

Brochette percolation

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ABSTRACT. We study bond percolation on the square lattice with one-dimensional inhomogeneities. Inhomogeneities are introduced in the following way: A vertical column on the square lattice is the set of vertical edges that project to the same vertex on \mathbb{Z} . Select vertical columns at random independently with a given positive probability. Keep (respectively remove) vertical edges in the selected columns, with probability p , (respectively $1 - p$). All horizontal edges and vertical edges lying in unselected columns are kept (respectively removed) with probability q , (respectively $1 - q$). We show that, if $p > p_c(\mathbb{Z}^2)$ (the critical point for homogeneous Bernoulli bond percolation) then q can be taken strictly smaller than $p_c(\mathbb{Z}^2)$ in such a way that the probability that the origin percolates is still positive.

1 Introduction

1.1 Definition of the model and statement of the result

Consider the square lattice $\mathbb{Z}^2 = (V(\mathbb{Z}^2), E(\mathbb{Z}^2))$ defined by

$$\begin{aligned} V(\mathbb{Z}^2) &:= \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \in \mathbb{Z}\}, \\ E(\mathbb{Z}^2) &:= \{\{x, y\} \subset V(\mathbb{Z}^2) : |x_1 - y_1| + |x_2 - y_2| = 1\}. \end{aligned}$$

A *percolation configuration* is an element of $\{0, 1\}^{E(\mathbb{Z}^2)}$ denoted generically by $\omega = (\omega(e) : e \in E(\mathbb{Z}^2))$. Note that ω can be seen as a subgraph of \mathbb{Z}^2 by setting $V(\omega) := V(\mathbb{Z}^2)$ and $E(\omega) := \{e \in E(\mathbb{Z}^2) : \omega(e) = 1\}$. The study of the connectivity properties of this subgraph obtained when ω is sampled at random is the main goal of percolation theory.

In the Bernoulli bond percolation model on \mathbb{Z}^2 , the $\omega(e)$'s are independent Bernoulli random variables with mean p_e . The model is said to be homogeneous when, for every e , $p_e = p$ for some $p \in [0, 1]$. Otherwise it is said to be inhomogeneous. There are many ways to introduce inhomogeneities. For instance, if one wants to preserve invariance under lattice shifts, p_e can be chosen according to a translation-invariant law. In this work we study one such model that contains one-dimensional inhomogeneities as we describe now.

For $\Lambda \subset \mathbb{Z}$, set

$$E_{\text{ver}}(\Lambda \times \mathbb{Z}) := \{\{(x_1, x_2), (x_1, x_2 + 1)\} : x_1 \in \Lambda, x_2 \in \mathbb{Z}\}.$$

For $p, q \in [0, 1]$, let $\mathbb{P}_{p,q}^\Lambda$ be the law on $\{0, 1\}^{E(\mathbb{Z}^2)}$ under which the ω_e ($e \in E(\mathbb{Z}^2)$) are independent Bernoulli random variables with mean p_e given by

$$p_e = \begin{cases} p & \text{if } e \in E_{\text{ver}}(\Lambda \times \mathbb{Z}) \\ q & \text{if } e \notin E_{\text{ver}}(\Lambda \times \mathbb{Z}) \end{cases}.$$

Observe that $\mathbb{P}_{p,p}^\Lambda$ is the *Bernoulli bond percolation* measure with edge-weight p , which will be denoted by \mathbb{P}_p .

We now wish to take Λ at random. For each $\rho \in [0, 1]$, define ν_ρ to be the probability measure on subsets of \mathbb{Z} under which $\{i \in \Lambda\}$ are independent events having probability

ρ . We are now in a position to state our main result. Denote by $\{0 \longleftrightarrow \infty\}$ the event that the origin is connected to infinity (see Section 1.3 for a precise definition). Let p_c be such that $\mathbb{P}_p(0 \leftrightarrow \infty)$ equals 0 if $p < p_c$, and is strictly positive if $p > p_c$ (Kesten proved in [9] that $p_c = 1/2$).

Theorem 1. *For every $\varepsilon \in (0, 1/2]$ and $\rho > 0$, there exists $\delta > 0$ such that for ν_ρ -almost every Λ ,*

$$\mathbb{P}_{p_c+\varepsilon, p_c-\delta}^\Lambda(0 \longleftrightarrow \infty) > 0.$$

In the remainder of this section we present some of our motivations for addressing this problem, mention some related works, and highlight the ideas and techniques to be used in the proof of the above theorem.

1.2 Motivation and related models

This work is motivated by the following general question: how do d -dimensional inhomogeneities in $(d+1)$ -dimensional lattice models shift the critical point or change the order of the phase transition? This question was raised before for a number of models and settings as we describe below.

In the context of percolation, a classical argument due to Aizenman and Grimmett [1] guarantees the validity of Theorem 1 in the case that Λ only contains bounded gaps (*i.e.* when there exists a $k \in (0, \infty)$ such that Λ intersects all sets of the type $[l, r] \cap \mathbb{Z}$ with $r - l = k$). Note that, for any $\rho \in (0, 1)$, the set of Λ 's that exhibit such a regularity condition, has zero measure under ν_ρ . We will use the framework developed in [1] as one of the elements of our proof.

In [21], Zhang addresses the case when $\Lambda = \{0\}$, and $q = p_c$. Relying on the idea of Harris [7] of constructing dual circuits around the origin together with the Russo [18] and Seymour and Welsh [20] techniques, he proves that $\mathbb{P}_{p, p_c}^{\{0\}}(0 \leftrightarrow \infty) = 0$ for any $p \in [0, 1)$. Monotonicity implies that $\mathbb{P}_{p, q}^{\{0\}}(0 \leftrightarrow \infty) = 0$ if $q < p_c$, and the results of Barsky, Grimmett and Newman on percolation in half-spaces [2] imply immediately that $\mathbb{P}_{p, q}^{\{0\}}(0 \leftrightarrow \infty) > 0$ whenever $q > p_c$.

For the Ising model on the square lattice, McCoy and Wu [15] considered the setting in which the coupling constants for horizontal edges are given by a fixed deterministic number whereas for vertical edges, all the coupling constants for edges connecting between sites in the j^{th} and $j+1^{\text{st}}$ rows are given by a random variable $E(j)$. There, the $E(j)$'s are assumed to be i.i.d. Their main motivation was to show how the presence of inhomogeneities leads to a model where the specific heat does not diverge (and not even its derivatives) close to the critical temperature. This contrasts with the classical results of Onsager for the Ising model on the square lattice with homogeneous coupling constants.

In [5], motivated by the study of the Ising model with a random transverse field, Campanino and Klein studied the decay of the two-point function for a $(d+1)$ -dimensional bond percolation (and also Ising and Potts models) with both d -dimensional and 1-dimensional disorder.

Another variation was studied by Hoffman [8]. That paper discussed percolation in a random environment where columns of horizontal edges were weakened independently, and so were rows of vertical edges.

More recently, in [11], the authors considered the $(1+1)$ -directed percolation model with inhomogeneities that are transversal to the “time direction” (in contrast with analogous results on the contact process in [4, 14] where the inhomogeneities are taken along lines

parallel to the “time direction”). We discuss their setting and their result in more detail in Section 1.4 since they are used as a fundamental step in our work (see Theorem 4 below).

