Absence of infinite cluster for critical Bernoulli percolation on slabs

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Abstract

We prove that for Bernoulli percolation on a graph $\mathbb{Z}^2 \times \{0, \ldots, k\}$ ($k \geq 0$), there is no infinite cluster at criticality, almost surely. The proof extends to finite range Bernoulli percolation models on $\mathbb{Z}^2$ which are invariant under $\pi/2$-rotation and reflection.

1 Introduction

Determining whether a phase transition is continuous or discontinuous is one of the fundamental questions in statistical physics. Bernoulli percolation has offered the mathematicians a setup to develop techniques to prove either continuity or discontinuity of the phase transition, which in the case of continuity corresponds to the absence of an infinite cluster at criticality. Harris [Har60] proved that the nearest neighbor bond percolation model with parameter $1/2$ on $\mathbb{Z}^2$ does not contain an infinite cluster almost surely. Viewed together with Kesten's result that $p_c \leq 1/2$ [Kes80], it provided the first proof of such type of statement. Since the original proof of Harris, a few alternative arguments have been found for planar graphs (See, for example, a short argument by Y. Zhang [Gri99, p 311]). In the late eighties, dynamic renormalization ideas were successfully applied to prove continuity in octants and half spaces of $\mathbb{Z}^d$, $d \geq 3$, [BGN91a, BGN91b]. The continuity was also proved for $\mathbb{Z}^d$ with $d \geq 19$ using the lace expansion technique [HS94], and for non-amenable Cayley graphs using mass-transport arguments [BLPS99]. Despite all these developments, a general argument to prove the continuity of the phase transition for the nearest neighbor Bernoulli percolation on arbitrary lattices is still missing, and the fact that the Bernoulli percolation undergoes a continuous phase transition on $\mathbb{Z}^3$ still represents one of the major open questions in the field.

This article provides the proof of continuity for Bernoulli percolation on a class of non-planar lattices, namely slabs. We wish to highlight that the lattices $\mathbb{Z}^d$ with $d \geq 3$ do not belong to this class of graphs.

Consider the graph $\mathbb{S}_k$, called *slab* of width k, given by the vertex set $\mathbb{Z}^2 \times \{0, \ldots, k\}$ and edges between nearest neighbors. In what follows, $\mathbf{P}_p$ denotes the Bernoulli bond percolation measure with parameter p on $\mathbb{S}_k$ defined as follows: every edge of $\mathbb{Z}^2 \times \{0, \ldots, k\}$ is *open* with probability p (if it is not open, it is said to be *closed*) independently of the other edges. Let $p_c(k)$ be the critical parameter of Bernoulli percolation on $\mathbb{S}_k$. Let B be a subset of $\mathbb{Z}^3$, the event $\{0 \overset{B}{\longleftrightarrow} \infty\}$ denotes the existence of an infinite path of open edges in B starting from 0.

Theorem 1. *For any $k > 0$, $\mathbf{P}_{p_c(k)}\big[0 \overset{\mathbb{S}_k}{\longleftrightarrow} \infty\big] = 0$.*

For *site* percolation on $\mathbb{S}_2$, an *ad hoc* argument was provided in [DNS12]. Nevertheless, one of the major difficulty of the present theorem is absent of [DNS12], namely the fact that "crossing paths do not necessarily intersect". This additional phenomenon, which is one of the main reasons why higher dimensional critical percolation is so difficult to study, requires the introduction of a new argument, based on the multi-valued map principle (see Lemma 6 below for further explanations).

Two generalizations The same proof works equally well (with suitable modifications) for any graph of the form $\mathbb{Z}^2 \times G$, where G is finite. This includes $G = \{0, \ldots, k\}^{d-2}$ for $d \geq 3$.

Similarly, symmetric finite range percolation on $\mathbb{Z}^2$ can be treated via the same techniques (once again, relevant modifications must be done). Let us state the result in this setting. Let $\mathbf{p} \in [0,1]^{\mathbb{Z}^2}$ be a set of edge-weight parameters, and $M > 0$. We consider functions $\mathbf{p}$'s that are M-supported (meaning $\mathbf{p}_z = 0$ for $|z| \geq M$) and invariant under reflection and $\pi/2$-rotation (meaning that for all z, $\mathbf{p}_{\mathrm{i}z} = \mathbf{p}_{\bar{z}} = \mathbf{p}_z$). Consider the graph with vertex set $\mathbb{Z}^2$ and edges between any two vertices and the percolation $\mathbf{P}_{\mathbf{p}}$ defined as follows: the edge (x,y) is open with probability $\mathbf{p}_{x-y}$, independently of the other edges.

Theorem 2. *Fix $M > 0$. The probability $\mathbf{P}_{\mathbf{p}}[0 \longleftrightarrow \infty]$ is continuous, when viewed as a function defined on the set of M-supported and invariant $\mathbf{p}$'s.*

From the slab to $\mathbb{Z}^3$? The fact that $\mathbb{Z}^2 \times \{0, \ldots, k\}^{d-2}$ is approximating $\mathbb{Z}^d$ when k tends to infinity suggests that the non-percolation on slabs could shed a new light on the problem of proving the absence of infinite cluster (almost

surely) for critical percolation on $\mathbb{Z}^d$. Nevertheless, we wish to highlight that this is not immediate. Indeed, while $p_c(k)$ is known to converge to $p_c(\mathbb{Z}^3)$ [GM90], passing at the limit requires a new ingredient. For instance, a uniform control (in k) on the explosion of the infinite-cluster density for p tending to the critical point would be sufficient.

