

Existence of phase transition for percolation using the Gaussian Free Field

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Abstract

In this paper, we prove that Bernoulli percolation on bounded degree graphs with isoperimetric dimension $d > 4$ undergoes a non-trivial phase transition (in the sense that $p_c < 1$). As a corollary, we obtain that the critical point of Bernoulli percolation on infinite quasi-transitive graphs (in particular, Cayley graphs) with super-linear growth is strictly smaller than 1, thus answering a conjecture of Benjamini and Schramm. The proof relies on a new technique consisting in expressing certain functionals of the Gaussian Free Field (GFF) in terms of connectivity probabilities for percolation model in a random environment. Then, we integrate out the randomness in the edge-parameters using a multi-scale decomposition of the GFF. We believe that a similar strategy could lead to proofs of the existence of a phase transition for various other models.

1 Introduction

Motivation. Whether a model undergoes a non-trivial phase transition or not is one of the most fundamental questions in statistical physics. In [Pei36], Peierls introduced a combinatorial technique, known as *Peierls argument*, to prove that the critical temperature of the Ising model is non-zero on \mathbb{Z}^d for $d \geq 2$, thus opening a new realm of questions concerning this important model of ferromagnetism. This argument found many applications to other models, including Potts models and models of random graphs such as Bernoulli percolation or the random-cluster model.

Peierls argument has two drawbacks. First, it often does not apply to continuous spin models, for instance the spin $O(n)$ models. In this case, the technique may sometimes be replaced by two other techniques: *Reflection Positivity* and the *Renormalization Group*. More precisely, Fröhlich, Simon and Spencer proved that the spin $O(n)$ model undergoes a non-trivial order/disorder phase transition on \mathbb{Z}^d with $d \geq 3$ [FSS76] using Reflection Positivity, and Balaban and coauthors (see [Bal99] and references therein) proved delicate properties of the large β regime using the Renormalization Group. Let us mention that in the special case of the XY model (i.e. when $n = 2$), there are special proofs relying on the Coulomb gas [FS82, KK86].

Another problem with Peierls argument is that it requires a precise understanding of so-called *cut sets*, i.e. sets of edges which disconnect certain sets of vertices from infinity. On planar graphs,

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this boils down to the understanding of circuits in the dual graph. On non-planar graphs, the question is much more complex combinatorial problem and it is not completely understood in general.

In this paper, we wish to present a new technique which we believe can be useful to prove the existence of a phase transition for various models. The object of interest of this paper will be Bernoulli percolation.

Main results. Consider a connected graph $G = (V, E)$ with vertex-set V and edge-set E . An edge with endpoints x and y will be denoted by xy . For every $x \in V$, let $d(x)$ be the number of y such that $xy \in E$. We will assume that the graph has *bounded degree*, that is $\sup\{d(x) : x \in V\} < +\infty$.

Bernoulli percolation is a model of a random subgraph of G with vertex-set V . The subgraph is encoded by a function ω from E into $\{0, 1\}$. We use the notation ω_{xy} to denote the value of ω at the edge xy and think of edges xy with $\omega_{xy} = 1$ as being the edges of the subgraph, that are called *open*. Those with $\omega_{xy} = 0$ are called *closed*. We are interested in the connectivity properties of the graph ω . We use the notation $S \longleftrightarrow T$ (resp. $S \longleftrightarrow \infty$) to denote the event that there is an open path in ω connecting a vertex in S to a vertex in T (resp. the event that S intersects an infinite connected component of ω). Also, let $S \nleftrightarrow T$ (resp. $S \nleftrightarrow \infty$) denote the complement of the event $S \longleftrightarrow T$ (resp. $S \longleftrightarrow \infty$).

The Bernoulli bond percolation of parameter $\mathbf{p} = (p_{xy} : xy \in E) \in [0, 1]^E$ is the probability measure on configurations ω for which the variables ω_{xy} form a family of independent Bernoulli random variables of respective parameters p_{xy} . We denote such a measure by $\mathbf{P}_{\mathbf{p}}$ and, when $p_{xy} = p$ for every $xy \in E$, we simply write \mathbf{P}_p . The main question of interest is whether the critical parameter

$$p_c(G) := \inf \{p \in [0, 1] : \mathbf{P}_p(\exists \text{ an infinite connected component in } \omega) > 0\}$$

is strictly smaller than 1 or not. Let us mention that proving $p_c(G) > 0$ is a much simpler task: Peierls argument actually implies $p_c(G) \geq \frac{1}{D}$, where D is the maximum degree of G .

In order to state the main result, we need another notion. Given a graph G , the *simple random walk* is the discrete-time Markov chain $(X_n)_n$ on V moving, at each time step, from its position to one of its neighbors in G chosen uniformly at random. Define the *heat kernel* and the *Green function* by the following formula: for every $x, y \in V$ and $n \geq 0$,

$$p_n(x, y) := \mathbb{P}[X_n = y | X_0 = x] \quad \text{and} \quad \mathbb{G}(x, y) := \frac{1}{d(y)} \sum_{n=0}^{\infty} p_n(x, y).$$

The main object of this paper is the following result.

Theorem 1.1. *Consider a graph G with bounded degree. Assume that there exist real numbers $d > 4$ and $c > 0$ such that*

$$p_n(x, x) \leq \frac{c}{n^{d/2}} \quad \forall x \in V, \forall n \geq 1. \tag{H_d}$$

Then, there exists $p < 1$ such that for every finite set $S \subset V$,

$$\mathbf{P}_p(S \nleftrightarrow \infty) \leq \exp[-\frac{1}{2}\text{cap}(S)], \tag{1.1}$$

where $\text{cap}(S) := \sum_{x \in S} d(x) \mathbb{P}[X_k \notin S \forall k \geq 1 | X_0 = x]$ is the capacity of S . In particular, $p_c(G) < 1$.

Let us mention a few applications of the theorem. We say that a graph G satisfies an *isoperimetric inequality of dimension d* if there exists some constant $c > 0$ such that

$$|\partial K| \geq c|K|^{\frac{d-1}{d}} \quad \text{for all finite } K \subset V. \quad (\text{I}_d)$$

The *isoperimetric dimension* of G , which we denote by $\text{Dim}(G)$, is defined as the supremum over all d such that (I_d) holds. In their celebrated paper [BS96], Benjamini and Schramm asked whether $\text{Dim}(G) > 1$ necessarily implies $p_c(G) < 1$. We give the following partial answer to this question:

Theorem 1.2. *If a bounded-degree graph G satisfies $\text{Dim}(G) > 4$, then $p_c(G) < 1$.*

This theorem follows directly from Theorem 1.1 and a result of Varopoulos [Var85] (see also [MP05] or [LP17, Corollary 6.32] for a proof relying on evolving sets) stating that (I_d) implies (H_d) .

