The Hodge locus

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- Based on a joint work with Bruno Klingler and Emmanuel Ullmo;
- Number Theory $\Rightarrow$ Hodge Theory $\Rightarrow$ Number Theory;
- Subtitle of the talk: ***Finiteness*** *theorems in variational Hodge theory*.

Motivation (from Algebraic/Arithmetic Geometry)

Let $f : X \to S$ be a smooth projective morphism of smooth irreducible $\mathbb{C}$-quasi-projective varieties.

Goal: Describe the *Motivic locus* of f:

$$\{s \in S(\mathbb{C}) : X_s = f^{-1}(s) \text{ is simpler than the very general fibre}\}$$

Simpler means: X_s, or possibly ${X_s}^n$, contains **more** algebraic cycles than the very general fibre (or possibly of its powers).

Example

Let $f : \mathbb{A}_g \to \mathcal{A}_{g,?}$ be the universal family of ppav of dimension g. The motivic locus of f contains:

- CM points: $s \in \mathcal{A}_g$ corresponding to CM abelian varieties A_s (cycles in $A_s \times A_s$);
- For any $k \leq g$, the set of $s = A_s \in \mathcal{A}_g$ where A_s contains a k-dim abelian subvariety.

Hodge linearisation

Problem: we know very little about algebraic cycles.

$$(X_s, \text{cycle of codim } i) \quad \rightsquigarrow \ (H^{2i}(X_s, \mathbb{Z}), \ \text{hodge class});$$

$$f : X \to S \quad \rightsquigarrow \ \mathbb{V} = R^{2i} f_* \mathbb{Z};$$

Motivic locus of $f \rightsquigarrow$ **Hodge locus**.

Remark

The Hodge conjecture "inverts" the first linearization, at least rationally.

Definitions: polarised Hodge structure

Let $V_{\mathbb{Z}}$ be a f.g. (torsion free) $\mathbb{Z}$-module. A **Hodge structure** on $V_{\mathbb{Z}}$ is a decomposition

$$V_{\mathbb{C}} := V_{\mathbb{Z}} \otimes_{\mathbb{C}} = \bigoplus_{p,q \in Z} V^{p,q}$$

such that: $\overline{V^{p,q}} = V^{q,p}$. This is the same as giving

$$x : \mathbb{S} = \mathrm{Res}_{\mathbb{R}}^{\mathbb{C}}(\mathbb{G}_m) \to \mathrm{GL}(V_{\mathbb{R}}).$$

A **polarization**

$$q_{\mathbb{Z}} : V_{\mathbb{Z}} \otimes V_{\mathbb{Z}} \to \mathbb{Z}(-n)$$

is a bilinear form such that the hodge form h is positive definite and the Hodge decomposition if h-orthogonal.

Definitions: Mumford–Tate group

- A (rational) **Hodge class** is a vector $v \in V_{\mathbb{Q}}$ invariant under the action of $\mathbb{S}$. If V has weight zero, it is the same as $V_{\mathbb{Q}} \cap V^{0,0}$. A **Hodge tensor** is a Hodge class of $\bigoplus_{a,b} V^{\otimes a}(\otimes V^{\vee})^{\otimes b}$.
- The **Mumford–Tate group** is the fixator in $\mathrm{GL}(V_{\mathbb{Q}})$ of all Hodge tensors of V. The same as the $\mathbb{Q}$-Zariski closure of $x(\mathbb{S})$, or also the Tannakian group associated to V. It is a reductive $\mathbb{Q}$-group.

Example

$\mathbb{Q}$-forms of real groups (whose derived subgroup look) like: $SU(p,q), SP_{2g}, SO^*(2r), EIII, EVII$, $SO(2p,r)$, $Sp(r_1,r_2), EII, EV, EVI, EVIII, EIX, FI, FII, G$. Non-example: SL_n, $n>3$.

Definitions: (integral polarised pure) variations of Hodge structures

Let S be a smooth quasi-projective variety. A VHS on S is

$$\mathbb{V} := (\mathbb{V}, (\mathcal{V}, \nabla, F^{\cdot}), Q)$$

where:

- $\mathbb{V}$ is a local system;
- $(\mathcal{V}, \nabla, F^{\cdot})$ is a filtered $\mathcal{O}_S$-module such that $\nabla(F^p) \subset F^{p-1} \otimes \Omega^1$;
- $Q : \mathcal{V} \times \mathcal{V} \to \mathbb{Z}_S$ bilinear form;

such that each fibre is a polarised Hodge structure.

