ for every archimedean place τ of F, Π_τ is the principal series of the characters $(\arg^{N-1}, \arg^{N-3}, \dots, \arg^{3-N}, \arg^{1-N})$, where $\arg\colon \mathbb{C}^\times \to \mathbb{C}^\times$ is the *argument character* defined by the formula $\arg(z) := z/\sqrt{z\overline{z}}$.

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Notation

In what follows, we will take an integer $n \geq 2$, and denote by n_0 and n_1 the unique even and odd numbers in $\{n, n+1\}$, respectively. For $\alpha = 0, 1$, we write $n_\alpha = 2r_\alpha + \alpha$ for a unique positive integer r_α.

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A *special inert prime* of F^+ is a prime $\mathfrak{p}$ that is of degree one over $\mathbb{Q}$, inert in F, and whose underlying rational prime p is unramified in F.

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Let Π_0 *and* Π_1 *be relevant representations of* $\mathrm{GL}_{n_0}(\mathbb{A}_F)$ *and* $\mathrm{GL}_{n_1}(\mathbb{A}_F)$, *respectively. Let* $E \subseteq \mathbb{C}$ *be a coefficient field of both* Π_0 *and* Π_1.

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If $L(\frac{1}{2}, \Pi_0 \times \Pi_1) \neq 0$, then for all but finitely many primes λ of E, the Bloch–Kato Selmer group $\mathrm{H}^1_f(F, \rho_{\Pi_0,\lambda} \otimes_{E_\lambda} \rho_{\Pi_1,\lambda}(n))$ vanishes.

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(c) is needed because two ingredients we need are not available when $F = \mathbb{Q}$ and $n \geq 3$ at this moment: First, the cohomology of the stable part of the unitary Shimura varieties. Second, a Caraiani–Scholze type result.

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This result follows from a series work by Jacquet–Rallis, Yun, W. Zhang, H. Xue, Waldspurger, Beuzart-Plessis, Zydor, Chaudouard–Zydor, and finally Beuzart-Plessis–L.–Zhang–Zhu.

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In what follows, we put $\mathrm{S}(\mathrm{V}_{n_\alpha}) := \mathrm{U}(\mathrm{V}_{n_\alpha})(F^+)\backslash\mathrm{U}(\mathrm{V}_{n_\alpha})(\mathbb{A}_{F^+}^\infty)$. We fix the choice of f_0 and f_1 as above. We also fix an open compact subgroup of $\mathrm{U}(\mathrm{V}_{n_\alpha})(\mathbb{A}_{F^+}^\infty)$ that fixes f_α for $\alpha = 0, 1$, and will carry them implicitly in the notation.

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In what follows, we put $\mathrm{S}(\mathrm{V}_{n_\alpha}) := \mathrm{U}(\mathrm{V}_{n_\alpha})(F^+)\backslash\mathrm{U}(\mathrm{V}_{n_\alpha})(\mathbb{A}_{F^+}^\infty)$. We fix the choice of f_0 and f_1 as above. We also fix an open compact subgroup of $\mathrm{U}(\mathrm{V}_{n_\alpha})(\mathbb{A}_{F^+}^\infty)$ that fixes f_α for $\alpha = 0, 1$, and will carry them implicitly in the notation.
We also fix a finite set Σ^+ of primes of F^+ outside which "everything is unramified".

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The generic fiber of $\mathbf{M}_\mathfrak{p}(\mathrm{V}_{n_\alpha})$ is the (base change to $F_\mathfrak{p} = \mathbb{Q}_{p^2}$ of the) Shimura variety associated to $\mathrm{U}(\mathrm{V}'_{n_\alpha})$ where V'_{n_α} is the hermitian space, unique up to isomorphism, satisfying that

- ✓ V'_{n_α} has signature $(n_\alpha - 1, 1)$ at some fixed archimedean place τ of F^+;
- ✓ $\mathrm{V}'_{n_\alpha} \otimes_{F^+} F^+_v \simeq \mathrm{V}_{n_\alpha} \otimes_{F^+} F^+_v$ for every place v of F^+ other than τ and $\mathfrak{p}$ (this implies that $\mathrm{V}'_{n_\alpha,\mathfrak{p}}$ does not admit a self-dual lattice);

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and the level structure at $\mathfrak{p}$ being the stabilizer of an almost self-dual lattice in $\mathrm{V}'_{n_\alpha,\mathfrak{p}}$.
The special fiber $\mathrm{M}_\mathfrak{p}(\mathrm{V}_{n_\alpha}) \coloneqq \mathbf{M}_\mathfrak{p}(\mathrm{V}_{n_\alpha}) \otimes_{\mathbb{Z}_{p^2}} \mathbb{F}_{p^2}$ is a union of $\mathrm{M}^\circ_\mathfrak{p}(\mathrm{V}_{n_\alpha})$ and $\mathrm{M}^\bullet_\mathfrak{p}(\mathrm{V}_{n_\alpha})$ in which