An important application of Theorem 1.2 is the following result answering the first two conjectures of Benjamini and Schramm from [BS96]. The graph G is called *quasi-transitive* if the action of the automorphism group $\text{Aut}(G)$ on V has only finitely many orbits. Typical examples of quasi-transitive graphs to keep in mind are the family of Cayley graphs of finitely generated groups. Let $B_r(x)$ be the ball of radius r centered at x with respect to the graph distance. We say that a graph G has *super-linear growth* if $\limsup \frac{1}{r}|B_r(x)| = +\infty$.

Theorem 1.3. *Let G be a quasi-transitive with super-linear growth, then $p_c(G) < 1$.*

The previous result can be deduced from Theorem 1.2 as follows:

- If $\liminf \frac{1}{r^d}|B_r(x)| < +\infty$ for some $d > 0$, then a result of Trofimov [Tro84] (see also [Woe00, Theorem 5.11]) implies that the graph is quasi-isometric to a Cayley graph with polynomial growth. This fact together with super-linear growth classically implies that $p_c(G) < 1$ (see the next section for details).
- If $\liminf \frac{1}{r^d}|B_r(x)| = +\infty$ for every $d > 0$, then in particular $\liminf \frac{1}{r^d}|B_r(x)| > 0$ for some $d > 4$. Therefore [LMS08, Lemma 7.2] (see also the proof of Corollary 7.3 of the same paper or [CSC93] for the special case of Cayley graphs) implies that the graph satisfies (I_d) , which in turn gives that $p_c(G) < 1$ by Theorem 1.2.

All the results in this paper can be extended to site percolation, finite dependent percolation and random-cluster models via classical comparisons, see respectively [GS98], [LSS97] and [Gri06]. The coupling between random-cluster models and the Ising/Potts model implies that the results translate into results on the latter as well.

Existing results. Our results should be put in context with the previous partial results regarding the Benjamini-Schramm questions.

As already mentioned, Peierls proved [Pei36] that the Ising phase transition is non-trivial for \mathbb{Z}^2 through bounding the number of cut sets of specific size disconnecting a vertex from infinity. The existence of a non-trivial phase transition for the Ising model implies, via monotonicity arguments, the existence of a non-trivial phase transition for Bernoulli percolation on \mathbb{Z}^d , $d \geq 2$. See also Lebowitz and Mazel [LM98] and Balister and Bollobás [BB07] for estimates on the number of cut sets of \mathbb{Z}^d . The cut set method was extended to Cayley graphs of finitely presented groups by Babson and Benjamini [BB99] (see also the work of Timar [Tim07]).

Say that a graph G has *polynomial growth* if $\limsup \frac{1}{r^d} |B_r(x)| < +\infty$ for some $d > 0$. As a consequence of Gromov’s celebrated theorem [Gro81], every infinite finitely generated group of polynomial growth, which is not virtually \mathbb{Z} , contains a subgroup isomorphic to \mathbb{Z}^2 [LP17, Proposition 7.18]. Hence, there exists a Cayley graph of such groups that has a subgraph isomorphic to \mathbb{Z}^2 . Since the property that $p_c(G) < 1$ is stable under quasi-isometries [LP17, Theorem 7.15], all the Cayley graphs of such groups have non-trivial phase transitions. This method was also used by Muchnik and Pak in [MP01] to prove $p_c(G) < 1$ for Grigorchuk groups which are a class of intermediate growth groups.

Lyons has proved [Lyo95] that every Cayley graph of *exponential growth* (i.e. satisfying that $\liminf \frac{1}{r} \log |B_r(x)| > 0$) has a non-trivial phase transition. As noted in [LP17, Page 264], the fact that $p_c(G) < 1$ for quasi-transitive graphs G with exponential growth can also be easily obtained from the finiteness of the susceptibility for subcritical percolation; see [AB87, Men86, DT16].

In [BPP98] Benjamini, Pemantle and Peres proved another criterion for $p_c(G) < 1$: the existence of an EIT measure for the graph. A measure on self-avoiding paths starting from a fixed vertex is an EIT measure if the number of intersections of two independent paths sampled according to the measure has an exponential tail. In [RY17], by constructing an EIT measure, it is proved that if G is a Cayley graph of indicable group which is not virtually \mathbb{Z} , then $p_c(G) < 1$. Indicable groups not only contain groups of polynomial growth, they also include some groups of intermediate growth. It is worth mentioning that the EIT method proves that for p sufficiently close to 1, there exists a transient infinite connected component almost surely.

The question of $p_c(G) < 1$ has also been approached by analyzing isoperimetric inequalities. Benjamini and Schramm proved in [BS96] that if G satisfies the isoperimetric inequality of “dimension ∞ ”, i.e. if G is non-amenable, then $p_c(G) < 1$. It was proved in [BLPS99] that every unimodular transitive non-amenable graph G has a threshold $\alpha < 1$ such that any automorphism invariant percolation measure on G with density higher than α has an infinite connected component with positive probability. Kozma proved in [Koz05] that planar graphs of polynomial growth with no vertex accumulation points and isoperimetric dimension greater than 1 have non-trivial phase transition.

In [Tei16], Teixeira showed that $p_c(G) < 1$ for graphs G with polynomial growth and isoperimetric dimension greater than 1 for a stronger version of the isoperimetric inequality, called *local isoperimetric inequality*. Teixeira’s proof relies on a clever renormalization argument using in a crucial way the (essential) uniqueness of large connected components in a box. It is important to note that this property is not satisfied under the sole assumption that $\text{Dim}(G) > 1$, as exemplified by the graph made of two copies of \mathbb{Z}^d connected to each other by a single edge between two of their vertices. Also, in contrast to Teixeira’s proof, our method *does not rely on uniqueness*: it works for graphs on which there may be any number of infinite connected components. The method of [Tei16] was further exploited in [CT15] to prove, without invoking Gromov’s theorem, that $p_c(G) < 1$ for quasi-transitive graphs G with super-linear but polynomial growth.

Discussion on the strategy of proof. The proof of Theorem 1.1 relies on a new connection between the Gaussian Free Field (GFF) and Bernoulli percolation. The connection goes through the observation that conditionally on the absolute value of the GFF at every point, the distribution of the signs of the GFF is the one of an Ising model with certain coupling constants. This observation already appeared in a paper of Lupu and Werner [LW16] where the random-current representation of the Ising model was related to the occupation time of a conditioned loop soup. Once the connection between the GFF and the Ising is made, we use the Edwards-Sokal cou-

pling to relate the Ising model to Bernoulli percolation. As a result, it is possible to express the expectations of particular observables of the GFF in terms of the probabilities of connections for a (annealed) percolation model on a random environment (i.e. random edge-parameters). Since the expectation of these observables can be explicitly computed in terms of the Green function and are therefore easy to study, one may bound from below the averaged probability of connections in this percolation model.

The second step of the proof consists in integrating out the randomness of the environment in order to compare the probabilities of connection in the previous model to those in a Bernoulli percolation with fixed edge-parameter. Since the environment is expressed in terms of the GFF, we will proceed step by step using a multi-scale decomposition of the GFF. More precisely, we will write $\psi = \sum_n \psi^n$ where the ψ^n are independent Gaussian fields with finite-range correlations. We will then remove the ψ^n one by one, increasing by an “independent” edge-parameter q in order to guarantee that the probabilities of connection keep increasing. At the end of the process, the randomness due to the ψ^n (and therefore to ψ) will have completely disappeared, and we will be facing a percolation model with constant edge-density.