Remark

Example to keep in mind: $\mathbb{V} = R^n f_* \mathbb{Z}_{\text{prim}}$

Finally the Hodge locus

$$\mathsf{HL}(S, \mathbb{V}^{\otimes}) := \{s \in S(\mathbb{C}) : \mathbb{V}_s \text{ has exceptional Hodge tenors}\}$$
$$= \{s \in S(\mathbb{C}) : MT(\mathbb{V}_s) \subsetneq MT(\mathbb{V})\}.$$

Easy to see that it is a countable (possibly finite) union of analytic subvarieties, and in fact

Theorem (Cattani-Deligne-Kaplan 1995)

$\mathsf{HL}(S, \mathbb{V}^{\otimes})$ *is a countable union of* ***algebraic*** *subvarieties of* S. *The so called (maximal) special subvarieties.*

Question

What is the distribution of HL? Can we predict whether it is big or small? Can we describe its Zariski closure? What is its arithmetic significance? . . .

Special subvarieties as intersections: **typical vs atypical**

A VHS $\mathbb{V} \to S$ is the same thing as a **period map**

$$\Psi : S(\mathbb{C}) \to \Gamma \backslash D.$$

Given $s \in S$, $\mathbb{V}_s$ gives to $[x_s : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})]$, which is a point $s \in D = G(\mathbb{R})/M$, where $G = MT(\mathbb{V})$ and M some compact subgroup. Finally Γ in an arithmetic lattice of $G(\mathbb{Q})_+$.

- $\Gamma \backslash D$ is a complex analytic variety, but, unless it is a **Shimura variety**, it is not algebraic;
- Ψ is holomorphic. It is not necessarily an immersion, but it can be assumed proper;
- Griffiths transversality says that $d\Psi$ maps the tangent bundle of S to the horizontal tangent bundle $T_h(D) \subset T(D)$

Functoriality

Let $Y \subset S$ a (smooth irreducible closed strict) subvariety. It supports $\mathbb{V}_{|Y}$, which corresponds to a period map

$$Y(\mathbb{C}) \to \Gamma_Y \backslash D_Y,$$

where D_Y is a homogeneous space under $H_Y = MT(\mathbb{V}_{|Y})$. But $H_Y \subset G = MT(\mathbb{V})$ and functoriality gives a commutative diagram

$$\begin{array}{ccc} Y(\mathbb{C}) & \xrightarrow{\Psi_{|Y}} & \Gamma_Y \backslash D_Y \\ \downarrow & & \downarrow \\ S(\mathbb{C}) & \xrightarrow{\Psi} & \Gamma \backslash D \end{array},$$

Special subvarieties

We say that Y is (strict) special if $Y = \Psi^{-1}(\Gamma_Y \backslash D_Y)^0$. Informally

$$\Psi(Y) = \Psi(S) \cap \Gamma_Y \backslash D_Y \subset \Gamma \backslash D.$$

Hodge locus = union of all special = union of maximal special.

Definition

A special subvariety Y is either typical or atypical:

TY $\operatorname{codim}_{\Gamma \backslash D}(\Psi(Y)) = \operatorname{codim}_{\Gamma \backslash D}(\Psi(S)) + \operatorname{codim}_{\Gamma \backslash D}(\Gamma_Y \backslash D_Y)$;

ATY $\operatorname{codim}_{\Gamma \backslash D}(\Psi(Y)) < \operatorname{codim}_{\Gamma \backslash D}(\Psi(S)) + \operatorname{codim}_{\Gamma \backslash D}(\Gamma_Y \backslash D_Y)$.

Conjecture (Zilber-Pink type conjecture for VHS)

$\mathrm{HL}(S, \mathbb{V}^{\otimes})_{\text{atyp}}$ *is **algebraic**, i.e. a finite union of maximal special subvarieties.*

Remarks and examples

- Our ZP conjecture refines the one proposed by Klingler, and contains the 'classical' ZP for Shimura varieties. So it implies André-Oort, ...
- HL$(S, \mathbb{V}^{\otimes})$ = HL$(S, \mathbb{V}^{\otimes})_{\text{atyp}} \amalg$ HL$(S, \mathbb{V}^{\otimes})_{\text{typ}}$. If HL$_{\text{typ}} = \emptyset$, then ZP predicts the algebraicity of the whole HL.
- Example. Let $C \subset \mathcal{A}_g$ be a Hodge generic curve, $g > 3$. Then the HL can only be atypical ($\mathcal{A}_g$ has no special divisors). So ZP predicts that HL$(C, \mathbb{V}^{\otimes})$ is a finite union of points!

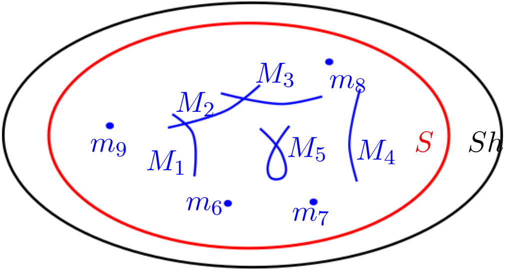
Special points are very difficult to understand and are reach in arithmetic, but we can understand the geometric part of the HL.

