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Août 2000

IHES/P/00/60

submitted to Nucl. Phys. B; hep-th/0007220

hep-th/0007220
July 2000
ULB-TH-00/14

Inconsistency of interacting, multi-graviton theories

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Abstract

We investigate, in any spacetime dimension ≥ 3, the problem of consistent couplings for a finite collection of massless, spin-2 fields described, in the free limit, by a sum of Pauli-Fierz actions. We show that there is no consistent (ghost-free) coupling, with at most two derivatives of the fields, that can mix the various "gravitons". In other words, there are no Yang-Mills-like spin-2 theories. The only possible deformations are given by a sum of individual Einstein-Hilbert actions. The impossibility of cross-couplings subsists in the presence of scalar matter. Our approach is based on the BRST-based deformation point of view and uses results on the so-called "characteristic cohomology" for massless spin-2 fields which are explained in detail.

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1 Introduction

A striking feature of the interactions observed in Nature is that most of them (if we weigh them by the number of helicity states) are described by nonlinearly interacting *multiplets* of massless spin-1 fields, i.e., by Yang-Mills' theory. By contrast, the gravitational interaction (Einstein's theory) involves only a *single* massless spin-two field. In this paper we shall show that there is a compelling theoretical reason underlying this fact: there exists no consistent (in particular, ghost-free) theory involving a (finite) multiplet of interacting massless spin-2 fields. In other words, there exists no spin-2 analog of Yang-Mills' theory. This no-go result gives a new argument (besides the usual one based on the problems of having particles of spin > 2) for ruling out $N > 8$ extended supergravity theories, since these would involve gravitons of different types.

It was shown by Pauli and Fierz [1] that there is a unique, consistent[2] action describing a pure spin-2, massless field. This action happens to be the linearized Einstein action. Therefore, the action for a collection $\{h^a_{\mu\nu}\}$ of N *non-interacting*, massless spin-2 fields in spacetime dimension n ($a = 1, \cdots, N$, $\mu, \nu = 0, \cdots, n-1$) must be (equivalent to) the sum of N separate Pauli-Fierz actions, namely[3]

$$S_0[h^a_{\mu\nu}] = \sum_{a=1}^{N} \int d^n x \left[-\frac{1}{2} (\partial_\mu h^a{}_{\nu\rho}) (\partial^\mu h^{a\nu\rho}) + (\partial_\mu h^{a\mu}{}_\nu) (\partial_\rho h^{a\rho\nu}) - \left(\partial_\nu h^{a\mu}{}_\mu\right) (\partial_\rho h^{a\rho\nu}) + \frac{1}{2} (\partial_\mu h^{a\nu}{}_\nu) \left(\partial^\mu h^{a\rho}{}_\rho\right) \right], \quad n > 2. \tag{1.1}$$

It is invariant under the following linear gauge transformations,

$$\delta_\epsilon h^a_{\mu\nu} = \partial_\mu \epsilon^a_\nu + \partial_\nu \epsilon^a_\mu \tag{1.2}$$

where the ϵ^a_ν are $n \times N$ arbitrary, independent functions. These transformations are abelian and irreducible.

The equations of motion are

$$\frac{\delta S_0}{\delta h^a_{\mu\nu}} \equiv -2 H^{\mu\nu}_a = 0 \tag{1.3}$$

where $H^a_{\mu\nu}$ is the linearized Einstein tensor,

$$H^a_{\mu\nu} = K^a_{\mu\nu} - \frac{1}{2} K^a \eta_{\mu\nu}. \tag{1.4}$$

Here, $K^a_{\alpha\beta\mu\nu}$ is the linearized Riemann tensor,

$$K^a_{\alpha\beta\mu\nu} = -\frac{1}{2} (\partial_{\alpha\mu} h^a_{\beta\nu} + \partial_{\beta\nu} h^a_{\alpha\mu} - \partial_{\alpha\nu} h^a_{\beta\mu} - \partial_{\beta\mu} h^a_{\alpha\nu}), \tag{1.5}$$

[2] All over this paper, we follow the standard field theory tenets which tell us that "consistent" theories should be free of: negative-energy (ghost) propagating excitations, algebraic inconsistencies among field equations, discontinuities in the degree-of-freedom content, etc.

[3] We use the signature "mostly plus": $- + + + \cdots$. Furthermore, spacetime indices are raised and lowered with the flat Minkowskian metric $\eta_{\mu\nu}$. Finally, we take the spacetime dimension n to be strictly greater than 2 since otherwise, the Lagrangian is a total derivative. Gravity in two dimensions needs a separate treatment.

$K^a_{\mu\nu}$ is the linearized Ricci tensor,

$$K^a_{\mu\nu} = K^{a\alpha}_{\ \cdot\mu\alpha\nu} = -\frac{1}{2}(\Box h^a_{\mu\nu} + \cdots), \tag{1.6}$$

and K^a is the linearized scalar curvature, $K^a = \eta^{\mu\nu} K^a_{\mu\nu}$. The Noether identities expressing the invariance of the free action (1.1) under (1.2) are

$$\partial_\nu H^{a\mu\nu} = 0 \tag{1.7}$$

(linearized Bianchi identities). The gauge symmetry removes unwanted unphysical states.

The problem of introducing consistent interactions for a collection of massless spin-2 fields is that of adding local interaction terms to the action (1.1) while modifying at the same time the original gauge symmetries if necessary, in such a way that the modified action be invariant under the modified gauge symmetries. We shall exclusively consider interactions that can formally be expanded in powers of a deformation parameter g ("coupling constant") and that are consistent order by order in g. The class of "consistent interactions" for (1.1) studied here could thus be called more accurately "perturbative, gauge-consistent interactions" (since we focus on compatibility with gauge-invariance order by order in g), but we shall just use the terminology "consistent interactions" for short.

Since we are interested in the classical theory, we shall also demand that the interactions contain at most two derivatives[4] so that the nature of the differential equations for $h^a_{\mu\nu}$ is unchanged. On the other hand, we shall make no assumption on the polynomial order of the fields in the Lagrangian or in the gauge symmetries.

In an interesting work [2], Cutler and Wald have proposed theories involving a multiplet of spin-2 fields, based on associative, commutative algebras. These authors arrived at these structures by focusing on the possible structures of modified gauge transformations and their algebra. However, they did not analyse the extra conditions that must be imposed on the modified gauge symmetries if these are to be compatible with a Lagrangian having the (unique, consistent) free field limit prescribed above. [Their work was subsequently extended to supergravity in [3].] Some explicit examples of Lagrangians that realize the Cutler-Wald algebraic structures have been constructed in [4] and [5], but none of these has an acceptable free field limit. Indeed, their free field limit does involve a sum of Pauli-Fierz Lagrangians, but some of the "gravitons" come with the wrong sign and thus, the energy of the theory is unbounded from below. To our knowledge, the question of whether other examples of (real) Lagrangians realizing the Cutler-Wald structure (with a finite number of gravitons) would exist and whether some of them would have a physically acceptable free field limit was left open.

Motivated by these developments, we have re-analyzed the question of consistent interactions for a collection of massless spin-2 fields by imposing from the outset that the deformed Lagrangian should have the free field limit (1.1). As we shall see, it turns out that this requirement forces one additional condition on the Cutler-Wald algebra defining

[4]in the sense of the usual power counting of perturbative field theory. Thus we allow only terms that are quadratic in the first derivatives of $h^a_{\mu\nu}$ or linear in their second derivatives.

the interaction, namely, that it be "symmetric" with respect to the scalar product defined by the free Lagrangian (see below for the precise meaning of "symmetric"). This extra constraint is quite stringent and implies that the algebra is the direct sum of one-dimensional ideals. This eliminates all the cross-interactions between the various gravitons[5]. Let us state the main (no-go) result of this paper, spelling out explicitly our assumptions :

Theorem 1.1 *Under the assumptions of: locality, Poincaré invariance, Eq.(1.1) as free field limit and at most two derivatives in the Lagrangian, the only consistent deformation of Eq.(1.1) involving a finite collection of spin-2 fields is (modulo field redefinitions) a sum of independent Einstein-Hilbert (or possibly Pauli-Fierz) actions,*

$$S[g^a_{\mu\nu}] = \sum_a \frac{2}{\kappa_a^2} \int d^n x (R^a - 2\Lambda^a)\sqrt{-g^a}, \quad g^a_{\mu\nu} = \eta_{\mu\nu} + \kappa^a h^a_{\mu\nu}\,, \tag{1.8}$$

where R^a is the scalar curvature of $g^a_{\mu\nu}$, g^a its determinant, $\kappa^a \geq 0$ a self-coupling constant and Λ^a independent cosmological constants. [A term with $\kappa^a = 0$ is a Pauli-Fierz action; the corresponding cosmological term reads $\lambda^a h^{a\mu}_{\ \ \mu}$.]

There are no other (perturbatively gauge-consistent) possibilities under the assumptions stated. Note, however, that Ref. [6] has shown that there exists a consistent, interacting theory involving an *infinite* number of spin-2 fields. We shall explicitly discuss below how, indeed, the case of an infinite collection can evade our no-go theorem.

We have also investigated how matter couplings affect the problem of the (non-)existence of cross-interactions between gravitons. We have taken the simplest example of a scalar field and have verified that the scalar field can only couple to one type of gravitons. Thus, even the existence of indirect cross-couplings (via intermediate interactions) between massless spin-2 particle is excluded. The interacting theory describes parallel worlds, and, in any given world, there is only one massless spin-2 field. This massless spin-2 field has (if it interacts at all) the standard graviton couplings with the fields living in its world (including itself), in agreement with the single massless spin-2 field studies of [7, 8, 9, 10, 11, 12, 13, 14, 15, 16].

The above theorem relies strongly on the assumption that the interaction contains at most two derivatives. If one allows more derivatives in the Lagrangian, one can construct cross-interactions involving the linearized curvatures, which are manifestly consistent with gauge invariance. An obvious cubic candidate is

$$g_{abc} K^a_{\alpha\beta\mu\nu} K^{b\alpha\beta}_{\rho\sigma} K^{c\mu\nu\rho\sigma} \tag{1.9}$$

where g_{abc} are arbitrary constants. This candidate can be added to the free Lagrangian and defines an interacting theory with the same abelian gauge symmetries as the original theory since (1.9) is invariant under (1.2). It contains six derivatives. Other deformations of the original free action that come to mind are obtained by going to the Einstein theory and adding then, in each sector, higher order polynomials in the curvatures and their covariant derivatives.

[5]The extra condition is in fact also derived by different methods in [3], (Eqs. (3.38) and (A.55)), but its full implications regarding the impossibility of cross-interactions have not been investigated.

All these deformations have the important feature of deforming the algebra of the gauge symmetries in a rather simple way: the deformed algebra is the direct sum of independent diffeomorphism algebras (in each sector with $\kappa_a \neq 0$) and abelian algebras. This is not an accident. The possibilities of deformations of the gauge algebra are in fact severely limited even in the more general context where no constraint on the number of derivatives is imposed (except that it should remain bounded). One has the theorem :

Theorem 1.2 *Under the assumptions of locality, Poincaré invariance and Eq.(1.1) as free field limit, the only consistent deformations of the action (1.1) involving a finite collection of spin-2 fields are such that the algebra of the deformed gauge-symmetries is given, to first order in the deformation parameter, by the direct sum of independent diffeomorphism algebras. [Some terms in the direct sum may remain undeformed, i.e., abelian.]*

This theorem strengthens previous results in that it does not assume off-shell closure of the gauge algebra (this is automatic) or any specific form of the gauge symmetries (which are taken to involve only one derivative in most treatments).

In order to prove these results, we shall begin the analysis without making any assumption on the number of derivatives, except that it is bounded. We shall see that this is indeed enough to completely control the algebra. We shall then point out where the derivative assumptions are explicitly needed, at the level of the gauge transformations and of the deformation of the Lagrangian. We shall discuss in section 11 the new features that appear in the absence of these assumptions.

Our approach is based on the BRST reformulation of the problem, in which consistent couplings define deformations of the solution of the so-called "master equation". The advantage of this approach is that it clearly organizes the calculation of the non-trivial consistent couplings in terms of cohomologies which are either already known or easily computed. These cohomologies are in fact interesting in themselves, besides their occurence in the consistent interaction problem. One of them is the "characteristic cohomology", which investigates higher order conservation laws involving antisymmetric tensors (see below). The use of BRST techniques somewhat streamlines the derivation, which would otherwise be more cumbersome.

In the next section, we review the master-equation approach to the problem of consistent interactions. We then recall some cohomological results necessary for solving the problem. In particular, we discuss at length the characteristic cohomology (section 4). Section 5 constitutes the hard core of our paper. We show how the structure of an associative, commutative algebra introduced first in this context by Cutler and Wald arises in the cohomological approach, and derive the further crucial condition of "symmetry" (explained in the text) that emerges from the requirement that the deformation not only defines consistent gauge transformations, but also can be extended to a consistent deformation of the Lagrangian. We then show that all the requirements on the algebra force it to be trivial (section 6), which implies that there can be no cross-interaction between the various spin-2 fields. In the next section (section 7), we complete the construction of the consistent Lagrangians and establish the validity of (1.8). In section 8, we discuss the possibility to evade the above no-go theorem by allowing for an infinite number of massless spin-2 fields. Section 9 shows that the coupling to matter does not allow the different types of gravitons to "see each other" through the matter. In section 10 we briefly generalize the discussion to the presumably physically unacceptable case of non-positive metrics in the internal space of the gravitons. This is done solely for the sake of comparison with the work of [2, 4], where there are propagating ghosts. Section 11 discusses the new features that arise when no restriction is imposed on the number of derivatives in the Lagrangian. A brief concluding section is finally followed by a technical appendix that collects the proofs of the theorems used in the core of the paper.

2 Cohomological reformulation

2.1 Gauge symmetries and master equation

The central idea behind the master equation approach to the problem of consistent deformations is the following. Consider an arbitrary irreducible gauge theory with fields Φ^i, action $S[\Phi^i]$, gauge transformations[6]

$$\delta_\varepsilon \Phi^i = R^i_\alpha(\Phi)\,\varepsilon^\alpha\,, \tag{2.1}$$

and gauge algebra

$$R^j_\alpha(\Phi)\,\frac{\delta R^i_\beta(\Phi)}{\delta\Phi^j} - R^j_\beta(\Phi)\,\frac{\delta R^i_\alpha(\Phi)}{\delta\Phi^j} = C^\gamma_{\alpha\beta}(\Phi)\,R^i_\gamma(\Phi) + M^{ij}_{\alpha\beta}(\Phi)\,\frac{\delta S}{\delta\Phi^j}. \tag{2.2}$$

We have allowed the gauge transformations to close only on-shell. The coefficient functions $M^{ij}_{\alpha\beta}$ are (graded) antisymmetric in both α, β and i, j. The Noether identities read

$$\frac{\delta S}{\delta\Phi^i} R^i_\alpha = 0. \tag{2.3}$$

One can derive higher order identities from (2.2) and (2.3) by differentiating (2.2) with respect to the fields and using the fact that second-order derivatives commute. These identities, in turn, lead to further identities by a similar process.

It has been established in [17, 18] that one can associate with S a functional W depending on the original fields Φ^i and on additional variables, called the ghosts C^α and the antifields Φ^*_i and C^*_α, with the following properties:

1. W starts like

$$W = S + \Phi^*_i R^i_\alpha C^\alpha + \frac{1}{2} C^*_\gamma C^\gamma_{\alpha\beta} C^\beta C^\alpha + \frac{1}{2}\Phi^*_i\Phi^*_j M^{ij}_{\alpha\beta} C^\alpha C^\beta + \text{ "more"} \tag{2.4}$$

where "more" contains at least three ghosts;

[6] Throughout this section, we use De Witt's condensed notation in which a summation over a repeated index implies also an integration. The $R^i_\alpha(\Phi)$ stand for $R^i_\alpha(x,x')$ and are combinations of the Dirac delta function $\delta(x,x')$ and some of its derivatives with coefficients that involve the fields and their derivatives, so that $R^i_\alpha\varepsilon^\alpha \equiv \int d^nx' R^i_\alpha(x,x')\varepsilon^\alpha(x')$ is a sum of integrals of ε^α and a finite number of its derivatives.

2. W fulfills the equation

$$(W, W) = 0 \tag{2.5}$$

in the antibracket $(,)$ that makes the fields and the antifields canonically conjugate to each other. This antibracket structure was first introduced by Zinn-Justin[7][19] and was denoted originally by a $\star$ ($(A, B) \equiv A \star B$). It is defined by

$$(A, B) = \frac{\delta^R A}{\delta \Phi^i} \frac{\delta^L B}{\delta \Phi_i^*} - \frac{\delta^R A}{\delta \Phi_i^*} \frac{\delta^L B}{\delta \Phi^i} + \frac{\delta^R A}{\delta C^\alpha} \frac{\delta^L B}{\delta C_\alpha^*} - \frac{\delta^R A}{\delta C_\alpha^*} \frac{\delta^L B}{\delta C^\alpha}, \tag{2.6}$$

where the superscript R (L) denotes a right (left) derivative, respectively.