It is interesting to notice that we will not prove, in our second step, that a percolation with some constant edge-parameter $p < 1$ stochastically dominates the one on the random environment, because this would be simply false. Rather, we will only compare the probabilities of connections, which is a weaker statement.

Open questions. The present results raise a number of interesting questions. The first natural one is to try to relax the requirements on the heat kernel decay. More precisely, we will see in the proof that we need to be able to define the GFF in infinite volume, which requires the graph to be transient. Still, this condition holds as soon as $d > 2$, thus raising the following question.

Question 1.4. *Prove Theorem 1.1 under the assumption that $d > 2$.*

The main step in which we lose information is in the rewiring estimate of Step 3 in the proof of Lemma 3.6. Replacing the exponential cost by a polynomial one would imply the result.

The proof uses the bounded degree assumption in one place only (in the last step of the proof of Lemma 3.6 again). It is therefore natural to ask the following question.

Question 1.5. *Prove Theorem 1.1 under the assumption that the graph is locally finite, meaning that $d(x) < \infty$ for every $x \in V$.*

Another natural question is to improve (1.1). This bound is not sharp, even when applied in a simple context. Indeed, for $G = \mathbb{Z}^d$ and S a ball of radius r , the upper bound provided by (1.1) is of the form $\exp(-cr^{d-2})$ while the truth, for p above $p_c(G)$, is rather $\exp(-cr^{d-1})$.

Question 1.6. *Improve the bound (1.1).*

This question is probably difficult with the current techniques, due to the following caveat. The percolation with random edge-densities introduced in this paper does not dominate any percolation model with a fixed positive edge-parameter. As a consequence, we believe that the probabilities of big but finite connected components do not have the same tail behavior as in standard Bernoulli percolation.

The last question is related to other models and is much more informal. In the next section, we will use that conditioned on the absolute value of the GFF, the signs of the GFF are sampled according to an Ising model. When conditioning the (Euclidean) norm of the n -component GFF,

the normalized field is sampled according to a spin $O(n)$ model. As a consequence, the first step of our proof can be extended to this context and it is believable that the second step (comparing the model in random environment to a model with fixed coupling constants) could be adapted, even though the lack of correlation inequalities makes it a challenge.

Question 1.7. *Use the techniques developed in the present paper to prove the existence of a phase transition for the spin $O(n)$ models.*

Notation. Set $u_+ = \max\{u, 0\}$ and $\text{sgn}(u) = +1$ if $u \geq 0$ and -1 otherwise. For a set $\Lambda \subset V$, set $\Lambda^c := V \setminus \Lambda$. Also, let $E(\Lambda)$ be the set of edges in E with *at least* one endpoint in Λ .

Organization of the paper. The next section presents the connection between the GFF and a percolation model with random edge-parameters. The third section implements the “integration” of the randomness in the edge-parameters.

2 GFF and Bernoulli percolation

Let Λ be a finite subset of V . The *Gaussian Free Field* (or GFF) with 0 boundary conditions on Λ is defined to be the random (Gaussian) field $\psi = (\psi_x : x \in \Lambda)$ in \mathbb{R}^Λ with distribution

$$d\mathbb{P}_\Lambda[\psi] := \frac{1}{Z_\Lambda} \exp[-\mathcal{D}_\Lambda(\psi)] d\psi,$$

where Z_Λ is a normalizing constant, $d\psi$ stands for the Lebesgue measure on \mathbb{R}^Λ and $\mathcal{D}_\Lambda(\psi)$ is the *Dirichlet energy* given by

$$\mathcal{D}_\Lambda(\psi) := \frac{1}{2} \sum_{xy \in E(\Lambda)} (\psi_x - \psi_y)^2,$$

where ψ_x is extended to every vertex of V by setting $\psi_x = 0$ for every $x \in \Lambda^c$. Under the assumption of transience of the graph G which follows from (H_d) , one can extend the GFF to $\Lambda = V$ by taking the weak limit \mathbb{P} of the measures \mathbb{P}_Λ as Λ tends to V . The measure \mathbb{P} is simply the centered Gaussian vector with covariance matrix given by the Green function \mathbf{G} . Expectation with respect to \mathbb{P} (resp. \mathbb{P}_Λ) is denoted by \mathbb{E} (resp. \mathbb{E}_Λ). The main result of this section is the following.

Proposition 2.1. *For any finite subset S of V one has:*

$$\mathbb{E}[\mathbf{P}_{\mathbf{p}(\psi)}(S \leftrightarrow \infty)] \leq \exp[-\frac{1}{2}\text{cap}(S)] \tag{2.1}$$

where $\mathbf{p}(\psi)_{xy} := 1 - \exp[-2(\psi_x + 1)_+(\psi_y + 1)_+]$ for every $xy \in E$.

Note that for $S = \{x\}$, one gets that x is connected to infinity with positive probability. One may wonder why we added 1 to the GFF: we refer to the remarks at the end of this section for a discussion of this technical trick. The key step in the proof of Proposition 2.1 is the following lemma.

Lemma 2.2. *Fix a finite subset S of V and $t \in \mathbb{R}^S$. If $X_S(\psi) := \exp[-\sum_{x \in S} t_x(\psi_x + 1)]$, then*

$$\mathbb{E}[\mathbf{P}_{\mathbf{p}(\psi)}(S \leftrightarrow \infty)] \leq \mathbb{E}[X_S(\psi)].$$

Before proving this lemma, let us show how it implies Proposition 2.1.

Proof of Proposition 2.1. Since $\sum_{x \in S} t_x(\psi_x + 1)$ is a Gaussian random variable with mean $\sum_{x \in S} t_x$ and variance $\sum_{x, y \in S} t_x t_y \mathbf{G}(x, y)$, we deduce that

$$\mathbb{E}[X_S(\psi)] = \exp\left(-\sum_{x \in S} t_x + \frac{1}{2} \sum_{x, y \in S} t_x t_y \mathbf{G}(x, y)\right).$$

Now, we choose t according to the *equilibrium measure* of S , namely

$$t_x = e_S(x) := d(x) \mathbb{P}[X_k \notin S \ \forall k \geq 1 | X_0 = x]$$

(which turns out to be the optimal choice of t). This gives the result by observing that $\text{cap}(S) = \sum_{x \in S} e_S(x)$ and that $\sum_{y \in S} e_S(y) \mathbf{G}(x, y) = 1$ for all $x \in S$ (which can be deduced in a straightforward way via a decomposition of the random walk started at x in terms of its last visit to S). \square

Let us now turn to the proof of the lemma.

Proof of Lemma 2.2. The proof proceeds in three steps. The first one relates the GFF on Λ to an Ising model on Λ with $+$ boundary conditions and random coupling constants. The second one relates the Ising model to Bernoulli percolation via the Edwards-Sokal coupling. The last step consists in taking the limit as Λ tends to V .