1.3 Notation

When there is no risk of ambiguity, we abuse notation and do not distinguish between $V(\mathbb{Z}^2)$ and \mathbb{Z}^2 , and similarly for other graphs. Let us write $x \sim y$ if x is a *neighbour* of y *i.e.*, if $\{x, y\} \in E(\mathbb{Z}^2)$. For $A \subset \mathbb{Z}^2$, we set $\partial A := \{x \in A : \exists y \notin A \text{ with } x \sim y\}$. A path in A is a sequence of sites $v_0 \sim v_1 \dots \sim v_n$ such that $v_i \in A$ for all i .

An edge e is said to be *open* (in ω) if $\omega(e) = 1$. Otherwise, it is said to be *closed*. For a set $A \subset \mathbb{Z}^2$ and two vertices $x, y \in \mathbb{Z}^2$, x and y are *connected in A* (denoted $x \xleftrightarrow{A} y$) if there exists a sequence $x = v_0 \sim \dots \sim v_n = y$ in A such that $\omega(\{v_i, v_{i+1}\}) = 1$ for every $0 \leq i < n$. If $A = \mathbb{Z}^2$, we omit it from the notation and simply say that x and y are connected. The cluster of a site x in a set A is the set of all sites y for which $\omega \in \{x \xleftrightarrow{A} y\}$. We denote by $\{x \longleftrightarrow \infty\}$ the event that there exists an unbounded sequence $(y_n) \subset \mathbb{Z}^2$ such that x is connected to each one of the y_n 's.

For $n < m \in \mathbb{Z}$, set $\llbracket n, m \rrbracket = \{n, n+1, \dots, m-1, m\}$. A set of this form will be called an interval of \mathbb{Z} and its diameter is defined to be equal to $m - n$. For $n \geq 1$ and $x \in \mathbb{Z}^2$, set $B_n(x) := x + \llbracket -n, n \rrbracket^2$, the box of size n centred at x . Define $\mathcal{A}_n(x)$ to be the event that there exists an open circuit in $B_{2n-1}(x)$ surrounding $B_n(x)$, *i.e.* that there exists a path $v_0 \sim v_1 \sim \dots \sim v_k = v_0$ such that:

- For all $0 \leq i \leq k$, v_i belongs to $B_{2n-1}(x) \setminus B_n(x)$;
- For all $0 \leq i < k$, $\omega(\{v_i, v_{i+1}\}) = 1$;
- The winding number of the path around x is non-zero.

If $x = 0$, we simply write \mathcal{A}_n instead of $\mathcal{A}_n(0)$.

In what follows, we denote by c a generic strictly positive constant whose value may change at each appearance. A numbered constant such as c_1, c_2, \dots will also have their value fixed at its first appearance.

1.4 Summary of the proof

Let us start by recalling the results of [11] and comparing them to ours. The problem analysed in [11] differs from ours in the order of quantifiers (and also in the choice of the two-dimensional lattice). Our result is that even if the “strong columns” are rare and just slightly strong, they still allow percolation. The result of [11] is that even if the “weak columns” are very weak, if they are sufficiently rare they do not disrupt percolation.

Our proof strategy is to reduce our problem to that of [11] using a one-step renormalisation procedure. This means that we find some n such that columns of width n are “good” with high probability, and inside each good column, each $n \times n$ block is good with high probability, while inside a bad column, each block is good with probability bigger than some constant independent of n . This will allow us to show that our renormalised model stochastically dominates that of [11], and hence percolates.

The choice of n is probably the interesting part in the procedure and requires some knowledge of near-critical percolation. It would be interesting to generalise our results to 3 dimensions, but our understanding of near-critical 3-dimensional percolation currently falls short of what is needed for the result. On the other hand, the results of [11], which are strictly 2-dimensional, are not necessary in the 3-dimensional case. It is only the near-critical behaviour that is missing.

We will now describe the renormalisation procedure, and then return to [11] and state their result in details. The first step is to compare $\mathbb{P}_{p,q}^\Lambda$ for Λ without big gaps to near-critical percolation. More precisely, given an integer $k \geq 1$, a subset $\Lambda \subset \mathbb{Z}$ is called *k-syndetic* if it intersects all intervals of \mathbb{Z} having diameter k . The following proposition shows that, starting from critical percolation, the effect of enhancing the parameter on $E_{\text{vert}}(\Lambda \times \mathbb{Z})$ for a *k-syndetic* set Λ is comparable to the effect of performing a certain homogeneous sprinkling.

Proposition 2. *Let $\varepsilon \in (0, 1/2)$. There exists $c_1 > 0$ such that for any k large enough (depending on ε),*

$$\mathbb{P}_{p_c + \varepsilon, p_c}^\Lambda(\mathcal{A}_n) \geq \mathbb{P}_{p_c + k^{-c_1}}(\mathcal{A}_n) \quad (1)$$

for any *k-syndetic* Λ and any $n \geq k$.

The proof is based on some quantitative estimates for non-local and non-translation invariant versions of enhancements. Local and translation invariant enhancements were studied by Aizenman and Grimmett in [1].

Let $\varepsilon \in (0, 1/2)$ be fixed. We wish to prove that for any $c_2 > 0$ and $\delta > 0$, there exists n large enough such that for any $(c_2 \log n)$ -syndetic set Λ ,

$$\mathbb{P}_{p_c + \varepsilon, p_c}^\Lambda(\mathcal{A}_n) \geq 1 - \delta. \quad (2)$$

In order to do that, we invoke general statements coming from the theory of near-critical percolation to prove the following proposition, which together with Proposition 2, implies (2).

Proposition 3. *For any $c_3 > 0$, we have*

$$\lim_{n \rightarrow \infty} \mathbb{P}_{p_c + (\log n)^{-c_3}}(\mathcal{A}_n) = 1.$$

We now return to the results of [11]. As already mentioned, they are on a different 2-dimensional (directed) lattice which we describe next.

Let ζ denote the lattice with sites given by $V(\zeta) = \{x = (x_1, x_2) \in \mathbb{Z} \times \mathbb{Z} : x_1 + x_2 \text{ is even}\}$ and oriented edges $E(\zeta) = \{[x, y] \subset V(\zeta) : y_1 - x_1 = 1 \text{ and } |x_2 - y_2| = 1\}$. Note that only edges oriented in the *north-east* or *south-east* direction are allowed. As before we denote $x \sim y$ if $[x, y] \in E(\zeta)$.

A *column* is a set of the type $\tilde{c}(i) = \{(i, j) \in V(\zeta); j \in \mathbb{Z}\}$. Fix p_B, p_G and ρ' in the interval $(0, 1)$. Let $\tilde{\Lambda} \subset \mathbb{Z}$ be a random set such that the events $\{i \in \tilde{\Lambda}\}$ are i.i.d. with probability ρ' and declare the column $\tilde{c}(i)$ to be *good* if $i \in \tilde{\Lambda}$. Columns that are not good are called *bad* columns. Conditionally on the state of the columns, we then declare each site in bad columns to be occupied or vacant with probability p_B and $1 - p_B$, respectively. Similarly we declare each site in a good column to be occupied or vacant with probability p_G and $1 - p_G$, respectively. The state of each site is decided independently of the others. We denote by $\tilde{\mathbb{P}}_{p_B, p_G}^{\tilde{\Lambda}}$ the law in $\{0, 1\}^{V(\zeta)}$ conditional on the state of the columns $\tilde{\Lambda}$.