Proposition 3. *Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $f(0) = 0$. If for any $k \geq 0$ and any $p \in (0,1)$,*

$$\mathbf{P}_p[0 \xleftrightarrow{\mathbb{S}_k} \infty] \leq f(p - p_c(k)),$$

then $\mathbf{P}_{p_c(\mathbb{Z}^3)}[0 \xleftrightarrow{\mathbb{Z}^3} \infty] = 0$.

It is natural to expect that proving the existence of f is roughly of the same difficulty as attacking the problem directly on $\mathbb{Z}^3$. Nevertheless, it could be that a suitable renormalization argument enables one to prove the existence of f.

Let us finish by recalling that several models undergo discontinuous phase transitions in high dimension and continuous phase transition in two dimensions (one may think of the 3 and 4-state Potts models). For most of these models, a discontinuous phase transition is expected to occur already in a slab. Theorem 1 shows that this is not the case for Bernoulli percolation.

What about other models? While this work is focused on the continuity of the phase transition for short range models, it is well known that the complete picture of phase transition for Bernoulli percolation is more complex. For one-dimensional long-range Bernoulli systems with power law decay, the transition may be discontinuous. Indeed, when the probabilities of edges of length r being open decay as $1/r^2$, the percolation density at criticality is strictly positive, see [AN86].

Also, one may consider more general percolation models with dependence. On $\mathbb{Z}^2$, the continuity of the phase transition was recently proven [DCST14] for dependent percolation models known as random-cluster models with cluster-weight $q \in [1,4]$ (the special case $q = 1$ corresponds to Bernoulli percolation). The continuity of the phase transition for $q = 1$ and 2 was previously established by Harris [Har60] and Onsager [Ons44] respectively. Furthermore, [LMMS+91] showed that the phase transition is discontinuous for q large enough.

Let us conclude this introduction by mentioning that the phase transition on $\mathbb{Z}^d$ is expected to be discontinuous for $q > 4$ when $d = 2$ (we refer to [DC13] for details on this prediction), and for $q > 2$ when $d \geq 3$. The best results

(for $q > 1$) in this direction are mostly restricted to integer values of q, for which the model is related to the Potts model. On the one hand, the fact that the phase transition is continuous for $q = 2$ (corresponding to the Ising model) is known for any $d \geq 3$ [ADCS13]. On the other hand for any $q \geq 3$, the random-cluster model undergoes a discontinuous phase transition above some dimension $d_c(q)$ [BCC06]. The proof of this result is based on Reflection-Positivity for the Potts model.

Notation. For a subset E of $\mathbb{Z}^2$, let $\overline{E}$ be the set of sites in $\mathbb{S}_k$ whose two first coordinates are in E. A *cluster in* $\overline{E}$ is a connected component of the graph given by all the vertices in $\overline{E}$ and the open edges with two endpoints in $\overline{E}$. Let n be a positive integer, B a subset of $\mathbb{Z}^2$, and $X, Y \subset B$. We define

$$X \overset{B}{\longleftrightarrow} Y = \{\text{there exists an open cluster in } \overline{B} \text{ connecting } \overline{X} \text{ to } \overline{Y}\},$$

$$X \overset{!B!}{\longleftrightarrow} Y = \{\text{there exists a } unique \text{ open cluster in } \overline{B} \text{ connecting } \overline{X} \text{ to } \overline{Y}\}.$$

Further we use the following notations: $B_n = [-n, n]^2$ and $\partial B_n = B_n \smallsetminus B_{n-1}$.

2 Proof

Outline of the proof. We follow a well known approach: we assume that $\mathbf{P}_p[0 \overset{\mathbb{S}_k}{\longleftrightarrow} \infty] > 0$, and using this, we construct a finite-size criterion which is sufficient for percolation to occur. By continuity, this finite-size criterion is satisfied for percolation with parameters sufficiently close to p. This immediately implies that $\mathbf{P}_{p_c}[0 \overset{\mathbb{S}_k}{\longleftrightarrow} \infty] = 0$.

The proof is divided in three steps:

- First, we prove that $\mathbf{P}_p[0 \overset{\mathbb{S}_k}{\longleftrightarrow} \infty] > 0$ implies the existence of a certain event with a large probability. This step is new, and in particular, we invoke a gluing lemma to estimate probability of connections between open paths.

- The second step is classical. It consists in applying a block argument to deduce that percolation occurs for any q sufficiently close to p.

- The last step provides the proof of the gluing lemma. This lemma provides an answer to a difficulty encountered when doing renormalization in 3-dimensions (e.g. in [GM90]) *in the case of slabs.* When trying to construct long open connections by connecting two open paths together,

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the conditioning on the first path creates negative information along the path. As a consequence, one may construct open paths coming at distance one of the existing path, but the last edge can potentially be already explored and closed. This difficulty is one of the major obstacles in using a renormalization scheme to prove that $\mathbf{P}_{p_c}[0 \overset{\mathbb{Z}^3}{\longleftrightarrow} \infty] = 0$. In our case, the fact that slabs are quasi-planar enables us to overcome this difficulty.

From now on in this section, we fix p and k and we assume that

$$\mathbf{P}_p[0 \overset{\mathbb{S}_k}{\longleftrightarrow} \infty] > 0.$$

Since the ambient space is fixed, we will not refer to $\mathbb{S}_k$ and will rather write $X \longleftrightarrow Y$ instead of $X \overset{\mathbb{S}_k}{\longleftrightarrow} Y$.