In the first two steps, we fix a finite subset Λ of V . We also define

$$\begin{aligned} |\psi + 1| &:= (|\psi_x + 1|)_{x \in V}, \\ \sigma(\psi) &:= (\text{sgn}(\psi_x + 1))_{x \in V}, \\ J(\psi)_{xy} &:= |\psi_x + 1| |\psi_y + 1|. \end{aligned}$$

Step 1: *Conditionally on $|\psi + 1|$, the random variable $\sigma(\psi)$ is distributed according to the Ising model on Λ with $+$ boundary conditions and coupling constants $J(\psi)$.*

Recall that the Ising model on Λ with $+$ boundary conditions and coupling constants $J = (J_{xy})$ is defined on configurations $\sigma = (\sigma_x : x \in \Lambda)$ in $\{-1, +1\}^\Lambda$ by

$$\mu_{\Lambda; J}^+(\sigma) := \frac{1}{\tilde{Z}_\Lambda} \exp[-\mathcal{H}_{\Lambda, J}(\sigma)]$$

where \tilde{Z}_Λ is a normalizing constant and $\mathcal{H}_{\Lambda, J}(\sigma)$ is the *Hamiltonian* given by

$$\mathcal{H}_{\Lambda, J}(\sigma) := - \sum_{xy \in E(\Lambda)} J_{xy} \sigma_x \sigma_y$$

where σ is extended to V by setting $\sigma_x = +1$ for every $x \in \Lambda^c$.

Now, the fact that $\psi_x = 0$ for every x outside Λ obviously implies $\sigma(\psi)_x = +1$. In addition, we have that

$$\mathcal{D}_\Lambda(\psi) = \frac{1}{2} \sum_{xy \in E(\Lambda)} (\psi_x - \psi_y)^2 = F(|\psi + 1|) + \mathcal{H}_{\Lambda, J(\psi)}(\sigma(\psi)),$$

where F is some function on \mathbb{R}^Λ . This implies that

$$d\mathbb{P}_\Lambda[\psi] = G(|\psi + 1|) \mu_{\lambda; J(\psi)}^+(\sigma(\psi)) d\psi$$

for all $\psi \in \mathbb{R}^\Lambda$, where G is some function on \mathbb{R}^Λ . Since $\psi \mapsto (|\psi + 1|, \sigma(\psi))$ is a bijection from $(\mathbb{R} \setminus \{-1\})^\Lambda$ (which has total Lebesgue measure) to $(\mathbb{R}_{>0} \times \{-1, +1\})^\Lambda$, the above equation implies Step 1 readily.

Step 2: *We have that $\mathbb{E}_\Lambda[\mathbf{P}_{\mathbf{p}(\psi)}(S \not\leftrightarrow \Lambda^c)] \leq \mathbb{E}_\Lambda[X_S(\psi)]$.*

This step relies on the Edwards-Sokal coupling (see [Gri06] for details), which we recall for completeness. Sample a configuration σ according to the Ising model with $+$ boundary conditions and coupling constants J_{xy} . Then, construct a configuration ω on the edges in E intersecting Λ as follows: for every edge xy , let ω_{xy} be a Bernoulli random variable with parameter $1 - \exp(-2J_{xy}\mathbf{1}_{\{\sigma_x = \sigma_y\}})$. Note that $\omega_{xy} = 0$ automatically if $\sigma_x \neq \sigma_y$. Below, \mathbf{P}_J denotes the law of (σ, ω) and \mathbb{E}_J the expectation with respect to \mathbf{P}_J . (We only use this notation in this step.)

The percolation process ω thus obtained is called the *random-cluster model with cluster-weight $q = 2$* , but this will be irrelevant for us. The important feature of this coupling will be that, conditionally on ω , σ is sampled as follows:

- every vertex connected to Λ^c receives the spin $+1$;
- for every connected component \mathcal{C} of ω not intersecting Λ^c , choose a spin $\sigma_{\mathcal{C}}$ equal to $+1$ or -1 with probability $1/2$, independently for each connected component, and set $\sigma_x = \sigma_{\mathcal{C}}$ for every $x \in \mathcal{C}$.

The construction above implies that, conditionally on ω , the law of the σ_x for the vertices which are not connected to Λ^c is symmetric by global flip. Using Step 1 and applying this observation, we deduce that

$$\mathbb{E}_\Lambda[X_S(\psi) \mid |\psi + 1|] \geq \mathbb{E}_{J(\psi)}[\mathbb{E}_{J(\psi)}(X_S(\psi) \mid \omega) \mathbf{1}_{\{S \not\leftrightarrow \Lambda^c \text{ in } \omega\}}] \geq \mathbf{P}_{J(\psi)}[S \not\leftrightarrow \Lambda^c \text{ in } \omega]. \quad (2.2)$$

In the second inequality, we used that, on the event that S is not connected to Λ^c , $X_S(\psi)$ and $1/X_S(\psi)$ have the same law, so that $\mathbb{E}(X_S(\psi) \mid \omega) \geq 1$.

Now, conditioned on σ , the only vertices that can potentially be connected to Λ^c in ω are those which are connected by a path of pluses in σ . For an edge xy with at least one endpoint of this type, one has

$$1 - \exp(-2J_{xy}(\psi)\mathbf{1}_{\{\sigma(\psi)_x = \sigma(\psi)_y\}}) = \mathbf{p}(\psi)_{xy}.$$

This observation together with the Edwards-Sokal coupling and Step 1 gives

$$\mathbf{P}_{J(\psi)}[S \not\leftrightarrow \Lambda^c \text{ in } \omega] = \mathbb{E}_\Lambda[\mathbf{P}_{\mathbf{p}(\psi)}(S \not\leftrightarrow \Lambda^c) \mid |\psi + 1|]. \quad (2.3)$$

Step 2 follows readily by putting (2.3) into (2.2) and then integrating with respect to $|\psi + 1|$.

Step 3: *Passing to the infinite volume.*

Step 2 implies that for every $S \subset T \subset \Lambda$,

$$\mathbb{E}_\Lambda[\mathbf{P}_{\mathbf{p}(\psi)}(S \not\leftrightarrow T^c)] \leq \mathbb{E}_\Lambda[X_S(\psi)]. \quad (2.4)$$

Since S and T are finite, the random variables considered in the previous inequality are continuous local observables of ψ . Letting Λ tend to V , by weak convergence we have

$$\mathbb{E}[\mathbf{P}_{\mathbf{p}(\psi)}(S \not\leftrightarrow T^c)] \leq \mathbb{E}[X_S(\psi)].$$

Letting T tend to V concludes the proof. \square

Remark 2.3. In this section, we did not use the bound (H_d) on the decay of return probabilities. The only property we needed from G was its transience, so that we could consider the GFF in infinite volume.

