Local to global principle for the moduli space of K3 surfaces

Gregorio Baldi

Workshop on Galois representations and K3 surfaces organised by Martin Orr and Alexei Skorobogatov

02/05/2018

Notations

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- K a number field, $\overline{K}$ a fixed algebraic closure, $\mathrm{Gal}(\overline{K}/K)$ its absolute Galois group;

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- a fixed embedding $\overline{K} \hookrightarrow \mathbb{C}$.

Motivation: section conjecture for the moduli space of abelian varieties

$\mathcal{A}_{\mathrm{g}} :=$ moduli space of p.p.a.v. of dimension g;
It is a Deligne-Mumford stack (or an orbifold) defined over $\mathbb{Q}$.

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Question

Are sections s of $\mathcal{A}_{\mathrm{g}} / K$ locally induced by points induced by global points?

Selmer set and family of Galois representations

When $g > 1$ we have:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(\mathcal{A}_{g,\mathbb{C}}) & \longrightarrow & \pi_1(\mathcal{A}_g) & \longrightarrow & \mathrm{Gal}(\overline{K}/K) & \to & 1 \\
 & & \downarrow \cong & & \downarrow & & \downarrow \chi & & \\
1 & \longrightarrow & \mathrm{Sp}_{2g}(\widehat{\mathbb{Z}}) & \longrightarrow & \mathrm{GSp}_{2g}(\widehat{\mathbb{Z}}) & \longrightarrow & \widehat{\mathbb{Z}}^* & \longrightarrow & 1
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Sections $\rightsquigarrow$ ℓ-adic families of Galois representations.

Question

Is it possible to find some 'local' representation-theoretical properties to ensure that a family of ℓ-adic reps comes from an abelian variety?

Weakly compatible family of ℓ-adic representations

Definition (Weakly compatible, after Serre)

A family $\{\rho_\ell : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_n(\mathbb{Q}_\ell)\}_\ell$ is *weakly compatible* if there exists a finite set of places Σ of K such that

(i) for all ℓ, ρ_ℓ is unramified outside the union of Σ and the places Σ_ℓ of K dividing ℓ;

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Example (Deligne)

If X is a smooth projective variety defined over K, $\{H^i_{\mathrm{et}}(X_{\overline{K}}, \mathbb{Q}_\ell(j))\}_\ell$ form a weakly compatible system.

Patrikis-Voloch-Zarhin's result (2016)

Let $\{\rho_\ell : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_{2N}(\mathbb{Q}_\ell)\}_\ell$ be a weakly compatible system such that for some primes ℓ_0, ℓ_1, ℓ_2 we have

- ρ_{ℓ_0} is de Rham at all places of K above ℓ_0;

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- ρ_{ℓ_1} is absolutely irreducible;
- there is at least one place $v \in \Sigma_{\ell_2}$, such that $\rho_{\ell_2 | \mathrm{Gal}(\overline{K_v}, K_v)}$ is de Rham with Hodge-Tate weights $-1, 0$ each with multiplicity N.

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Then, assuming some well known conjectures, there exists an abelian variety A defined over K such that $\rho_\ell \cong V_\ell(A)$ for all ℓ.

Formalism of motives

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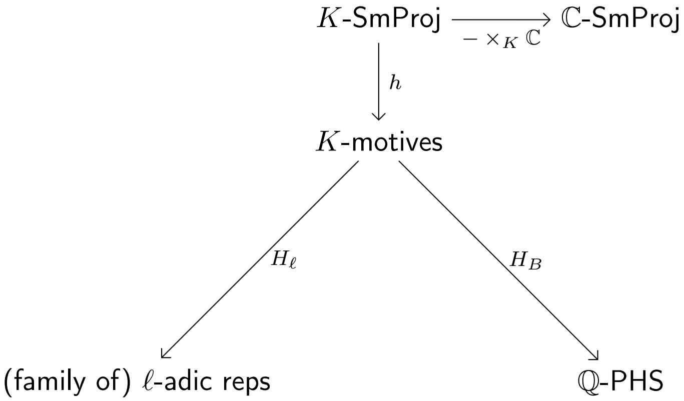
We fix a family of embeddings $\iota_\ell : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell$ and write

$$H_\ell : \mathcal{M}_{K,E} \to \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\mathrm{Gal}(\overline{K}/K))$$

for the ℓ-adic realisation functors associated to ι_ℓ.