2.1   The finite-size criterion

The infinite cluster in $\mathbb{S}_k$ being unique almost surely [AKN87, BK89], one can construct a sequence $(u_n)_{n \geq 1}$ such that $u_n \leq n/3$ and

$$\lim_{n \to \infty} \mathbf{P}_p\left[B_{u_n} \overset{!B_n!}{\longleftrightarrow} \partial B_n\right] = 1. \tag{2.1}$$

For simplicity, we set $S_n = B_{u_n}$. For $0 \leq \alpha \leq \beta \leq n$, we define the following event:

$$\mathcal{E}_n(\alpha, \beta) = \left\{S_n \overset{B_n}{\longleftrightarrow} \{n\} \times [\alpha, \beta]\right\}.$$

Lemma 4. *There exist two sequences (y_n) and (α_n) with values in $[0, n]$, such that*

$$\lim_{n \to \infty} \mathbf{P}_p\left[\mathcal{E}_n(\alpha_n, n)\right] = 1,$$
$$\lim_{n \to \infty} \mathbf{P}_p\left[\mathcal{E}_n(y_n - \alpha_n/4, y_n + \alpha_n/4)\right] = 1.$$

The proof relies on the following classical inequality, which is a straightforward consequence of the Harris-FKG inequality. Let $\mathcal{A}_1, \ldots, \mathcal{A}_m$ be m increasing events. Then

$$\max_{i=1,\ldots,m} \mathbf{P}_p\left[\mathcal{A}_i\right] \geq 1 - (1 - \mathbf{P}_p\left[\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_m\right])^{1/m}. \tag{2.2}$$

When the events are of equal probability, this inequality is known as "square-root trick". We use the same name for the generalization given by (2.2).
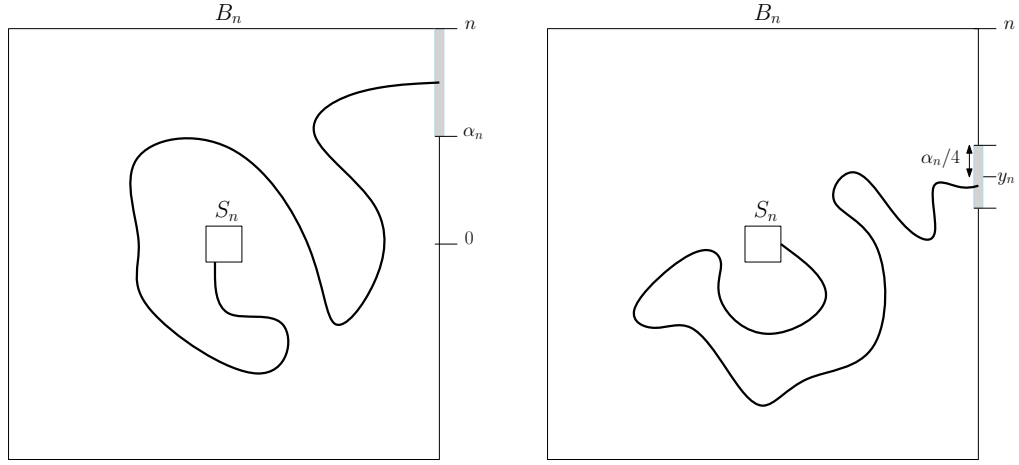
(a) The event $\mathcal{E}_n(\alpha_n, n)$.   (b) The event $\mathcal{E}_n(y_n - \alpha_n/4, y_n + \alpha_n/4)$.

Figure 1: The two events of Lemma 4.

Proof of Lemma 4. Applying the square-root trick and using the symmetries of the box, we obtain

$$\mathbf{P}_p\left[\mathcal{E}_n(0,n)\right] \geq 1 - \left(1 - \mathbf{P}_p\left[S_n \overset{B_n}{\longleftrightarrow} \partial B_n\right]\right)^{1/8}$$

which implies that $\mathbf{P}_p\left[\mathcal{E}_n(0,n)\right]$ also tends to 1 as n goes to infinity. Now, for $\alpha \in \{0, \dots, n-1\}$ we will use the decomposition

$$\mathcal{E}_n(0,n) = \mathcal{E}_n(0,\alpha) \cup \mathcal{E}_n(\alpha + 1, n).$$

The probability of the event $\mathcal{E}_n(0,0)$ is smaller than some constant $c < 1$ uniformly in n and $\mathbf{P}_p\left[\mathcal{E}_n(0,n)\right]$ tends to 1, providing that for n large enough:

$$\mathbf{P}_p\left[\mathcal{E}_n(0,0)\right] < \mathbf{P}_p\left[\mathcal{E}_n(1,n)\right].$$

In the same way, we also have for n large enough

$$\mathbf{P}_p\left[\mathcal{E}_n(0,n-1)\right] > \mathbf{P}_p\left[\mathcal{E}_n(n,n)\right].$$

The two inequalities above ensure that the inequality between $\mathbf{P}_p\left[\mathcal{E}_n(0,\alpha-1)\right]$ and $\mathbf{P}_p\left[\mathcal{E}_n(\alpha,n)\right]$ reverses for a non-trivial α. More precisely we can define $\alpha_n \in \{1, \dots, n-1\}$ by

$$\alpha_n = \max\left\{\alpha \leq n-1 : \mathbf{P}_p\left[\mathcal{E}_n(0,\alpha-1)\right] < \mathbf{P}_p\left[\mathcal{E}_n(\alpha,n)\right]\right\},$$