Remark 2.4. If we were only interested in proving $p_c(G) < 1$, but not the quantitative bound (1.1), we could have proceeded as follows. The Edward-Sokal coupling implies that for any x

$$\mathbb{P}_{J(\psi)}[x \longleftrightarrow \Lambda^c \text{ in } \omega] = \mathbb{E}_{J(\psi)}[\sigma_x].$$

Using the above, in place of (2.2), and subsequently applying (2.3) and integrating with respect to $|\psi + 1|$, we deduce that

$$\mathbb{E}_\Lambda[\mathbf{P}_{\mathbf{p}(\psi)}(x \longleftrightarrow \Lambda^c)] = \mathbb{E}_\Lambda[\text{sgn}(\psi_x + 1)].$$

Proceeding as in the third step, we obtain $\mathbb{E}[\mathbf{P}_{\mathbf{p}(\psi)}(x \longleftrightarrow \infty)] \geq \mathbb{E}[\text{sgn}(\psi_x + 1)] > 0$.

Remark 2.5. By comparing with [Lup16], one can deduce that the clusters of the annealed percolation with random parameters $\mathbf{p}(\psi)$ exactly correspond to the connected components (when restricted to the vertices of G) of the super level-set $\{y \in \tilde{G} : \tilde{\psi}_y > -1\}$ of the (extended) GFF $\tilde{\psi}$ on the metric graph \tilde{G} constructed by interpreting each edge of G as an interval where the field takes values continuously. This connection follows from the following observations: first, the field $\tilde{\psi}$ can be constructed from ψ by simply putting independent Brownian bridges on each edge, interpolating between the values on its endpoints; second, the probability that a Brownian bridge between a and b stays above -1 is exactly $1 - \exp[-2(a+1)_+(b+1)_+]$ (see [Lup16] for details).

Remark 2.6. In the same spirit as in the previous remark, Bernoulli percolation with random parameters given by $\mathbf{q}(\psi)_{xy} := 1 - \exp[-2(\psi_x)_+(\psi_y)_+]$ corresponds to the 0 super level-set $\{y \in \tilde{G} : \tilde{\psi}_y > 0\}$. Also, using the strong Markov property for $\tilde{\psi}$, Lupu proved in [Lup16] that the sign of ψ can be sampled by assigning independent uniform signs to each excursion of $|\tilde{\psi}|$. As a consequence, one has

$$\mathbb{E}[\mathbf{P}_{\mathbf{q}(\psi)}(x \longleftrightarrow y)] = \frac{1}{2} \mathbb{E}[\text{sgn}(\psi_x) \text{sgn}(\psi_y)] = \frac{1}{\pi} \arcsin \left(\frac{\mathbf{G}(x, y)}{\sqrt{\mathbf{G}(x, x) \mathbf{G}(y, y)}} \right) \quad (2.5)$$

for every $x, y \in V$. One can easily deduce from this identity that the (annealed) percolation on the random environment $\mathbf{q}(\psi)$ has infinite *susceptibility*, i.e. $\sum_{y \in V} \mathbb{E}[\mathbf{P}_{\mathbf{q}(\psi)}(x \longleftrightarrow y)] = +\infty$ for any $x \in V$.

Remark 2.7. The previous remark shows that (a priori) there is no infinite connected component for the model with edge-parameters $\mathbf{q}(\psi)$. This is the reason why we shift the GFF by 1 to obtain the edge-parameters $\mathbf{p}(\psi)$ for which the bound on $\mathbb{E}[\mathbf{P}_{\mathbf{p}(\psi)}(x \longleftrightarrow \infty)]$ guarantees the fact that there is at least one infinite connected component.

3 Integrating out the random environment

If $\mathbf{p}(\psi)$ was bounded away from 1, the result would follow by comparison between different Bernoulli percolations. Yet, the GFF is unbounded, and places where the field is large are places for which $\mathbf{p}(\psi)$ is very close to 1, so that the vertices in these regions are almost automatically connected. As a consequence, we will need to consider the annealed probabilities.

Remark 3.1. Let us mention that we were very inspired by the beautiful paper of Rodriguez and Sznitman [RS13] on the study of the super level-set percolation of GFF.

If we could prove that the annealed percolation on the random environment $\mathbf{p}(\psi)$ was stochastically dominated by a Bernoulli percolation \mathbf{P}_p with $p < 1$, then we would be done. Unfortunately, this is not true (for example, one can prove that in \mathbb{Z}^d , the probability that all the edges inside a ball are open in the former decays slower than in the later for any $p < 1$). On the other hand, we are able to compare the probabilities for “connectivity events” such as $\{S \longleftrightarrow \infty\}$.

Proposition 3.2. *There exists $p < 1$ such that for every finite subset S of V ,*

$$\mathbf{P}_p[S \longleftrightarrow \infty] \geq \mathbb{E}[\mathbf{P}_{\mathbf{p}(\psi)}(S \longleftrightarrow \infty)].$$

This proposition, together with Proposition 2.1, implies Theorem 1.1 readily. We therefore focus on the proof of the proposition.

Remark 3.3. It will be evident in the proof that we could also get the result of Proposition 3.2 for all events of the form $\{A \longleftrightarrow B\}$ where $A, B \subset V$ are finite. It will also be clear that the same proof works for $\mathbf{q}(\psi)$ (see Remark 2.6) instead of $\mathbf{p}(\psi)$. This, together with Remark 2.6, would imply the existence of $p < 1$ such that the susceptibility of Bernoulli percolation with parameter p is infinite. Since for quasi-transitive graphs the susceptibility is finite in the whole subcritical phase (see [AB87, Men86, DT16]), we would deduce that $p_c(G) \leq p < 1$. Therefore, if we only wanted to prove Theorem 1.3, it would be enough to consider the (perhaps more intuitive) random environment $\mathbf{q}(\psi)$.

The fact that the GFF has long-range dependencies is a difficulty here. In order to overcome this problem, the key tool used in the proof of Proposition 3.2 is a multi-scale decomposition of the GFF in terms of finite-range-dependent Gaussian fields. Such decompositions appear naturally in rigorous implementations of the Renormalization Group. In this context, the spin-spin correlations of a spin system (for instance the Ising model or the φ_d^4 lattice models) with a certain set of parameters β, λ, \dots are expressed in terms of the GFF ψ , which itself is decomposed into a sum of fields with finite-range dependencies $\psi = \sum_n \psi^n$. Then, one *integrates out* the fields ψ^n one by one by changing the parameters β, λ, \dots into parameters $\beta_1, \lambda_1, \dots$, then $\beta_2, \lambda_2, \dots$, etc. We will do the same in our context. The parameter that will vary in each step to compensate for the integration of the field ψ^n will be called q_n . A main difference with the Renormalization Group is that we will only be interested in inequalities; see (3.3) below.