A picture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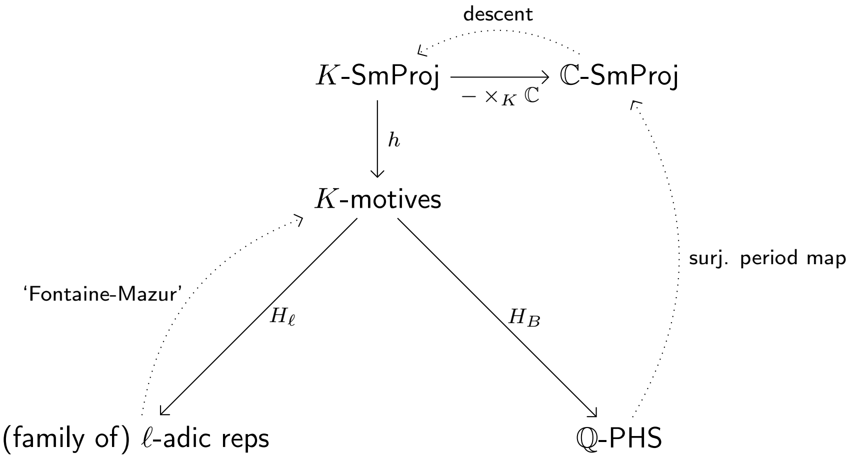
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The conjecture

Conjecture

Let $r_\ell : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_n(\mathbb{Q}_\ell)$ *be an irreducible geometric Galois representation. Then there exists an object* $M \in \mathcal{M}_{K,\overline{\mathbb{Q}}}$ *such that*

$$r_\ell \otimes \overline{\mathbb{Q}}_\ell \cong H_\ell(M) \in \mathit{Rep}_{\overline{\mathbb{Q}}_\ell}(\mathrm{Gal}(\overline{K}/K)).$$

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Remark

We work with compatible systems of ℓ-adic reps, rather than a fixed ρ_ℓ, to produce an object in $\mathcal{M}_K$, rather than $\mathcal{M}_{K,\overline{\mathbb{Q}}}$.

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K3 surfaces and Galois representations

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Question

Given $\{\rho_\ell\}_\ell$ a weakly compatible system of ℓ-adic representations of $\mathrm{Gal}(\overline{K}/K)$ that looks like the transcendental part of a K3 surface, can we construct a K3 surface X (defined over K) such that $T(X_{\overline{K}})_{\mathbb{Q}_\ell} \cong \rho_\ell$ for all ℓs?

Motive of a surface (after Murre-Pedrini)

We can isolate the transcendental part of the motive of a surface X:

$$h_2(X) = \left(h^2_{alg}(X) \oplus t_2(X)\right),$$

where $h^2_{alg}(X) = (X, \pi_2^{alg}, 0)$ and $t_2(X) = (X, \pi_2^{tr}, 0)$, for a refined Künneth decomposition $\pi_2 = \pi_2^{alg} + \pi_2^{tr}$.

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$$H_B(h^2_{alg}(X) \oplus t_2(X)) = \mathrm{NS}(X)_{\mathbb{Q}} \oplus T(X)_{\mathbb{Q}},$$

$$H_\ell(h^2_{alg}(X) \oplus t_2(X)) = \mathrm{NS}(X_{\overline{K}})_{\mathbb{Q}_\ell} \oplus T(X_{\overline{K}})_{\mathbb{Q}_\ell}.$$

Local conditions

For a *refined Fontaine-Mazur* we need to work with the following local conditions:

1. For some prime ℓ_0, ρ_{ℓ_0} is de Rham at all places of K above ℓ_0;

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3. For some prime ℓ_2 and at least one place $v \in \Sigma_{\ell_2}$, $\rho_{\ell_2 | \mathrm{Gal}(\overline{K}_v/K_v)}$ is de Rham with Hodge-Tate weights of a K3 surface, and multiplicities, respectively, $1, 20-\rho, 1$.

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Note that condition (3) is satisfied if there exists a $K3$ surface X_v/K_v of Picard rank ρ and $\rho_{\ell_2 | \mathrm{Gal}(\overline{K}_v/K_v)}$ is isomorphic to the representation induced by $T(X_{\overline{K_v}})_{\mathbb{Q}_\ell}$.

Main theorem

Theorem

Let $\rho \in \mathbb{N}$ *be such that* $2 < 22 - \rho \leq 19$. *Assume the Tate, Fontaine-Mazur and the Hodge conjectures. Let*

$$\{\rho_\ell : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_{22-\rho}(\mathbb{Q}_\ell)\}_\ell$$

be a weakly compatible family of ℓ*-adic representations satisfying the conditions (1), (2), (3).*

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be a weakly compatible family of ℓ-adic representations satisfying the conditions (1), (2), (3).
Then there exists a simple motive M defined over K inducing the representations ρ_ℓ and a finite extension L/K, such that the base change of M to L is isomorphic to the transcendental part of the motive of a K3 surface defined over L.

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- We do not have enough information to reconstruct the algebraic part of the H^2. This is why we need a finite extension...
- the transcendental part determines the full H^2 only in particular cases (Nikulin)...

Proof

Choosing a place ℓ_0 as in (1), our conjectural description of the essential image of H_{ℓ_0} ensures the existence of a motivic Galois representation

$$\rho : \mathcal{G}_{K,E} \to \mathsf{GL}_{22-\rho,E}$$

for some number field E, such that $H_{\ell_0}(\rho) \cong \rho_{\ell_0} \otimes \overline{\mathbb{Q}}_{\ell_0}$ (the same holds for every ℓ).