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and this choice implies that

$$\mathbf{P}_p\left[\mathcal{E}_n(0, \alpha_n - 1)\right] < \mathbf{P}_p\left[\mathcal{E}_n(\alpha_n, n)\right] \text{ and } \mathbf{P}_p\left[\mathcal{E}_n(0, \alpha_n)\right] \geq \mathbf{P}_p\left[\mathcal{E}_n(\alpha_n + 1, n)\right].$$

Therefore, two other uses of the square-root trick imply that $\mathbf{P}_p\left[\mathcal{E}_n(0, \alpha_n)\right]$ and $\mathbf{P}_p\left[\mathcal{E}_n(\alpha_n, n)\right]$ are larger than $1 - (1 - \mathbf{P}_p\left[\mathcal{E}_n(0, n)\right])^{1/2}$ and thus tends to 1 when n goes infinity. Finally, we decompose

$$\mathcal{E}_n(0, \alpha_n) = \mathcal{E}_n(0, \alpha_n/2) \cup \mathcal{E}_n(\alpha_n/2, \alpha_n)$$

and a last application of the square root trick allows to define $y_n = \alpha_n/4$ or $y_n = 3\alpha_n/4$ such that

$$\mathbf{P}_p\left[\mathcal{E}_n(y_n - \alpha_n/4, y_n + \alpha_n/4)\right] \geq 1 - \sqrt{1 - \mathbf{P}_p\left[\mathcal{E}_n(0, \alpha_n)\right]},$$

which concludes the proof of the lemma. □

Lemma 5. *There exist infinitely many n such that $\alpha_{3n} \leq 4\alpha_n$.*

Proof. A sequence of positive integers such that $\alpha_{3n} > 4\alpha_n$ for n large enough grows super-linearly. Since $\alpha_n \leq n$, we obtain the result. □

Let $n \geq 1$. Write $y = y_{3n}$ and define the following five subsets of $\mathbb{Z}^2$ (see Fig. 2 for an illustration):

$$\begin{aligned}
B'_n &= (2n, y) + B_n, \\
S'_n &= (2n, y) + S_n, \\
Y^+_n &= \{3n\} \times [y + \alpha_n, y + n], \\
Y^-_n &= \{3n\} \times [y - n, y - \alpha_n], \\
Z_n &= \{3n\} \times [y - \alpha_n, y + \alpha_n].
\end{aligned}$$

When n is such that $\alpha_{3n}/4 \leq \alpha_n$, we have

$$\mathbf{P}_p\left[S_{3n} \overset{B_{3n}}{\longleftrightarrow} Z_n\right] \geq \mathbf{P}_p\left[\mathcal{E}_{3n}(y_{3n} - \alpha_{3n}, y_{3n} + \alpha_{3n})\right],$$

and Lemmata 4 and 5 imply that

$$\limsup_{n \to \infty} \mathbf{P}_p\left[S_{3n} \overset{B_{3n}}{\longleftrightarrow} Z_n\right] = 1. \tag{2.3}$$

Using Harris inequality and the invariance of $\mathbf{P}_p$ under reflection, we deduce that

$$\mathbf{P}_p\left[S_{3n} \overset{B_{3n}}{\longleftrightarrow} Z_n, S'_n \overset{B'_n}{\longleftrightarrow} Y^-_n, S'_n \overset{B'_n}{\longleftrightarrow} Y^+_n\right] \geq \mathbf{P}_p\left[S_{3n} \overset{B_{3n}}{\longleftrightarrow} Z_n\right]\mathbf{P}_p\left[\mathcal{E}_n(0, \alpha_n)\right]^2.$$

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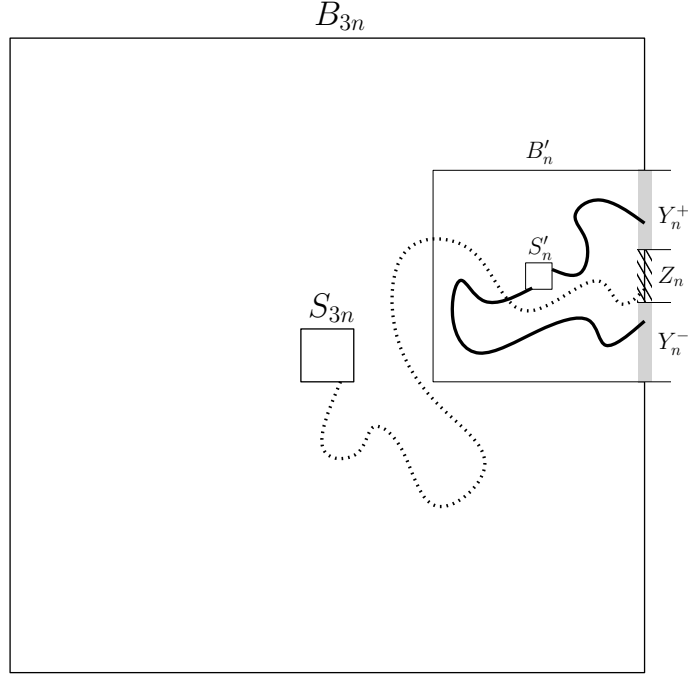
Figure 2: The events $S_{3n} \overset{B_{3n}}{\longleftrightarrow} Z_n$ (the path is depicted by dots) and $\{S'_n \overset{B'_n}{\longleftrightarrow} Y_n^-\} \cap \{S'_n \overset{B'_n}{\longleftrightarrow} Y_n^+\}$ (the paths are depicted in bold).