We now describe the decomposition that we are going to use in our proof. Let $q_n(x, y)$ be the heat kernel associated with the *lazy* random walk in G , i.e. the Markov chain which stays put with probability 1/2, and moves to one of the neighbors chosen uniformly at random with probability 1/2. For any $x, y \in V$, set $\mathbf{G}_0(x, y) := \frac{1}{2d(y)} q_0(x, y)$ and

$$\mathbf{G}_n(x, y) := \frac{1}{2d(y)} \sum_{2^{n-1} \leq k < 2^n} q_k(x, y)$$

for all $n \geq 1$. The matrices $(\mathbf{G}_n)_n$ satisfy the following properties:

1. $\mathbf{G}(x, y) = \sum_{n \geq 0} \mathbf{G}_n(x, y)$ for all $x, y \in V$,
2. \mathbf{G}_n is a covariance matrix (i.e. symmetric positive semi-definite) for every $n \geq 0$,
3. $\mathbf{G}_n(x, y) \geq 0$ for any $x, y \in V$ and $n \geq 0$,
4. $\mathbf{G}_n(x, y) = 0$ for any $x, y \in V$ with $d(x, y) \geq 2^n$,
5. there exists $c' > 0$ such that, for every $n \geq 0$ and $x \in V$, one has

$$\mathbf{G}_n(x, x) \leq c' 2^{-\left(\frac{d-2}{2}\right)n}. \quad (3.1)$$

Properties 1, 3 and 4 are evident. Property 2 follows from the fact q_n is positive semi-definite for every n (this is why we take the lazy random walk instead of the simple one). Property 5 is a direct consequence of our assumption (\mathbf{H}_d) on the heat kernel decay.

It follows from Properties 1 and 2 above that, if $\psi^n \sim \mathcal{N}(0, \mathbf{G}_n)$ are *independent* Gaussian fields, then

$$\psi = \sum_{n \geq 0} \psi^n \quad (3.2)$$

in law. Property 4 is called the *finite-range* property (the value 2^n should be understood as the scale at which correlations occur in ψ^n). Property 3 implies that each field ψ^n is positively associated. Property 5, which bounds the value of $\mathbf{G}_n(x, x)$, will be used to show that ψ^n is small.

Remark 3.4. We will use the assumption $d > 4$ only to guarantee that the exponent $\frac{d-2}{2}$ in the bound (3.1) is strictly larger than 1. Let us mention that in [Bau13], it was proved that there is a decomposition such that the bound (3.1) holds with exponent $d - 2$ instead of $\frac{d-2}{2}$. Unfortunately, this decomposition does not seem to satisfy Property 3.

From now on, we write \mathbb{P} (resp. \mathbb{E}) for the probability (resp. expectation) with respect to $(\psi^n)_{n \geq 0}$, and set $\psi := \sum \psi^n$. By construction, ψ has the law of the GFF. For convenience, we introduce the normalized Gaussian processes $\phi^n := \frac{\pi(n+1)}{\sqrt{12}} \psi^n$.

For the proof, we add three copies of the edge xy of G , that we denote \overline{xy} , \overrightarrow{xy} , \overleftarrow{xy} and call the new graph with all these edges \overline{G} (it has the same set of vertices and four edges between every pair of neighbors in G). Fix some $h \geq 0$ to be determined below. For each realization of $(\psi^n)_{n \geq 0}$, define a Bernoulli percolation model $\mathbf{P}_{q,n,\lambda}$ on \overline{G} with parameters

$$\begin{aligned} \mathbf{P}_{xy} &:= q, \\ \mathbf{P}_{\overline{xy}} &:= 1 - \exp\left(-h - \sum_{k > n} (\phi_x^k)_+^2 + (\phi_y^k)_+^2\right), \\ \mathbf{P}_{\overrightarrow{xy}} &:= 1 - \exp\left(-(\phi_x^n)^2 \mathbf{1}_{\{\phi_x^n \geq \lambda\}}\right), \\ \mathbf{P}_{\overleftarrow{xy}} &:= 1 - \exp\left(-(\phi_y^n)^2 \mathbf{1}_{\{\phi_y^n \geq \lambda\}}\right). \end{aligned}$$

The edge-density of \overline{xy} depends on the ϕ^k with $k > n$ only, those of \overrightarrow{xy} and \overleftarrow{xy} depend on ϕ^n only, and that of xy is deterministic. Also, the parameter λ enables us to interpolate between $\mathbf{P}_{q,n,0}$ and $\mathbf{P}_{q,n+1,0}$. (Notice that the dependence on h is omitted for the sake of the notational convenience, especially since it will be fixed once and for all in Lemma 3.5.)

We now integrate out the randomness coming from the Gaussian processes by showing that there exists h large enough and an increasing sequence (q_n) such that $\lim_n q_n < 1$ and

$$\mathbb{E}[\mathbf{P}_{q_n, n, 0}(S \longleftrightarrow \infty)] \geq \mathbb{E}[\mathbf{P}_{\mathbf{p}(\psi)}(S \longleftrightarrow \infty)]$$

for all n . We prove this by induction. The first lemma initiates the induction.

Lemma 3.5. *For every $n_0 \geq 0$, there exists $h = h(n_0) > 0$ such that*

$$\mathbb{E}[\mathbf{P}_{0, n_0, 0}(S \longleftrightarrow \infty)] \geq \mathbb{E}[\mathbf{P}_{\mathbf{p}(\psi)}(S \longleftrightarrow \infty)]$$

for every finite subset S of V .

Proof. Using that $(1+a)(1+b) \leq 2+a^2+b^2$, we find that

$$\begin{aligned} 2(1+\psi_x)_+(1+\psi_y)_+ &\leq 2\left(1+\sum_{n \geq 0}(\psi_x^n)_+\right)\left(1+\sum_{n \geq 0}(\psi_y^n)_+\right) \\ &\leq 4+2\left(\sum_{n \geq 0}(\psi_x^n)_+\right)^2+2\left(\sum_{n \geq 0}(\psi_y^n)_+\right)^2. \end{aligned}$$

Cauchy-Schwarz inequality (twice) together with the identity $\psi_x^n = \frac{\sqrt{12}}{\pi(n+1)}\phi_x^n$ gives that

$$2(1+\psi_x)_+(1+\psi_y)_+ \leq 4 + \sum_{n \geq 0} [(\phi_x^n)_+^2 + (\phi_y^n)_+^2]$$

Define $K_{xy} := 4 + \sum_{k < n_0} [(\phi_x^k)_+^2 + (\phi_y^k)_+^2]$ and $\mathbf{q}_{xy} := 1 - \exp(-K_{xy})$. We only need to show that there exists $h > 0$ such that the annealed percolation model with (random) parameters \mathbf{q} given by is stochastically dominated by a Bernoulli percolation with parameter $1 - e^{-h}$. Notice that, for every $M > 0$, this model is clearly dominated by the superposition of $\omega_{xy} := \mathbf{1}_{\{K_{xy} > M\}}$ and an independent Bernoulli percolation with parameter $1 - e^{-M}$. As each ϕ^k has finite range of dependence, ω also does. This observation together with the result [LSS97, Theorem 1.3] implies that, provided that M is chosen large enough (depending on n_0), ω is dominated by a Bernoulli percolation with parameter $1 - e^{-1}$. Taking $h = M + 1$ gives the result. \square

The second lemma is used for the induction step. More precisely, it will allow us to remove continuously the field ψ^n using a reasoning similar to the Aizenman-Grimmett paper [AG91] on essential enhancements.