For a configuration $\omega \in \{0, 1\}^{V(\zeta)}$, we say that the origin belongs to an infinite connected component for oriented percolation in ζ if there exists an infinite sequence $0 = v_0 \sim v_1 \sim v_2 \sim \dots$ with $v_i \neq v_j$ when $i \neq j$ and such that $\omega(v_i) = 1$ for all $i \geq 0$. The critical percolation on ζ will be denoted by $p_c(\zeta)$.

We are ready to state the main input to our renormalisation scheme.

Theorem 4 (Kesten, Sidoravicius, Vares [11]). *Assume that $p_B > 0$ and that $p_G > p_c(\zeta)$. Then, there exists a $\rho' < 1$ such that, for almost all realizations of $\tilde{\Lambda}$,*

$$\tilde{\mathbb{P}}_{p_B, p_G}^{\tilde{\Lambda}}(0 \text{ belongs to an infinite oriented connected component}) > 0.$$

We finally get to the renormalisation scheme. Let n be an integer. We think about $2n\triangleleft$ as a subset of \mathbb{Z}^2 and examine the events $\mathcal{A}_n(2nv)$ for $v \in \triangleleft$ (see Figure 3 below). We say that the i^{th} column of \triangleleft is *good* if Λ intersects every subinterval of $\llbracket 2n(i-1), 2n(i+1) \rrbracket$ that has diameter $\lceil \frac{2}{p} \log(2n) \rceil$. Otherwise, the column is said to be *bad*.

Now, $v \in \triangleleft$ is said to be *occupied* if $\mathcal{A}_n(2nv)$ occurs. On the one hand, for v in a bad column, classic crossing estimates at criticality imply that the probability of such v being occupied is larger than some constant $c > 0$ independent of n . On the other hand, for v in a good column, (2) implies that the probability of being occupied can be made as close to 1 as we wish, provided that n is chosen large. Denote $X(v) = \mathbf{1}[\mathcal{A}_n(2nv)]$ for brevity.

Note that $X(v)$ and $X(w)$ are not independent if v and w are neighbours in \triangleleft . They are only 1-dependent i.e. each $X(v)$ is independent of $\{X(w) : |v-w| > 2\}$. Similarly, the events that columns i and $i+1$ are good are not independent. Nevertheless, one may compare these 1-dependent events with independent percolation using standard methods such as Liggett-Schonmann-Stacey [13].

The renormalisation scheme is now clear: By choosing n large enough, one may guarantee that each column is good with probability close to 1, and that every vertex in a good column is occupied with good probability, move from 1-independent events to truly independent events, and then apply Theorem 4. The details occupy the remainder of the paper.

2 Crossing estimates for k -syndetic sets

In this section we prove Proposition 2. We start with the approach of Aizenman and Grimmett [1] which allows to reduce the problem to a problem about comparison of pivotality probabilities. In other words, to reduce Proposition 2 to the following.

Proposition 5. *Let $\varepsilon \in (0, 1/2)$. There exists $c_4 > 0$ such that for any k large enough, any k -syndetic set $\Lambda \subset \mathbb{Z}$, any $n \geq k$ and any $(p, q) \in [p_c, p_c + \varepsilon] \times [p_c - k^{-2}, p_c + k^{-2}]$,*

$$\frac{\partial}{\partial q} \mathbb{P}_{p,q}^\Lambda(\mathcal{A}_n) \leq k^{c_4} \cdot \frac{\partial}{\partial p} \mathbb{P}_{p,q}^\Lambda(\mathcal{A}_n). \quad (3)$$

Before proving this result, let us show how it implies Proposition 2 (as mentioned above, our argument is similar to the one in [1]).

Proof of Proposition 2. Choose $c_1 > \max\{c_4, 2\}$ and let k be large enough so that $k^{-c_1} < \min\{k^{-2}, \varepsilon/(2k^{c_4})\}$ and that the previous proposition applies. Let Λ be a k -syndetic set and $n \geq k$. For any $t \in [0, 1]$, let us define

$$p(t) = p_c + (1-t)k^{-c_1} + t\varepsilon \quad \text{and} \quad q(t) = p_c + (1-t)k^{-c_1}.$$

With the notation $f(p, q) := \mathbb{P}_{p,q}^\Lambda(\mathcal{A}_n)$, which is a polynomial in p and q and in particular differentiable, we find

$$\frac{d}{dt} f(p(t), q(t)) = p'(t) \frac{\partial}{\partial p} f(p(t), q(t)) + q'(t) \frac{\partial}{\partial q} f(p(t), q(t)) \quad (4)$$

$$= (-k^{-c_1} + \varepsilon) \frac{\partial}{\partial p} f(p(t), q(t)) - k^{-c_1} \frac{\partial}{\partial q} f(p(t), q(t)). \quad (5)$$

Since $(p(t), q(t)) \in [p_c, p_c + \varepsilon] \times [p_c, p_c + k^{-2}]$ for all $t \in [0, 1]$, Proposition 5 implies

$$\frac{d}{dt} f(p(t), q(t)) \geq (-k^{-c_1} + \varepsilon - k^{c_4 - c_1}) \frac{\partial}{\partial p} f(p(t), q(t)) \geq 0$$

from which we conclude $f(p(0), q(0)) \leq f(p(1), q(1))$, a fact which gives (1). \square

We now focus on the proof of Proposition 5.

Proof of Proposition 5. Let Λ be k -syndetic for some $k \geq 100$ and $(p, q) \in [p_c, p_c + \varepsilon] \times [p_c - k^{-2}, p_c + k^{-2}]$.

Define E to be the set of edges of \mathbb{Z}^2 with both endpoints in $B_{2n-1} \setminus B_n$. Define also $F = E \cap E_{\text{vert}}(\Lambda \times \mathbb{Z})$. For an edge $e = \{x, y\} \in E \setminus F$, let $f(e) \in F$ be a minimiser of the $\|\cdot\|_1$ -distance between $\{x, y\}$ and F . Note that $f(e)$ may not be defined uniquely (there may be up to six such edges). In case there is more than one choice for $f(e)$, select one of them according to some arbitrary rule. For $f \in F$, let $E(f) = \{e \in E \setminus F : f(e) = f\}$. Russo's Formula (see [6, Section 2.4]) implies that

$$\frac{\partial}{\partial q} \mathbb{P}_{p,q}^\Lambda(\mathcal{A}_n) = \sum_{e \in E \setminus F} \mathbb{P}_{p,q}^\Lambda(e \text{ is pivotal for } \mathcal{A}_n) = \sum_{f \in F} \sum_{e \in E(f)} \mathbb{P}_{p,q}^\Lambda(e \text{ is pivotal for } \mathcal{A}_n).$$

Now, the fact that Λ is k -syndetic implies that $\text{card}(E(f)) \leq 10k$. If one assumes that there exists $c_5 > 0$ such that for any $f \in F$ and $e \in E(f)$,

$$\mathbb{P}_{p,q}^\Lambda(e \text{ is pivotal for } \mathcal{A}_n) \leq k^{c_5} \cdot \mathbb{P}_{p,q}^\Lambda(f \text{ is pivotal for } \mathcal{A}_n), \quad (6)$$

then we may deduce that for k large enough,

$$\begin{aligned} \frac{\partial}{\partial q} \mathbb{P}_{p,q}^\Lambda(\mathcal{A}_n) &\leq \sum_{f \in F} \text{card}(E(f)) \cdot \max \{ \mathbb{P}_{p,q}^\Lambda(e \text{ is pivotal for } \mathcal{A}_n) : e \in E(f) \} \\ &\leq 10k \cdot \sum_{f \in F} \max \{ \mathbb{P}_{p,q}^\Lambda(e \text{ is pivotal for } \mathcal{A}_n) : e \in E(f) \} \\ &\leq 10k \cdot k^{c_5} \sum_{f \in F} \mathbb{P}_{p,q}^\Lambda(f \text{ is pivotal for } \mathcal{A}_n) = 10k^{c_5+1} \frac{\partial}{\partial p} \mathbb{P}_{p,q}^\Lambda(\mathcal{A}_n), \end{aligned}$$

where we used Russo's Formula in the last equality. This implies the claim with $c_4 > c_5 + 1$ and k large enough.