3. W is bosonic and has ghost number zero.

To explain this last statement, we recall that all fields belong to a Grassmann algebra $\mathcal{G}$: the fields Φ^i and C_α^* belong to the even part of $\mathcal{G}$ (i.e. they commute with everything), while the fields C^α and Φ_i^* belong to the odd part of $\mathcal{G}$ (i.e. they anticommute among themselves). [Instead of "commuting" or "anticommuting", we shall simply say "bosonic", or "fermionic", respectively. Note, however, that we work in a purely classical framework.] Moreover, in addition to the above "fermionic" Z_2 grading (odd or even) one endows the algebra of the dynamical variables with a Z-valued "ghost grading" defined such that the original fields Φ^i, the ghosts C^α, the antifields Φ_i^* and the antifields Φ_α^* have ghost number zero, one, minus one and minus two, respectively. The statement that W has ghost number zero means that each term in W has a zero ghost number. Note that the antibracket increases the ghost number by one unit, i.e., $gh((A, B)) = gh(A) + gh(B) + 1$ (we refer to the book [20] for more information).

It is also useful to introduce a second Z-valued grading for the basic variables, called the "antifield" (or "antighost") number [20]. This grading is defined by assigning antifield number zero to the fields Φ^i and the ghosts C^α, antifield number one to the antifields Φ_i^* and antifield number two to the antifields C_α^*. The antifield number thus counts the number of antifields Φ_i^* and C_α^*, with weight two given to the antifields C_α^* conjugate to the ghosts. There are different ways to achieve a fixed ghost number by combining the antifields and the ghosts. For instance, $\Phi_i^* C^\alpha$, $C_\alpha^* C^b C^c$, $\Phi_i^* C_\alpha^* C^b C^c C^d$ all have ghost number zero; but the first term has antifield number one, the second has antifield number two and the third has antifield number three. The antifield number keeps track of these different possibilities. By introducing it, one can split an equation with definite ghost number into simpler equations at each value of the antifield number. This procedure will be amply illustrated in the sequel.

In our irreducible case where there is only one type of ghosts, the antifield number can also be viewed as an indirect way of keeping track of the number of explicit ghost fields C_α^a entering any expression. Indeed, if we define the "pureghost number" of any expression as the number of explicit C_α^a's in it, it is easy to see from the antighost attributions above that the (net) ghost number is given by: $gh = puregh - antigh$.

The equation (2.5) is called the master equation while the function W is called the (minimal) solution of the master equation. It is easily seen that, because of the Z_2-grading

[7] In Zinn-Justin's work the antifields appear as "sources" K_i, L_α.

of the various fields (the "canonically conjugate" fields in the antibracket have opposite fermionic gradings), (A, B) is *symmetric* for bosonic functions A and B, $(A, B) = (B, A)$. One can also check that the antibracket satisfies the (graded) Jacobi identity (see, e.g., [20]). This fact will play an important role in the work below.

The master equation is fulfilled as a consequence of the Noether identities (2.3), of the gauge algebra (2.2) and of all the higher order identities alluded to above that one can derive from them. Conversely, given some W, solution of (2.5), one can recover the gauge-invariant action as the term independent of the ghosts in W, while the gauge transformations are defined by the terms linear in the antifields Φ^*_i and the structure functions appearing in the gauge algebra can be read off from the terms quadratic in the ghosts. The Noether identities (2.3) are fulfilled as a consequence of the master equation (the left-hand side of the Noether identities is the term linear in the ghosts in (W, W); the gauge algebra (2.2) is the next term in $(W, W) = 0$).

In other words, there is complete equivalence between gauge invariance of S and the existence of a solution W of the master equation. For this reason, one can reformulate the problem of consistently introducing interactions for a gauge theory as that of deforming W while maintaining the master equation [21].

2.2 Perturbation of the master equation

Let W_0 be the solution of the master equation for the original free theory,

$$W_0 = S_0 + \Phi^*_i R^i_{0\alpha} C^\alpha, \; (W_0, W_0) = 0. \tag{2.7}$$

Because the gauge transformations are abelian, there is no further term in W_0 ($C^\gamma_{\alpha\beta} = 0, M^{ij}_{\alpha\beta} = 0$). Let W be the solution of the master equation for the searched-for interacting theory,

$$\begin{aligned} W &= S + \Phi^*_i R^i_\alpha C^\alpha + O(C^2), && (2.8)\\ S &= S_0 + \text{interactions}, && (2.9)\\ R^i_\alpha &= R^i_{0\alpha} + \text{deformation terms}, && (2.10)\\ (W, W) &= 0. && (2.11) \end{aligned}$$

As we have just argued, W exists if and only if $S = S_0 +$ "interactions" is a consistent deformation of S_0.

Let us now expand W and the master equation for W in powers of the deformation parameter g. With

$$W = W_0 + gW_1 + g^2 W_2 + O(g^3) \tag{2.12}$$

the equation $(W, W) = 0$ yields, up to order g^2

$$\begin{aligned} O(g^0): &\quad (W_0, W_0) = 0 && (2.13)\\ O(g^1): &\quad (W_0, W_1) = 0 && (2.14)\\ O(g^2): &\quad (W_0, W_2) = -\frac{1}{2}(W_1, W_1). && (2.15) \end{aligned}$$

The first equation is fulfilled by assumption since the starting point defines a consistent theory. To analyse the higher order equations, one needs further information about the meaning of W_0.

2.3 BRST transformation, first order deformations, obstructions

It turns out that W_0 is in fact the generator of the BRST transformation s of the free theory through the antibracket[8], i.e.

$$sA = (W_0, A). \tag{2.16}$$

The nilpotency $s^2 = 0$ follows from the master equation (2.13) for W_0 and the (graded) Jacobi identity for the antibracket. Thus, Eq. (2.14) simply expresses that W_1 is a BRST-cocycle, i.e. that it is "closed" under s: $sW_1 = 0$.

Now, not all consistent interactions are relevant. Indeed, one may generate "fake" interactions by making non-linear field redefinitions. Such interactions are trivial classically and quantum-mechanically [22]. One can show [21] that the physically trivial interactions generated by field-redefinitions that reduce to the identity at order g^0,

$$\Phi^i \to \Phi'^i = \Phi^i + g\,\Xi^i(\Phi, \partial\Phi, \cdots) + O(g^2) \tag{2.17}$$

precisely correspond to cohomologically trivial solutions of (2.16), i.e.,correspond to "exact" A's (also called "coboundaries") of the form

$$A = sB \tag{2.18}$$

for some B. We thus come to the conclusion that the non-trivial consistent interactions are characterized to first order in g by the *cohomological group*[9] $H(s)$ at ghost number zero. In fact, since W_1 must be a local functional, the cohomology of s must be computed in the space of local functionals. Because the equation $s \int a = 0$ is equivalent to $sa + dm = 0$ (where d denotes Cartan's exterior differential) for some m, and $\int a = s \int b$ is equivalent to $a = sb + dn$ for some n, one denotes the corresponding cohomological group by $H^{0,n}(s|d)$[10] (0 is the ghost number and n the form-degree: a and b are n-forms).

The redundancy in W_1 is actually slightly bigger than the possibility of adding trivial cocycles, since one can admit changes of field variables $\Phi^i \to \Phi'^i$ that do not reduce to the identity at zeroth order in g, but reduce to a global symmetry of the original theory, i.e., leave the free action invariant. Two distinct BRST cocycles W_0 and W_0' that can be obtained from one another under such a transformation should be identified. In practice,

[8]We denote the BRST transformation for the free theory by s, rather than s_0 because this is the only BRST symmetry we shall consider so no confusion can arise.

[9] We recall that, given some nilpotent s, $s^2 = 0$, $H(s)$ denotes the equivalence classes of "closed" A's, modulo "exact" ones, i.e. the solutions of $sA = 0$, modulo the equivalence relation $A' = A + sB$.

[10]More generally, we shall use in this paper the notation $H^{i,p}_j$ to denote a cohomological group for $p-$forms having a fixed ghost number i, and a fixed "antifield" number j (see below). If we indicate only one superscript, it will always refer to the form degree p.

however, only a few of these transformations are to be taken into account (if any) since only a few of them preserve the condition on the number of derivatives of the deformation. This will be explicitly illustrated in the graviton case.

Once a first-order deformation is given, one must investigate whether it can be extended to higher orders. It is a direct consequence of the Jacobi identity for the antibracket that (W_1, W_1) is BRST-closed, $(W_0, (W_1, W_1)) = 0$. However, it may not be BRST-exact (in the space of local functionals). In that case, the first-order deformation W_1 is obstructed at second-order, so, it is not a good starting point. If, on the other hand, (W_1, W_1) is BRST-exact, then a solution W_2 to (2.15), which may be rewritten

$$sW_2 = -\frac{1}{2}(W_1, W_1), \tag{2.19}$$

exists. As (W_1, W_1) has ghost number one (since the antibracket increases the ghost number by one unit), we see that obstructions to continuing a given, first-order consistent interaction are measured by the cohomological group $H^{1,n}(s|d)$. Furthermore, the ambiguity in W_2 (when it exists) is a solution of the homogeneous equation $(W_0, W_2) = 0$. Among these solutions, those that are equivalent through field redefinitions should be identified. $O(g^2)$-redefinitions of the fields yield trivial BRST-cocycles, so again, the space of equivalent W_2's is a quotient of $H^{0,n}(s|d)$. Further identifications follow from $O(g^0)$ and $O(g^1)$-redefinitions that leave the previous terms invariant. These identifications will be discussed in more details below.

The same pattern is found at higher orders : obstructions to the existence of W_k are elements of $H^{1,n}(s|d)$, while the ambiguities in W_k (when it exists) are elements of appropriate quotient spaces of $H^{0,n}(s|d)$.

Since the identifications of equivalent solutions will play an important role in the sequel, let us be more explicit on the precise form that the equations describing these identifications take. Two solutions of the master equation are equivalent if they differ by an anti-canonical transformation in the antibracket. These correspond indeed to field and gauge parameter (ghost) redefinitions [23, 24, 20]. Infinitesimally, two solutions W and $W + \Delta W$ are thus equivalent if

$$\Delta W = (W, K) \tag{2.20}$$

for some K of ghost number -1. If we expand this equation in powers of g, we get

$$\begin{aligned} \Delta W_0 &= (W_0, K_0), & (2.21)\\ \Delta W_1 &= (W_0, K_1) + (W_1, K_0), & (2.22)\\ \Delta W_2 &= (W_0, K_2) + (W_1, K_1) + (W_2, K_0), & (2.23)\\ &\dots \end{aligned}$$

Since W_0 is given, one must impose $\Delta W_0 = 0$, and thus, from (2.21),

$$(W_0, K_0) = 0 : \tag{2.24}$$

K_0 defines a global symmetry of the free theory [25, 26]. The first term on the right-hand side of (2.22) is a BRST-coboundary and shows that indeed, one must identify two

BRST-cocycles that are in the same cohomological class of $H^{0,n}(s|d)$. There is a further identification implied by the term (W_1, K_0). Similarly, besides the BRST-coboundary (W_0, K_2), there are extra terms in the right-hand side of (2.23).

The cohomological considerations that we have just outlined are equivalent to the conditions for consistent interactions derived in [27] without use of ghosts or antifields. The interest of the master equation approach is that it organizes these equations in a rather neat way. Also, one can use cohomological tools, available in the literature, to determine these interactions and their obstructions.

In the sequel, we shall compute explicitly $H^{0,n}(s|d)$ for a collection of free, massless spin-2 fields, i.e., we shall determine all possible first-order consistent interactions. We shall then determine the conditions that these must fulfill in order to be unobstructed at order g^2. These conditions turn out to be extremely strong and prevent cross interactions between the various types of gravitons.

2.4 Solution of the master equation for a collection of free, spin-2, massless fields

We rewrite the free action (1.1) as

$$S_0 = \int d^n x\, k_{ab} \Big[-\frac{1}{2} (\partial_\mu h^a{}_{\nu\rho}) \left(\partial^\mu h^{b\nu\rho}\right) + (\partial_\mu h^{a\mu}_{\ \nu}) \left(\partial_\rho h^{b\rho\nu}\right) - \left(\partial_\nu h^{a\mu}_{\ \mu}\right) \left(\partial_\rho h^{b\rho\nu}\right) + \frac{1}{2} (\partial_\mu h^{a\nu}_{\ \nu}) \left(\partial^\mu h^{b\rho}_{\ \rho}\right) \Big] , \quad (2.25)$$

with a quadratic form k_{ab} defined by the kinetic terms. In the way of writing the Pauli-Fierz free limit above, Eq.(1.1), k_{ab} was simply the Kronecker delta δ_{ab}. What is essential for the physical consistency of the theory (absence of negative-energy excitations, or stability of the Minkowski vacuum) is that k_{ab} defines a positive-definite metric in internal space; it can then be normalized to be δ_{ab} by a simple linear field redefinition.

Following the previous prescriptions, the fields, ghosts and antifields are found to be

- the fields $h^a_{\alpha\beta}$, with ghost number zero and antifield number zero;
- the ghosts C^a_α, with ghost number one and antifield number zero;
- the antifields $h^{*\alpha\beta}_a$, with ghost number minus one and antifield number one;
- the antifields $C^{*\alpha}_a$, with ghost number minus two and antifield number two.

The solution of the master equation for the free theory is, reverting to notations where integrals are all explicitly written,

$$W_0 = S_0 + \int d^n x\, h^{*\alpha\beta}_a (\partial_\alpha C + \partial_\beta C^a_\alpha), \quad (2.26)$$

from which we get the BRST differential s of the free theory as

$$s = \delta + \gamma \quad (2.27)$$

where the action of γ and δ on the variables is zero except (note in particular that $\gamma C^a_\alpha = \delta C^a_\alpha = 0$)[11]

$$\gamma h^a_{\alpha\beta} = 2\partial_{(\alpha} C^a_{\beta)} \tag{2.28}$$

$$\delta h^{*\alpha\beta}_a = \frac{\delta S_0}{\delta h^a_{\alpha\beta}} \tag{2.29}$$

$$\delta C^{*\alpha}_a = -2\partial_\beta h^{*\beta\alpha}_a . \tag{2.30}$$

The decomposition of s into δ plus γ is dictated by the antifield number: δ decreases the antifield number by one unit, while γ leaves it unchanged. Combining this property with $s^2 = 0$, one concludes that,

$$\delta^2 = 0, \; \delta\gamma + \gamma\delta = 0, \; \gamma^2 = 0. \tag{2.31}$$

3 Cohomology of γ

To compute the consistent, first order deformations, i.e., $H(s|d)$, we shall see in Section 5 that we need $H(\gamma)$ and $H(\delta|d)$. We start with $H(\gamma)$, which is rather easy.

As it is clear from its definition, γ is related to the gauge transformations. Acting on anything, it gives zero, except when it acts on the spin-2 fields, on which it gives a gauge transformation with gauge parameters replaced by the ghosts.

The only gauge-invariant objects that one can construct out of the gauge fields $h^a_{\mu\nu}$ and their derivatives are the linearized curvatures $K^a_{\alpha\beta\mu\nu}$ and their derivatives.

The antifields and their derivatives are also γ-closed. The ghosts and their derivatives are γ-closed as well but their symmetrized first order derivatives are γ-exact (see Eq. (2.28)), as are all their subsequent derivatives since

$$\partial_{\alpha\beta} C^a_\gamma = \frac{1}{2}\gamma\left(\partial_\alpha h^a_{\beta\gamma} + \partial_\beta h^a_{\alpha\gamma} - \partial_\gamma h^a_{\alpha\beta}\right) . \tag{3.1}$$

It follows straightforwardly from these observations that the γ-cohomology is generated by the linearized curvatures, the antifields and all their derivatives, as well as by the ghosts C^a_μ and their antisymmetrized first-order derivatives $\partial_{[\mu} C^a_{\nu]}$. More precisely, let $\{\omega^I\}$ be a basis of the space of polynomials in the C^a_μ and $\partial_{[\mu} C^a_{\nu]}$ (since these variables anticommute, this space is finite-dimensional). One has:

$$\gamma a = 0 \Rightarrow a = \alpha_J\left([K],[h^*],[C^*]\right)\omega^J\left(C^a_\mu, \partial_{[\mu} C^a_{\nu]}\right) + \gamma b, \tag{3.2}$$

where the notation $f([m])$ means that the function f depends on the variable m and its subsequent derivatives up to a finite order. If a has a fixed, finite ghost number, then a can

[11] We denote $t_{(\alpha\beta)} \equiv \frac{1}{2}(t_{\alpha\beta} + t_{\beta\alpha})$, and $t_{[\alpha\beta]} \equiv \frac{1}{2}(t_{\alpha\beta} - t_{\beta\alpha})$.

only contain a finite number of antifields. If we assume in addition that a has a bounded number of derivatives, as we shall do from now on, then, the α_J are polynomials[12].

In the sequel, the polynomials $\alpha_J([K],[h^*],[C^*])$ in the linearized curvature $K^a_{\alpha\beta\mu\nu}$, the antifields $h^{*\mu\nu}_a$ and $C^{*\mu}_a$, as well as all their derivatives, will be called "invariant polynomials". They may of course have an extra, unwritten, dependence on dx^μ, i.e., be exterior forms. At zero antifield number, the invariant polynomials are the polynomials in the linearized curvature $K^a_{\alpha\beta\mu\nu}$ and its derivatives.

We shall need the following theorem on the cohomology of d in the space of invariant polynomials.