Lemma 3.6. *There exist $\alpha > 0$ and $n_0 \geq 1$ depending only on G , such that for every two finite subsets $S \subset \Lambda$ of V , every $n \geq n_0$, $\lambda \geq n^{-1}$ and $q \geq \frac{1}{2}$, we have*

$$-\frac{d}{d\lambda} \mathbb{E}[\mathbf{P}_{q, n, \lambda}(S \longleftrightarrow \Lambda^c)] \leq \exp(-\alpha 2^n \lambda^2) \frac{d}{dq} \mathbb{E}[\mathbf{P}_{q, n, \lambda}(S \longleftrightarrow \Lambda^c)].$$

Before proving this lemma, let us show how Proposition 3.2 follows from it.

Proof of Proposition 3.2. Take n_0 and $h = h(n_0) > 0$ given by Lemmas 3.6 and 3.5 respectively. Define q_n inductively by setting $q_n := 1/2$ for all $n \leq n_0$ and $q_{n+1} := \lim_{\lambda \rightarrow \infty} q_n(\lambda)$ for all $n \geq n_0$, where

$$q_n(\lambda) := \begin{cases} q_n + 2n^{-2}, & \text{if } \lambda \leq n^{-1} \\ q_n + 2n^{-2} + \int_{n^{-1}}^{\lambda} \exp(-\alpha 2^n t^2) dt, & \text{if } \lambda \geq n^{-1}. \end{cases}$$

First, notice that

$$q := \lim_{n \rightarrow \infty} q_n = \frac{1}{2} + \sum_{n \geq n_0} \left(2n^{-2} + \int_{n^{-1}}^{\infty} \exp(-\alpha 2^n t^2) dt \right) < \infty,$$

so that we can assume without loss of generality (increasing n_0 if necessary) that $q < 1$. Fix two finite subsets $S \subset \Lambda$ of V .

On the one hand, using that $(\phi_x^n)^2 \mathbf{1}_{\{\phi_x^n \geq 0\}} \leq \lambda^2 + (\phi_x^n)^2 \mathbf{1}_{\{\phi_x^n \geq \lambda\}}$, one can easily deduce that

$$\mathbb{E}[\mathbf{P}_{q_n(\lambda), n, \lambda}(S \longleftrightarrow \Lambda^c)] \geq \mathbb{E}[\mathbf{P}_{q_n, n, 0}(S \longleftrightarrow \Lambda^c)]$$

for all $\lambda \leq n^{-1}$. On the other hand, Lemma 3.6 with the choice of $q_n(\lambda)$ tells us that the function $\lambda \mapsto \mathbb{E}[\mathbf{P}_{q_n(\lambda), n, \lambda}(S \longleftrightarrow \Lambda^c)]$ is increasing on $[n^{-1}, \infty)$. Taking λ to infinity implies that

$$\mathbb{E}[\mathbf{P}_{q_{n+1}, n+1, 0}(S \longleftrightarrow \Lambda^c)] \geq \mathbb{E}[\mathbf{P}_{q_n, n, 0}(S \longleftrightarrow \Lambda^c)] \quad (3.3)$$

for all $n \geq n_0$. This, together with Lemma 3.5, gives

$$\mathbf{P}_p(S \longleftrightarrow \Lambda^c) = \mathbb{E}[\mathbf{P}_p(S \longleftrightarrow \Lambda^c)] = \lim_{n \rightarrow \infty} \mathbb{E}[\mathbf{P}_{q_n, n, 0}(S \longleftrightarrow \Lambda^c)] \geq \mathbb{E}[\mathbf{P}_{\mathbf{p}(\psi)}(S \longleftrightarrow \Lambda^c)]$$

where $p := 1 - (1 - q)e^{-h}$. The result follows by letting Λ tend to V . \square

We now go back to the proof of Lemma 3.6. Let us first recall classical expressions for derivatives of events in Bernoulli percolation. Consider an increasing event A depending on finitely many edges. A set F of edges in \overline{G} is *pivotal* (in ω) for A if the configuration is in A when one opens all the edges in F and not in A when one closes these edges. We say that F is *open* (resp. *closed*) *pivotal* if in addition $\omega \in A$ (resp. $\omega \notin A$). Notice that F being open pivotal *does not* necessarily imply that *all* the edges in F are open. Of course, all these definitions apply when F consists of a single edge to recover the standard notion of pivotality. Russo's formula states that

$$\frac{d}{dq} \mathbb{E}[\mathbf{P}_{q, n, \lambda}(A)] = \sum_{xy \in E} \mathbb{E}[\mathbf{P}_{q, n, \lambda}(xy \text{ pivotal for } A)]. \quad (3.4)$$

For the derivative in λ , a quick analysis of $\frac{1}{\delta} \mathbb{E}[\mathbf{P}_{q, n, \lambda + \delta}(A) - \mathbf{P}_{q, n, \lambda}(A)]$ gives that

$$-\frac{d}{d\lambda} \mathbb{E}[\mathbf{P}_{q, n, \lambda}(A)] = \sum_x \rho_x^n(\lambda) \mathbb{E}[\mathbf{P}_{q, n, \lambda}(N_x \text{ open pivotal for } A) | \phi_x^n = \lambda], \quad (3.5)$$

where $\rho_x^n(\lambda)$ is the density of ϕ_x^n and $N_x := \{\overrightarrow{xy} : xy \in E(G)\}$ is the directed edge neighborhood of x .

Proof of Lemma 3.6. To lighten the notation, write $L = 2^n$ and $\mathbf{P}_{q, \lambda}$ instead of $\mathbf{P}_{q, n, \lambda}$ and keep in mind that $\mathbf{P}_{q, \lambda}$ is a function of $(\psi^k)_{k \geq n}$. Below, we apply the notions defined in the last paragraphs for A being equal to $\{S \longleftrightarrow \Lambda^c\}$, where Λ is a finite set of vertices containing S . In order to lighten the notation, we write ‘‘pivotal’’ instead of ‘‘pivotal for $\{S \longleftrightarrow \Lambda^c\}$ ’’.

The proof proceeds as follows. We start from the quantity obtained on the right of (3.5), and try to compare it to the one obtained in (3.4). We do it in three steps. The first one consists in going from open to closed pivotal. The second one enables us to get rid of the conditioning on $\phi_x^n = \lambda$, at the cost of comparing to the probability that the ball $B_L(x)$ of radius L around x is pivotal. The third step brings us back from the probability of the latter to probabilities of being pivotal for individual edges.