We therefore focus on the proof of (6). Fix $f = \{u, v\} \in F$ and $e = \{x, y\} \in E(f)$. By definition of $E(f)$, there exist z and $\ell \leq k$ such that $B := B_\ell(z)$ satisfies

- $B \subset B_{2n-1} \setminus B_n$,
- f has both endpoints in ∂B ,
- e has both endpoints in B ,
- $\Lambda \times \mathbb{Z}$ does not intersect $B \setminus \partial B$.

The proof is going to be based on surgery in the box B (and its immediate neighbourhood). For $a, b \in \partial B$, let $\mathcal{Q}_{a,b}$ be the event that there is an open path γ from a to b in $B_{2n-1} \setminus (B_n \cup B)$ which surrounds B_n , or to be more precise, that can be completed to a path surrounding B_n by adding a path from a to b contained in B . Let also

$$\mathcal{P}_{a,b} = \{e \text{ is pivotal for } \mathcal{A}_n\} \cap \mathcal{Q}_{a,b}, \quad (7)$$

$$\mathcal{G}_{a,b} = \{f \text{ is pivotal for } a \overset{B}{\longleftrightarrow} b\} \cap \{a \leftrightarrow \mathbb{Z}^2 \setminus B\} \cap \{b \leftrightarrow \mathbb{Z}^2 \setminus B\}. \quad (8)$$

Let $C_{a,b} = C_{a,b}(\omega)$ be the union of the clusters of a and b in B for the configuration ω and also denote

$$E_{a,b} = \{ \{u, v\} \in E(\mathbb{Z}^2) : u = a \text{ or } b, \text{ and } v \notin B \}.$$

For a pair $(\omega, \xi) \in \{0, 1\}^{E(\mathbb{Z}^2)} \times \{0, 1\}^{E(\mathbb{Z}^2)}$, let $\Phi(\omega, \xi)$ be defined as follows: For $e' \in E(\mathbb{Z}^2)$, set

$$\Phi(\omega, \xi)(e') := \begin{cases} 0 & \text{if } e' = e \text{ and } e \notin C_{a,b}(\xi), \\ \xi(e') & \text{if } e' \notin E_{a,b} \text{ and } e' \text{ has at least one endpoint in } C_{a,b}(\xi), \\ \omega(e') & \text{otherwise.} \end{cases} \quad (9)$$

We now claim that $\Phi(\omega, \xi) \in \{f \text{ is pivotal for } \mathcal{A}_n\}$ for $(\omega, \xi) \in \mathcal{P}_{a,b} \times \mathcal{G}_{a,b}$. To see this, first note that when f is open in ξ , $\Phi(\omega, \xi)$ must contain an open circuit in B_{2n-1} that surrounds B_n . In fact this circuit can be taken as the union of the connection between a and b outside B from ω and the connection inside B from ξ . To see that, if f is closed, there is no such open circuit note that, because e is pivotal for \mathcal{A}_n in ω , after e is closed, ω no longer contains an open path surrounding B_n . Superimposing $C_{a,b}$ from ξ over ω does not change this because $C_{a,b}$ comes with all the closed edges that surround it apart from the ones in $E_{a,b}$. So the only open path it could possibly add inside B is from a to b . However, recalling that f is pivotal for $\{a \xleftrightarrow{B} b\}$ for ξ , there can be no such path when f is closed. As a consequence,

$$\mathbb{P}_{p,q}^\Lambda(f \text{ is pivotal for } \mathcal{A}_n) \geq \mathbb{P}_{p,q}^\Lambda(\Phi(\mathcal{P}_{a,b} \times \mathcal{G}_{a,b})) \geq (1-p)\mathbb{P}_{p,q}^\Lambda(\mathcal{P}_{a,b})\mathbb{P}_{p,q}^\Lambda(\mathcal{G}_{a,b}), \quad (10)$$

where $1-p$ accounts for the eventual price for closing e when necessary, and the inequality is due to the fact that the law of $\Phi(\omega, \xi)$ in $E(\mathbb{Z}^2) \setminus \{e\}$ coincides with $\mathbb{P}_{p,q}^\Lambda$ (since $C_{a,b}(\xi) = C$ is measurable in terms of the states of edges with one endpoint in C).

On the one hand, if e is pivotal for \mathcal{A}_n , one of the $\mathcal{P}_{a,b}$ must occur, allowing us to choose a and b so that

$$\mathbb{P}_{p,q}^\Lambda(\mathcal{P}_{a,b}) \geq \frac{1}{(8k)^2} \mathbb{P}_{p,q}^\Lambda(e \text{ pivotal for } \mathcal{A}_n).$$

(we use here that $|\partial B| \leq 8k$). On the other hand, Lemma 6 below implies that $\mathbb{P}_{p,q}^\Lambda(\mathcal{G}_{a,b}) \geq k^{-c_6}$. Putting these two inequalities in (10) implies (6) for k large enough. This concludes the proof. \square

In order to have a complete proof of Proposition 5, we only need to prove the following lemma. Recall the definition of $\mathcal{G}_{a,b}$ in (8).

Lemma 6. *There exists $c_6 > 0$ such that for any $k \geq 2$, if $\Lambda \times \mathbb{Z}$ does not intersect $B_\ell(z) \setminus \partial B_\ell(z)$ with $\ell \leq k$ and $z \in \mathbb{Z}^2$, then*

$$\mathbb{P}_{p,q}^\Lambda(\mathcal{G}_{a,b}) \geq k^{-c_6}$$

for any $a, b \in \partial B_\ell(z)$, any $f \in E_{\text{vert}}(\Lambda \times \mathbb{Z}) \cap \partial B_\ell(z)$ and any $(p, q) \in [p_c, p_c + \varepsilon] \times [p_c - k^{-2}, p_c + k^{-2}]$.

Some readers may want to skip the proof of this lemma since it relies on very standard arguments involving the Russo-Seymour-Welsh theory at criticality [18, 20] (see also [6, Section 11.7] for a comprehensive exposition). For completeness, we include a proof here.

We need the notion of dual configuration. Let $(\mathbb{Z}^2)^* = (\frac{1}{2}, \frac{1}{2}) + \mathbb{Z}^2$. Vertices and edges of $(\mathbb{Z}^2)^*$ are called dual vertices and dual edges. Each edge e of \mathbb{Z}^2 corresponds to a dual edge e^* of $(\mathbb{Z}^2)^*$ that it intersects in its middle. As before, we write $u \sim v$ if u and v are endpoints of a dual edge. Also define B^* to be the subset of $(\mathbb{Z}^2)^*$ of endpoints of dual edges of the form $\{x, y\}^*$ with $x, y \in B$.

