Lieu de Hodge et applications : quelques résultats et conjectures

Gregorio Baldi

26/05/2025

- Habilitation à Diriger des Recherches : synthèse de l'activité scientifique
- Discipline : Mathématiques

SORBONNE UNIVERSITÉ

Plan: unifications via typical-vs-atypical intersections

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1. *Opening.* A guiding example: smooth hypersurfaces
   - Surfaces and the Noether-Lefschetz locus
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4. *Post Mortem.* Ideas behind some proofs

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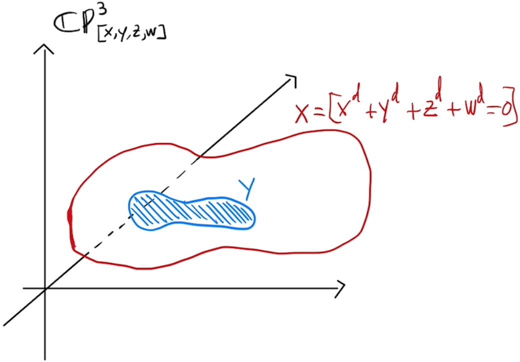
Example (Fermat curve)

$$Y = \{[X, Y, Z] \in \mathbb{CP}^2 : X^d + Y^d + Z^d = 0\}$$

Higher dimension

Goal

Understand higher dimensional varieties by looking at their lower dimensional subvarieties.



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Parameter space: $U_{n,d} := V - \Delta.$

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*What can we say about a **very general** hypersurface X?*

I.e. $[X]$ in the complement of a countable union of subvarieites of $U_{n,d}$.

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$$\boxed{U_{2,d} - \bigcup_{n,g} \rho(W(n,g))}$$

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- $\dim \mathbb{G}r(1,3) = 4$, so
  $\operatorname{codim} W = d+1-4 = d-3 > 0$, since $d \geq 4$.

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Theorem (Noether-Lefschetz theorem)

If $d \geq 4$, NL_d *is a countable union of strict subvarieties of* $U_{2,d}$. *Any* $X \notin \mathrm{NL}_d$ *has* $\mathrm{Pic}\, X = \langle \mathcal{O}_X(1) \rangle$.

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Exceptional components exist only for $d \geq 5$.

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Theorem (B.-Klingler-Ullmo)

The exceptional components are not Zariski dense in $U_{2,d}$.

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$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$$

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$$\boxed{\mathrm{HL}(U_{n,d},\mathbb{V}^{\otimes}) := \{x \in U_{n,d} : \mathbb{V}_{|x} \text{ has extra Hodge tensors}\}}$$

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Example

For $d \geq 4$, $\mathrm{NL}_d \subset \mathrm{HL}(U_{2,d}, \mathbb{V}^{\otimes})$. Are they equal?

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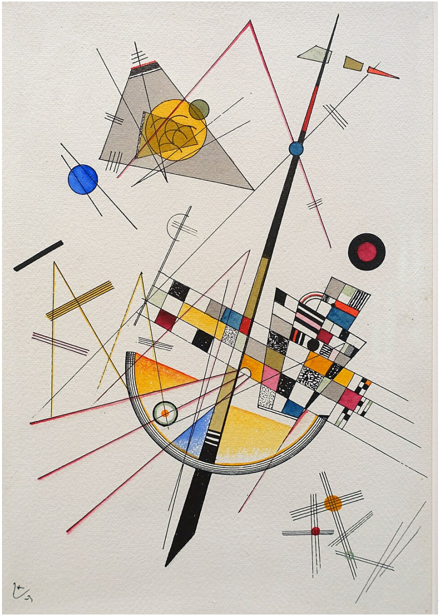
- CDK is **predicted** by the Hodge conjecture

Delicate tension (Kandinsky 1923) of $\mathrm{HL}(S, \mathbb{V}^{\otimes})$



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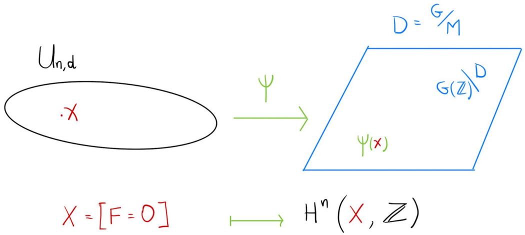

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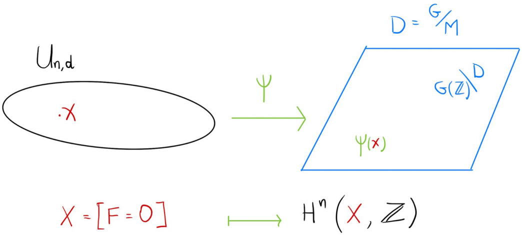

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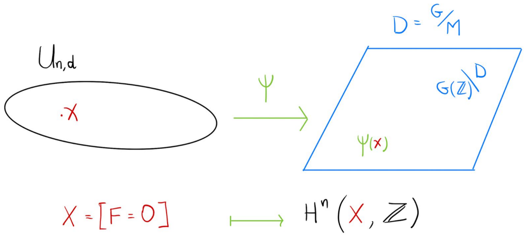

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(P.p.) abelian scheme over $S \rightsquigarrow S \to \mathcal{A}_g = \mathrm{Sp}_{2g}(\mathbb{Z})\backslash \mathbb{H}_g$
Notable example: $\mathcal{M}_g \to \mathcal{A}_g, C \mapsto \mathrm{Jac}(C)$.