Step 1. *From x open pivotal to x closed pivotal.*

Since N_x being pivotal is independent of the state at N_x , we deduce that

$$\begin{aligned} \mathbb{E}[\mathbf{P}_{q,\lambda}(N_x \text{ closed pivotal})|\phi_x^n = \lambda] &\geq \mathbb{E}[\mathbf{P}_{q,\lambda}(N_x \text{ closed}) \cdot \mathbf{P}_{q,\lambda}(N_x \text{ pivotal})|\phi_x^n = \lambda] \\ &= \exp(-d(x)\lambda^2) \mathbb{E}[\mathbf{P}_{q,\lambda}(N_x \text{ pivotal})|\phi_x^n = \lambda]. \end{aligned} \quad (3.6)$$

Step 2. *Removing the conditioning on $\phi_x^n = \lambda$.*

For N_x to be closed pivotal, the ball $B_L(x)$ must be closed pivotal. We therefore find that

$$\mathbb{E}[\mathbf{P}_{q,\lambda}(N_x \text{ closed pivotal})|\phi_x^n = \lambda] \leq \mathbb{E}[\mathbf{P}_{q,\lambda}(B_L(x) \text{ closed pivotal})|\phi_x^n = \lambda].$$

Conditionally on $\phi_x^n = \lambda$, ϕ^n is a Gaussian process with means and covariances given, respectively, by

$$m_z := \lambda \frac{\tilde{\mathbf{G}}_n(x, z)}{\tilde{\mathbf{G}}_n(x, x)} \quad \text{and} \quad \mathbf{G}'_n(z, w) := \tilde{\mathbf{G}}_n(z, w) - \frac{\tilde{\mathbf{G}}_n(z, x)\tilde{\mathbf{G}}_n(x, w)}{\tilde{\mathbf{G}}_n(x, x)}$$

for every $z, w \in V$ (recall that $\tilde{\mathbf{G}}_n$ is the covariance matrix of ϕ^n). In particular, for every $\mu \leq \lambda$, ϕ^n conditioned on $\phi_x^n = \lambda$ and ϕ^n conditioned on $\phi_x^n = \mu$ are shifts of the same centered Gaussian process. Since the shift $(\lambda - \mu)\tilde{\mathbf{G}}_n(x, z)/\tilde{\mathbf{G}}_n(x, x)$ is non-negative for $z \in B_L(x)$ and equal to 0 for $z \notin B_L(x)$ (by Properties 3 and 4 of (\mathbf{G}_n) , respectively), we deduce that

$$\mathbb{E}[\mathbf{P}_{q,\lambda}(B_L(x) \text{ closed pivotal})|\phi_x^n = \lambda] \leq \mathbb{E}[\mathbf{P}_{q,\lambda}(B_L(x) \text{ closed pivotal})|\phi_x^n = \mu].$$

Integrating on $\mu \leq \lambda$ gives that

$$\mathbb{E}[\mathbf{P}_{q,\lambda}(B_L(x) \text{ closed pivotal})|\phi_x^n = \lambda] \leq \frac{\mathbb{E}[\mathbf{P}_{q,\lambda}(B_L(x) \text{ closed pivotal})]}{\mathbb{P}[\phi_x^n \leq \lambda]}.$$

Using $\mathbb{P}[\phi_x^n \leq \lambda] \geq \frac{1}{2}$ and (3.6) gives that

$$\mathbb{E}[\mathbf{P}_{q,\lambda}(N_x \text{ open pivotal})|\phi_x^n = \lambda] \leq 2 \exp(d(x)\lambda^2) \mathbb{E}[\mathbf{P}_{q,\lambda}(B_L(x) \text{ closed pivotal})]. \quad (3.7)$$

Step 3. *Bound on $\mathbb{E}[\mathbf{P}_{q,\lambda}(B_L(x) \text{ closed pivotal})]$ in terms of probabilities of being pivotal.*

Fix an order for vertices and edges of G and consider a configuration ω for which $B_L(x)$ is closed pivotal. Let y and z be the smallest vertices in $B_L(x)$ such that $S \longleftrightarrow y$ and $z \longleftrightarrow \Lambda^c$ both without using any edge contained in $B_L(x)$. Take γ in G to be the earliest (in lexicographical order) path contained in $B_L(x)$ of length at most $2L$ between y and z , and define a configuration ω' by opening the edges of γ one by one (in order) until the first time that an edge uv of $B_L(x)$ becomes pivotal. By construction, ω' contains a pivotal edge in $B_L(x)$, and it is elementary to check that

$$\mathbb{E}[\mathbf{P}_{q,\lambda}(\omega')] \geq q^{2L} \mathbb{E}[\mathbf{P}_{q,\lambda}(\omega)].$$

Furthermore, the map from ω to ω' is at most $2L$ -to-one (since the configuration is not altered outside $B_L(x)$, the sites y and z can be reconstructed, and so can γ). We deduce that

$$\mathbb{E}[\mathbf{P}_{q,\lambda}(B_L(x) \text{ closed pivotal})] \leq 2Lq^{-2L} \sum_{\substack{u,v \in B_L(x): \\ uv \in E}} \mathbb{E}[\mathbf{P}_{q,\lambda}(uv \text{ pivotal})]. \quad (3.8)$$

Conclusion of the proof. Let $D := \max_x d(x)$ be the maximum degree of G . Combining the two inequalities (3.7) and (3.8) enables us to compare the right-hand side of (3.5) and (3.4):

$$\begin{aligned} -\frac{d}{d\lambda} \mathbb{E}[\mathbf{P}_{q,\lambda}(S \longleftrightarrow \Lambda^c)] &\leq \left(\sup_{x \in V} \rho_x^n(\lambda) \right) \cdot 2L \left(\frac{D}{q^2} \right)^L \exp(D\lambda^2) \sum_{uv \in E} \mathbb{E}[\mathbf{P}_{q,\lambda}(uv \text{ pivotal})] \\ &\leq \left(\sup_{x \in V} \rho_x^n(\lambda) \right) \cdot \exp(C2^n + D\lambda^2) \frac{d}{dq} \mathbb{E}[\mathbf{P}_{q,\lambda}(S \longleftrightarrow \Lambda^c)] \end{aligned}$$

for some constant $C > 0$. We replaced L by 2^n and used that $q \geq 1/2$ and that the number of times that an edge uv appears in the summation is $|\{x \in V : u, v \in B_L(x)\}| \leq D^L$. Reminding that $\rho_x^n(\lambda) := \frac{1}{\sqrt{2\pi\hat{\mathbf{G}}_n(x,x)}} \exp[-\frac{1}{2}\lambda^2/\hat{\mathbf{G}}_n(x,x)]$, where $\hat{\mathbf{G}}_n$ is the covariance matrix of ϕ^n , and using that

$$\hat{\mathbf{G}}_n(x,x) = \frac{\pi^2 n^2}{12} \mathbf{G}_n(x,x) \stackrel{(3.1)}{\leq} c'' 2^{-\beta n}$$

for some $c'' > 0$ and $\beta > 1$ (here is the only place where we use $d > 4$), one can find $n_0 \geq 0$ and $\alpha > 0$ such that

$$\left(\sup_{x \in V} \rho_x^n(\lambda) \right) \cdot \exp(C2^n + D\lambda^2) \leq \exp[-\alpha 2^{\beta n} \lambda^2]$$

for every $n \geq n_0$, $\lambda \geq n^{-1}$ and $q \geq 1/2$, thus concluding the proof. \square

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