Define the dual configuration $\omega^* \in \{0, 1\}^{E((\mathbb{Z}^2)^*)}$ of $\omega \in \{0, 1\}^{E(\mathbb{Z}^2)}$ by $\omega^*(e^*) = 1 - \omega(e)$. Two dual vertices u and v of $(\mathbb{Z}^2)^*$ are *dual-connected* in $V \subset (\mathbb{Z}^2)^*$ if there exists

$u = v_0 \sim \dots \sim v_k = v$ such that $v_i \in V$ for every $0 \leq i \leq k$ and $\omega^*({v_i, v_{i+1}}) = 1$ for every $0 \leq i < k$. We denote this event by $u \xleftrightarrow{*,V} v$.

Proof of Lemma 6. For simplicity, assume that $k \geq 200$ is divisible by 50. Assume that $B := \llbracket -k, k \rrbracket^2$. Also consider $\widehat{B} = \llbracket -k + 100, k - 100 \rrbracket^2$ and $\widetilde{B} = \llbracket -k/2, k/2 \rrbracket^2$. Set $f = \{c, d\}$, so that a, b, c and d are all lying on ∂B . Assume that a, b, c and d are distinct vertices (the proof is similar in the other cases).

First, observe that $E_{\text{vert}}(\Lambda \times \mathbb{Z})$ does not intersect $B \setminus \partial B$. Therefore, the fact that $|q - p_c| \leq k^{-2}$ and that B has at most ck^2 edges, guarantees that there exists $c_7 > 0$ (independent of k) such that for any configuration ω in $B \setminus \partial B$, $\mathbb{P}_{p,q}^\Lambda(\omega) \leq c_7 \mathbb{P}_{p_c}(\omega)$. So that we may focus on $p = q = p_c$.

Partition each of the sides of $\partial \widetilde{B}$ into 25 intervals of length $k/50$. From this collection of intervals, select 11 intervals I_1, \dots, I_{11} arranged in increasing index order counter-clockwise along $\partial \widetilde{B}$ with the following properties:

- intervals are distant of $k/50$ from each other and from the corners of \widetilde{B} ,
- the intervals I_1, I_2 and I_3 are on the top (respectively left, right, bottom) side of $\partial \widetilde{B}$ if a is on the top (respectively left, right, bottom) side of ∂B ,
- the intervals I_4, \dots, I_8 are on the left (respectively right) side of $\partial \widetilde{B}$ if c and d are on the left (respectively right) side of ∂B ,
- the intervals I_9, I_{10} and I_{11} are on the top (respectively left, right, bottom) side of $\partial \widetilde{B}$ if b is on the top (respectively left, right, bottom) side of ∂B .

Define $\overline{C}_1, \overline{C}_2$ and \overline{C}_3 to be the cones from a with basis I_1, I_2 and I_3 . Similarly, define $\overline{C}_4, \dots, \overline{C}_8$ from c with basis I_4, \dots, I_8 and $\overline{C}_9, \overline{C}_{10}$ and \overline{C}_{11} from b to I_9, I_{10} and I_{11} . We set $C_i = \overline{C}_i \cap \widehat{B} \setminus (\widetilde{B} \setminus \partial \widetilde{B})$ and $C_i^* = \overline{C}_i \cap \widehat{B}^* \setminus (\widetilde{B}^* \setminus \partial \widetilde{B}^*)$ (see Figure 2).

For $i = 1, \dots, 11$, select x_i and x_i^* in $C_i \cap \partial \widehat{B}$ and $C_i^* \cap \partial \widehat{B}^*$ respectively. Also select z_i and z_i^* in $C_i \cap \partial \widetilde{B}$ and $C_i^* \cap \partial \widetilde{B}^*$ respectively. Define

$$\mathcal{G}_i = \{x_i \xleftrightarrow{C_i} z_i\} \text{ and } \mathcal{G}_i^* = \{x_i^* \xleftrightarrow{*,C_i^*} z_i^*\}.$$

Also set

$$\mathcal{H} = \{z_1^* \xleftrightarrow{*,\widetilde{B}^*} z_6^*, z_2 \xleftrightarrow{\widetilde{B}} z_5, z_3^* \xleftrightarrow{*,\widetilde{B}^*} z_4^*, z_6^* \xleftrightarrow{*,\widetilde{B}^*} z_{11}^*, z_7 \xleftrightarrow{\widetilde{B}} z_{10}, z_8^* \xleftrightarrow{*,\widetilde{B}^*} z_9^*\}.$$

By a standard application of the Russo-Seymour-Welsh arguments, there exists $c_8 > 0$ (not depending on the choice of the I_i, x_i, z_i , etc.) such that for k large enough

$$\mathbb{P}_{p_c}(\mathcal{G}_i) \geq k^{-c_8}, \quad \mathbb{P}_{p_c}(\mathcal{G}_i^*) \geq k^{-c_8}.$$

Now, the sites z_i and z_i^* are all well separated so that one may again employ Russo-Seymour-Welsh arguments in order to check that

$$\mathbb{P}_{p_c}(\mathcal{H}) \geq k^{-c_8}$$

(where the value constant c_8 may need to be modified).

We deduce that

$$\mathbb{P}_{p_c}(\mathcal{H} \cap \mathcal{G}_1^* \cap \mathcal{G}_2 \cap \mathcal{G}_3^* \cap \mathcal{G}_4^* \cap \mathcal{G}_5 \cap \mathcal{G}_6^* \cap \mathcal{G}_7 \cap \mathcal{G}_8^* \cap \mathcal{G}_9^* \cap \mathcal{G}_{10} \cap \mathcal{G}_{11}^*) \geq k^{-12c_8}.$$

One concludes the proof by noticing that a local surgery near a, b, c and d implies the existence of $c_9 > 0$ such that

$$\mathbb{P}_{p_c}(\mathcal{A}_{a,b}) \geq c_9 \mathbb{P}_{p_c}(\mathcal{H} \cap \mathcal{G}_1^* \cap \mathcal{G}_2 \cap \mathcal{G}_3^* \cap \mathcal{G}_4^* \cap \mathcal{G}_5 \cap \mathcal{G}_6^* \cap \mathcal{G}_7 \cap \mathcal{G}_8^* \cap \mathcal{G}_9^* \cap \mathcal{G}_{10} \cap \mathcal{G}_{11}^*) \geq c_9 k^{-12c_8}.$$