Theorem 3.1 *In form degree less than n and in antifield number strictly greater than* 0*, the cohomology of d is trivial in the space of invariant polynomials.*

That is to say, if α is an invariant polynomial, the equation $d\alpha = 0$ with $antigh(\alpha) > 0$ implies $\alpha = d\beta$ where β is also an invariant polynomial. To see this, treat the antifields as "foreground fields" and the curvatures as "background fields", as in [28]. Namely, split d as $d = d_1 + d_0$, where d_1 acts only on the antifields and d_0 acts only on the curvatures. The so-called "algebraic Poincaré lemma" states that d_1 has no cohomology in form degree less than n (and in antifield number strictly greater than 0) because there is no relation among the derivatives of the antifields. By contrast, d_0 has some cohomology in the space of polynomials in the curvatures because these are constrained by the Bianchi identities. From the triviality of the cohomology of d_1, one easily gets $d\alpha = 0 \Rightarrow \alpha = d\beta + u$, where β is an invariant polynomial, and where u is an invariant polynomial that does not involve the antifields. However, since $antigh(\alpha) > 0$, u must vanish. qed.

4 Characteristic cohomology – cohomology of δ modulo d

4.1 Characteristic cohomology

It has been shown in [29] that $H(\delta|d)$ is trivial in the space of forms with positive pure ghost number. Thus the next cohomology that we shall need is $H(\delta|d)$ in the space of local forms that do not involve the ghosts, i.e., having $puregh = 0$. This cohomology has an interesting interpretation in terms of conservation laws, which we first review [25] (see also [26] for a recent review).

Conserved currents j^μ are defined through the condition

$$\partial_\mu j^\mu \approx 0 \tag{4.1}$$

[12] A term like $\exp \kappa K$, where K is the linearized scalar curvature, does not have a bounded number of derivatives since it contains arbitrarily high powers of K, and since the number of derivatives in K^m is $2m$. Note, however, that the coefficient at each order in the coupling κ is of bounded derivative order - just by dimensional analysis - so that in our perturbative approach where we expand the interactions in powers of the coupling constant and work order by order, the assumption of bounded derivative order is not a restriction.

where $\approx$ means "equal when the equations of motion hold", or, as one also says, "weakly equal to". These currents may carry further internal or spacetime indices that we shall not write explicitly. Among the conserved currents, those of the form

$$j^{\mu}_{\text{triv}} \approx \partial_\nu S^{\mu\nu} \tag{4.2}$$

where $S^{\mu\nu}$ is antisymmetric in μ and ν, $S^{\mu\nu} = -S^{\nu\mu}$, are sometimes called (mathematically) trivial (although they may not be physically trivial), because they can be constructed with no information on the equations of motion. We shall adopt this terminology here. If we call k the $(n-1)$-form dual to j^μ, and r the $(n-2)$-form dual to $S^{\mu\nu}$, the conditions (4.1) and (4.2) can be rewritten as

$$dk \approx 0 \tag{4.3}$$

and

$$k_{\text{triv}} \approx dr, \tag{4.4}$$

respectively. These conditions define the characteristic cohomology $H^{n-1}_{\text{char}}(d)$ in degree $n-1$ [30, 31]. One may define more generally the characteristic cohomology $H^{p}_{\text{char}}(d)$ in any form degree $p \leq n$, by the same conditions (4.3) and (4.4). Again, k may have extra internal or spacetime unspecified indices.

4.2 Cohomology of δ modulo d

A crucial aspect of the differential δ defined through (2.29) and (2.30) is that it is related to the dynamics of the theory. This is obvious since $\delta h^{*\ m\nu}_{a}$ reproduces the Euler-Lagrange derivatives of the Lagrangian. In fact, one has the following important (and rather direct) results about the cohomology of δ [32, 29, 20]

1. Any form of zero antifield number which is zero on-shell is δ-exact;
2. $H^p_i(\delta) = 0$ for $i > 0$, where i is the antifield number, in any form-degree p. [The antifield number is written as a lower index; the ghost number is not written because it is irrelevant here.]

Because of the first property, one can rewrite the cocycle condition and coboundary condition of the characteristic cohomology as

$$dk^p_0 + \delta k^{p+1}_1 = 0 \tag{4.5}$$

and

$$k^p_{\text{triv}0} = dr^{p-1}_0 + \delta r^p_1, \tag{4.6}$$

respectively, where all relevant degrees have been explicitly written (recall that there is no ghost here, i.e., *puregh* $= 0$ throughout section 4). Thus, we see that the characteristic cohomology is just $H_0^p(d|\delta)$. Using $H_i^p(\delta)=0$ for $i>0$, one can then easily establish the isomorphisms $H_0^p(d|\delta)\simeq H_1^{p+1}(\delta|d)$ $(n>p>0$) and $H_0^0(d|\delta)/R\simeq H_1^1(\delta|d)$ [28, 25][13]

Finally, using the isomorphism $H_j^i(\delta|d)\simeq H_{j+1}^{i+1}(\delta|d)$ [25], we conclude

$$H_{\text{char}}^{n-p}(d) \simeq H_p^n(\delta|d),\ 0<p<n \tag{4.7}$$

$$H_{\text{char}}^{0}(d)/R \simeq H_n^n(\delta|d). \tag{4.8}$$

The following vanishing theorem on $H_p^n(\delta|d)$ (and thus also on $H_{\text{char}}^{n-p}(d)$ or $H_{\text{char}}^0(d)/R$) can be proven:

Theorem 4.1 *The cohomology groups $H_p^n(\delta|d)$ vanish in antifield number strictly greater than 2,*

$$H_p^n(\delta|d)=0 \ \textit{for}\ p>2. \tag{4.9}$$

The proof of this theorem is given in [25] and follows from the fact that linearized gravity is a linear, irreducible, gauge theory. In terms of the characteristic cohomology, this means that all conservation laws involving antisymmetric objects of rank >2 are trivial, $\partial_{\mu_1}S^{\mu_1\mu_2\cdots\mu_k}\approx 0\Rightarrow S^{\mu_1\mu_2\cdots\mu_k}\approx\partial_{\mu_0}R^{\mu_0\mu_1\cdots\mu_k}$ with $k>2$, $S^{\mu_1\mu_2\cdots\mu_k}=S^{[\mu_1\mu_2\cdots\mu_k]}$, $R^{\mu_0\mu_1\cdots\mu_k}=R^{[\mu_0\mu_1\cdots\mu_k]}$. [This result holds whether or not $S^{\mu_1\mu_2\cdots\mu_k}$ carries extra indices.]

In antifield number two, the cohomology is given by the following theorem (which will be proven below),

Theorem 4.2 *A complete set of representatives of $H_2^n(\delta|d)$ is given by the antifields $C_a^{*\mu}$ conjugate to the ghosts, i.e.,*

$$\delta a_2^n+da_1^{n-1}=0\Rightarrow a_2^n=\lambda_\mu^a C_a^{*\mu}dx^0dx^1\cdots dx^{n-1}+\delta b_3^n+db_2^{n-1} \tag{4.10}$$

where the λ_μ^a are constant.

In order to interpret this theorem in terms of the characteristic cohomology (using Eq.(4.7) and recalling that $n>2$), we note that the equations of motion $H_a^{\mu\alpha}=0$ of the linearized theory can be rewritten as

$$H_a^{\mu\alpha}\equiv\partial_\nu\Phi_a^{\mu\nu\alpha} \tag{4.11}$$

with

$$\Phi_a^{\mu\nu\alpha}\equiv\partial_\beta\Psi_a^{\mu\nu\alpha\beta}=-\Phi_a^{\nu\mu\alpha}. \tag{4.12}$$

The tensor $\Psi_a^{\mu\nu\alpha\beta}$ is explicitly given by

$$\Psi_a^{\mu\nu\alpha\beta}=-\eta^{\mu\alpha}h^{a\nu\beta}-\eta^{\nu\beta}h^{a\mu\alpha}+\eta^{\mu\beta}h^{a\nu\alpha}+\eta^{\nu\alpha}h^{a\mu\beta}+\eta^{\alpha\mu}\eta^{\beta\nu}h^a-\eta^{\alpha\nu}\eta^{\beta\mu}h^a \tag{4.13}$$

[13]The quotient $H_0^0(d|\delta)/R$ is taken here in the sense of vector spaces: the set R of real numbers is naturally identified with a vector subspace of $H_0^0(d|\delta)$ since $dc=0$ for any constant c and $c\neq\delta$(something) $+d$(something') unless $c=0$. The constants occur in the isomorphism $H_0^0(d|\delta)/R\simeq H_1^1(\delta|d)$ because the cohomology of d is non trivial in form degree zero, $H^0(d)\simeq R$ (see [28] for details). The relation $H_0^0(d|\delta)/R\simeq H_1^1(\delta|d)$ implies in particular that if $H_1^1(\delta|d)=0$, then $H_0^0(d|\delta)\simeq R$.

(where h^a is the trace $h^{a\mu}{}_{\mu}$) and has the symmetries of the Riemann tensor. The equations of motion can thus be viewed as conservation laws involving antisymmetric tensors $S^{\mu\nu}$ of rank two, parametrized by further indices (α and a). These conservation laws are non-trivial because one cannot write $\Phi_a^{\mu\nu\alpha}$ as the divergence $\partial_\lambda \Theta_a^{\mu\nu\lambda\alpha}$ of a tensor $\Theta_a^{\mu\nu\lambda\alpha}$ that would be completely antisymmetric in μ, ν and λ ($\Psi_a^{\mu\nu\alpha\beta}$ does not have the required symmetries). Theorem 4.2 states that these are the only non-trivial conservation laws, i.e.,

$$\partial_\nu S^{\mu\nu} \approx 0,\ S^{\mu\nu} = -S^{\nu\mu} \Rightarrow S^{\mu\nu} \approx \lambda^a_\alpha \Phi_a^{\mu\nu\alpha} + \partial_\lambda U^{\mu\nu\lambda},\ U^{\mu\nu\lambda} = U^{[\mu\nu\lambda]}. \tag{4.14}$$

Let us now turn to the proof of Theorem 4.2. Let a be a solution of the cocycle condition for $H_2^n(\delta|d)$, written in dual notations,

$$\delta a + \partial_\mu V^\mu = 0. \tag{4.15}$$

Without loss of generality, one can assume that a is linear in the undifferentiated antifields, since the derivatives of $C_a^{*\mu}$ can be removed by integrations by parts (which leaves one in the same cohomological class of $H_2^n(\delta|d)$). Thus,

$$a = f^a_\mu C_a^{*\mu} + \mu \tag{4.16}$$

where μ is quadratic in the antifields $h_a^{*\mu\nu}$ and their derivatives, and where the f^a_μ are functions of $h^a_{\mu\nu}$ and their derivatives. Because $\delta\mu \approx 0$, the equation (4.15) implies the linearized Killing equations for f^a_μ,

$$\partial_\nu f^a_\mu + \partial_\mu f^a_\nu \approx 0. \tag{4.17}$$

If one differentiates this equation and uses the similar equations obtained by appropriate permutations of the spacetime indices, one gets, in the standard fashion

$$\partial_\lambda \partial_\nu f^a_\mu \approx 0. \tag{4.18}$$

This implies, using the isomorphism $H_0^0(d|\delta)/R \simeq H_n^n(\delta|d)$ and the previous theorem $H_n^n(\delta|d) = 0$ $(n > 2)$

$$\partial_\nu f^a_\mu \approx t^a_{\mu\nu} \tag{4.19}$$

where the $t^a_{\mu\nu}$ are constants. If one splits $t^a_{\mu\nu}$ into symmetric and antisymmetric parts, $t^a_{\mu\nu} = s^a_{\mu\nu} + a^a_{\mu\nu}$, $s^a_{\mu\nu} = s^a_{\nu\mu}$, $a^a_{\mu\nu} = -a^a_{\nu\mu}$, one gets from the linearized Killing equation (4.17) $s^a_{\mu\nu} \approx 0$ and thus $s^a_{\mu\nu} = 0$ (any constant weakly equal to zero is strongly equal to zero). Let $\bar{f}^a_\mu$ be $\bar{f}^a_\mu = f^a_\mu - a^a_{\mu\nu}x^\nu$. One has from (4.19) $\partial_\nu \bar{f}^a_\mu \approx 0$ and thus, using again $H_0^0(d|\delta) \simeq R$, $\bar{f}^a_\mu \approx \lambda^a_\mu$ for some constant λ^a_μ. This implies $f^a_\mu \approx \lambda^a_\mu + a^a_{\mu\nu}x^\nu$: f^a_μ is one-shell equal to a Killing field of the flat metric. If one does not allow for an explicit coordinate dependence, as one should in the context of constructing Poincaré invariant Lagrangians, one has $f^a_\mu \approx \lambda^a_\mu$. Substituting this expression into (4.16), and noting that the term proportional to the equation of motion can be absorbed through a redefinition of μ, one gets

$$a = \lambda^a_\mu C_a^{*\mu} + \mu' \tag{4.20}$$

(up to trivial terms). Now, the first term in the right-hand side of (4.20) is a solution of $\delta a + \partial_\mu V^\mu = 0$ by itself. This means that μ', which is quadratic in the $h_a^{*\mu\nu}$ and their derivatives, must be also a δ-cocyle modulo d. But it is well known that all such cocycles are trivial [25]. Thus, a is given by

$$a = \lambda^a_\mu C^{*\mu}_a + \text{ trivial terms} \tag{4.21}$$

as we claimed. This proves the theorem.

Comments

(1) The above theorems provide a complete description of $H^n_k(\delta|d)$ for $k > 1$. These groups are zero ($k > 2$) or finite-dimensional ($k = 2$). In contrast, the group $H^n_1(\delta|d)$, which is related to ordinary conserved currents, is infinite-dimensional since the theory is free. To our knowledge, it has not been completely computed. Fortunately, we shall not need it below.

(2) One can define a generalization of the characteristic cohomology using the endomorphism defined in [33], which fulfills $D^3 = 0$ (rather than $d^2 = 0$; for more information, see [34]). In the language of [33], the Bianchi identities can be written as $D \cdot H = 0$ and follow from the fact that $H = D^2 \cdot \Psi$ (just as the Noether identities $dM = 0$ for the Maxwell equations $M \approx 0$ follow from $M = d^*F$). The equations of motion read $D^2 \cdot \Psi \approx 0$ and define a non-trivial element of a generalized characteristic cohomology involving D rather than d, since one cannot write Ψ as the D of a local object (just as one cannot write $^*F$ as the d of a local object). There is thus a close analogy between gravity and the Maxwell theory provided one replaces the standard exterior derivative d by D, and the standard cohomology of d by the cohomologies of D. Note, however, that Ψ is not gauge-invariant, while $^*F$ is.

4.3 Invariant cohomology of δ modulo d

We have studied above the cohomology of δ modulo d in the space of arbitary functions of the fields $h^a_{\mu\nu}$, the antifields, and their derivatives. One can also study $H^n_k(\delta|d)$ in the space of invariant polynomials in these variables, which involve $h^a_{\mu\nu}$ and its derivatives only through the linearized Riemann tensor and its derivatives (as well as the antifields and their derivatives). The above theorems remain unchanged in this space. This is a consequence of

Theorem 4.1 *Let a be an invariant polynomial. Assume that a is δ trivial modulo d in the space of all (invariant and non-invariant) polynomials, $a = \delta b + dc$. Then, a is δ trivial modulo d in the space of invariant polynomials, i.e., one can assume without loss of generality that b and c are invariant polynomials.*

The proof is given in the appendix A.2.

5 Construction of the general gauge theory of interacting gravitons by means of cohomological techniques

Having reviewed the tools we shall need, we now come to grips with our main problem: to compute $H^{0,n}(s|d)$. To do this, the main technique is to expand according to the antifield number, as in [35]. Let a be a solution of

$$sa + db = 0 \tag{5.1}$$

with ghost number zero. One can expand a as

$$a = a_0 + a_1 + \cdots a_k \tag{5.2}$$

where a_i has antifield number i (and ghost number zero). [Equivalently, a_i has $puregh = i$, i.e. contains i's explicit ghost fields C^a_α's.] Without loss of generality, one can assume that the expansion (5.2) stops at some finite value of the antifield number. This was shown in [35] (section 3), under the sole assumption that the first-order deformation of the Lagrangian a_0 has a finite (but otherwise arbitrary) derivative order.

The previous theorems on the characteristic cohomology imply that one can remove all components of a with antifield number greater than or equal to 3. Indeed, the (invariant) characteristic cohomology in degree k measures precisely the obstruction for removing from a the term a_k of antifield number k (see appendix A.3). Since $H^n_k(\delta|d)$ vanishes for $k \geq 3$ by Theorem 4.1, one can assume

$$a = a_0 + a_1 + a_2. \tag{5.3}$$

Similarly, one can assume (see appendix A.3)

$$b = b_0 + b_1. \tag{5.4}$$

Inserting the expressions (5.3) and (5.4) in (5.1) we get

$$\delta a_1 + \gamma a_0 = db_0 \tag{5.5}$$

$$\delta a_2 + \gamma a_1 = db_1 \tag{5.6}$$

$$\gamma a_2 = 0\,. \tag{5.7}$$

Recall the meaning of the various terms in a : a_0 is the deformation of the Lagrangian; a_1 captures the information about the deformation of the gauge transformations; while a_2 contains the information about the deformation of the gauge algebra. We shall first deal with a_2, and then "descend" to a_1 and a_0.