From Lemma 4 and (2.3), we finally obtain

$$\limsup_{n \to \infty} \mathbf{P}_p \left[S_{3n} \overset{B_{3n}}{\longleftrightarrow} Z_n, S'_n \overset{B'_n}{\longleftrightarrow} Y_n^-, S'_n \overset{B'_n}{\longleftrightarrow} Y_n^+ \right] = 1. \qquad (2.4)$$

We now intend to construct a path from $\overline{S_{3n}}$ to $\overline{S'_n}$. Projections of paths from $\overline{S_{3n}}$ to $\overline{Z_n}$ and from $\overline{S'_n}$ to $\overline{Y_n^-}$ and $\overline{Y_n^+}$ must intersect (as illustrated on Fig. 2), but the paths themselves have no reason to do so. This is one of the main difficulties when working with non-planar graphs. Let us assume for a moment that we have the following lemma at our disposition and let us finish the proof. Note that this lemma is a crucial ingredient of the proof, since it solves the problem of the intersection of paths on slabs.

Lemma 6 (Gluing Lemma). *For any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, k) > 0$ such that for any n,*

$$\mathbf{P}_p \left[S_{3n} \overset{B_{3n}}{\longleftrightarrow} Z_n, S'_n \overset{B'_n}{\longleftrightarrow} Y_n^-, S'_n \overset{B'_n}{\longleftrightarrow} Y_n^+ \right] \geq 1 - \delta$$

implies

$$\mathbf{P}_p \left[S_{3n} \overset{B_{3n} \cup B'_n}{\longleftrightarrow} S'_n \right] \geq 1 - \varepsilon.$$

Lemma 6 and (2.4) imply that

$$\limsup_{n\to\infty} \mathbf{P}_p\left[S_{3n} \overset{B_{4n}}{\longleftrightarrow} S'_n\right] = 1. \tag{2.5}$$

Observe that

$$\mathbf{P}_p\left[S_{3n} \overset{(2n,0)+B_{6n}}{\longleftrightarrow} (4n,0)+S_{3n}\right]$$

$$\geq \mathbf{P}_p\left[\{S_{3n} \overset{B_{4n}}{\longleftrightarrow} S'_n\} \cap \{S'_n \overset{(4n,0)+B_{4n}}{\longleftrightarrow} (4n,0)+S_{3n}\} \cap \{S'_n \overset{!B'_n!}{\longleftrightarrow} \partial B'_n\}\right]$$

$$\geq \mathbf{P}_p\left[\{S_{3n} \overset{B_{4n}}{\longleftrightarrow} S'_n\} \cap \{S'_n \overset{(4n,0)+B_{4n}}{\longleftrightarrow} (4n,0)+S_{3n}\}\right] + \mathbf{P}_p\left[S'_n \overset{!B'_n!}{\longleftrightarrow} \partial B'_n\right] - 1$$

$$\geq \mathbf{P}_p\left[S_{3n} \overset{B_{4n}}{\longleftrightarrow} S'_n\right]^2 + \mathbf{P}_p\left[S'_n \overset{!B'_n!}{\longleftrightarrow} \partial B'_n\right] - 1.$$

The first inequality followed from the fact that paths coming from $\overline{S_{3n}}$ and $\overline{(4n,0)+S_{3n}}$ and going to $\overline{S'_n}$ must be connected to each other in $\overline{B'_n}$ by uniqueness of the cluster in $\overline{B'_n}$ from $\overline{S'_n}$ to $\overline{\partial B'_n}$. The Harris inequality and the reflection across the axis $\{2n\} \times \mathbb{R}$ were used in the last inequality.

Using (2.5) and (2.1), we find

$$\limsup_{n\to\infty} \mathbf{P}_p\left[S_{3n} \overset{(2n,0)+B_{6n}}{\longleftrightarrow} (4n,0)+S_{3n}\right] = 1. \tag{2.6}$$

2.2 The renormalization step

Fix $n \in \mathbb{N}$ to be chosen below. Call an edge $\{z, z'\}$ of $4n\mathbb{Z}^2$ *good* if

- $z + S_{3n} \overset{R_n}{\longleftrightarrow} z' + S_{3n}$, with $R_n = \frac{z+z'}{2} + B_{6n}$,

- $z + S_{3n} \overset{!z+B_{3n}!}{\longleftrightarrow} z + \partial B_{3n}$ and $z' + S_{3n} \overset{!z'+B_{3n}!}{\longleftrightarrow} z' + \partial B_{3n}$.

Notice that the set of good edges follows a percolation law which is 4-dependent. In particular, there exists $\eta > 0$ such that whenever the probability to be good exceeds $1 - \eta$, the set of good edges percolates (this fact follows from a Peierls argument presented for example in [BBW05, Lemma 1], or from the classical result of [LSS97] comparing 4-dependent percolation to Bernoulli percolation).

Equations (2.6) and (2.1) guarantee the existence of n such that the $\mathbf{P}_p$-probability that an edge is good is larger than $1-\eta$. Since being good depends only on the state of the edges in a finite box, there exists $q < p$ such that an edge is good with $\mathbf{P}_q$-probability larger than $1 - \eta$, and the set of good edges percolates for the percolation of parameter q.