The proof follows by choosing $c_6 > 0$ large enough. \square

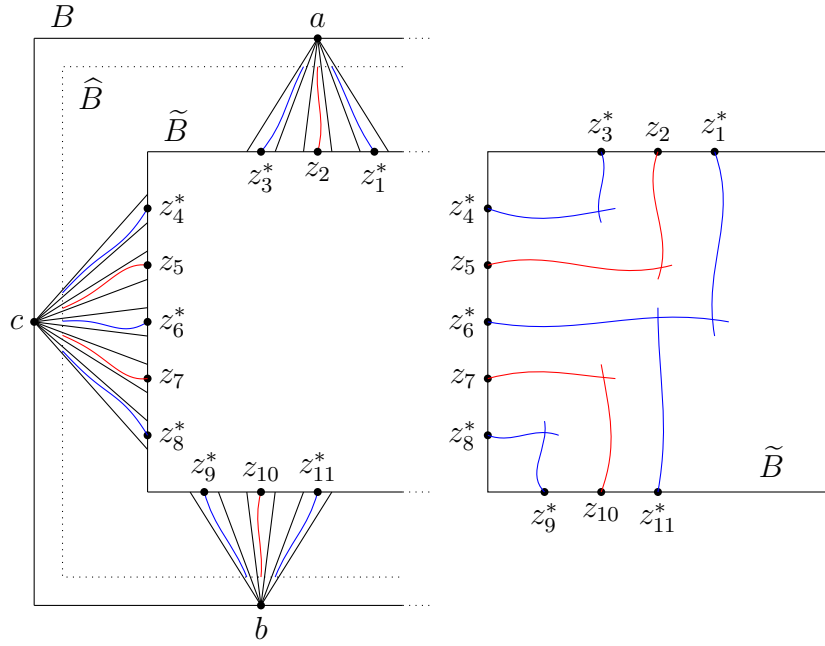


FIGURE 1. Dual paths are represented in blue and primal paths in red. On the left we show the events \mathcal{G}_i (respectively \mathcal{G}_i^*) whose occurrence is assured by the existence of the blue (respectively red) paths inside the cones C_i (respectively C_i^*). On the right we show a simple way of constructing the event \mathcal{H} once the z_i^* and z_i are well separated.

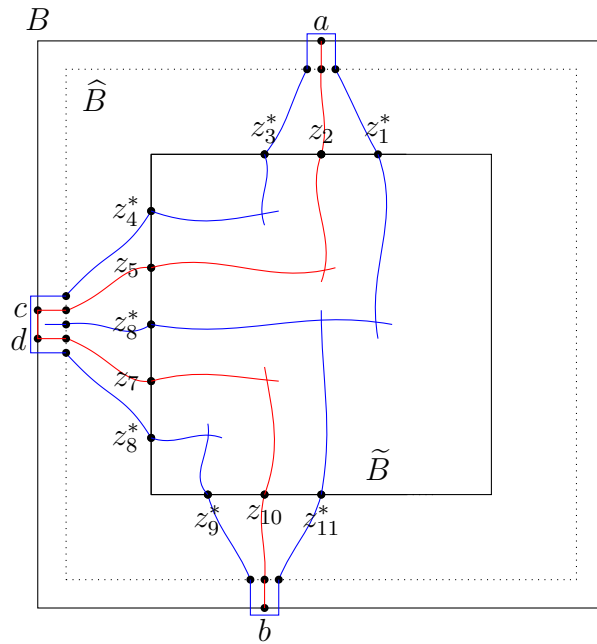


FIGURE 2. The event \mathcal{H} , together with the appropriate \mathcal{G}_i 's and \mathcal{G}_i^* 's and a local surgery in the neighbourhoods of a , b and c , implies in the occurrence of the event $\mathcal{A}_{a,b}$.

Remark. Under the conjecture that 2-dimensional percolation is conformally invariant, the best constant in Lemma 6 may be calculated. The worst case is when a and b are both in the same corner, and then one gets a half-plane 5-arm exponent at f and a quarter-plane 5-arm exponent at the corner (which is twice the half-plane exponent, by conformal invariance). Using the determination of these exponents in [19] gives $c_6 = 15$.

3 Input from near-critical percolation

In this section, we recall general facts on planar Bernoulli percolation which imply Proposition 3 (recall that it claimed that $\mathbb{P}_{p_c + (\log n)^{-c}}(\mathcal{A}_n) \rightarrow 1$ as $n \rightarrow \infty$). Let $1 \gg \varepsilon > 0$. For $p > p_c$, introduce

$$L_\varepsilon(p) := \min \left\{ n \geq 1; \mathbb{P}_p \left(\{0\} \times \llbracket 0, n \rrbracket \xleftrightarrow{\llbracket 0, 2n \rrbracket \times \llbracket 1, n-1 \rrbracket} \{2n\} \times \llbracket 0, n \rrbracket \right) \geq 1 - \varepsilon \right\}.$$

This quantity, sometimes called *characteristic* or *correlation* length, was proved [10, 16] to satisfy the following facts:

P1 (Probability for hard-way crossings). For any $p > p_c$ and $n \geq L_\varepsilon(p)$,

$$\mathbb{P}_p \left(\{0\} \times \llbracket 0, n \rrbracket \xleftrightarrow{\llbracket 0, 2n \rrbracket \times \llbracket 1, n-1 \rrbracket} \{2n\} \times \llbracket 0, n \rrbracket \right) \geq 1 - \varepsilon. \quad (11)$$

P2 (Probability for 4 arms). There exist $c_{10}, c_{11} > 0$ such that for any $p \in (p_c, 1 - \varepsilon)$,

$$c_{10} \leq (p - p_c) L_\varepsilon(p)^2 \mathbb{P}_{p_c}(\mathcal{E}_4(L_\varepsilon(p))) \leq c_{11}, \quad (12)$$

where (below, x is a fixed neighbour of the origin)

$$\mathcal{E}_4(n) := \{0 \leftrightarrow \partial B_n\} \cap \{x \leftrightarrow \partial B_n\} \cap \{0 \xleftrightarrow{B_n} x\}^c.$$

Let us recall the following fact, of which we provide a sketch of proof for completeness.

Lemma 7. *There exists $c_{12} < 2$ such that for any n large enough, $\mathbb{P}_{p_c}(\mathcal{E}_4(n)) \geq n^{-c_{12}}$.*

Sketch of proof. Let

$$\mathcal{E}_5(n) := \mathcal{E}_4(n) \cap \{\text{there exist two open paths from } 0 \text{ to } \partial B_n \text{ intersecting at } 0 \text{ only}\}.$$

It was proved in [12, Lemma 5] that there exists $c_{13} > 0$ such that

$$\mathbb{P}_{p_c}[\mathcal{E}_5(n)] \geq \frac{c_{13}}{n^2}. \quad (13)$$

Since the occurrence of $\mathcal{E}_5(n)$ implies the disjoint occurrence (see [6, Section 2.3] for a definition of disjoint occurrence) of $\mathcal{E}_4(n)$ and $\{0 \leftrightarrow \partial B_n\}$, Reimer's inequality [17] implies that

$$\mathbb{P}_{p_c}(\mathcal{E}_5(n)) \leq \mathbb{P}_{p_c}(\mathcal{E}_4(n)) \cdot \mathbb{P}_{p_c}(0 \leftrightarrow \partial B_n). \quad (14)$$

Now, a simple application of the Russo-Seymour-Welsh theory [18, 20] implies that

$$\mathbb{P}_{p_c}(0 \leftrightarrow \partial B_n) \leq n^{-c_{14}}$$

for all $n \geq 1$. Plugging this estimate in (14) and using (13) implies the claim readily. \square

Proof of Proposition 3. Fix $c_3 > 0$ and $\varepsilon > 0$. Lemma 7 and (12) show that

$$L_\varepsilon(p) \leq \left(\frac{c_{11}}{p - p_c} \right)^{1/(2-c_{12})}.$$

Thus, there exists $n_0 = n_0(\varepsilon) > 0$ such that for all $n > n_0$,

$$L_\varepsilon(p_c + (\log n)^{-c_3}) \leq (c_{15} \log n)^{c_3/(2-c_{12})} \leq n.$$