5.1 Determination of a_2

As we have seen in section 3, the general solution of (5.7) reads, modulo trivial terms,

$$a_2 = \sum_J \alpha_J \omega^J \tag{5.8}$$

where the α_J are invariant polynomials (see (3.2)). A necessary (but not sufficient) condition for a_2 to be a (non-trivial) solution of (5.6), so that a_1 exists, is that α_J be a (non-trivial) element of $H^n_2(\delta|d)$ (see appendix A.3) Thus, by Theorem 4.2, the polynomials α_J must be linear combinations of the antifields $C^*_{\alpha a}$. The monomials ω^J have ghost number two; so they can be of only three possible types

$$C^a_\alpha C^b_\beta, \quad C^a_\alpha \partial_{[\beta} C^b_{\gamma]}, \quad \partial_{[\alpha} C^a_{\beta]} \partial_{[\gamma} C^b_{\delta]}. \tag{5.9}$$

They should be combined with C^{*a}_α to form a_2. By Poincaré invariance, the only possibility is to take $C^a_\alpha \partial_{[\beta} C^b_{\gamma]}$, which yields[14]

$$a_2 = -C^{*\beta}_a C^{\alpha b} \partial_{[\alpha} C^c_{\beta]} a^a_{bc} + \gamma b_2. \tag{5.10}$$

Here we have introduced constants a^a_{bc} that parametrize the general solution a_2 of equations (5.6), (5.7). The trivial "γ-exact" additional term in Eq.(5.10) will be normalized to a convenient value below.

The a^a_{bc} can be identified with the structure constants of a N-dimensional real algebra $\mathcal{A}$. Let V be an "internal" (real) vector space of dimension N; we define a product in V through

$$(x \cdot y)^a = a^a_{bc} x^b y^c, \quad \forall x, y \in V. \tag{5.11}$$

The vector space V equipped with this product defines the algebra $\mathcal{A}$. At this stage, $\mathcal{A}$ has no particular further structure. Extra conditions will arise, however, from the demand that a (and not just a_2) exists and defines a deformation that can be continued to all orders. We shall recover in this manner the conditions found in [2], plus one additional condition that will play a crucial role.

It is convenient (to simplify later developments) to fix the γ–exact term in Eq.(5.10) to the value $b_2 = \frac{1}{2} C^{*\beta}_a C^{\alpha b} h^c_{\alpha\beta} a^a_{bc}$. Using $\gamma h^a_{\alpha\beta} = 2\partial_{(\alpha} C^a_{\beta)}$, we then get,

$$a_2 = C^{*\beta}_a C^{\alpha b} \partial_\beta C^c_\alpha a^a_{bc}\,. \tag{5.12}$$

In terms of the algebra of the gauge transformations, this term a_2 implies that the gauge parameter $\zeta^{a\mu}$ corresponding to the commutator of two gauge transformations with parameters $\xi^{a\mu}$ and $\eta^{a\mu}$ is given by

$$\zeta^{a\mu} = a^a_{bc} [\xi^b, \eta^c]^\mu \tag{5.13}$$

where $[,]$ is the Lie bracket of vector fields. It is worth noting that at this stage, we have not used any a priori restriction on the number of derivatives (except that it is finite). The assumption that the interactions contain at most two derivatives will only be needed below. Thus, the fact that a stops at a_2, and that a_2 is given by (5.12) is quite general.

[14]Actually, for particular values of the dimension n, there are also solutions of (5.6), (5.7) built with the ε tensor. If one imposes PT invariance, these possibilities are excluded. Furthermore, they lead to interaction terms with three derivatives. The corresponding theories will be studied elsewhere [36]. As often in the sequel, we shall switch back and forth between a form and its dual without changing the notation when no confusion can arise. So the same equation for a is sometimes written as $sa + db = 0$ and sometimes written as $sa + \partial_\mu b^\mu = 0$.

Differently put: to first-order in the coupling constant, the deformation of the algebra of the spin-2 gauge symmetries is universal and given by (5.12). There is no other possibility. In particular, there is no room for deformations of the algebra such that the new gauge transformations would close only on-shell (terms quadratic in h^* are absent from (5.12)). This strengthens the analysis of [2] where assumptions on the number of derivatives in the gauge transformations were made. No such assumption is in fact needed.

5.2 Determination of a_1

In order to find a_1 we have to solve equation (5.6),

$$\delta a_2 + \gamma a_1 = d b_1 \,. \tag{5.14}$$

We have

$$\begin{aligned} \delta a_2 &= -2\partial_\gamma h^{*\beta\gamma}_a C^{\alpha b}\partial_\beta C^c_\alpha a^a_{bc} = -2\partial_\gamma\left(h^{*\beta\gamma}_a C^{\alpha b}\partial_\beta C^c_\alpha a^a_{bc}\right) + \\ &\quad 2h^{*\beta\gamma}_a \partial_\gamma C^{\alpha b}\partial_\beta C^c_\alpha a^a_{bc} + 2h^{*\beta\gamma}_a C^{\alpha b}\partial_{\beta\gamma} C^c_\alpha a^a_{bc} \,. \end{aligned} \tag{5.15}$$

The term with two derivatives of the ghosts is γ-exact (see Eq.(3.1), thus, for a_1 to exist, the term $2h^{*\beta\gamma}_a \partial_\gamma C^{\alpha b}\partial_\beta C^c_\alpha a^a_{bc}$ should be γ-exact modulo d. But this can happen only if is zero. Indeed, we can rewrite it in terms of the generators of $H(\gamma)$ by adding a γ-exact term, as

$$2h^{*\beta\gamma}_a \partial_\gamma C^{\alpha b}\partial_\beta C^c_\alpha a^a_{bc} = 2h^{*\beta}_a{}_\gamma \partial^{[\gamma} C^{\alpha]b}\partial_{[\beta} C^c_{\alpha]} a^a_{bc} + \gamma(\ldots) \,. \tag{5.16}$$

It is shown in appendix B that this term is trivial only if it vanishes. Since

$$2h^{*\beta}{}_{\gamma a}\partial^{[\gamma} C^{\alpha]b}\partial_{[\beta} C^c_{\alpha]} a^a_{bc} = 2h^{*\beta}{}_{\gamma a}\partial^{[\gamma} C^{\alpha]b}\partial_{[\beta} C^c_{\alpha]} a^a_{[bc]} \tag{5.17}$$

the vanishing of this term yields

$$a^a_{bc} = a^a_{(bc)} \,, \tag{5.18}$$

namely, the *commutativity* of the algebra $\mathcal{A}$ defined by the a^a_{bc}'s. This result is not surprising in view of the form of the commutator of two gauge transformations since (5.13) ought to be antisymmetric in ξ^a and η^a. When (5.18) holds, δa_2 becomes

$$\delta a_2 = -2\partial_\gamma\left(h^{*\beta\gamma}_a C^{\alpha b}\partial_\beta C^c_\alpha a^a_{bc}\right) + \gamma\left(h^{*\beta\gamma}_a C^{\alpha b}(\partial_\gamma h^c_{\alpha\beta} + \partial_\beta h^c_{\alpha\gamma} - \partial_\alpha h^c_{\gamma\beta}) a^a_{bc}\right) \tag{5.19}$$

which yields a_1

$$a_1 = -h^{*\beta\gamma}_a C^{\alpha b}\left(\partial_\gamma h^c_{\alpha\beta} + \partial_\beta h^c_{\alpha\gamma} - \partial_\alpha h^c_{\gamma\beta}\right) a^a_{bc} \tag{5.20}$$

up to a solution of the "homogenous" equation $\gamma a_1 + db_1 = 0$.

As we have seen, the solutions of the homogeneous equation do not modify the gauge algebra (since they have a vanishing a_2), but they do modify the gauge transformations. By a reasoning analogous to the one given in the appendix, one can assume $b_1 = 0$ in $\gamma a_1 + db_1 = 0$. Thus, a_1 is a γ-cocycle. It must be linear in $h^{*\mu\nu}_\alpha$ and in C^a_μ or $\partial_{[\mu} C^a_{\nu]}$. By Lorentz invariance, it must contain at least one linearized curvature since the Lorentz-invariant $h^{*\mu\nu}_\alpha \partial_{[\mu} C^b_{\nu]}$ vanishes. But this would lead to an interaction term a_0 that would contain at least three derivatives and which is thus excluded by our derivative assumptions. Thus, the most general a_1 compatible with our requirements is given by (5.20). This is the first place where we do need the derivative assumption. [We believe that this derivative assumption is in fact not needed here in generic spacetime dimensions, if one takes into account the other conditions on a_1: Poincaré invariance, existence of a_0, etc. However, we do not have a proof. More information on this in section 11.]

5.3 Determination of a_0

We now turn to the determination of a_0, that is, to the determination of the deformed Lagrangian at first order in g. The equation for a_0 is (5.5),

$$\delta a_1 + \gamma a_0 = d b_0 \,. \tag{5.21}$$

We have

$$\begin{aligned}\delta a_1 &= -\frac{\delta S_0}{\delta h^a_{\alpha\beta}} C^{\gamma b}\left(\partial_\alpha h^c_{\beta\gamma} + \partial_\beta h^c_{\alpha\gamma} - \partial_\gamma h^c_{\alpha\beta}\right) a^a_{bc} = \\ &-(\Box h^a_{\alpha\beta} + \partial_{\alpha\beta} h^a - \partial_\alpha \partial^\rho h^a_{\rho\beta} - \partial_\beta \partial^\rho h^a_{\rho\alpha} + \eta_{\alpha\beta}\partial_{\sigma\rho} h^{a\sigma\rho} \\ &-\eta_{\alpha\beta}\Box h^a) C^b_\gamma \left(\partial^\alpha h^{c\beta\gamma} + \partial^\beta h^{c\alpha\gamma} - \partial^\gamma h^{c\alpha\beta}\right) a_{abc} \,,\end{aligned} \tag{5.22}$$

where we have defined

$$a_{abc} \equiv k_{ad} a^d_{bc} \,, \tag{5.23}$$

where k_{ab} is the quadratic form defined by the free kinetic terms. Now we prove that (as in Yang-Mills theory) these "structure constants" with all indices down, a_{abc}, must be *fully symmetric*, $a_{abc} = a_{(abc)}$, for (5.21) to have a solution.

The polynomial δa_1 is trilinear in $\partial_{\alpha_1\alpha_2} h^a_{\alpha_3\alpha_4}$, $\partial_{\alpha_5} h^b_{\alpha_6\alpha_7}$ and $C^c_{\alpha_8}$. There exist twenty-three different ways to contract the Lorentz indices in the product $\partial_{\alpha_1\alpha_2} h^a_{\alpha_3\alpha_4}\, \partial_{\alpha_5} h^b_{\alpha_6\alpha_7}\, C^c_{\alpha_8}$ to form a Lorentz scalar. These are, in full details (and dropping the internal indices),

$$\begin{aligned}\{Q_\Delta\} = \{&\Box h\,\partial_\alpha h\, C^\alpha, \Box h\,\partial^\beta h_{\alpha\beta}\, C^\alpha, \Box h_{\beta\gamma}\,\partial^\gamma h^\beta{}_\alpha\, C^\alpha, \Box h_{\beta\gamma}\,\partial_\alpha h^{\beta\gamma}\, C^\alpha, \Box h_{\beta\gamma}\,\partial_\alpha h^{\beta\alpha}\, C^\gamma, \\ &\Box h_{\beta\gamma}\,\partial^\beta h\, C^\gamma, \partial_{\alpha\beta} h^{\alpha\beta}\,\partial_\gamma h\, C^\gamma, \partial_{\alpha\beta} h^{\alpha\beta}\,\partial^\mu h_{\mu\gamma}\, C^\gamma, \partial_{\alpha\beta} h^{\alpha\gamma}\,\partial_\gamma h^\beta{}_\mu\, C^\mu, \\ &\partial_{\alpha\beta} h^{\alpha\gamma}\,\partial_\mu h^\beta{}_\gamma\, C^\mu, \partial_{\alpha\beta} h^{\alpha\gamma}\,\partial_\mu h^{\beta\mu}\, C_\gamma, \partial_{\alpha\beta} h^{\alpha\gamma}\,\partial^\beta h_{\gamma\mu}\, C^\mu, \partial_{\alpha\beta} h^{\alpha\gamma}\,\partial^\beta h\, C_\gamma, \\ &\partial_{\alpha\beta} h^{\alpha\gamma}\,\partial^\gamma h\, C^\beta, \partial_{\alpha\beta} h^{\alpha\gamma}\,\partial^\mu h_{\gamma\mu}\, C^\beta, \partial_{\alpha\beta} h\,\partial_\gamma h^{\alpha\beta}\, C^\gamma, \partial_{\alpha\beta} h_{\gamma\mu}\,\partial^\gamma h^{\alpha\beta}\, C^\mu, \\ &\partial_{\alpha\beta} h\,\partial^\beta h^\alpha{}_\gamma C^\gamma, \partial_{\alpha\beta} h_{\gamma\mu}\,\partial^\beta h^{\alpha\gamma} C^\mu, \partial_{\alpha\beta} h\,\partial_\gamma h^{\alpha\gamma}\, C^\beta, \partial_{\alpha\beta} h_{\gamma\mu}\,\partial^\gamma h^{\alpha\mu}\, C^\beta, \\ &\partial_{\alpha\beta} h\,\partial^\alpha h\, C^\beta, \partial_{\alpha\beta} h_{\gamma\mu}\,\partial^\alpha h^{\gamma\mu}\, C^\beta\}\end{aligned} \tag{5.24}$$

($\Delta = 1, \ldots, 23$). These polynomials are independent: if $\alpha^\Delta Q_\Delta = 0$, then $\alpha^\Delta = 0$; this can be easily verified. Consequently, these polynomials form a basis of the vector space under consideration. In particular, two polynomials $\alpha^\Delta Q_\Delta$ and $\beta^\Delta Q_\Delta$ are equal if and only if all their coefficients are equal, $\alpha^\Delta = \beta^\Delta$.

Let us single out the terms in (5.22) containing two traces h^a; there is only one such term, along the first element of the basis,

$$-\Box h^a\; C^b_\gamma\; \partial^\gamma h^c\, a_{abc} \tag{5.25}$$

By counting derivatives and ghost number, one easily sees that the solution a_0 of (5.21) must be a sum of terms cubic in the fields $h^a_{\alpha\beta}$, with two derivatives (γ brings in one derivative). The only monomials which give terms with two traces h^a by applying the γ operator are $h^d h^e \Box h^f$, $h^d \partial^\mu h^e \partial_\mu h^f$, $\partial^{\mu\nu} h^d_{\mu\nu} h^e h^f$, $\partial^\mu h^d_{\mu\nu} \partial^\nu h^e h^f$, $h^d_{\mu\nu} \partial^\mu h^e \partial^\nu h^f$ and $h^d h^e_{\mu\nu} \partial^{\mu\nu} h^f$. Some of these terms are equivalent modulo integrations by parts; only three of them are independent, which can be taken to be $h^d \partial^\mu h^e \partial_\mu h^f$, $h^d_{\mu\nu} \partial^\mu h^e \partial^\nu h^f$ and $h^d h^e_{\mu\nu} \partial^{\mu\nu} h^f$. The piece in a_0 that we are considering is then

$$a_0 = \ldots + h^d \partial^\mu h^e \partial_\mu h^f \, b^1_{def} + h^d_{\mu\nu} \partial^\mu h^e \partial^\nu h^f \, b^2_{def} + h^d h^e_{\mu\nu} \partial^{\mu\nu} h^f \, b^3_{def} \,, \tag{5.26}$$

with b^i_{def} being constants with the symmetries

$$b^1_{def} = b^1_{dfe} \quad b^2_{def} = b^2_{dfe} \,. \tag{5.27}$$

Then we apply γ to a_0, and integrate by parts. The rationale behind the integrations by parts that we perform is to require that the ghosts, which occur linearly, should carry no derivatives, as in δa_1. Proceeding in this manner and focusing only on the terms with two traces h^a in $\gamma a_0 - db_0 = -\delta a_1$, we easily get the condition

$$\begin{aligned} &\Box h^a C^b_\nu \partial^\nu h^c \left(4b^1_{cba} - 2b^2_{bac}\right) + C^a_\mu \partial^{\mu\nu} h^b \partial_\nu h^c \left(-4b^1_{abc} + 4b^1_{bac} + 4b^1_{cab} - 2b^2_{acb} - 2b^3_{cab}\right) + \\ &h^a C^b_\nu \Box \partial^\nu h^c \left(4b^1_{abc} - 2b^3_{abc}\right) = \Box h^a C^b_\nu \partial^\nu h^c \, a_{abc} \,. \end{aligned} \tag{5.28}$$

From this equation, we obtain

$$2b^2_{abc} = 4b^1_{cab} - a_{bac}, \quad b^3_{abc} = 2b^1_{abc}, \quad -4b^1_{abc} + a_{cab} = 0 \,. \tag{5.29}$$

In particular we find that

$$-\, a_{abc} = -4b^1_{bca} = -4b^1_{bac} = -a_{cba} \,, \tag{5.30}$$

and thus

$$a_{abc} = a_{(abc)} \tag{5.31}$$

where we have used the symmetry relations of $b^1_{abc} = b^1_{a(bc)}$ and $a_{abc} = a_{a(bc)}$ previously derived. An algebra which fulfills $a_{abc} = a_{cba}$ is called Hilbertian, or, in the real case considered here, "symmetric".