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By construction, an infinite path of good edges in the coarse-grained lattice immediately implies the existence of an infinite path of open edges in the original lattice. As a consequence, $q \geq p_c(k)$ and therefore $p > p_c(k)$. This concludes the proof of $\mathbf{P}_{p_c(k)}[0 \longleftrightarrow \infty] = 0$ conditionally on Lemma 6.

2.3 The proof of Lemma 6 (the gluing Lemma)

First, observe that the lemma holds trivially for $k = 0$ by setting $\delta(\varepsilon, 0) = \varepsilon$. *We therefore assume from now on that $k \geq 1$.* We will be using the following lemma.

Lemma 7. *Let $s, t > 0$. Consider two events $\mathcal{A}$ and $\mathcal{B}$ and a map Φ from $\mathcal{A}$ into the set $\mathfrak{P}(\mathcal{B})$ of subevents of $\mathcal{B}$. We assume that:*

1. for all $\omega \in \mathcal{A}$, $|\Phi(\omega)| \geq t$,

2. for all $\omega' \in \mathcal{B}$, there exists a set S with less than s edges such that $\{\omega : \omega' \in \Phi(\omega)\} \subset \{\omega : \omega_{|S^c} = \omega'_{|S^c}\}$.

Then,

$$\mathbf{P}_p[\mathcal{A}] \leq \frac{(2/\min\{p, 1-p\})^s}{t} \mathbf{P}_p[\mathcal{B}].$$

This lemma will enable us to bound from above the probability of $\mathcal{A}$ when s is small and t is large.

Proof. It follows from exchanging the order of the summation on ω and on $\omega' \in \Phi(\omega)$:

$$\sum_{\omega \in \mathcal{A}} \mathbf{P}_p[\omega] \leq \frac{1}{t(\min\{p, 1-p\})^s} \sum_{\omega \in \mathcal{A}} \mathbf{P}_p[\Phi(\omega)]$$

$$= \frac{1}{t(\min\{p, 1-p\})^s} \sum_{\omega' \in \mathcal{B}} \mathrm{Card}\{\omega : \omega' \in \Phi(\omega)\} \cdot \mathbf{P}_p[\omega']$$

$$\leq \frac{2^s}{t(\min\{p, 1-p\})^s} \sum_{\omega' \in \mathcal{B}} \mathbf{P}_p[\omega'].$$

$\square$

Let us now explain how the previous statement can be used to prove Lemma 6. Fix an arbitrary order $\prec$ on edges emanating from each vertex of $\mathbb{S}_k$, which is invariant under translations of $\mathbb{Z}^2$. Also fix an arbitrary order $\ll$ on vertices of $\mathbb{S}_k$. Then, define a total order on self-avoiding paths from $\overline{S_{3n}}$ to $\overline{Z_n}$ by taking the lexicographical order: for two paths $\gamma = (\gamma_i)_{i \leq r}$ and $\gamma' = (\gamma'_i)_{i \leq r'}$, we set $\gamma < \gamma'$ if one of the following conditions occurs:

- $r < r'$ and $\gamma = (\gamma_i')_{i \le r}$,

- $\gamma_0 \ll \gamma_0'$,

- there exists $k < \min\{r, r'\}$ such that $\gamma_j = \gamma_j'$ for $j \le k$ and $(\gamma_k, \gamma_{k+1}) \prec (\gamma_k', \gamma_{k+1}')$.

Definition. *Consider ω with at least one open path from $\overline{S_{3n}}$ to $\overline{Z_n}$. Define $\gamma_{\min}(\omega)$ to be the minimal (for the order defined above) open self-avoiding path from $\overline{S_{3n}}$ to $\overline{Z_n}$. Let $U(\omega)$ be the set of points z in B_n' with*

P1 $\overline{\{z\}} \cap \gamma_{\min}(\omega) \ne \varnothing$,

P2 $\overline{z + B_1}$ *is connected to $\overline{S_n'}$ by an open path π, such that the distance between the canonical projections of π and $\gamma_{\min}$ onto $\mathbb{Z}^2$ is exactly 1.*

Write $\mathcal{X} = \{S_{3n} \overset{B_{3n}}{\longleftrightarrow} Z_n, S_n' \overset{B_n'}{\longleftrightarrow} Y_n^-, S_n' \overset{B_n'}{\longleftrightarrow} Y_n^+\} \cap \{S_{3n} \overset{B_{3n} \cup B_n'}{\longleftrightarrow} S_n'\}^c$. Proving Lemma 6 corresponds to proving that the probability of $\mathcal{X}$ is small whenever the probability of $\{S_{3n} \overset{B_{3n}}{\longleftrightarrow} Z_n, S_n' \overset{B_n'}{\longleftrightarrow} Y_n^-, S_n' \overset{B_n'}{\longleftrightarrow} Y_n^+\}$ is close to 1. We proceed in two steps, depending on whether the cardinality of $U(\omega)$ is large or not.

Fact 1. *Fix $\varepsilon > 0$ and $t > 0$. There exists $\delta > 0$ so that*

$$\mathbf{P}_p\left[S_n' \overset{B_n'}{\longleftrightarrow} Y_n^-, S_n' \overset{B_n'}{\longleftrightarrow} Y_n^+\right] > 1 - \delta$$

implies $\mathbf{P}_p[\mathcal{X} \cap \{|U| < t\}] \le \varepsilon$.

Proof of Fact 1. Let $\omega \in \mathcal{X}$ such that $|U(\omega)| < t$. Define ω' to be the configuration obtained from ω by closing, for any $z \in U(\omega)$, all the edges $\{u, v\}$ such that $u \in \overline{\{z\}}$ and v is connected to $\overline{S_n'}$ by an open path.