- ✓ $\mathrm{M}^\circ_\mathfrak{p}(\mathrm{V}_{n_\alpha})$ is a $\mathbb{P}^{n_\alpha - 1}$-fibration over $\mathrm{S}(\mathrm{V}_{n_\alpha})$;
- ✓ $\mathrm{M}^\bullet_\mathfrak{p}(\mathrm{V}_{n_\alpha})$ is smooth, whose "basic locus" is a Deligne–Lusztig variety fibration of dimension r_α over (essentially) $\mathrm{S}(\mathrm{V}_{n_\alpha})$.
- ✓ the intersection $\mathrm{M}^\circ_\mathfrak{p}(\mathrm{V}_{n_\alpha}) \cap \mathrm{M}^\bullet_\mathfrak{p}(\mathrm{V}_{n_\alpha})$ is a Fermat hypersurface in $\mathrm{M}^\circ_\mathfrak{p}(\mathrm{V}_{n_\alpha})$.

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For $\alpha = 0, 1$, let $\mathbb{T}_\alpha$ be the abstract spherical Hecke algebra of $\mathrm{U}(\mathrm{V}_{n_\alpha})$ away from $\mathfrak{p}$ and Σ^+.
The representation Π_α gives rise to a homomorphism $\phi_\alpha \colon \mathbb{T}_\alpha \to \mathbb{Z}$. For every ℓ, we denote by $\mathfrak{m}_{\alpha,\ell}$ the kernel of the composition of ϕ_α with the quotient map $\mathbb{Z} \to \mathbb{F}_\ell$, which is a maximal ideal of $\mathbb{T}_\alpha$.

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In what follows, we will stop explaining further steps toward the main theorems, but explaining more on the arithmetic level-raising phenomenon, which employs new ideas from the theory of Galois deformation.

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Proposition (Level-raising isomorphism)

Suppose ℓ (effectively) sufficiently large, and that $\mathfrak{p}$ is a level-raising prime with respect to ℓ. Then we have a canonical isomorphism

$$\mathrm{H}^1_{\mathrm{sing}}(\mathbb{Q}_{p^2}, \mathrm{H}^{n_0-1}(\mathrm{M}_{\mathfrak{p}}(\mathrm{V}_{n_0})_{\overline{\mathbb{F}}_p}, \mathrm{R}\Psi\mathbb{Z}_\ell(r_0))/\mathfrak{m}_{0,\ell}) \simeq \mathbb{Z}_\ell[\mathrm{S}(\mathrm{V}_{n_0})]/\mathfrak{m}_{0,\ell} \tag{1}$$

of $\mathbb{F}_\ell$-vector spaces of finite dimension.

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Through studying the geometry and the intersection theory on $\mathrm{M}_{\mathfrak{p}}(\mathrm{V}_{n_0})$, we can show that RHS of (1) is canonically a *subquotient* of LHS of (1). Thus, to obtain (1), it suffices to compare the cardinality.

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Now we assume that $\rho_{\Pi_0,\ell}\colon \Gamma_F \to \mathrm{GL}_{n_0}(\mathbb{Q}_\ell)$ is residually absolutely irreducible (which is the case for ℓ sufficiently large). Let $\bar{\rho}_{\Pi_0,\ell}\colon \Gamma_F \to \mathrm{GL}_{n_0}(\mathbb{F}_\ell)$ be the residue representation.

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We put $\mathsf{R}^{\mathrm{unr}} := \mathsf{R}^{\mathrm{mix}}/(\mathtt{x})$, $\mathsf{R}^{\mathrm{ram}} := \mathsf{R}^{\mathrm{mix}}/(\mathtt{s} - p^{-n_0})$, and $\mathsf{R}^{\mathrm{cong}} := \mathsf{R}^{\mathrm{unr}} \otimes_{\mathsf{R}^{\mathrm{mix}}} \mathsf{R}^{\mathrm{ram}}$.