Definition

A subvariety $Z \subset S$ is of **positive period dimension** if $\dim(\Psi(Z)) > 0$.

André–Oort in one picture (intersections=inclusions)

S Hodge generic in a Shimura Sh



- Blue are the most atypical intersections: $\operatorname{codim}_{Sh}(M_n = M_n \cap S) < \operatorname{codim}_{Sh} S + \operatorname{codim}_{Sh} M_n$.
- Announced for every Shimura variety by Pila, Shankar and Tsimerman.

Main results, I

We proved the geometric part of Zilber–Pink for VHS ($+\epsilon$):

Theorem (B.-Klingler-Ullmo)

The maximal atypical special subvarieties of positive period dimension arise in a finite number of families, and each family lies in a typical intersection.

Informally: $\mathsf{HL}(S, \mathbb{V}^{\otimes})_{\mathsf{atyp,pos}}$ is algebraic. Generalises work of Daw-Ren regarding Shimura varieties.

Theorem (B.-Klingler-Ullmo)

Suppose that the level of $\mathbb{V}$ is at least 3. Then $\mathsf{HL}(S, \mathbb{V}^{\otimes})_{\mathsf{typ}} = \emptyset$.

The level of a VHS refines the "(normalised) weight" and measures how complicated $\mathbb{V}$ is ($\sim$ how far it is from a family of abelian motives). The biggest k for which $\mathfrak{g}^{k,-k} \neq 0$.

Main results, II 'algebraicity'

Putting things together in level ≥ 3:

$$\mathsf{HL}(S, \mathbb{V}^{\otimes}) = \mathsf{HL}(S, \mathbb{V}^{\otimes})_{\mathsf{atyp}}.$$

ZP then predicts that $\mathsf{HL}(S, \mathbb{V}^{\otimes})_{\mathsf{atyp}}$ is algebraic, we get

Theorem (B.-Klingler-Ullmo)

*If $\mathbb{V}$ has **level** at least 3, then $\mathsf{HL}(S, \mathbb{V}^{\otimes})_{\textsf{f-pos}}$ is algebraic.*

A concrete example is given by the moduli spaces of ample, smooth hypersurfaces/complete intersections of degree big enough in a projective variety.

Corollary (B.-Klingler-Ullmo)

Let $\mathbb{P}_{\mathbb{C}}^{N(n,d)}$ *be the projective space parametrising hypersurfaces* $X \subset \mathbb{P}_{\mathbb{C}}^{n+1}$ *of degree* d. *Let* $U_{n,d} \subset \mathbb{P}_{\mathbb{C}}^{N(n,d)}$ *be the Zariski-open subset parametrising the smooth hypersurfaces and let* $\mathbb{V} \to U_{n,d}$ *be the* $\mathbb{Z}$*VHS corresponding to the primitive cohomology* $H^n(X, \mathbb{Z})_{\mathsf{prim}}$.
If $n \geq 3$, $d \geq 5$ *and* $(n, d) \neq (4, 5)$ *then* $\mathsf{HL}(U_{n,d}, \mathbb{V}^{\otimes})_{\mathsf{pos}} \subset U_{n,d}$ *is algebraic.*

To complete the picture:

Theorem (B.-Klingler-Ullmo)

In level 1 and 2, if $\mathsf{HL}(S, \mathbb{V}^{\otimes})_{\mathsf{typ}} \neq \emptyset$ *(possibly the* ***zero*** *dimensional part), then* $\mathsf{HL}(S, \mathbb{V}^{\otimes})$ *is dense.*

See also the work of Tayou and Tholozan. Can we predict when $\mathsf{HL}(S, \mathbb{V}^{\otimes})_{\mathsf{typ}}$ is non-empty?

Some applications

- A final (geometric) step in the Lawrence-Venkatesh method, to obtained refined results.
- Serre/Gross question on the existence of Jacobians with a given Mumford-Tate group (the Hodge locus of $\mathcal{M}_4$);

Moduli space of hypersurfaces and integral points

Let K be a number field not containing a CM field.

Theorem (Lawrence-Venkatesh)

There exist $n_0 \geq 3$ and a function $d_0 : \mathbb{N} \to \mathbb{N}$ such that,

$$\text{for every } n \geq n_0 \text{ and } d \geq d_0(n), \tag{0.1}$$

the set $U_{n,d}(\mathcal{O}_{K,S})$ is not Zariski dense in $U_{n,d,\mathbb{C}}$, for every K and S.

They actually prove that each positive period dimensional component of $\overline{U_{n,d}(\mathcal{O}_{K,S})}^{\mathrm{Zar}}$ is in the Hodge locus (since *the monodromy drops*).