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Remark (Refined Bombieri-Lang conjecture)

The above can be plugged into the Lawrence-Venkatesh method to get finer results on integral points of $U_{n,d}$.

Components of the Hodge locus as intersections: Functoriality for $(S, \mathbb{V})$

$Y \subset S$ supports $\mathbb{V}_{|Y}$, which corresponds to a period map

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$$\begin{tikzcd} Y(\mathbb{C}) \arrow[r, "\Psi_{|Y}"] \arrow[d] & \Gamma_Y \backslash D_Y \arrow[d] \\ S(\mathbb{C}) \arrow[r, "\Psi"] & G(\mathbb{Z}) \backslash D \end{tikzcd}$$

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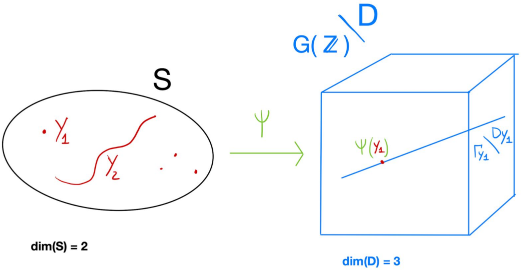
$$\mathrm{HL} = \mathrm{HL}_{\mathrm{typ}} \cup \mathrm{HL}_{\mathrm{atyp}}$$

Schematic: typical components



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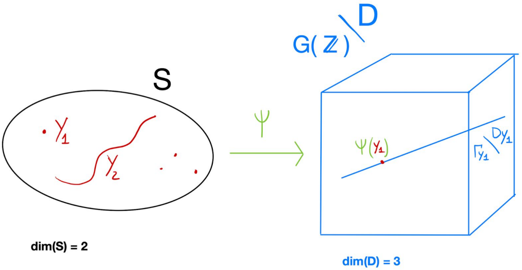
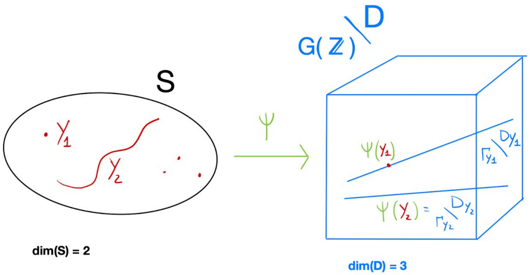


$$2 = \mathrm{Codim}_S(Y_1) = \mathrm{Codim}_{G(\mathbb{Z})\backslash D}(\Gamma_{Y_1}\backslash D_{Y_1})$$

Vs atypical components



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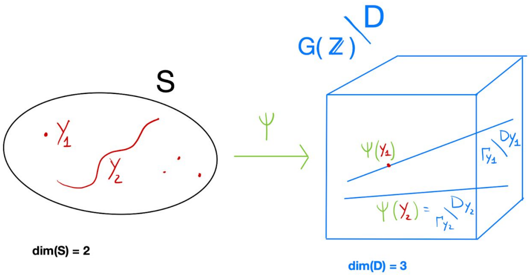

$$1 = \mathrm{Codim}_S(Y_2) < \mathrm{Codim}_{G(\mathbb{Z})\backslash D}(\Gamma_{Y_2}\backslash D_{Y_2}) = 2$$

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Remark

1. $\Rightarrow$ 2., and generalization of **André-Oort** for arbitrary VHS (Pila-Shankar-Tsimerman '21, for Shimura varieties)

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- Gukov-Vafa, Moore 2004: CM Calabi-Yau 3-folds and relation to string theory
- de Jong, *Beyond the André-Oort conjecture*. 6 page personal note, 2004. Pure VHS
- Klingler 2017. Mixed VHS but weaker version *Hodge codimension*

Tightly related to work of Pila, Bakker-Tsimerman ...

