Special subvarieties of non-arithmetic ball quotients

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- Lattices are Zariski dense (Borel);
- They contain a finite index subgroup which is torsion free (Selberg);
- If Γ is discrete and $\Gamma\backslash G$ is compact, then Γ is a lattice (but this is not the only way).

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Example: $\mathbf{G} = \mathbf{GSp}_{2g}/\mathbb{Q}$, $G/K = \mathbb{H}_g$ the g-dim Siegel space, and

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is naturally the moduli space of principally polarised g-dimensional abelian varieties.

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For each $n > 1$ how many non-arithmetic lattices in $\mathrm{PU}(1,n)$ are there? Can arithmeticity be detected at the topological level?

Motivation: generalising the theory of Shimura varieties

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What is a *special subvariety* of S_Γ?

Main result

Theorem (B.-Ullmo)

If S_Γ contains infinitely many maximal complex totally geodesic subvarieties, then $\Gamma \subset G$ is arithmetic.

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1. and 2. build on superrigidity theorems and use results on equidistribution from homogeneous dynamics.

Main steps of the strategy

Goal: parametrise maximal special subvarieties by a countable and definable set (in some o-minimal structure)!

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Theorem (B.-Ullmo)

New proof of Margulis commensurator theorem for lattices in $\mathrm{PU}(1, n)$, $n > 1$ (and some lattices in $\mathrm{PU}(1, 1)$).

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$\Rightarrow \Gamma \subset \mathbf{G}(K)$.

Simpson's theory and Weil restriction

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- Eventually $\bigoplus_\sigma \mathbb{V}^\sigma$ has a natural structure of $\mathbb{Z}$-VHS!

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It may happen that $\widehat{\mathbf{G}}(\mathbb{Z})\backslash D$ is a Shimura variety. But in general $\widehat{\mathbf{G}}(\mathbb{Z})\backslash D$ is not algebraic, and ψ is holomorphic.

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- All the G_{σ_i} are isomorphic over $\mathbb{C}$, so they are $\mathrm{PU}(p_{\sigma_i}, q_{\sigma_i})$, $p_{\sigma_i} + q_{\sigma_i} = n + 1$;

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- $\tilde{\psi}$ detects arithmeticity: X' is finite iff Γ is arithmetic (Mostow-Vinberg).

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- What about the other implication? Can we take all $\mathbf{M}$s to be Weil restriction?

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The crux in our strategy:

Theorem (B.-Ullmo)

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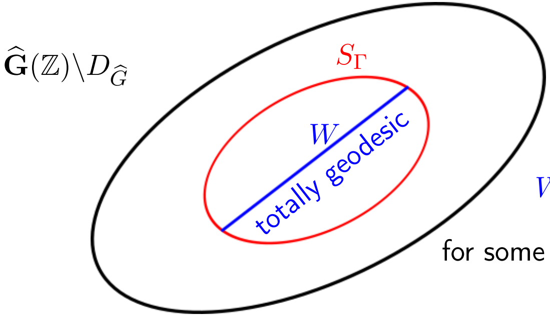
5. *W is a component of $\psi^{-1}(\pi(Y))$ for some algebraic subvariety Y of $D^\vee$.*

A picture



$W = S_\Gamma \cap \widehat{\mathbf{H}}(\mathbb{Z})\backslash D_{\widehat{H}}$

for some $\widehat{\mathbf{H}}(\mathbb{Z})\backslash D_{\widehat{H}} \hookrightarrow \widehat{\mathbf{G}}(\mathbb{Z})\backslash D_{\widehat{G}}$

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Two objects of codimension 2 in a 4-dimensional space, should intersect in a finite number of points, not in a curve!...

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$$S_{x,\hat{g}} := \psi^{-1}\pi(\hat{g}\widehat{H}\hat{g}^{-1}.\tilde{\psi}(x)) \subset {S_\Gamma}^{\mathrm{an}}$$

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such that the projection of $\widehat{U}$ to S_Γ contains $S_{x,\hat{g}}$

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We conclude by applying Bakker-Tsimerman's version of Ax-Schanuel for $(S_\Gamma, \widehat{\mathbb{V}})$: $S_{x,\hat{g}}$ is $\mathbb{Z}$-special.

THANKS FOR YOUR ATTENTION!