Now we prove that $a_{abc} = a_{(abc)}$ is a *sufficient* condition for the (5.21) to have solution. This is simply done by explicitly exhibiting a solution. Substituting the expression

$$\begin{aligned} a_0 \;=\; & \Big(\frac{1}{4} h^a \partial^\mu h^b \partial_\mu h^c - \partial_\mu h^a \partial^\mu h^{b\alpha\beta} h^c_{\alpha\beta} - \frac{1}{4} \partial_\mu h^{a\alpha\beta} \partial^\mu h^b_{\alpha\beta} h^c \\ & + \partial_\mu h^a_{\alpha\beta} \partial^\mu h^{c\beta}_{\ \ \gamma} h^{c\gamma\alpha} - \partial^{\mu\nu} h^{a\alpha\beta} h^b_{\mu\nu} h^c_{\alpha\beta} - \frac{1}{2} \partial^\mu h^{a\alpha\beta} \partial^\nu h^b_{\alpha\beta} h^c_{\mu\nu} \\ & + \frac{1}{2} \partial^{\mu\nu} h^a h^b_{\mu\nu} h^c + \frac{1}{2} h^a \partial_\beta h^{b\beta\gamma} \partial^\alpha h^c_{\gamma\alpha} - \frac{1}{2} \partial^{\mu\nu} h^a h^{b\alpha}_{\ \ \nu} h^c_{\alpha\mu} \\ & + \partial_\mu h^a h^{b\mu\nu} \partial^\alpha h^c_{\nu\alpha} - \partial_\mu h^{a\mu\alpha} \partial_\nu h^{b\nu\beta} h^c_{\alpha\beta} + h^{a\mu\alpha} h^{b\nu\beta} \partial_{\mu\nu} h^c_{\alpha\beta}\Big) \, a_{abc} \end{aligned} \tag{5.32}$$

with $a_{abc} = a_{(abc)}$ in the equation (5.21) one finds that it is satisfied. The expression (5.32) has been derived by considering initially the case with one spin two field. In this case, general relativity with $g_{\alpha\beta} = \eta_{\alpha\beta} + g h_{\alpha\beta}$ is a solution and the corresponding a_0 is the term of the Einstein–Hilbert lagrangian cubic in $h_{\alpha\beta}$. We verified that this expression satisfies $\delta a_1 + \gamma a_0 = d b_0$, and found that the proof remains valid if we take the same expression with different fields contracted by a symmetric tensor.

We have therefore proven that a gauge theory of interacting spin two fields, with a non trivial gauge algebra, is first-order consistent if and only if the algebra $\mathcal{A}$ defined by a^a_{bc}, which characterizes a_2, is commutative and symmetric.

Again, there is some ambiguity in a_0 since we can add to (5.32) any solution of the "homogeneous" equation $\gamma a_0 + d b_0 = 0$ without a_1. If one requires that a_0 has at most two derivatives, there is only one possibility, namely

$$- 2\tilde{\Lambda}^{(1)}_a h^{a\mu}_{\ \ \mu} \tag{5.33}$$

where the $\tilde{\Lambda}^{(1)}_a$'s are constant. This term fulfills

$$\gamma(\tilde{\Lambda}^{(1)}_a h^{a\mu}_{\ \ \mu}) = \partial_\mu(2\tilde{\Lambda}^{(1)}_a C^{a\mu}) \tag{5.34}$$

and is of course the (linearized) cosmological term. There is no other non-trivial term. Indeed, the Euler-Lagrange derivatives $S^{\mu\nu} \equiv \delta a_0 / \delta h_{\mu\nu}$ of any a_0 fulfilling $\gamma a_0 + \partial_\mu b^\mu_0 = 0$ is an invariant, symmetric tensor fulfilling the contracted Bianchi identities $\partial_\mu S^{\mu\nu} = 0$ and containing at most two derivatives. Now, the only such tensors are $\eta^{\mu\nu}$ and the linearized Einstein tensor. The first corresponds to the cosmological term; the second vanishes on-shell and derives from a piece in the Lagrangian that can be absorbed through redefinitions of the fields; it is trivial.

If one does not restrict the derivative order of a_0, there are further possibilities, e.g., any polynomial in the linearized Riemann tensor and its derivatives is a solution. This is the second place where the derivative assumption is explicitly used in the analysis. We shall come back to this point in section 11.

The extra consistency condition (5.31) arises because we demand that a_0, the first-order deformation of the Lagrangian, should exist. Its form explicitly depends on the original Lagrangian through the metric k_{ab} defined in internal space by the kinetic term.

The condition (5.31) does not appear in [2] (although it is mentioned in [3], but not discussed in the context of the free limit). As we shall see, it is this condition that is responsible for the impossibility to have consistent cross-couplings between a finite collection of (non-ghost) gravitons.

It is interesting to note that a similar phenomenon appears in the construction of the Yang-Mills theory from a collection of free spin-1 particles. If one focuses only on a_1 and a_2, one finds that the deformations are characterized by a Lie algebra [16]. But if one requires also that a_0 exist, the Lie algebra should have a further property: it should admit an invariant metric, and that metric should be the metric defined by the Lagrangian of the free theory (see e.g. [26] and references therein). In the spin-1 case, of course, this extra condition does not prevent cross-interactions.

5.4 The associativity of the algebra from the absence of obstructions at second order

The master equation at order two is

$$(W_1, W_1) = -2sW_2 \tag{5.35}$$

with

$$W_1 = \int d^n x \; (a_0 + a_1 + a_2) \; . \tag{5.36}$$

One can expand (W_1, W_1) according to the antifield number. One finds

$$(W_1, W_1) = \int d^n x(\alpha_0 + \alpha_1 + \alpha_2) \tag{5.37}$$

where the term of antifield number two α_2 comes from the antibracket of $\int d^n x \, a_2$ with itself and reads explicitly (using (5.12))

$$\alpha_2 = -\left(2C_a^{*\beta}\partial_\beta C_\sigma^b + \partial_\beta C_a^{*\beta} C_\sigma^b\right) C^{\alpha f}\partial^\sigma C_\alpha^c (a_{db}^a a_{fc}^d). \tag{5.38}$$

If one also expands W_2 according to the antifield number, one gets from (5.35) the following condition on α_2 (it is easy to see, by using the arguments given in the appendix, that the expansion of W_2 can be assumed to stop at antifield number three, $W_2 = \int d^n x(c_0 + c_1 + c_2 + c_3)$ and that c_3 may be assumed to be invariant, $\gamma c_3 = 0$)

$$\alpha_2 = -2(\gamma c_2 + \delta c_3) + db_2 \, . \tag{5.39}$$

It is impossible to get an expression with three ghosts, one $C_a^{*\beta}$ and no fields, by applying δ to c_3, so we can assume without loss of generality that c_3 vanishes, which implies that α_2 should be γ–exact modulo total derivatives.

Integrating by parts and adding γ–exact terms, one finds

$$\alpha_2 = -2C_a^{*\beta}\partial_{[\beta}C_{\sigma]}^b C_\alpha^f \partial^{[\sigma}C^{\alpha]c} a_{d[b}^a a_{f]c}^d + \text{ trivial terms.} \tag{5.40}$$

This expression has the standard form (3.2). It is simple to prove, as in the proof of appendix B, that it is not a mod-d γ-coboundary unless it vanishes. This happens if and only if

$$a_{d[b}^a a_{f]c}^d = 0 \, , \tag{5.41}$$

which is the associative property for the algebra $\mathcal{A}$ defined by the a_{bc}^a. Thus, $\mathcal{A}$ must be *commutative, symmetric and associative.*

It is quite important to note that this result holds even if we allow more general a_1's or a_2's involving more derivatives, since these terms will not contribute to α_2. So, the absence of obstructions at order g^2 will lead to the same associativity condition and the same triviality of the algebra which we establish now.

6 Impossibility of cross-interactions

Finite-dimensional real algebras that are commutative, symmetric and associative have a trivial structure: they are the direct sum of one-dimensional ideals.

To see this, one proceeds as follows. The algebra operation allows us to associate to every element of the algebra $u \in \mathcal{A}$ a linear operator

$$A(u) : \mathcal{A} \longrightarrow \mathcal{A} \tag{6.1}$$

defined by

$$A(u)v \equiv u \cdot v \,. \tag{6.2}$$

In a basis $(e_1, \ldots, e_m)$, one has $v = v^a e_a$ and

$$A(u)^c{}_b = u^a a^c_{ab} \,. \tag{6.3}$$

Because of the associativity property, the operators $A(u)$ provide a representation of the algebra

$$A(u)A(v) = A(u \cdot v) \tag{6.4}$$

and so, since the algebra is commutative,

$$[A(u), A(v)] = 0 \,. \tag{6.5}$$

Now, the free Lagrangian endows the algebra $\mathcal{A}$ (viewed as an N-dimensional vector space) with an Euclidean structure, defined by the scalar product $(u, v) = k_{ab} u^a v^b$. At this point, it is convenient to normalize the Euclidean metric k_{ab} in the standard way, $k_{ab} = \delta_{ab}$, i.e. to endow $\mathcal{A}$ with the usual Euclidean scalar product

$$(u, v) = \delta_{ab} u^a v^b \,. \tag{6.6}$$

The symmetry property

$$a_{abc} = a_{(abc)} \tag{6.7}$$

expresses that the operators $A(u)$ are all symmetric

$$(u, A(v)w) = (A(v)u, w) \,, \tag{6.8}$$

that is,

$$A(u) = A(u)^T \,. \tag{6.9}$$

Then the real, symmetric operators $A(u)$, $u \in \mathcal{A}$ are diagonalizable by a rotation in $\mathcal{A}$, viewed as an N-dimensional Euclidean space . Since they are commuting, they are simultaneously diagonalizable. In a basis $\{e_1, \ldots, e_m\}$ in which they are all diagonal, one has $A(e_a)e_b = \alpha(a, b)e_b$ for some numbers $\alpha(a, b)$ and thus

$$e_a \cdot e_b = A(e_a)e_b = \alpha(a, b)e_b = e_b \cdot e_a = A(e_b)e_a = \alpha(b, a)e_a \,. \tag{6.10}$$

So $\alpha(a, b) = 0$ unless $a = b$. We set $\alpha(a, a) \equiv \tilde{\kappa}^{(1)}_a$. By using the discrete symmetry $h^a_{\mu\nu} = -h^a_{\mu\nu}$ of the free theory, we can always enforce that $\tilde{\kappa}^{(1)}_a \geq 0$.

Consequently, the structure constants a^a_{bc} of the algebra $\mathcal{A}$ vanish whenever two indices are different. There is no term in W_1 coupling the various spin-2 sectors, which are therefore completely decoupled. Only self-interactions are possible. The first-order deformation W_1 is in fact the sum of Einstein cubic vertices (one for each spin-2 field with $\alpha(a,a) \neq 0$) + (first-order) cosmological terms.

Technically, the passage from an arbitrary orthonormal basis in internal space to the basis where the $A(u)$'s are all diagonal is achieved by exponentiating a transformation $\Delta W_1 = (W_1, K_0)$ (see (2.22)), where K_0 defines an infinitesimal rotation in internal space. It is clear that these rotations leave the free Lagrangian invariant ($\Leftrightarrow (W_0, K_0) = 0$). So we see that the extra identifications of the form $\Delta W_1 = (W_1, K_0)$ have a rather direct and natural meaning in the present case.

When none of the $\tilde{\kappa}^{(1)}_a$ vanishes, which is in a sense the "generic case", the basis $\{e_a\}$ is unique. The allowed redefinitions $\Delta W = (W, K)$ must fulfill

$$(W_0, K_0) = 0, \;\; (W_0, K_1) + (W_1, K_0) = 0 \tag{6.11}$$

in order to preserve the given W_0 and W_1. The term (W_1, K_0) modify the structure constants a^a_{bc} by a rotation and so, cannot be BRST-exact unless it is zero. So, we must have separately $(W_0, K_1) = 0$ and $(W_1, K_0) = 0$. Since the basis $\{e_a\}$ in which the a^a_{bc} take their canonical form is unique, we infer from $(W_1, K_0) = 0$ that K_0 is zero. We can thus conclude that given W_0 and W_1, the redefinition freedom is characterized by a $K = K_0 + gK_1 + \cdots$ with $K_0 = 0$ and $(W_0, K_1) = 0$.

7 Complete Lagrangian

With the above information, it is easy to complete the construction of the full Lagrangian to all orders in the coupling constant. This is because one knows already one solution , namely the Einstein-Hilbert action. So, the only point that remains to be done is to check that there are no others. In other words, given W_0 and W_1, equal to the standard Einstein terms, how unique are W_2, W_3 etc?

One has

$$W = W^E_0 + gW^E_1(\tilde{\kappa}^{(1)}_a) + g^2 W_2 + \cdots \tag{7.1}$$

where we emphasize the dependence of W^E_1 on the constants $\tilde{\kappa}^{(1)}_a$. The equation determining W_2 is, as we have seen, $sW_2 = -(1/2)(W^E_1, W^E_1)$. A particular solution is the functional $W^E_2((\tilde{\kappa}^{(1)}_a)^2)$ corresponding to the sum of second-order Einstein deformations, which we know exists. Thus, $W_2 = W^E_2 + W'_2$, where W'_2 is a solution of the homogeneous equation $sW'_2 = 0$. The general solution to that equation is $W'_2 = \tilde{W}_1(b^a_{bc})$, where $\tilde{W}_1(b^a_{bc}) \equiv \tilde{W}_1$ has been determined in section 5 and involves at this stage arbitrary constants b^a_{bc} fulfilling $b^a_{bc} = b^a_{cb}$ and $b_{abc} = b_{(abc)}$.

The equation for W_3 is then

$$sW_3 = -(W_2, W^E_1) \tag{7.2}$$

i.e., setting $W_3 = W^E_3 + W'_3$, where W^E_3 is the Einsteinian solution of $sW^E_3 = -(W^E_2, W^E_1)$,

$$sW'_3 = -(\tilde{W}_1, W^E_1). \tag{7.3}$$

Now, $(\tilde{W}_1, W_1^E)$ is s-exact if and only if the constants b^a_{bc} are subject to $a^a_{d[b}b^d_{f]c}+b^a_{d[b}a^d_{f]c} = 0$. But this condition expresses that $a^a_{bc} + g b^a_{bc}$ defines an associative algebra (to the relevant order). Therefore, one can repeat the argument of the previous section: by making an order-g rotation of the fields, i.e., by choosing appropriately the term K_1 in K, one can arrange that the only non-vanishing components of b^a_{bc} are those with three equal indices, and we set $b^a_{aa} = \tilde{\kappa}^{(2)}_a$. When this is done, we see that the term W_2' is equal to $W_1^E(\tilde{\kappa}^{(2)}_a)$ and that W_3' is equal to $W_2^E(2\tilde{\kappa}^{(1)}_a\tilde{\kappa}^{(2)}_a)$ plus a solution W_3'' of the homogeneous equation $sW_3'' = 0$. Continuing in the same way, one easily sees that $W_3'' = W_1^E(\tilde{\kappa}^{(3)}_a)$ and the higher order terms are determined to follow the same pattern.

Regrouping all the terms in W, one finds that W is a sum of Einsteinian solutions, one for each massless spin-2 field, with coupling constants

$$\kappa_a = g\tilde{\kappa}^{(1)}_a + g^2\tilde{\kappa}^{(2)}_a + g^3\tilde{\kappa}^{(3)}_a + \cdots \tag{7.4}$$

For simplicity of notation, we assumed that the cosmological constant was vanishing at each order. Had we included it, we would have found possible cosmological terms for each massless, spin-2 field, with cosmological constant given by

$$\Lambda_a = g\tilde{\Lambda}^{(1)}_a + g^2\tilde{\Lambda}^{(2)}_a + g^3\tilde{\Lambda}^{(3)}_a + \cdots \tag{7.5}$$

We can thus conclude that indeed, the most general deformation of the action for a collection of free, massless, spin-2 fields is the sum of Einstein-Hilbert actions, one for each field,

$$S[g^a_{\mu\nu}] = \sum_a \frac{2}{\kappa_a^2}\int d^nx(R^a - 2\Lambda_a)\sqrt{-g^a},\ g^a_{\mu\nu} = \eta_{\mu\nu} + \kappa^a h^a_{\mu\nu}. \tag{7.6}$$

as we announced. There is thus no cross-interaction, to all orders in the coupling constants. This action is invariant under independent diffeomorphisms,

$$\frac{1}{\kappa^a}\delta_\epsilon g^a_{\mu\nu} = \epsilon^a_{\mu;\nu} + \epsilon^a_{\nu;\mu} \tag{7.7}$$

and so has manifestly the required number of independent gauge symmetries (as many as in the free limit). Cosmological terms can arise in the deformation because they are compatible with the gauge symmetries. One may view the diffeomorphisms (7.7) as algebra-valued diffeomorphisms of a manifold of the type considered by Wald [4], but in the present case where the algebra is completely reducible and given by the direct sum of one-dimensional ideals, the structure of the manifold is rather trivial. In the case of a single massless spin-2 field, we recover the known results on the uniqueness of the Einstein construction.

If some coupling constants κ^a vanish, the corresponding free action is undeformed at each order in g and the full action coincides, in those sectors, with the free action plus a possible linearized cosmological term $-2\lambda_a h^{a\mu}{}_\mu$; the gauge symmetry (7.7) reduces of course to the original one. This situation is non-generic and unstable under arbitrary deformations. By contrast, the Einstein action is stable under arbitrary deformations (with at most two derivatives) [37].