In particular

$$\bigcup_{K,S} \overline{U_{n,d}(\mathcal{O}_{K,S})}_{\mathsf{pos}} \subset \mathsf{HL}(U_{n,d}, \mathbb{V}^{\otimes})_{\mathsf{pos}}.$$

By our main theorem, the latter is an **algebraic subvariety** of $U_{n,d}$ (rather than a countable union of such). So

$$\overline{\bigcup_{K,S} \overline{U_{n,d}(\mathcal{O}_{K,S})}_{\mathsf{pos}}} \subset \mathsf{HL}(U_{n,d}, \mathbb{V}^{\otimes})_{\mathsf{pos}}.$$

I.e. we proved the following special case of the *refined Bombieri-Lang*.

Theorem (B.-Klingler-Ullmo)

There exists a closed strict subvariety $E \subset U_{n,d}$ such that, for all K and all S, we have

$$\overline{U_{n,d}(\mathcal{O}_{K,S})}_{\mathsf{pos}} \subset E.$$

Otherwise stated: the Zariski closure of $U_{n,d}(\mathcal{O}_{K,S}) - E(\mathcal{O}_{K,S})$ has period dimension zero.

THANKS FOR YOUR ATTENTION!

If there's still time: Serre's open image Theorem

Let K be a number field, A/K be a g-dimensional (pp) abelian variety

$$\rho = \rho_{A,\ell} : \mathrm{Gal}(\overline{K}/K) \to \mathbf{GSp}_{2g}(\mathbb{Z}_\ell).$$

Assume that $\mathrm{End}(A/\mathbb{C}) = \mathbb{Z}$:

- If $g = 1$, then ρ has open image in $\mathrm{GL}_2(\mathbb{Z}_\ell)$ (and for ℓ big enough it is actually surjective);
- If $g \not\equiv 0 \mod 4$, then ρ has open image (Serre);
- If $g = 4$ this fails (Mumford).

Mumford–Tate group of an abelian variety

- Given an abelian variety A, $H^1(A,\mathbb{Q})$ is a "Hodge structure"

$$x_A : \mathbb{S} = \mathrm{Res}_{\mathbb{R}}^{\mathbb{C}}(\mathbb{G}_m) \to \mathrm{GL}(H^1(A,\mathbb{R})).$$

  The **Mumford-Tate group** of A is the $\mathbb{Q}$-Zariski closure of x_A.
- Mumford constructed examples of 4-dim abelian varieties A with $\mathrm{End}(A)=\mathbb{Z}$ and $\mathbf{MT}(A)$ strictly contained in $\mathbf{GSp}_8/\mathbb{Q}$.
- Gross/Serre: can we find a Jacobian "of Mumford's type"?

Theorem (B.-Klingler-Ullmo)

There exists a smooth projective curve C/K of genus 4 whose Jacobian is of Mumford's type (i.e. $\mathbf{MT}(J(C))$ is isogenous to a $\mathbb{Q}$-form of the complex group $\mathbb{G}_m \times \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$).

Sketch of the proof (typical and atypical)

- Let $\mathcal{M}_4$ be the moduli space of curves of genus 4;
- $j : \mathcal{M}_4 \hookrightarrow \mathcal{A}_4$, $C \mapsto J(C)$;
- $\dim \mathcal{A}_4 = 10$ and $\dim \mathcal{M}_4 = 9$;
- Mumford constructed special curves $(M_n)_{n\in\mathbb{N}} \subset \mathcal{A}_4$ whose group is some $\mathbb{Q}$-from of $\mathbb{G}_m \times \mathrm{SL}_2^3$;
- We have to find a **typical** point in $M_n \cap \mathcal{M}_4$.

Since **10=9+1**, some M_n should intersect $\mathcal{M}_4$ in a zero dimensional set.

- $P \in \mathcal{M}_4 \cap M_n$ is Jacobian with CM; or
- $P \in \mathcal{M}_4 \cap M_n$ is a Jacobian with $\mathbf{MT}_{\mathbb{C}} = \mathbb{G}_m \times \mathrm{SL}_2^3$.

Almost all M_n should cut $\mathcal{M}_4$.
The first case is an atypical intersection, and so it should not happen for all n, and all P. We "found" the desired genus 4 curve (∞-many)!

General Theorem

Theorem (B.-Klingler-Ullmo)

Let $(\mathbf{G}, X)$ be a Shimura datum such that $\mathbf{G}$ is absolutely simple and containing a one dimensional Shimura sub-datum $(\mathbf{H}, X_H)$. Let $S \subset \Gamma \backslash X$ be an irreducible subvariety of codimension one. Then the typical Hodge locus of S is (analytically) dense.