Proposition

(1) There exists an $\mathsf{R}^{\mathrm{ram}}$*-module* $\mathsf{H}^{\mathrm{ram}}$ *such that* $\mathsf{H}^{\mathrm{ram}} \otimes_{\mathsf{R}^{\mathrm{ram}}} (\mathsf{R}^{\mathrm{ram}})^{\oplus n_0} \simeq \mathrm{H}^{n_0-1}(\mathrm{M}_{\mathfrak{p}}(\mathrm{V}_{n_0})_{\overline{\mathbb{F}}_p}, \mathrm{R}\Psi\mathbb{Z}_\ell(r_0))_{\mathfrak{m}_{0,\ell}}$, *and*

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(3) *for* $v \in \Sigma^+$, *every polarized local lifting of* $(\bar{\rho}_{\Pi_0,\ell})_v$ *is minimally ramified.*

Proposition

(1) There exists an $\mathsf{R}^{\mathrm{ram}}$*-module* $\mathsf{H}^{\mathrm{ram}}$ *such that* $\mathsf{H}^{\mathrm{ram}} \otimes_{\mathsf{R}^{\mathrm{ram}}} (\mathsf{R}^{\mathrm{ram}})^{\oplus n_0} \simeq \mathrm{H}^{n_0-1}(\mathrm{M}_{\mathfrak{p}}(\mathrm{V}_{n_0})_{\overline{\mathbb{F}}_p}, \mathrm{R}\Psi\mathbb{Z}_\ell(r_0))_{\mathfrak{m}_{0,\ell}}$*, and*

$$\mathrm{H}^1_{\mathrm{sing}}(\mathbb{Q}_{p^2}, \mathrm{H}^{n_0-1}(\mathrm{M}_{\mathfrak{p}}(\mathrm{V}_{n_0})_{\overline{\mathbb{F}}_p}, \mathrm{R}\Psi\mathbb{Z}_\ell(r_0))/\mathfrak{m}_{0,\ell}) \simeq \mathsf{H}^{\mathrm{ram}} \otimes_{\mathsf{R}^{\mathrm{ram}}} \mathsf{R}^{\mathrm{cong}} \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell.$$

(2) The $\mathbb{Z}_\ell$*-module* $\mathsf{H}^{\mathrm{unr}} := \mathbb{Z}_\ell[\mathrm{S}(\mathrm{V}_{n_0})]_{\mathfrak{m}_{0,\ell}}$ *is naturally an* $\mathsf{R}^{\mathrm{unr}}$*-module, and*

$$\mathbb{Z}_\ell[\mathrm{S}(\mathrm{V}_{n_0})]/\mathfrak{m}_{0,\ell} \simeq \mathsf{H}^{\mathrm{unr}} \otimes_{\mathsf{R}^{\mathrm{unr}}} \mathsf{R}^{\mathrm{cong}} \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell.$$

Proposition

Suppose

(1) $\rho_{\Pi_0,\ell}$ *is residually absolutely irreducible (which has been supposed);*

(2) $\bar{\rho}_{\Pi_0,\ell}|_{\mathrm{Gal}(\overline{F}/F(\zeta_\ell))}$ *remains absolutely irreducible;*

(3) for $v \in \Sigma^+$*, every polarized local lifting of* $(\bar{\rho}_{\Pi_0,\ell})_v$ *is minimally ramified.*

Then both $\mathsf{H}^{\mathrm{ram}}$ *and* $\mathsf{H}^{\mathrm{unr}}$ *are finite free modules over* $\mathsf{R}^{\mathrm{ram}}$ *and* $\mathsf{R}^{\mathrm{unr}}$*, respectively, and of the same rank. In particular, the level-raising isomorphism* (1) *holds.*

Arithmetic level-raising and Galois deformation

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We explain this in a simplified situation: Assume that $(\bar{\rho}_{\Pi_0,\ell})_v \colon \Gamma_{F_v} \to \mathrm{GL}_{n_0}(\mathbb{F}_\ell)$ is tamely ramified and that the tame generator t acts by a unipotent element of Jordan type $(m_1, \dots, m_s)$. Then a lifting ρ_v of $(\bar{\rho}_{\Pi_0,\ell})_v$ is *minimally ramified* if ρ_v is tamely ramified, and $\rho_v(t)$ is conjugate to a unipotent element in $\mathrm{GL}_{n_0}(\mathbb{Z}_\ell)$ of the same Jordan type $(m_1, \dots, m_s)$.

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We have extended this notion to every place v of F^+, which is quite technical if v ramifies in F. In the symplectic or orthogonal case, this was recently studied by Booher; and our results rely on his work.

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Proposition

Suppose $\ell > n_0$. For every $v \in \Sigma^+$, the local deformation problem classifying minimally ramified polarized liftings of $(\bar{\rho}_{\Pi_0,\ell})_v$ is formally smooth over $\mathbb{Z}_\ell$ of pure relative dimension n_0^2.

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Remark

Originally, we also need Π to be a twist of Steinberg at some place not above Σ^+, in order to deal with (2). But recently, Toby Gee told us an argument to remove this restriction.

Thank you! Stay safe!