8 Infinite-dimensional algebras

The absence of cross-interactions between the various massless spin-2 fields relies heavily on the fact that all *finite-dimensional* associative, commutative and symmetric algebras are trivial. This property, demonstrated in section 6 is no longer valid in the *infinite-dimensional* case, for which the operators $A(u)$ may not be diagonalizable. Thus for a system with an infinite number of massless spin-2 fields, one may construct cross-interactions that are not removable by field redefinitions. Actually, an example where this happens was given in [6]. The infinite-dimensional algebra that occurs in that precise example is that of the complex functions on the 2-sphere. This example was arrived at by dimensionally reducing the six-dimensional Lovelock theory to four dimensions with a sphere as internal space. A simpler example would be the algebra of real functions on the circle endowed with the natural L^2 metric. One uses as algebra-product the usual point-wise product of functions: $(f \cdot g)(\varphi) \equiv f(\varphi)g(\varphi)$. This algebra is clearly commutative and associative. It is also symmetric for the scalar product

$$(f,g) \equiv \int d\varphi\, f(\varphi)\, g(\varphi) \tag{8.1}$$

since

$$(A(f)g,h) = \int d\varphi(f(\varphi)g(\varphi))h(\varphi) = \int d\varphi\, g(\varphi)(f(\varphi)h(\varphi)) = (g, A(f)h). \tag{8.2}$$

To work with a discrete basis it is enough to use any orthonormal basis for $\mathcal{A}$, such as the functions $\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos m\varphi, \frac{1}{\sqrt{\pi}}\sin m\varphi\}$ $(m > 0)$. The operators $A(f)$ are not diagonal in that basis. To be able to diagonalize $A(f)$, one should find *square-integrable* eigenfunctions $g(\varphi)$ such that

$$(A(f)g)(\varphi) \equiv f(\varphi)g(\varphi) = \lambda_f\, g(\varphi). \tag{8.3}$$

When $f(\varphi) \neq const$, there is no non identically vanishing function $g(\varphi)$ belonging to L^2 that fulfills this condition. Thus, one cannot diagonalize[15] the $A(f)$. Consequently, one can avoid the no-go theorem and construct an algebraically consistent interacting theory by considering an infinite number of massless, spin-2 fields [6].

9 Coupling to matter

We have shown that a (finite) collection of massless spin-2 fields alone cannot have *direct* cross-interactions. One may wonder whether the inclusion of matter fields could change this picture: if a given matter field was able to couple to two different gravitons simultaneously, we would have, at least, some *indirect* (non local) cross-interactions. It is of course impossible to consider exhaustively all possible types of matter fields. We shall consider here only the couplings to a scalar field and show that within this framework, cross-interactions remain impossible. Our analysis does not exclude possibilities based

[15] The formal solutions of Eq.(8.3) are Dirac delta functions, $\{\delta(\varphi - \varphi_0)\}$, which are not square integrable.

on a more complicated matter sector, but we feel that the simple scalar case is a good illustration of the general situation and of the difficulties that should be overcome in order to get consistent cross-interactions through matter couplings.

So, we want to consistently deform the free theory consisting in N copies of linearised gravity plus a scalar field

$$\mathcal{L} = \sum_a \mathcal{L}^a_{PF} - \frac{1}{2}\partial^\mu\phi\partial_\mu\phi. \tag{9.1}$$

The BRST differential in the spin-2 sector is unchanged while, for the scalar field, it reads

$$\gamma\phi = 0 = \delta\phi, \quad \delta\phi^* = \frac{\delta S_0}{\delta\phi} = \Box\phi, \quad \gamma\phi^* = 0. \tag{9.2}$$

Because the matter does not carry a gauge invariance of its own, Theorems 4.1 and 4.2 on the characteristic cohomology remain valid. This implies that a_2 is unchanged and still given by

$$a_2 = a_2^{\text{old}} = C_a^{*\beta} C^{\alpha b} \partial_\beta C^c_\alpha a^a_{bc} \tag{9.3}$$

even in the presence of the scalar field. The scalar field variables can occur only in a_1 and a_0.

Because a_2 is unchanged, a_1 will be given by the expression found above plus the general solution $\bar{a}_1$ of the homogeneous equation $\gamma\bar{a}_1 + db_1 = 0$,

$$a_1 = a_1^{\text{old}} + \bar{a}_1 \tag{9.4}$$

with

$$a_1^{\text{old}} = -h_a^{*\beta\gamma} C^{\alpha b} \left(\partial_\gamma h^c_{\alpha\beta} + \partial_\beta h^c_{\alpha\gamma} - \partial_\alpha h^c_{\gamma\beta}\right) a^a_{bc}. \tag{9.5}$$

Without loss of generality, we can assume $\gamma\bar{a}_1 = 0$ (see appendix). The only possibility compatible with Lorentz-invariance and leading to an interaction with no more than two derivatives is

$$\bar{a}_1 = -\phi^*\partial^\beta\phi C^a_\beta U^a(\phi). \tag{9.6}$$

Indeed, by integrations by parts, one can assume that no derivative of ϕ^* occurs, while the term $\partial^\beta C^a_\beta$ is γ-exact. Also, the term $h_a^{*\alpha\beta} C^b_\alpha \partial_\beta V^a_b(\phi) \sim -\partial_\beta h_a^{*\alpha\beta} C^b_\alpha V^a_b(\phi) - h_a^{*\alpha\beta}\partial_\beta C^b_\alpha V^a_b(\phi)$ is trivial.

Requiring a_0 to exist forces the functions $U^a(\phi)$ to be constants, so we set $U^a(\phi) = 2\xi_a$, where the ξ_a are constants. Indeed, in the equation $\delta\bar{a}_1 + \gamma\bar{a}_0 + \partial_\mu k^\mu = 0$, one may assume that $\bar{a}_0$ is linear in $h_{\alpha\beta}$ since $\bar{a}_1$ is linear in the variables of the gravitational sector (ghosts). One may also assume that $h_{\alpha\beta}$ appears undifferentiated since derivatives can be absorbed through integrations by parts. This yields $\bar{a}_0 = h^a_{\alpha\beta}\Psi_a^{\alpha\beta}$ where $\Psi_a^{\alpha\beta}$ involves the scalar field and two of its derivatives, $\Psi_a^{\alpha\beta} = \partial^\alpha\phi\partial^\beta\phi P_a(\phi) + \eta^{\alpha\beta}\partial^\mu\phi\partial_\mu\phi Q_a(\phi) + \partial^{\alpha\beta}\phi R^a(\phi) + \eta^{\alpha\beta}\Box\phi S^a(\phi)$, where $P_a(\phi)$, $Q_a(\phi)$, $R^a(\phi)$ and $S^a(\phi)$ are some functions of the undifferentiated scalar field. Substituting this expression into $\delta\bar{a}_1 + \gamma\bar{a}_0 + \partial_\mu k^\mu = 0$ and taking the variational derivative with respect to the ghosts gives the desired result $U'^a = 0$.

This leads to the following expression for the complete a_0,

$$a_0 = a_0^{\rm old} + \bar{a}_0 \tag{9.7}$$
$$\bar{a}_0 = t^{\alpha\beta} h^a_{\alpha\beta} \xi_a \tag{9.8}$$

where $t^{\alpha\beta}$ is the stress-energy tensor of the scalar field

$$t^{\alpha\beta} = \left(\partial^\alpha \phi \partial^\beta \phi - \frac{1}{2}\eta^{\alpha\beta}\partial^\mu \phi \partial_\mu \phi\right). \tag{9.9}$$

We thus see that the coupling to the gravitons takes the form $t^{\alpha\beta}h_{\alpha\beta}$. This is not an assumption, but follows from the general consistency conditions. Of course, we can also add to the deformation of the Lagrangian non-minimal terms of the form $V_a(\phi)K^a$, which are solutions of the "homogeneous equation" $\gamma a_0 + db_0 = 0$ without source δa_1. However, such terms vanish on (free) shell and thus can be absorbed through field-redefinitions in the adopted perturbative scheme.

The previous discussion completely determines the consistent interactions to first order. In order not to have an obstruction at order 2 in the deformation parameter, (W_1, W_1) should be BRST exact. Now, one has

$$(W_1, W_1) = \int d^n x ((a_1, a_1) + (a_2, a_2) + 2(a_0, a_1) + 2(a_1, a_2)) \tag{9.10}$$

with obvious meaning for the notation (a_i, a_j). This should be equal to $-2sW_2$ and again, without loss of generality, we can assume that W_2 stops at antifield number 2, $-2W_2 = \int d^n x(b_0 + b_1 + b_2)$. When expanded according to antifield number, the condition $(W_1, W_1) = -2sW_2$ yields (in this precise case)

$$(a_2, a_2) = \gamma b_2 + dm_2, \tag{9.11}$$
$$(a_1, a_1) + 2(a_1, a_2) = \delta b_2 + \gamma b_1 + dm_1, \tag{9.12}$$
$$2(a_0, a_1) = \delta b_1 + \gamma b_0 + dm_0. \tag{9.13}$$

Taking into account the fact that $a_2 \equiv a_2^{\rm old}$ and $a_1^{\rm old}$ fulfill these conditions, one gets the following requirement on $\bar{a}_1$

$$2(\bar{a}_1, a_2) + (\bar{a}_1, \bar{a}_1) = \delta \tilde{b}_2 + \gamma \tilde{b}_1 + d\tilde{m}_1, \tag{9.14}$$

where $\tilde{b}_2$ can be assumed to fulfill $\gamma \tilde{b}_2 = 0$. Computing the left-hand side of (9.14) we get

$$\begin{aligned} & 8\partial^\beta \phi C^a_\beta \xi_a \partial^\gamma \left(\phi^* C^b_\beta \xi_b\right) - 4\phi^* \partial^\beta \phi C^{\alpha b} \partial_\beta C^d_\alpha a^c_{bd} \xi_c \\ & = \partial_\mu j^\mu - 8\phi^* \partial^\beta \phi C^b_\alpha \partial^\alpha C^a_\beta \xi_a \xi_b - 4\phi^* \partial^\beta \phi C^{\alpha b} \partial_\beta C^d_\alpha a^c_{bd} \xi_c. \end{aligned} \tag{9.15}$$

Inserting in this expression $\partial_\beta C^d_\alpha = \partial_{(\beta} C^d_{\alpha)} + \partial_{[\beta} C^d_{\alpha]}$, we see that the term with symmetrized derivatives is γ-exact, while the term with antisymmetrized derivatives defines a cocycle of the γ-cohomology which reads explicitly

$$-4\phi^* 2\partial^{[\beta} \phi C^{\alpha]b} \partial_{[\alpha} C^b_{\beta]} \left(2\xi_a \xi_b - a^c_{ba} \xi_c\right). \tag{9.16}$$

This term is trivial in $H(\gamma|d)$ if and only if its coefficient is zero,

$$2\xi_a\xi_b - a^c_{ba}\xi_c = 0 \tag{9.17}$$

(the term $\delta\tilde{b}_2$ contains more derivatives and cannot play a role here). In the basis where $a^c_{ba} = 0$ if $a \neq b$, one gets $\xi_a\xi_b = 0$ when $a \neq b$, which means that ϕ can couple to only one graviton, as announced.

10 Non positive-definite metric in internal space

A crucial assumption in the above derivation of the absence of couplings mixing two different massless spin-2 fields was that the metric in internal space is positive-definite. This requirement follows from the basic tenets of (perturbative) field theory, as it is necessary for the stability of the Minkowski vacuum (absence of negative-energy excitations, or of negative-norm states). However, for completeness (and for making a link with Ref.[2]), we shall now formally discuss the case where δ_{ab} is replaced by a non positive-definite, but still non-degenerate, metric k_{ab} in internal space. In this case, the algebra $\mathcal{A}$ does not need to be trivial, and one can construct interacting multi-"graviton" theories, as first shown by Cutler and Wald in the paper [2] that initiated our study. As proven above, these are determined by a commutative, associative and symmetric algebra $\mathcal{A}$ (where "symmetric" refers to the condition $a_{abc} = a_{(abc)}$, the index a being lowered with a non-positive-definite k_{ab}).

As shown in [4], irreducible, commutative, associative algebras can be of either three types:

1. $\mathcal{A}$ contains no identity element and every element of $\mathcal{A}$ is nilpotent ($v^m = 0$ for some m).

2. $\mathcal{A}$ contains one (and only one) identity element e and no element j such that $j^2 = -e$. In that case, $\mathcal{A}$ contains a $(N-1)$-dimensional ideal of nilpotent elements and one may choose a basis $\{e, v_k\}$ $(k = 1, \cdots, N-1)$ such that all v_k's are nilpotent.

3. $\mathcal{A}$ contains one identity element e and an element j such that $j^2 = -e$. The algebra $\mathcal{A}$ is then of even dimension $N = 2m$, and there exists a $(2(m-1))$-dimensional ideal of nilpotent elements. One can choose a basis $\{e, v_k, j, j\cdot v_k\}$ $(k = 1, \cdots, m-1)$ such that all v_k's are nilpotent.

One can view the third case as a m-dimensional complex algebra with basis $\{e, v_k\}$. This is what we shall do in the sequel to be able to cover simultaneously both cases 2 and 3. So, when we refer to the dimension, it will be understood that this is the complex dimension in case 3.

We now show that in cases 2 and 3, the symmetry condition on the algebra implies that the most nilpotent subspace must be at most one-dimensional. This condition was used in [38] in order to write down Lagrangians.

The most nilpotent subspace of $\mathcal{A}$ is the subspace of elements x that have a vanishing product with everything else, except the identity. More precisely, one has

$$e \cdot x = x, \;\; v_k \cdot x = 0. \tag{10.1}$$

Let us now compute the scalar product (v_k, x). One has $(v_k, x) = (v_k \cdot e, x) = (A(v_k)e, x)$. Using the symmetry property, this becomes $(A(v_k)e, x) = (e, A(v_k)x) = (e, v_k \cdot x) = (e, 0) = 0$. Thus, one has

$$(v_k, x) = 0, \;\; (e, x) \neq 0 \tag{10.2}$$

where the last equality follows from the fact that the scalar product defined by k_{ab} must be non-degenerate. However, if the most nilpotent subspace has a dimension greater than or equal to two, one gets a contradiction since if $(e, x_1) = m_1$ and $(e, x_2) = m_2$, the non-zero vector $m_2 x_1 - m_1 x_2$ has a vanishing scalar product with everything else, implying that k_{ab} is degenerate. QED.

When the most nilpotent subspace is precisely one-dimensional, one can write real Lagrangians [2, 3, 4, 38], so there exist interacting theories with cross-interactions which are consistent from the point of view of gauge invariance but which do not have the free field limit (1.1). We refer to these works for further information.

11 Analysis without derivative assumptions

The derivative assumption was used at two places in the derivation. First, in the determination of a_1; second, in the determination of a_0. In both cases, the solution was found to be unique only if one restricts the number of derivatives.

11.1 Ambiguities in a_1

Let us examine first a_1. If one allows more derivatives in a_1, one can add to a_1 terms of the form $\Theta^\alpha_a C^a_\alpha + \Theta^{\alpha\beta}_a \partial_{[\alpha} C^a_{\beta]}$ where Θ^α_a and $\Theta^{\alpha\beta}_a = -\Theta^{\beta\alpha}_a$ have antifield number one and are annihilated by γ.

For such additional terms, say $\tilde{a}_1$, to be still compatible with the existence of an a_0, one must have

$$\delta \tilde{a}_1 + \gamma \tilde{a}_0 + \partial_\mu k^\mu = 0. \tag{11.1}$$

One may expand k^μ in derivatives of the ghosts as follows,

$$k^\mu = t^{\mu\rho}_a C^a_\rho + t^{\mu\rho\sigma}_a \partial_{[\rho} C^a_{\sigma]} + \text{ more} \tag{11.2}$$

where "more" contains $\partial_{(\rho} C^a_{\sigma)}$ and higher derivatives of C^a_ρ. Using the ambiguity $k^\mu \rightarrow k^\mu + \partial_\nu S^{\mu\nu}$ with $S^{\mu\nu} = -S^{\nu\mu}$, one can assume $t^{\mu\rho\sigma}_a = 0$ (take $S^{\mu\nu} = \Phi^{\mu\nu\rho}_a C^a_\rho$ and adjust $\Phi^{\mu\nu\rho}_a = -\Phi^{\nu\mu\rho}_a$ appropriately). Substituting this expression for k^μ (with $t^{\mu\rho\sigma}_a$ equal to zero) in (11.1) yields the following conditions upon equating the coefficients of C^a_ρ and $\partial_{[\rho} C^a_{\sigma]}$ (which do not occur in $\gamma \tilde{a}_0$),

$$\begin{aligned} \partial_\beta t^{\alpha\beta}_a &= -\delta \Theta^\alpha_a, & (11.3)\\ t^{[\alpha\beta]} &= -\delta \Theta^{\alpha\beta}_a. & (11.4) \end{aligned}$$

The second of these equations implies that one can get rid of $t^{[\alpha\beta]}$ by adding trivial terms.

So we see that the interactions defined by the new terms in a_1 are determined by symmetric tensors $t_a^{\alpha\beta}$ which are conserved modulo the equations of motion ($\partial_\beta t_a^{\alpha\beta} = -\delta\Theta_a^\alpha \approx 0$) and which are such that $\partial_\beta t_a^{\alpha\beta}$ is gauge-invariant[16].

Equivalently, in view of Noether's theorem, these interactions are determined by rigid symmetries with a vector index and an internal index,

$$\delta_\eta h^a_{\alpha\beta} = \eta^b_\gamma \Delta^{a\gamma}_{b\alpha\beta}([K]) \tag{11.5}$$

which commutes with the gauge transformations since the coefficients $\Delta^{a\gamma}_{b\alpha\beta}([K])$ involve the gauge-invariant linearized curvatures and their derivatives. The connection between $\Delta^{a\gamma}_{b\alpha\beta}([K])$ and Θ_b^γ is simply [25] $\Theta_b^\gamma = h_a^{*\alpha\beta}\Delta^{a\gamma}_{b\alpha\beta}([K])$. To be compatible with Lorentz invariance, the $\Delta^{a\gamma}_{b\alpha\beta}([K])$ should transform as indicated by their Lorentz indices. Furthermore, the corresponding Noether charges $t_b^{\alpha\beta}$ should be symmetric in α and β, and two sets of $\Delta^{a\gamma}_{b\alpha\beta}$'s that differ on-shell by a gauge transformation (with gauge parameters involving the curvatures and their derivatives) should be identified, since they lead to a_1's that differ by trivial terms.

