Independence of points on elliptic curves coming from modular curves

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- $E/\overline{\mathbb{Q}}$ an elliptic curve;
- a (non-constant) $\overline{\mathbb{Q}}$-morphism

$$\phi : X \to E.$$

Motivation

Theorem (Gross-Zagier, Kolyvagin $\sim$ 1990)

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The crux in (2) is to construct a non-torsion point in $E(\mathbb{Q})$. This is done constructing (special) points on X: it is easier to construct points on a moduli space such as X. Especially CM points. . .

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Theorem (Nekovar, Schappacher 1999)

There are only finitely many torsion $\phi(P_{\mathfrak{a}})$ on any elliptic curve E over $\mathbb{Q}$.

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All together: Let Γ be a finite rank subgroup of $E(\overline{\mathbb{Q}})$, and for every $\epsilon \geq 0$ define

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We want to find Σs such that for some $\epsilon > 0$, $\phi(\Sigma) \cap \Gamma_\epsilon$ is finite.

Natural choices: Zilber-Pink conjecture for mixed Shimura varieties

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Let C be a curve over an algebraically closed field of characteristic zero. The curve C, seen in its Jacobian variety J, can only contain a finite number of points that are of finite order in J, unless $C = J$.

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- Subvarieties of abelian varieties having *large* intersection with the subgroups described before, are *quite special*: Manin-Mumford, Mordell, Bogomolov conjectures.
- Subvarieties of Shimura varieties having *large* intersection with Σ are *quite special*, whenever Σ consists of CM points or an isogeny class.

Zilber-Pink and Bogomolov

Conjecture

Let S be a Shimura variety with $\Sigma \subset S$ be either an isogeny class or the set of CM points, A an abelian variety and $\Gamma \subset A(\overline{\mathbb{Q}})$ a finite rank subgroup. An irreducible subvariety $V \subset S \times A$ containing a dense set of points lying in $\Sigma \times \Gamma_\epsilon$ for every $\epsilon > 0$, is weakly special.

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By weakly special we mean an irreducible algebraic subvariety of $S \times A$ that can be written as a product $S' \times A'$, where S' is such that its smooth locus is totally geodesic in S and A' is a translate of an algebraic subgroup of A.

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Remark

There is a more general conjecture about unlikely intersection, the *Zilber-Pink conjecture*, for mixed Shimura varieties that indeed implies the above one when $\epsilon = 0$.

CM points

Recall that Heegner points on elliptic curves are particular points coming from $X(\mathsf{CM})$, i.e. they correspond to elliptic curves with CM by $\mathcal{O}_K$ for some quadratic imaginary field K (satisfying the Heegner hypothesis).

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Theorem (Buium-Poonen, 2007)

For some $\epsilon > 0$, the set $\phi(X(\mathit{CM})) \cap \Gamma_\epsilon$ is finite.

Main theorem

Let $x \in X(\overline{\mathbb{Q}})$ be a non-cuspidal point corresponding to a pair (E_x, Ψ_x). By isogeny class Σ_x we mean the subset of $X(\overline{\mathbb{Q}})$ corresponding to elliptic curves admitting an isogeny to E_x (possibly without respecting the extra structure Ψ_x).

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Theorem (G.B.)

Let $E/\overline{\mathbb{Q}}$ be an elliptic curve and $\Gamma \subset E(\overline{\mathbb{Q}})$ a finite rank subgroup. Let $\phi : X \to E$ be a non-constant morphism defined over $\overline{\mathbb{Q}}$. For some $\epsilon > 0$, the image of an isogeny class $\Sigma_x \subset X(\overline{\mathbb{Q}})$ intersects Γ_ϵ in only finitely many points.

Remarks about O-minimality

- O-minimality, via the *Pila-Wilkie counting theorem*, is a powerful tool often used to (re)prove results of this kind. Indeed it can be used to prove Manin-Mumford, André-Oort, and many instances of the Zilber-Pink conjecture;

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- Independently Gabriel Dill proved the version with $\epsilon = 0$ of the theorem, using O-minimality and the Pila-Zannier strategy;
- Recently Pila and Tsimermann have also obtained a generalization of the $\epsilon = 0$ part of the theorem;
- It seems that the Bogomolov part of the theorem ($\epsilon > 0$) can not be proven using such strategy. Indeed our proof relays on equidistribution results, as in the proof of the Bogomolov conjecture (Ullmo, Zhang 1990)...

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- Hecke orbits are equidistributed with respect to the hyperbolic measure on X;
- The corresponding Galois orbits on the abelian variety side equidistribute to the Haar measure on $E(\mathbb{C})$;
- The two measures are "incomparable".

Probability measures

- On $X/\mathbb{C}$ we have the hyperbolic measure μ_X: it is the measure whose pullback to $\mathbb{H}$ equals a multiple of the hyperbolic measure $y^{-2}\mathrm{d}x\mathrm{d}y$;

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For example for every continuous real function f over E we have

$$\int_E f(x)d\mu_E(x) = \lim_{n\to\infty} \frac{1}{n^2} \sum_{p\in E[n]} f(p).$$

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- Given a point p we denote by δ_p the Dirac measure supported on p.

Preliminary assumptions

- We may assume X, E, ϕ, x are all defined over a number field K and that Γ is contained in the division hull of $E(K)$;

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Heading for a contradiction we may assume that that, for every $\epsilon > 0$, the set $\Sigma_x \times \Gamma_\epsilon$ is dense in the graph of ϕ. Therefore we may find a generic infinite sequence of points $(x_n, a_n)_n$ such that $x_n \in \Sigma_x$, $\phi(x_n) = a_n$ and $a_n \in \Gamma_{\epsilon_i}$ where $\epsilon_i \to 0$.

Sequence of measures on X

Consider the sequence of measures on $X(\mathbb{C})$

$$\Delta(x_n) := \frac{1}{|\operatorname{Gal}(\overline{K}/K).x_n|} \sum_{p \in \operatorname{Gal}(\overline{K}/K).x_n} \delta_p.$$

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In particular we can translate a result of Clozel-Ullmo about the equidistribution of Hecke points on Shimura varieties, in a equidistribution result about the Galois conjugates of x:

$$\Delta(x_n) \to \mu_X, \text{ as } n \to +\infty.$$

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$$[K(x_n):K] \leq \deg(\phi)[K(a_n):K],$$

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and $[K(x_n) : K]$ tends to infinity since the x_ns lie in an infinite isogeny class and the boundedness of such degree would prevent the equidistribution of the $\Delta(x_n)$s.
The last property can be also seen using the Masser-Wüstholz Isogeny Theorem.

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But this violates the condition that the two measures are incomparable.

THANKS FOR YOUR ATTENTION!