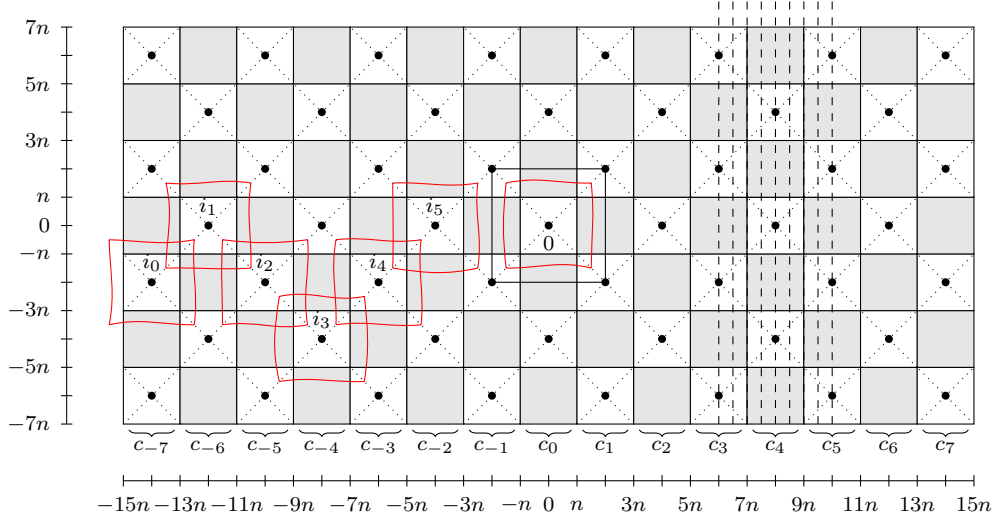


FIGURE 3. The lattice L is composed of the sites on the centre of the white boxes and the edges represented by the dotted segments connecting them. We illustrate the event $\mathcal{A}_n(0)$ that the origin is occupied, with the red path in the annulus $B_{2n-1} \setminus B_n$. On the left there is an oriented path of occupied sites i_0, \dots, i_5 . The dashed vertical lines around $c_4(n)$ cover the region where one needs to inspect $\Lambda \times \mathbb{Z}$ in order to verify that it is a good column.

As a consequence, (11) implies that for $n \geq n_0$,

$$\mathbb{P}_{p_c + \log n - c_3} \left(\{0\} \times \llbracket 0, n \rrbracket \xleftarrow{\llbracket 0, 2n \rrbracket \times \llbracket 1, n-1 \rrbracket} \{2n\} \times \llbracket 0, n \rrbracket \right) \geq 1 - \varepsilon. \quad (15)$$

Now, assume that the following events occur simultaneously for $i, j \in \{-2, -1, 0, 1\}$,

- $\{in\} \times \llbracket jn, (j+1)n \rrbracket \xleftarrow{\llbracket in, (i+2)n \rrbracket \times \llbracket jn+1, (j+1)n-1 \rrbracket} \{(i+2)n\} \times \llbracket jn, (j+1)n \rrbracket$,
- $\llbracket in, (i+1)n \rrbracket \times \{jn\} \xleftarrow{\llbracket in+1, (i+1)n-1 \rrbracket \times \llbracket jn, (j+2)n \rrbracket} \llbracket in, (i+1)n \rrbracket \times \{(j+2)n\}$.

In such case \mathcal{A}_n occurs. (Note that we have been wasteful in the number of events involved above, some of them are not necessary in order to guarantee the occurrence of \mathcal{A}_n .) Therefore, the FKG inequality combined with (15) implies that for $n \geq n_0$,

$$\mathbb{P}_{p_c + \log n - c_3}(\mathcal{A}_n) \geq (1 - \varepsilon)^{32}$$

which implies the claim readily. \square

4 The renormalisation scheme

Recall that Λ is a random subset of \mathbb{Z} having law ν_ρ under which the events $\{i \in \Lambda\}$ are mutually independent and have probability ρ . Also recall that the i^{th} column of ζ is called good if Λ intersects every subinterval of $\llbracket 2n(i-1), 2n(i+1) \rrbracket$ that has diameter $\lceil \frac{2}{\rho} \log(2n) \rceil$. We start by proving that columns are good with high probability.

Lemma 8. *Let $\rho > 0$. For every $i \in \mathbb{Z}$, $\lim_{n \rightarrow \infty} \nu_\rho(i \text{ is good}) = 1$.*

Proof. For any $x \in \mathbb{Z}$ and $\Lambda \subset \mathbb{Z}$ let

$$\ell(x) = \ell(x, \Lambda) := \inf\{w - x : w \in \Lambda, w > x\}.$$

A calculation gives $\nu_\rho(\ell(x) > k) = (1 - \rho)^k \leq e^{-\rho k}$. It follows that

$$\nu_\rho(\ell(x) > k \text{ for some } x \text{ between } 2i(n-1) \text{ and } 2i(n+1)) \leq 4ne^{-\rho k}.$$

For $k = \lceil \frac{2}{\rho} \log(2n) \rceil$, the right-hand side is at most equal to $1/n$ while the left-hand side contains the event that i is bad. This proves the result. \square

Let us dedicate a paragraph to the nature of the last remaining obstacle. We already established that every column is good with high probability; that in good columns $\mathbb{E}(X(v))$ can be made as close to 1 as we wish; and that in bad columns $\mathbb{E}(X(v)) > c$ (recall that $X(v) = \mathbf{1}[\mathcal{A}_n(2nv)]$). The problem is that the $X(v)$ at different v are not independent, they are only 1-dependent. Now, the boxes in good columns do not pose any problem: By [13] 1-dependent events with sufficiently high probability stochastically dominate independent events with lower probability. It is the boxes in the bad column that we must worry about. Liggett, Schonmann and Stacey give a simple and highly instructive example of 1-dependent events with probability $1/2$ which do not dominate p -independent events, no matter how small p is taken (see the bottom of page 73 in [13]). Hence, if we want to show that the collection of $X(v)$ dominates independent percolation, we need to use some specific property of them. We will use the FKG inequality.

In the rest of this article, we will drop n from the notation. While it is probably true that $\mathbb{E}(X(v) | X(w) \forall w \neq v) > c$, we found it easier to use external randomization. Let us therefore introduce a parameter $\eta \leq \frac{1}{2}$ (to be fixed later) and a family of i.i.d. Bernoulli($1 - \eta$) random variables $(Y(z) : z \in \triangleleft)$ which is independent of everything else. Define $W(z) := X(z)Y(z)$.