The determination of all the non trivial rigid symmetries with these properties (if any) appears to be a rather complicated problem whose resolution goes beyond the scope of this paper. Let us simply point out that there exists a similar problem in the case of massless spin-1 fields, where these conditions turn out to be so restrictive that they admit no non-trivial solution in spacetime dimension 4 (and presumably > 4 also). The corresponding problem there is that of determining the gauge-invariant conserved currents $j^\mu([F])$, which are Lorentz-vectors. These lead to interactions of the form $A_\mu j^\mu$ which do modify the gauge transformations but not their algebra ($a_2 = 0$ because $j^\mu([F])$ is gauge invariant). Equivalently, one must determine the non-trivial rigid symmetries which commute with the gauge transformations and the Lorentz transformations. In 3 spacetime dimensions, there is a solution, which yields the Freedman-Townsend vertex (with $j^\mu_a \sim f_{abc}\epsilon^{\mu\alpha\beta}\,{}^*F^b_\alpha\,{}^*F^c_\beta$ where F^a_α is the 1-form dual to the 2-form $F^a_{\alpha\beta}$) [39, 40]. In four (and presumably higher) spacetime dimensions, there is no solution according to a theorem by Torre [41]. If one believes that the spin-1 case is a good analogy, one would expect no non-trivial $\tilde{a}_1$ of the type discussed in this section except perhaps in particular spacetime dimensions (furthermore, there are further restrictions at order g^2 that these $\tilde{a}_1$'s would have to satisfy). If this expectation is correct, the most general a_1 would be the one given above (subsection 5.2), associated with the unique a_2 determined in subsection 5.1.

[16] Presumably, this implies that $t_a^{\alpha\beta}$ itself can be assumed to be gauge-invariant, so that the corresponding interaction is just $h^a_{\alpha\beta}t_a^{\alpha\beta}$. This interaction has the same form as the Einstein self-coupling $h^a_{\alpha\beta}t^{\alpha\beta}_{Ga}$, where $t^{\alpha\beta}_{Ga}$ is the energy-momentum tensor of the a-th massless spin-2 field. But neither $t^{\alpha\beta}_{Ga}$ nor $\partial_\beta t^{\alpha\beta}_{Ga}$ is gauge-invariant. This is why the Einstein self-coupling leads to a non-vanishing a_2, i.e., modifies the algebra of the gauge transformations. As we have seen, it is the only coupling with this property (up to redefinitions). Note that couplings of the form $h^a_{\alpha\beta}t_a^{\alpha\beta}$, with $t_a^{\alpha\beta}$ gauge-invariant (if they exist), are equivalent to strictly gauge-invariant couplings that do not modify the gauge transformations (i.e., are such that $\tilde{a}_1$ can be redefined away) if $t_a^{\alpha\beta} = \partial_\mu\partial_\nu Q_a^{\alpha\mu\beta\nu}$ for some $Q_a^{\alpha\mu\beta\nu}$ with the symmetries of the Riemann tensor, since then $\int h^a_{\alpha\beta}t_a^{\alpha\beta} = \int K^a_{\alpha\mu\beta\nu}Q_a^{\alpha\mu\beta\nu}$. So we see that the (gauge-invariant) generalized (characteristic) cohomology of [33] is also relevant here.

11.2 Ambiguities in a_0

We now turn to the ambiguity in a_0. Assuming, in view of the previous discussion, that a_1 is given by (5.20), we see that the most general a_0 is given by the particular solution (5.32) plus the general solution $\tilde{a}_0$ of the equation without a_1-source

$$\gamma\tilde{a}_0 + \partial_\mu p^\mu = 0 \,. \tag{11.6}$$

The addition of such deformations to the Lagrangian do not deform the gauge transformations.

There are two types of solutions to (11.6): those for which p^μ vanishes (or can be made to vanish by redefinitions); and those for which the divergence term $\partial_\mu p^\mu$ is unremovable. Examples of the second type are the cosmological term, the Lagrangian itself and, more generally, the leading non-trivial orders of the Lovelock terms [42]. The first type is given by all strictly gauge-invariant expressions, i.e., by the polynomials in the linearized Riemann tensors $K^a_{\mu\nu\alpha\beta}$ and their derivatives (without inner contractions since the linearized Ricci tensors vanish on-shell and can be eliminated by field redefinitions).

If some of the a^a_{aa}'s occuring in a_2 vanish, it is clear that cross-interactions involving any polynomial in the corresponding curvatures are consistent to all orders. If all a^a_{aa}'s are not vanishing, however, - which is in some sense the "generic case" -, there appear non trivial consistency conditions at order g^2. These conditions read

$$(\bar{a}_0, a_1) = \gamma f + dh + \delta h.. \tag{11.7}$$

Although we have not investigated in detail this equation for all possible $\bar{a}_0$'s, we anticipate that it prevents cross-terms. Only terms of the form $\sum_a f_a$ where f_a involves only the curvature $K^a_{\mu\nu\alpha\beta}$ and its derivatives, are expected to be allowed. These lead to consistent interactions to all orders, obtained by mere covariantization.

12 Conclusions

In this paper, we have established no-go results on cross-interactions between a collection of massless spin-2 fields. Our method relies on the antifield approach and uses cohomological techniques.

First, we have shown that the only possible deformation of the algebra of the gauge symmetries is given by the direct sum of diffeomorphism algebras, one in each spin-2 field sector (Eq. (7.7) with some κ^a's possibly equal to zero). This result holds independently of any assumption on the number of derivatives present in the deformation and is our main achievement. It goes beyond previous studies which restricted the number of derivatives in the modified gauge transformations and hence in the modified gauge algebra.

Under the assumption that the number of derivatives in the interactions does not exceed two, we have then derived the most general deformation of the Lagrangian, which is a sum of independent Einstein-Hilbert actions (with possible cosmological terms and again with some κ^a's possibly equal to zero), one for each spin-2 field (Eq. (7.6)). This prevents cross-interactions. The impossibility to introduce even indirect cross-interactions (via the exchange of another sector) remains valid if one couples a scalar field (but we have not explored all possible matter sectors). Thus, there is only one type of graviton that one can see in each "parallel", non-interacting world. In that sense, there is effectively only one massless spin-2 field. The fact that the Einstein theory involves only one type of graviton is therefore not a choice but a necessity that adds to its great theoretical appeal.

We have then discussed how this picture could change if one did not restrict the derivative order of the interactions. Although the analysis gets then technically more involved, we have provided arguments that cross-interactions remain impossible (apart from the obvious interactions that do not modify the gauge transformations and involve polynomials in the linearized curvatures and their derivatives). The only modification appears to be the possible addition of higher order curvature terms in each sector.

Restricted to the case of a single massless spin-2 field, our study recovers and somewhat generalizes previous results on the inevitability of the Einstein vertex and of the diffeomorphism algebra by relaxing assumptions usually made on the number of derivatives in the gauge transformations and on the coupling of matter to the graviton through the energy-momentum tensor.

The main virtue of no-go theorems is to put into clear light the assumptions that underlie the negative result under focus. In our case, the key assumptions are, besides locality: finite number of massless spin-2 fields and positive-definite metric in the internal space of the gravitons. If either of these assumptions is relaxed, cross-interactions become mathematically possible [6, 2]. While we think that, in the case of a finite collection of fields, it is physically unacceptable to have a theory involving negative-energy (or negative-norm) states, it would be interesting to study further the infinite-collection case (such as [6], or its simpler "circle" analog sketched above) to check whether it defines a fully consistent theory.

Acknowledgements

N.B. and L.G. are grateful to Glenn Barnich and Xavier Bekaert for discussions. M.H. thanks the Institut des Hautes Etudes Scientifiques for its kind hospitality. The work of N. B., L.G. and M.H. is partially supported by the "Actions de Recherche Concertées" of the "Direction de la Recherche Scientifique - Communauté Française de Belgique", by IISN - Belgium (convention 4.4505.86) and by Proyectos FONDECYT 1970151 and 7960001 (Chile).

A Cohomological results

The content of this appendix is based on [43].

A.1 A consequence of Theorem 3.1

The following useful result follows from Theorem 3.1. If a has strictly positive antifield number, the equation

$$\gamma a + db = 0 \tag{A.1}$$

is equivalent, up to trivial redefinitions, to

$$\gamma a = 0. \tag{A.2}$$

That is, one can add d-exact terms to a, $a \rightarrow a' = a + dv$ such that $\gamma a' = 0$.

In order to prove this theorem, we consider the descent associated with $\gamma a + db = 0$: from this equation, one infers, by using the properties $\gamma^2 = 0$, $\gamma d + d\gamma = 0$ and the triviality of the cohomology of d, that $\gamma b + dc = 0$ for some c. Going on in the same way, we introduce a "descent" $\gamma c + de = 0$, $\gamma e + df = 0$, etc, in which each successive equation has one less unit of form-degree. The descent ends with last two equations $\gamma m + dn = 0$, $\gamma n = 0$ (the last equation is $\gamma n = 0$ either because n is a zero-form, or because one stops earlier with a γ-closed term).

Now, because n is γ-closed, one has, up to trivial, irrelevant terms, $n = \alpha_J \omega^J$. Inserting this into the previous equation in the descent yields

$$d(\alpha_J)\omega^J \pm \alpha_J d\omega^J + \gamma m = 0. \tag{A.3}$$

In order to analyse this equation, we introduce a new differential D, whose action on $h_{\mu\nu}$, $h^\star_{\mu\nu}$, $C^\star_\alpha$ and all their derivatives is the same as the action of d, but whose action on the ghosts is given by :

$$\begin{aligned} DC_\mu &= \frac{1}{2} dx^\nu C_{[\mu,\nu]} \\ D(\partial_{\rho_1 \dots \rho_s} C_\mu) &= 0 \text{ if } s \geq 1. \end{aligned} \tag{A.4}$$

The operator D coincides with d up to γ-exact terms. It follows from the definitions that $D\omega^J = A^J_I \omega^I$ for some constant matrix A^J_I that involves dx^μ.

One can rewrite (A.3) as

$$d(\alpha_J)\omega^J \pm \alpha_J D\omega^J + \gamma m' = 0 \tag{A.5}$$

which implies,

$$d(\alpha_J)\omega^J \pm \alpha_J D\omega^J = 0 \tag{A.6}$$

since a term of the form $\beta_J \omega^J$ (with β_J invariant) is γ-exact if and only if it is zero. It is convenient to further split D as the sum of an operator D_0 and an operator D_1. D_0 has the same action as D on $h_{\mu\nu}$, $h^\star_{\mu\nu}$, $C^\star_\alpha$ and all their derivatives, and gives 0 when acting on the ghosts. D_1 gives 0 when acting on all the variables but the ghosts on which it reproduces the action of D. The operator D_1 comes with a grading : the number of $C_{[\mu,\nu]}$. D_1 raises the number of $C_{[\mu,\nu]}$ by one unit, while D_0 leaves it unchanged. We call this grading the D-degree. The D-degree is bounded because there is a finite number of $C^a_{[\mu,\nu]}$, which are anticommuting.

Let us expand (A.3) according to the D-degree. At lowest order, we get

$$d\alpha_{J_0} = 0 \tag{A.7}$$

where J_0 labels the ω^J that contain zero derivative of the ghosts ($D\omega^J = D_1\omega^J$ contains at least one derivative). This equation implies, according to theorem 3.1, that $\alpha_{J_0} = d\beta_{J_0}$ where β_{J_0} is an invariant polynomial. Accordingly, one can write

$$\alpha_{J_0}\omega^{J_0} = d(\beta_{J_0}\omega^{J_0}) \mp \beta_{J_0} D\omega^{J_0} + \ \gamma\text{-exact terms.} \tag{A.8}$$

The term $\beta_{J_0} D\omega^{J_0}$ has D-degree equal to 1. Thus, by adding trivial terms to the last term n in the descent, we can assume that n contains no term of D-degree 0. One can then successively removes the terms of D-degree 1, D-degree 2, etc, until one gets $n = 0$. One then repeats the argument for m and the previous terms in the descent until one gets $b = 0$, i.e., $\gamma a = 0$, as requested.

A.2 Invariant cohomology of δ modulo d.

Throughout this subsection, there will be no ghost; i.e., the objects that appear involve only the fields, the antifields and their derivatives.

Theorem A.1 *Assume that the invariant polynomial a^p_k (p = form-degree, k = antifield number) is δ-trivial modulo d,*

$$a^p_k = \delta\mu^p_{k+1} + d\mu^{p-1}_k \quad (k \geq 1). \tag{A.9}$$

Then, one can always choose μ^p_{k+1} and μ^{p-1}_k to be invariant.

To prove the theorem, we need the following lemma:

Lemma A.1 *If a is an invariant polynomial that is δ-exact, $a = \delta b$, then, a is δ-exact in the space of invariant polynomials. That is, one can take b to be also invariant.*

Demonstration of the lemma : Any function $f([h],[h^*],[C^*])$ can be viewed as a function $f(\tilde{h},[K],[h^*],[C^*])$, where $[K]$ denotes the linearized curvatures and their derivatives, and where the $\tilde{h}$ denote a complete set of non-invariant derivatives of $h^a_{\mu\nu}$ ($\{\tilde{h}\} = \{h^a_{\mu\nu}, \partial_\rho h^a_{\mu\nu}, \cdots\}$). (One can put the $\tilde{h}$ in bijective correspondence with the ghosts and their derivatives through γ.) The K's are not independent because of the linearized Bianchi identities, but this does not affect the argument. An invariant function is just a function that does not involve $\tilde{h}$, so one has (if f is invariant), $f = f_{|\tilde{h}=0}$. Now, the differential δ commutes with the operation of setting $\tilde{h}$ to zero. So, if $a = \delta b$ and a is invariant, one has $a = a_{|\tilde{h}=0} = (\delta b)_{|\tilde{h}=0} = \delta(b_{|\tilde{h}=0})$, which proves the lemma since $b_{|\tilde{h}=0}$ is invariant. $\diamond$

Demonstration of the theorem : We first derive a chain of equations with the same structure as (A.9) [35]. Acting with d on (A.9), we get $da^p_k = -\delta d\mu^p_{k+1}$. Using the lemma and the fact that da^p_k is invariant, we can also write $da^p_k = -\delta a^{p+1}_{k+1}$ with a^{p+1}_{k+1}

invariant. Substituting this in $da_k^p = -\delta d\mu_{k+1}^p$, we get $\delta\left[a_{k+1}^{p+1} - d\mu_{k+1}^p\right] = 0$. As $H(\delta)$ is trivial in antifield number > 0, this yields

$$a_{k+1}^{p+1} = \delta\mu_{k+2}^{p+1} + d\mu_{k+1}^p \tag{A.10}$$

which has the same structure as (A.9). We can then repeat the same operations, until we reach form-degree n,

$$a_{k+n-p}^n = \delta\mu_{k+n-p+1}^n + d\mu_{k+n-p}^{n-1}. \tag{A.11}$$

Similarly, one can go down in form-degree. Acting with δ on (A.9), one gets $\delta a_k^p = -d(\delta\mu_k^{p-1})$. If the antifield number $k-1$ of δa_k^p is greater than or equal to one (i.e., $k > 1$), one can rewrite, thanks to Theorem 3.1, $\delta a_k^p = -da_{k-1}^{p-1}$ where a_{k-1}^{p-1} is invariant. (If $k = 1$ we cannot go down and the bottom of the chain is (A.9) with $k = 1$, namely $a_1^p = \delta\mu_2^p + d\mu_1^{p-1}$.) Consequently $d\left[a_{k-1}^{p-1} - \delta\mu_k^{p-1}\right] = 0$ and, as before, we deduce another equation similar to (A.9) :

$$a_{k-1}^{p-1} = \delta\mu_k^{p-1} + d\mu_{k-1}^{p-1}. \tag{A.12}$$

Applying δ on this equation the descent continues. This descent stops at form degree zero or antifield number one, whichever is reached first, i.e.,

$$\begin{array}{ll} \text{either} & a_{k-p}^0 = \delta\mu_{k-p+1}^0 \\ \text{or} & a_1^{p-k+1} = \delta\mu_2^{p-k+1} + d\mu_1^{p-k}. \end{array} \tag{A.13}$$

Putting all these observations together we can write the entire descent as

$$\begin{array}{rcl} a_{k+n-p}^n &=& \delta\mu_{k+n-p+1}^n + d\mu_{k+n-p}^{n-1} \\ &\vdots& \\ a_{k+1}^{p+1} &=& \delta\mu_{k+2}^{p+1} + d\mu_{k+1}^p \\ a_k^p &=& \delta\mu_{k+1}^p + d\mu_k^{p-1} \\ a_{k-1}^{p-1} &=& \delta\mu_k^{p-1} + d\mu_{k-1}^{p-2} \\ &\vdots& \\ \text{either }\ a_{k-p}^0 &=& \delta\mu_{k-p+1}^0 \\ \text{or }\ a_1^{p-k+1} &=& \delta\mu_2^{p-k+1} + d\mu_1^{p-k} \end{array} \tag{A.14}$$

where all the $a_{k\pm i}^{p\pm i}$ are invariants.