Main results

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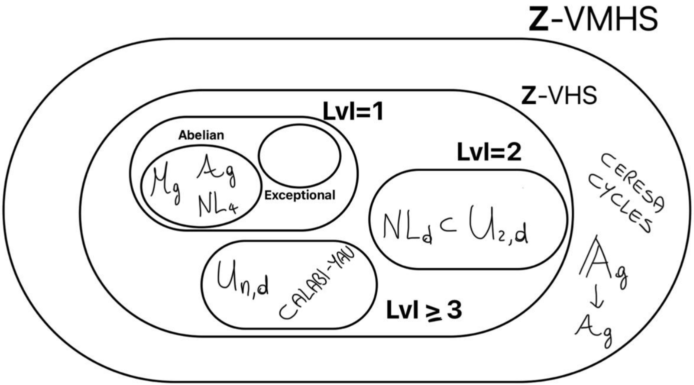
Smallest k : $\underbrace{[-[-[\ldots[\mathfrak{g}^{-1},\mathfrak{g}^{-1}],\mathfrak{g}^{-1}]\ldots],\mathfrak{g}^{-1}]}_{k \text{ times}} = 0.$

Roadmap for the level



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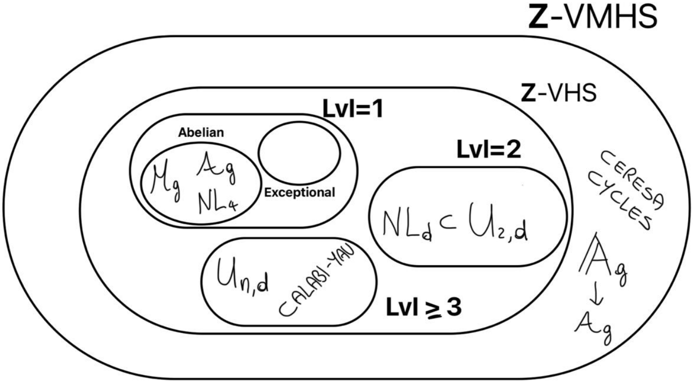

Question

What do the conjecture/results say about the above moduli spaces? Concrete applications of such viewpoint?

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- $\mathbb{Q}$-forms of M define Shimura curves $\mathcal{S}$ in $\mathcal{A}_4$ (discovered by Mumford)
- **Main Conj.** implies that *usually* $\mathcal{M}_4 \cap \mathcal{S}$ is a point with $\mathbf{MT}_{\mathbb{C}} = M$.

Teichmüller dynamics. . .

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- Orbit closures are **algebraic**, defined over $\overline{\mathbb{Q}}$, and admit a Hodge theoretic description (Filip, Möller).

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Motivations after Reid & McMullen 2000s

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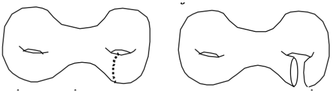

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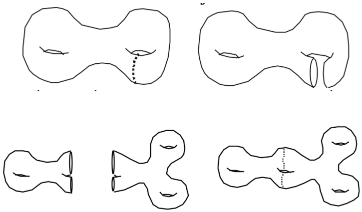

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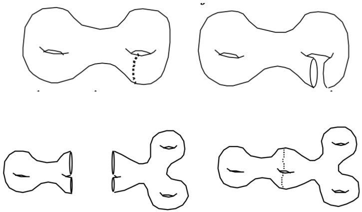

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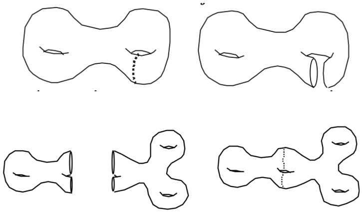

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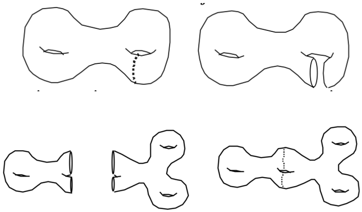

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How to prove all this?

Functional transcendence for foliated bundles

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then the projection of U to S is contained in a ***strict weakly-special subvariety****.*

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describes, for e in the atypical range, the projection of $\Sigma(f, e)$ to S, and therefore $\mathrm{HL}_{\mathrm{pos,aty}}$.

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In the style of Kostant's results on root systems of Levi factors for complex semi-simple Lie algebras.

THANKS FOR YOUR ATTENTION!

MERCI! GRAZIE!

Precise statement of Geometric Z.P.

Theorem (B.-Urbanik)

There is a finite set $\Sigma = \Sigma_{(S,\mathbb{V})}$ *of triples* $(\mathbf{H}, D_H, \mathbf{N})$*, where* $(\mathbf{H}, D_H)$ *is some sub-Hodge datum of the generic Hodge datum* $(\mathbf{G}_S, D_S)$*,* $\mathbf{N}$ *is a normal subgroup of* $\mathbf{H}$ *whose reductive part is semisimple, and such that the following property holds:*

- *For each monodromically atypical maximal (among all monodromically atypical subvarieties)* $Y \subset S$ *there is some* $(\mathbf{H}, D_H, \mathbf{N}) \in \Sigma$ *such that, up to the action of* Γ*,* D_Y^0 *is the image of* $\mathbf{N}(\mathbb{R})^+\mathbf{N}(\mathbb{C})^u \cdot y$*, for some* $y \in D_H$*.*