Lemma 9. *For any $\Lambda \subset \mathbb{Z}$ and $p, q \in [0, 1]$, the following inequality holds almost surely*

$$\mathbb{P}_{p,q}^\Lambda(W(0) = 1 | W(z), z \in \triangleleft \setminus \{0\}) \geq \eta^4 \mathbb{P}_{p,q}^\Lambda(W(0) = 1).$$

Proof. For simplicity we will remove Λ, p and q from the notation. Let z_1, \dots, z_4 be the 4 neighbours of 0 in \triangleleft (since \triangleleft is directed, we should specify that we mean either $0 \sim z_i$ or $z_i \sim 0$). Let $\zeta \in \{0, 1\}^4$, let $m > 2n$ and let $\xi \in \{0, 1\}^{B_m \setminus B_{2n}}$. We define the event

$$\mathcal{B} = \mathcal{B}_{\zeta, \xi} = \{W(z_i) = \zeta_i \forall i \in \llbracket 1, 4 \rrbracket\} \cap \{\omega(e) = \xi(e) \forall e \in E(B_m \setminus B_{2n})\}$$

where ω is as before the percolation configuration. It is enough to show

$$\mathbb{P}(W(0) = 1 | \mathcal{B}) \geq \eta^4 \mathbb{P}(W(0) = 1) \quad \text{a.s.} \quad \forall \zeta, \xi. \quad (16)$$

Indeed, once (16) is shown, it is possible to add an arbitrary conditioning on $\{Y(v) | v \notin \{0, z_1, \dots, z_4\}\}$ as these are independent of both $W(0)$ and of E . Then taking $m \rightarrow \infty$ would give the statement of the lemma but on a finer σ -field. Integrating would give the exact claim of the lemma.

We thus focus on the proof of (16). Fix some ζ, m and ξ for the rest of the proof. Define

$$\mu(\cdot) = \mathbb{P}(\cdot | \omega(e) = \xi(e) \forall e \in E(B_m \setminus B_{2n})).$$

Define $I(\zeta) = \{i : \zeta_i = 1\}$ and then write

$$\begin{aligned} & \mu(X(0) = 1, W(z_i) = \zeta_i \forall i \in \llbracket 1, 4 \rrbracket) \\ & \geq \mu(X(0) = 1, W(z_i) = \zeta_i, Y(z_i) = \zeta_i \forall i \in \llbracket 1, 4 \rrbracket) \\ & = \mu(X(0) = 1, X(z_i) = 1 \forall i \in I(\zeta), Y(z_i) = \zeta_i \forall i \in \llbracket 1, 4 \rrbracket) \\ & = \mu(X(0) = 1, X(z_i) = 1 \forall i \in I(\zeta)) \cdot \mathbb{P}(Y(z_i) = \zeta_i \forall i \in \llbracket 1, 4 \rrbracket) \\ & \geq \eta^4 \mu(X(0) = 1, X(z_i) = 1 \forall i \in I(\zeta)) \\ & \geq \eta^4 \mu(X(0) = 1, W(z_i) = 1 \forall i \in I(\zeta)). \end{aligned}$$

Noting that

$$\mu(W(z_i) = 1 \forall i \in I(\zeta)) \geq \mu(W(z_i) = \zeta_i \forall i \in \llbracket 1, 4 \rrbracket),$$

one can conclude that

$$\begin{aligned} & \mu(X(0) = 1 | W(z_i) = \zeta_i \forall i \in \llbracket 1, 4 \rrbracket) \\ & \geq \eta^4 \mu(X(0) = 1 | W(z_i) = 1 \forall i \in I(\zeta)) \\ \text{by FKG} \quad & \geq \eta^4 \mu(X(0) = 1) = \eta^4 \mathbb{P}(X(0) = 1). \end{aligned}$$

(It is easy to check that the FKG inequality holds for μ). Since $Y(0)$ is independent of $X(0)$ and $W(z_i)$ for $i = 1, \dots, 4$ (and also after the conditioning on E), we have that

$$\mu(W(0) = 1 | W(z_i) = \zeta_i \forall i \in \llbracket 1, 4 \rrbracket) \geq \eta^4 \mathbb{P}(W(0) = 1).$$

which is equivalent to (16), proving the lemma. \square

We are now ready for:

Proof of Theorem 1. As parameter dependency is a little complicated here, let us start by setting all parameters formally. First choose η (from the definition of Y and W) to be $1/3(1 - p_c(\zeta))$. Next choose

$$p_B = \eta^4(1 - \eta) \inf\{\mathbb{P}_{p_c}(\mathcal{A}_n) : n \geq 1\}$$

which is strictly positive by the Russo-Seymour-Welsh theorem (see [18, 20] again). Define $p_G = 1 - 2\eta$ (so still $p_G > p_c(\zeta)$). Theorem 4 proves the existence of $\rho' < 1$ such that oriented percolation in ζ with density of good lines ρ' and probability p_G and p_B in good and bad lines respectively percolates a.s.

Next use the theorem of Liggett, Schonmann and Stacey [13, Theorem 0.0] to find some σ such that any 1-dependent family of variables $\{X_i : i \in V(\zeta)\}$ with $\mathbb{P}(X_i = 1) > \sigma$ stochastically dominates $(1 - \eta)$ -Bernoulli independent variables. Use the theorem again to find some ρ such that any 1-dependent family of variables $\{G_i : i \in \mathbb{Z}\}$ with $\mathbb{P}(G_i) > \rho$ stochastically dominates ρ' -Bernoulli independent variables.

Finally, we claim that for n sufficiently large (depending on the p from the statement of the theorem), the probability that a column is good is more than ρ , while the probability that $\{X(v) = 1\} = \mathcal{A}_n(2nv)$ occurs in a good column is more than σ . Indeed, the first follows from Lemma 8 while the second follows from (2). Fix n to satisfy this property. Finally, use continuity to choose some $q < p_c$ such that $\mathbb{P}_{p,q}^\Lambda(X(v) = 1) > \sigma$ in any good column.

With all parameters defined, let us start with the columns. The definitions of n and ρ allow to define ρ' -independent variables $\Xi(i)$, depending only on Λ , such that if $\Xi(i) = 1$ then the column $c(i)$ is good. It will be convenient to define, for a vertex v in a column i , $\Xi(v) = \Xi(i)$. Fix one realization of Λ and Ξ .

By the choice of n and q , we know that for every v , $\Xi(v) = 1 \Rightarrow \mathbb{P}_{p,q}^\Lambda(X(v) = 1) > \sigma$. Since these events are 1-dependent, they dominate $(1 - \eta)$ -independent variables. Therefore the variables $\{W(v) : \Xi(v) = 1\}$ dominate $(1 - \eta)^2$ -independent variables. When $\Xi = 0$ we use Lemma 9 and get that, for any realisation of W on the $\{\Xi = 1\}$, $\{W(v) : \Xi(v) = 0\}$ dominates i.i.d. Bernoulli variables with probability

$$\eta^4 \mathbb{P}(W(0) = 1) = \eta^4(1 - \eta) \mathbb{P}(X(0) = 1) \geq p_B.$$

All in all we get that W dominates a family of independent variables with probability p_G where $\Xi = 1$ and p_B where $\Xi = 0$. Denote a realisation of these independent variables by $\Psi = \{\Psi(v) : v \in \zeta\}$.

We get that Ξ and Ψ have exactly the distribution of variables on ζ such that Ξ are ρ' -independent and Ψ have probability p_G where $\Xi = 1$ and p_B where $\Xi = 0$. Hence Theorem 4 applies and we get that the Ψ percolate. Since $X(v) \geq \Psi(v)$ so do the $\{X(v); v \in \zeta\}$. But if $X(v) = 1$, then $\mathcal{A}_n(2nv)$ occurs, and the geometric setup (see Figure 3) requires that these loops connect to one infinite cluster, proving the theorem. \square

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