Observe that ω' cannot contain two open paths in $\overline{B_n'}$ from $\overline{S_n'}$ to $\overline{Y_n^-}$ and $\overline{Y_n^+}$ respectively. Indeed, an open path in ω' must be in ω. Furthermore, two paths from $\overline{S_n'}$ to $\overline{Y_n^-}$ and $\overline{Y_n^+}$ respectively must intersect at least one set of the form $\overline{\{z\}}$ with z in $U(\omega)$. But this implies that one edge of one of these two paths was turned to closed in ω', which is a contradiction. We therefore constructed a map

$$\Phi: \quad \mathcal{X} \cap \{|U| < t\} \quad \longrightarrow \quad \{S_n' \overset{B_n'}{\longleftrightarrow} Y_n^-, S_n' \overset{B_n'}{\longleftrightarrow} Y_n^+\}^c$$

mapping a configuration ω to ω'. For any ω' in the image of Φ, the set $\{\omega : \Phi(\omega) = \omega'\}$ contains only configurations that are equal to ω' except possibly

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on the edges adjacent to $U(\omega')$. Here, we use the fact that $U(\omega') = U(\omega)$ and $\gamma_{\min}(\omega') = \gamma_{\min}(\omega)$ for any pre-image of ω' (since **P1** guarantees that no edge of $\gamma_{\min}(\omega')$ was closed in the process). Lemma 7 can be applied to obtain

$$\mathbf{P}_p[\mathcal{X} \cap \{|U| < t\}] \leq (2/\min\{p, 1-p\})^{6kt} \mathbf{P}_p\left[\left\{S_n' \overset{B_n'}{\longleftrightarrow} Y_n^-, S_n' \overset{B_n'}{\longleftrightarrow} Y_n^+\right\}^c\right].$$

Fact 1 follows immediately. $\qquad\square$

Fact 2. *Fix $\varepsilon > 0$. For t large enough,*

$$\mathbf{P}_p[\mathcal{X} \cap \{|U| \geq t\}] \leq \varepsilon \mathbf{P}_p\left[S_{3n} \overset{B_{3n} \cup B_n'}{\longleftrightarrow} S_n'\right].$$

Proof of Fact 2. For $R \geq 1$ and $z = (z_1, z_2, z_3) \in \mathbb{S}_k$, we write $\overline{B_R}(z)$ for $(z_1, z_2) + B_R$. Fix $R \geq 2$ in such a way that for any site $z \in \mathbb{S}_k$, for any three distinct neighbors u, v, w of z and any three distinct sites u', v', w' on the boundary of $\overline{B_R}(z)$, there exist three disjoint self-avoiding paths in $\overline{B_R}(z) \smallsetminus \{z\}$ connecting u to u', v to v' and w to w'. Note that such an R exists since in this section, k is assume to be strictly larger than 0.

Remark. *For the slab, one could take $R = 2$. Nevertheless, taking larger R becomes necessary when dealing with finite range percolation. Since the proof is not more complicated, we choose to present it with an arbitrary R.*

Fix $\omega \in \mathcal{X}$ such that $|U(\omega)| \geq t$ and pick $z \in U(\omega)$. Construct the configuration $\omega^{(z)}$ as follows (see Fig. 3 for an illustration of the construction):

1. Choose u, v, w in such a way that (z, u), (z, v) and (z, w) are three distinct edges with $(z, v) \prec (z, w)$.

   Define u' and v' to be respectively the first and last (when going from S_{3n} to $Y_n^+ \cup Y_n^-$) vertices of $\gamma_{\min}(\omega)$ which are in $\overline{B_R}(z)$ (these two vertices exist and are distinct since $\gamma_{\min}(\omega)$ intersects the set $\overline{B_1}(z)$ by **P2**).

   Choose w' on the boundary of $\overline{B_R}$ in such a way that there exists an open self-avoiding path π from w' to $\overline{S_n'}$, all the edges of which lie outside $\overline{B_R}(z)$ (this path exists by **P1**). Since $\omega \in \mathcal{X}$, we also have that w' is different from u' and v' (otherwise $S_{3n} \longleftrightarrow S_n'$ in ω).

2. Close all edges of ω in $\overline{B_{R+1}}(z)$ at the exception of the edges of $\overline{B_{R+1}}(z) \smallsetminus \overline{B_R}(z)$ which are in $\gamma_{\min}(\omega)$ or π.

3. Open the edges (z, u), (z, v) and (z, w), together with three disjoint self-avoiding paths γ_u, γ_v and γ_w in $\overline{B_R}(z) \smallsetminus \{z\}$ connecting u to u', v to v', and w to w'.
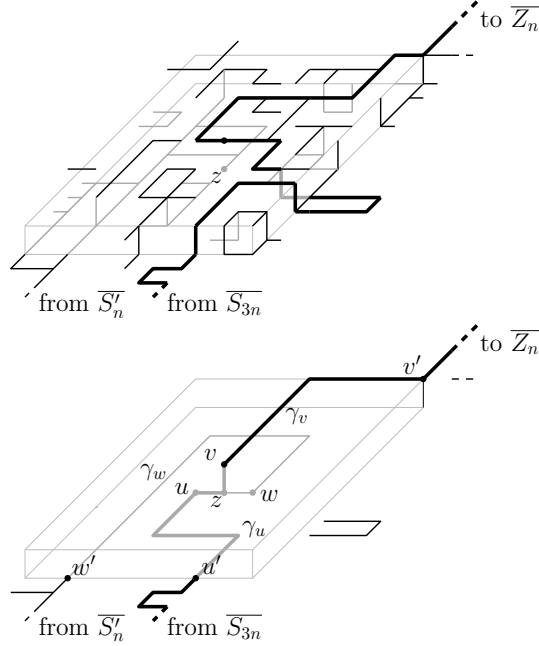
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Figure 3: Two configurations ω and $\omega^{(z)}$. In both cases, $\gamma_{\min}$ is depicted in bold, and closed edges are not drawn for clarity. Note that at the end of the construction, there are exactly three open edges connecting a vertex of $\overline{B_R}(z)$ to a vertex in the complement of $\overline{B_R}(z)$.