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for some number field E, such that $H_{\ell_0}(\rho) \cong \rho_{\ell_0} \otimes \overline{\mathbb{Q}}_{\ell_0}$ (the same holds for every ℓ). The obstruction to descending ρ to a $\mathbb{Q}$-rational representation of $\mathcal{G}_K$ is an element $\mathsf{obs}_\rho \in H^1(\mathrm{Gal}(E/\mathbb{Q}), \mathrm{PGL}_{22-\rho}(E))$.

Lemma (P-V-Z)

In fact obs_ρ lies in

$$\ker(H^1(\mathrm{Gal}(E/\mathbb{Q}), \mathrm{PGL}_{22-\rho}(E)) \to \prod_\ell (\mathrm{Gal}(E_\lambda/\mathbb{Q}_\lambda), \mathrm{PGL}_{22-\rho}(E_\lambda))).$$

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$$H_{dR}(M) \otimes_K K_v \cong D_{dR,K_v}(H_{\ell_2}(M)).$$

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Tanks to the Betti-de Rham comparison isomorphism we conclude that $H_B(M_{|\mathbb{C}})$ is a polarizable rational Hodge structure of weight two and with Hodge numbers $1-(20-\rho)-1$, since $\rho_{\ell_2|\,\mathrm{Gal}(\overline{K}_v/K_v)}$ has such multiplicities.

Surjectivity of the period map

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Proposition

Let (V, h, ψ) be a $\mathbb{Q}$-PHS of K3 type of dimension $22-\rho$. If $2 \leq 22-\rho \leq 19$, then there exists a complex K3 surface X with $T(X)_{\mathbb{Q}}$ isomorphic to (V, h, ψ) as rational Hodge structures.

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Such proposition requires ρ to be different from 1 and 2, where some restrictions on the square class of the determinant of (V, ψ) and its Hasse invariant appear.

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Thanks to the Hodge conjecture we can lift the isomorphism of Hodge structures to get an isomorphism at the level of motives:

$$t_2(X) \cong M_{|\mathbb{C}} \in \mathcal{M}_{\mathbb{C}},$$

where $t_2(X)$ is the transcendental part of the motive of X

Since M is defined over a number field, for all $\sigma \in \mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$, we have the following chain of isomorphisms:

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It follows that, for all$\sigma \in \mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$ we have an isomorphism of $\mathbb{Q}$-PHS

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Descent

We are left to prove the following.

Theorem

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Then X admits a model defined over $\overline{\mathbb{Q}}$.

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Proof.

For the first point, use the fact there X admits only finitely many Fourier-Mukai partners (Mukai).

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- If all the conjugates of X fall into countably many isomorphism classes, then X descends to a number field.

Proof.

For the first point, use the fact there X admits only finitely many Fourier-Mukai partners (Mukai).
For the second, use that K3s (with some extra structure) have a fine moduli space defined over $\overline{\mathbb{Q}}$ (Rizov). □

How to get rid of the extension L/K?

Question

Assume that $\mathcal{M}_K$ is a semisimple neutral Tannakian category over $\mathbb{Q}$. Let $M \in \mathcal{M}_K$ be a simple motive defined over some number field K. Assume there exists a finite extension L/K such that M_L is isomorphic to the transcendental part of the motive of Y_L, a K3 surface defined over L. Is there a K3 surface X defined over K such that

$$t_2(X) \cong M \in \mathcal{M}_K.$$

Answer for abelian varieties

Proposition

Let K be a number field, and assume that the category $\mathcal{M}_K$ is a semisimple neutral Tannakian category over $\mathbb{Q}$. Let $M \in \mathcal{M}_K$ be a simple motive such that, after a finite extension L/K,

$$M_L \cong H_1(A_L) \in \mathcal{M}_L$$

for some abelian variety A_L defined over L.

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Then there exists an abelian variety A/K such that

$$M \cong H_1(A) \in \mathcal{M}_K.$$

Proof

Faltings proved that the following functor is full (and faithful):

$$H_1(-) : \mathsf{AV}_K^0 \to \mathcal{M}_K, \quad B \mapsto H_1(B).$$

Consider the K-ab. var. $\mathrm{Res}_{L,K}(A_L)$ and notice that $H_1(\mathrm{Res}_{L,K}(A_L))$ corresponds to $\mathsf{Ind}_L^K(H_1(A_L))$.

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$$\mathrm{Hom}_{\mathcal{M}_K}(M, \mathsf{Ind}_L^K(H_1(A_L))) \neq 0.$$

Since M is simple, an element in such Hom-set realizes M as a direct summand of $H_1(\mathrm{Res}_{L,K}(A_L))$ in $\mathcal{M}_K$, therefore in AV_K^0.

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Otherwise stated there exists an endomorphism of $\mathrm{Res}_{L,K}(A_L)$ whose image is an abelian variety A/K such that $H_1(A) \cong M \in \mathcal{M}_K$.

THANKS FOR YOUR ATTENTION!