Now let us show that when one of the μ's in the chain is invariant, we can actually choose all the other μ's in such a way that they share this property. Let us thus assume that μ_b^{c-1} is invariant. This μ_b^{c-1} appears in two equations of the descent :

$$\begin{array}{rcl} a_b^c &=& \delta\mu_{b+1}^c + d\mu_b^{c-1}, \\ a_{b-1}^{c-1} &=& \delta\mu_b^{c-1} + d\mu_b^{c-2} \end{array} \tag{A.15}$$

(if we are at the bottom or at the top, μ_b^{c-1} occurs in only one equation, and one should just proceed from that one). The first equation tells us that $\delta\mu_{b+1}^c$ is invariant. Thanks to

Lemma A.1 we can choose μ^c_{b+1} to be invariant. Looking at the second equation, we see that $d\mu^{c-2}_b$ is invariant and by virtue of theorem 3.1, μ^{c-2}_b can be chosen to be invariant since the antifield number b is positive. These two μ's appear each one in two different equations of the chain, where we can apply the same reasoning. The invariance property propagates then to all the μ's. Consequently, it is enough to prove the theorem in form degree n.

Now, let us prove the following lemma :

Lemma A.2 *If a^n_k is of antifield number $k > n$, then the "μ"s in (A.9) can be taken to be invariant.*

Demonstration : Indeed, if $k > n$, the last equation of the descent is $a^0_{k-n} = \delta\mu^0_{k-n+1}$. We can, using Lemma A.1, choose μ^0_{k-n+1} invariant, and so, all the μ's can be chosen to have the same property.$\diamond$

It remains therefore to demonstrate Theorem A.1 in the case where the antifield number satisfies $k \leq n$. Rewriting the top equation (i.e. (A.9) with $p = n$) in dual notation, we have

$$a_k = \delta b_{k+1} + \partial_\rho j^\rho_k, \; (k \geq 1). \tag{A.16}$$

We will work by induction on the antifield number, showing that if the property is true for $k+2$ (with $k > 0$), then it is true for k. As we already know that it is true in the case $k > n$, the theorem will be demonstrated. Let us take the Euler-Lagrange derivatives of (A.16). Since the E.L. derivatives with respect to the C^*_α commute with δ, we get first :

$$\frac{\delta^R a_k}{\delta C^*_\alpha} = \delta Z^\alpha_{k-1} \tag{A.17}$$

with $Z^\alpha_{k-1} = \frac{\delta^R b_{k+1}}{\delta C^*_\alpha}$. For the E.L. derivatives of b_{k+1} with respect to $h^*_{\mu\nu}$ we obtain, after a direct computation,

$$\frac{\delta^R a_k}{\delta h^*_{\mu\nu}} = -\delta X^{\mu\nu}_k + 2\partial^{(\mu} Z^{\nu)}_{k-1}. \tag{A.18}$$

where $X^{\mu\nu}_k = \frac{\delta b_{k+1}}{\delta h^*_{\mu\nu}}$. Finally, let us compute the E.L. derivatives of a_k with respect to the fields. We get :

$$\frac{\delta^R a_k}{\delta h_{\mu\nu}} = \delta Y^{\mu\nu}_{k+1} - \eta^{\mu\nu}\partial_{\alpha\beta}X^{\alpha\beta}_k - \partial^\rho\partial_\rho X^{\mu\nu}_k + \partial^\mu\partial_\rho X^{\rho\nu}_k + \partial^\nu\partial_\rho X^{\rho\mu}_k - \eta_{\alpha\beta}\partial^{\mu\nu}X^{\alpha\beta}_k + \eta_{\alpha\beta}\eta^{\mu\nu}\partial^\rho\partial_\rho X^{\alpha\beta}_k \tag{A.19}$$

where $Y^{\mu\nu}_{k+1} = \frac{\delta^R b_{k+1}}{\delta h_{\mu\nu}}$.

The E.L. derivatives of an invariant object are invariant. Thus, $\frac{\delta^R a_k}{\delta C^*_\alpha}$ is invariant. Therefore, by our lemma A.1 and Eq. (A.17), we have also

$$\frac{\delta^R a_k}{\delta C^*_\alpha} = \delta Z'^\alpha_{k-1} \tag{A.20}$$

for some invariant Z'^{α}_{k-1}. Similarly, one easily verifies that

$$\frac{\delta^R a_k}{\delta h^*_{\mu\nu}} = -\delta X'^{\mu\nu}_k + 2\partial^{(\mu} Z'^{\nu)}_{k-1}. \qquad \text{(A.21)}$$

and

$$\frac{\delta^R a_k}{\delta h_{\mu\nu}} = \delta Y'^{\mu\nu}_{k+1} - \eta^{\mu\nu}\partial_{\alpha\beta}X'^{\alpha\beta}_k - \partial^\rho\partial_\rho X'^{\mu\nu}_k + \partial^\mu\partial_\rho X'^{\rho\nu}_k + \partial^\nu\partial_\rho X'^{\rho\mu}_k - \eta_{\alpha\beta}\partial^{\mu\nu}X'^{\alpha\beta}_k + \eta_{\alpha\beta}\eta^{\mu\nu}\partial^\rho\partial_\rho X'^{\alpha\beta}_k \qquad \text{(A.22)}$$

for some invariant $X'^{\mu\nu}_k$ and $Y'^{\mu\nu}_{k+1}$.

Now, since a_k is invariant, it depends on the fields only through the linearized Riemann tensor and its derivatives. We can thus write

$$\frac{\delta^R a_k}{\delta h_{\mu\nu}} = 4\partial_{\alpha\beta} A^{\alpha\mu\beta\nu} \qquad \text{(A.23)}$$

where $A^{\alpha\mu\beta\nu}$ has the symmetries of the Riemann tensor. This implies

$$\delta Y'^{\mu\nu}_{k+1} = \partial_{\alpha\beta} M^{\alpha\mu\beta\nu} \qquad \text{(A.24)}$$

with $M^{\alpha\mu\beta\nu}$ having the symmetries of the Riemann tensor. The equation (A.24) tells us that the $Y'^{\mu\nu}_{k+1}$ for given ν are δ-cocycles modulo d, in form degree $n-1$ and antifield number $k+1$. There are thus δ-exact modulo d ($H^{n-1}_{k+1}(\delta|d) \simeq H^n_{k+2}(\delta|d) \simeq 0$), $Y'^{\mu\nu}_{k+1} = \delta A^{\mu\nu}_{k+2} + \partial_\rho T^{\rho\mu\nu}_{k+1}$ where $T^{\rho\mu\nu}_{k+1}$ is antisymmetric in ρ and μ. By our hypothesis of induction, $A^{\mu\nu}_{k+2}$ and $T^{\rho\mu\nu}_{k+1}$ can be assumed to be invariant. Since $Y^{\mu\nu}_{k+1}$ is symmetric in μ and ν, we have also $\delta A^{[\mu\nu]}_{k+2} + \partial_\rho T^{\rho[\mu\nu]}_{k+1} = 0$. The triviality of $H^n_{k+2}(d|\delta)$ implies again that $A^{[\mu\nu]}_{k+2}$ and $T^{\rho[\mu\nu]}_{k+1}$ are trivial, in particular, $T^{\rho[\mu\nu]}_{k+1} = \delta Q^{\rho\mu\nu}_{k+2} + \partial_\alpha S^{\alpha\rho\mu\nu}_{k+1}$, where $S^{\alpha\rho\mu\nu}_{k+1}$ is antisymmetric in (α,ρ) and in (μ,ν), respectively. The induction assumption allows us to choose $Q^{\rho\mu\nu}_{k+2}$ and $S^{\alpha\rho\mu\nu}_{k+1}$ to be invariant. Writing $E^{\mu\alpha\nu\beta}_{k+1} = -[S^{\mu\alpha\nu\beta}_{k+1} + S^{\nu\beta\mu\alpha}_{k+1}]$ and computing $\partial_{\alpha\beta}E^{\mu\alpha\nu\beta}_{k+1}$, we observe finally that

$$Y'^{\mu\nu}_{k+1} = \delta F^{\mu\nu}_{k+1} + \partial_{\alpha\beta}E^{\mu\alpha\nu\beta}_{k+1} \qquad \text{(A.25)}$$

with $E^{\mu\alpha\nu\beta}_{k+1} = E^{\nu\beta\mu\alpha}_{k+1}$, $E^{\mu\alpha\nu\beta}_{k+1} = E^{[\mu\alpha]\nu\beta}_{k+1}$ and $E^{\mu\alpha\nu\beta}_{k+1} = E^{\mu\alpha[\nu\beta]}_{k+1}$.

We can now complete the argument. Using the homotopy formula

$$a_k = \int_0^1 dt[\frac{\delta^R a_k}{\delta C^*_\alpha}(t)C^*_\alpha + \frac{\delta^R a_k}{\delta h^*_{\mu\nu}}(t)h^*_{\mu\nu} + \frac{\delta^R a_k}{\delta h_{\mu\nu}}(t)h_{\mu\nu}], \qquad \text{(A.26)}$$

that enables one to reconstruct a_k from its E.L. derivatives, as well as the expressions (A.17), (A.18), (A.19) for these E.L. derivatives, we get

$$a_k = \delta[\int_0^1 [Z'^{\alpha}_{k-1}C^*_\alpha + X'^{\alpha\beta}_k h^*_{\alpha\beta} + Y'^{\mu\nu}_{k+1}h_{\mu\nu}]] + \partial_\rho k^\rho. \qquad \text{(A.27)}$$

The first two terms in the argument of δ are manifestly invariant. As to the third term, we use (A.25). The δ-exact term disappears ($\delta^2 = 0$) while the second one yields $\delta[\int_0^1 dt[\partial_{\alpha\beta}E^{\mu\alpha\nu\beta}_{k+1}h_{\mu\nu}]]$. Integrating by part twice gives $E^{\mu\alpha\nu\beta}_{k+1}$ times the linearized Riemann tensor, which is also invariant. This proves the theorem.

A.3 Cohomology of s modulo d

We have now developed all the necessary tools for the study of the cohomology of s modulo d in form degree n. A cocycle of $H(s|d)$ must obey

$$sa + db = 0. \tag{A.28}$$

Let us expand a and b according to the antifield number :

$$\begin{aligned} a &= a_0 + a_1 + ... + a_k \\ b &= b_0 + b_1 + ... + b_l \end{aligned} \tag{A.29}$$

where, as shown in [35], the expansion stops at some finite antifield number.

Writing s as the sum of γ and δ, the equation $sa + db = 0$ is equivalent to the system of equations :

$$\begin{aligned} \delta a_1 + \gamma a_0 + db_0 &= 0 \\ \delta a_2 + \gamma a_1 + db_1 &= 0 \\ &\vdots \\ \delta a_k + \gamma a_{k-1} + db_{k-1} &= 0 \\ &\vdots \end{aligned} \tag{A.30}$$

Where the system ends depends on k and l, but, without loss of generality, we can assume that $l = k - 1$. Indeed, if $l > k - 1$ the last equations look like $db_i = 0$, (with $i > k$) and imply that (because b is of form degree $(n-1)$) $b_i = dc_i$. We can thus absorb these terms in a redefinition of b. The last equation is then $\gamma a_k + db_k = 0$ which, using the consequence of theorem 3.1 discussed in appendix A.1, can be written $\gamma a_k = 0$.

We have then the system of equations (where some b_i could be zero):

$$\begin{aligned} \delta a_1 + \gamma a_0 + db_0 &= 0 \\ &\vdots \\ \delta a_k + \gamma a_{k-1} + db_{k-1} &= 0 \\ \gamma a_k &= 0. \end{aligned} \tag{A.31}$$

The last equation enables us to write $a_k = \alpha_J \omega^J$. Acting with γ on the second to last equation and using $\gamma^2 = 0$, $\gamma a_k = 0$, we get $d\gamma b_{k-1} = 0$; and then, thanks to the consequence of theorem 3.1, b_{k-1} can also be assumed to be invariant, $b_{k-1} = \beta_J \omega^J$. Substituting the invariant forms of a_k and b_{k-1} in the second to last equation, we get :

$$\delta[\alpha_J \omega^J] + D[\beta_J \omega^J] = \gamma(\ldots). \tag{A.32}$$

As above, this equation implies

$$\delta[\alpha_J \omega^J] + D[\beta_J \omega^J] = 0. \tag{A.33}$$

We now expand this equation according to the D-degree. The term of degree zero reads

$$[\delta\alpha_{J_0} + D_0\beta_{J_0}]\omega^{J_0} = 0. \tag{A.34}$$

This equation implies that the coefficient of ω^J must be zero, and as D_0 acts on the objects upon which β_J depends in the same way as d, we get :

$$\delta\alpha_{J_0} + d\beta_{J_0} = 0. \tag{A.35}$$

If the antifield number of α_{J_0} is strictly greater than 2, the solution is trivial, thanks to our results on the cohomology of δ modulo d:

$$\alpha_{J_0} = \delta\mu_{J_0} + d\nu_{J_0}. \tag{A.36}$$

Furthermore, theorem A.1 tells us that μ_{J_0} and ν_{J_0} can be chosen invariants. We thus get :

$$\begin{aligned} a_k^0 &= (\delta\mu_{J_0} + D_0\nu_{J_0})\omega^{J_0} \\ &= s(\mu_{J_0}\omega^{J_0}) + d(\nu_{J_0}\omega^{J_0}) + \text{ "more"} \end{aligned} \tag{A.37}$$

where "more" arises from $d\omega^{J_0}$, which can be written as $d\omega^{J_0} = A^{J_0}_{J_1}\omega^{J_1} + su^{J_0}$. The term $\nu_{J_0}A^{J_0}_{J_1}\omega^{J_1}$ has D-degree one, while the term $\nu_{J_0}su^{J_0}$ differs from the s-exact term $s(\pm\nu_{J_0}u^{J_0})$ by the term $\pm\delta(\nu_{J_0})u^{J_0}$, which is of lowest antifield number. Thus, trivial redefinitions enable one to assume that a_k^0 vanishes. Once this is done, β_{J_0} must fulfill $d\beta_{J_0} = 0$ and thus be d-exact in the space of invariant polynomials by theorem 3.1 , which enables one to set it to zero through appropriate redefinitions.

We can then successively remove the terms of higher D-degree by a similar procedure, until one has completely redefined away a_k and b_{k-1}. One can next repeat the argument for antifield number $k-1$, etc, until one reaches antifield number 2. Consequently, we can indeed assume that the expansion of a in Eq. (5.1) stops at antifield number 2 and takes the form $a = a_0 + a_1 + a_2$ with $b = b_0 + b_1$, as in (5.3) and (5.4). Furthermore, the last term a_2 can be assumed to involve only non-trivial elements of the characteristic cohomology $H_2^n(\delta|d)$.

B Proof of statement made in subsection 5.2

We answer in this appendix the question raised in subsection 5.2 as to whether the term $h^{\star\beta}_{(a)\gamma}\partial^{[\gamma}C^{(b)\alpha]}\partial_{[\beta}C^{(c)}_{\alpha]}a^a_{[bc]}$ is γ-exact modulo d,

$$h^{\star\beta}_{(a)\gamma}\partial^{[\gamma}C^{(b)\alpha]}\partial_{[\beta}C^{(c)}_{\alpha]}a^a_{[bc]} = \gamma(u) + \partial_\mu k^\mu. \tag{B.1}$$

Both u and k^μ have antifield number one. Without loss of generality, we can assume that u contains $h^{\star\beta}_{(a)\gamma}$ undifferentiated, since derivatives can be removed through integration by parts. As the Euler derivative of a total divergence is zero, we can reformulate the question as to whether the following identity holds,

$$\frac{\delta^L}{\delta h^{\star\beta}_{(a)\gamma}}(h^{\star\beta}_{(a)\gamma}\partial^{[\gamma}C^{(b)\alpha]}\partial_{[\beta}C^{(c)}_{\alpha]}a^a_{[bc]}) = \frac{\delta^L}{\delta h^{\star\beta}_{(a)\gamma}}(\gamma u) \tag{B.2}$$

i.e.

$$\partial^{[\gamma}C^{(b)\alpha]}\partial_{[\beta}C^{(c)}_{\alpha]}a^a_{[bc]} = \frac{\delta^L}{\delta h^{\star\beta}_{(a)\gamma}}\left[\text{linear combination of } \gamma\left\{\begin{array}{c} h^\star\partial C^{(b)}h^{(c)} \\ h^\star C^{(b)}\partial h^{(c)}\end{array}\right\}\right]. \tag{B.3}$$

The notations $h^\star\partial C^{(b)}h^{(c)}$ and $h^\star C^{(b)}\partial h^{(c)}$ stand for all terms having these structures.

Now, since $h^\star$ appears undifferentiated in u and hence also in γu, the Euler-Lagrange derivative with respect to $h^\star$ of γu can be read off straightforwardly and is just the coefficient of $h^\star$ in γu, i.e., a linear combination of $\gamma(\partial C^{(b)}h^{(c)})$ and $\gamma(C^{(b)}\partial h^{(c)})$. But none of these terms has the required form to match $\partial^{[\gamma}C^{(b)\alpha]}\partial_{[\beta}C^{(c)}_{\alpha]}a^a_{[bc]}$ since $\gamma(C^{(b)}\partial h^{(c)})$ contains second derivatives of the ghosts while $\gamma(\partial C^{(b)}h^{(c)})$ contains the product of symmetrized derivatives with anntisymmetrized derivatives. This establishes the result that $h^{\star\beta}_{(a)\gamma}\partial^{[\gamma}C^{(b)\alpha]}\partial_{[\beta}C^{(c)}_{\alpha]}a^a_{[bc]}$ is not γ-exact modulo d, unless it vanishes.

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