By construction, $\omega^{(z)}$ is in $\{S_{3n} \xleftrightarrow{B_{3n} \cup B'_n} S'_n\}$ and we can define the map

$$\Psi: \quad \mathcal{X} \cap \{|U| > t\} \quad \longrightarrow \quad \mathfrak{P}(S_{3n} \xleftrightarrow{B_{3n} \cup B'_n} S'_n)$$
$$\omega \quad \longmapsto \quad \{\omega^{(z)}, z \in U(\omega)\}.$$

We wish to apply Lemma 7. In order to do so, the following observation will be useful.

Working with the lexicographical order implies that $\gamma_{\min}(\omega^{(z)})$ and $\gamma_{\min}(\omega)$ necessarily coincide up to u'. Thanks to the second step, the degree of u' in $\omega^{(z)}$ is 2. This fact forces any self-avoiding open path from $\overline{S_{3n}}$ to $\overline{Z_n}$ containing the minimal path up to u' to contain γ_u. Now (this is the crucial point of the construction), we have that $(z, v) \prec (z, w)$. Therefore, even though there could exist an open path from z to $\overline{Z_n}$ passing by w, *the minimal path will still be going through v*. Hence, the continuation of the minimal path goes through v and thus contains γ_v for the same reason that it was including γ_u. From v', the minimality of $\gamma_{\min}(\omega)$ implies that $\gamma_{\min}(\omega^{(z)})$ and $\gamma_{\min}(\omega)$ coincide from this vertex up to the end.

Since no site of $\gamma_{\min}(\omega)$ is connected to $\overline{S'_n}$ in ω (simply because $\omega \in \mathcal{X}$),

the previous paragraph implies that z is *the only site on $\gamma_{\min}(\omega^{(z)})$ to be connected to $\overline{S'_n}$ without using any edge in $\gamma_{\min}(\omega^{(z)})$.*

We are now in a position to apply Lemma 7. The configurations $\omega^{(z)}$ are all distinct since either $\gamma_{\min}(\omega^{(z)}) \neq \gamma_{\min}(\omega^{(z')})$ (which readily implies that the configurations are distinct), or $\gamma_{\min}(\omega^{(z)}) = \gamma_{\min}(\omega^{(z')})$ but then $z = z'$ by the characterization of z (and z') above.

Furthermore, consider a pre-image ω of ω' and assume that $\omega' = \omega^{(z)}$ for some $z \in \mathbb{S}_k$. The discussion above shows that z is determined uniquely. Beside, the configurations ω and $\omega^{(z)}$ differ only in $\overline{B_{R+1}}(z)$.

In conclusion, the map Φ verifies the hypotheses of Lemma 7 with s equal to the number of edges in $\overline{B_{R+1}}$. This gives

$$\mathbf{P}_p\left[\mathcal{X} \cap \{|U| > t\}\right] \leq \frac{(2/\min\{p, 1-p\})^C}{t}\mathbf{P}_p\left[S_{3n} \xleftrightarrow{B_{3n} \cup B'_n} S'_n\right].$$

Choosing t large enough concludes the proof. $\qquad\square$

Fix $\varepsilon > 0$. Choosing first t as in Fact 2 and then δ as in Fact 1 conclude the proof of Lemma 6.

2.4   The proof of Proposition 3

Proof of Proposition 3. Recall the result of [GM90] yielding that $p_c(k)$ tends to $p_c(\mathbb{Z}^3)$ as k tends to infinity.

Let $p > p_c(\mathbb{Z}^3)$. Since the infinite cluster is unique almost surely, and since there exits an infinite cluster in Slab_k for any k sufficiently large (simply choose k so that $p_c(k) < p$), we obtain that

$$\mathbf{P}_p\left[0 \xleftrightarrow{\mathbb{Z}^3} \infty\right] = \mathbf{P}_p\left[\bigcup_{k \geq 0}\{0 \xleftrightarrow{\mathbb{S}_k} \infty\}\right],$$

from which we deduce that

$$\mathbf{P}_p\left[0 \xleftrightarrow{\mathbb{Z}^3} \infty\right] = \lim_{k \to \infty}\mathbf{P}_p\left[0 \xleftrightarrow{\mathbb{S}_k} \infty\right] \leq \lim_{k \to \infty} f(p - p_c(k)) = f(p - p_c(\mathbb{Z}^3)).$$

As p tends to $p_c(\mathbb{Z}^3)$, the continuity of f implies that

$$\mathbf{P}_{p_c(\mathbb{Z}^3)}\left[0 \xleftrightarrow{\mathbb{Z}^3} \infty\right] = 0.$$

$\square$

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