Marginal triviality of the scaling limits of critical 4D Ising and ϕ_4^4 models

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Abstract

We prove that the scaling limits of spin fluctuations in four-dimensional Ising-type models with nearest-neighbor ferromagnetic interaction at or near the critical point are Gaussian. A similar statement is proven for the $\lambda\phi^4$ fields over $\mathbb{R}^4$ with a lattice ultraviolet cutoff, in the limit of infinite volume and vanishing lattice spacing. The proofs are enabled by the models' random current representation, in which the correlation functions' deviation from Wick's law is expressed in terms of intersection probabilities of random currents with sources at distances which are large on the model's lattice scale. Guided by the analogy with random walk intersection amplitudes, the analysis focuses on the improvement of the so-called tree diagram bound by a logarithmic correction term, which is derived here through multi-scale analysis.

1 Introduction

The results presented below address questions pertaining to two distinct research agendas: one aims at Constructive Field Theory and the other at the understanding of the critical behaviour in Statistical Mechanics. While the goals may appear as somewhat different, the questions and the answers are related. We start with their brief presentation.

1.1 Constructive Quantum Field Theory and Functional Integration

Quantum field theories with local interaction play an important role in the physics discourse, where they appear in subfields ranging from high energy to condensed matter physics. The mathematical challenge of proper formulation of this concept led to programs of Constructive Quantum Field Theory (CQFT). A path towards that goal was charted through the proposal to define quantum fields as operator valued distributions whose essential properties are formulated as the Wightman axioms [55]. Wightman's reconstruction theorem allows one to recover this structure from the collection of the corresponding correlation functions, defined over the Minkowski space-time. By the Osterwalder-Schrader theorem [44, 45], correlation functions with the required properties may potentially be obtained through analytic continuation from those of random distributions defined over the corresponding Euclidean space that meet a number of conditions: suitable analyticity, permutation symmetry, Euclidean covariance, and reflection positivity.

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Seeking natural candidates for such *Euclidean fields*, one ends up with the task of constructing probability averages over random distributions $\Phi(x)$, for which the expectation value of functionals $F(\Phi)$ would have properties fitting the formal expression:

$$\langle F(\Phi)\rangle \approx \frac{1}{\text{norm}} \int F(\Phi) \exp[-H(\Phi)] \prod_{x\in\mathbb{R}^d} d\Phi(x), \tag{1.1}$$

where $H(\Phi)$ is the Hamiltonian. In this context, it seems natural to consider expressions of the form

$$H(\Phi) :\approx (\Phi, A\Phi) + \int_{\mathbb{R}^d} P(\Phi(x))\, dx \tag{1.2}$$

with $(\Phi, A\Phi)$ a positive definite and reflection positive quadratic form, and $P(\Phi(x))$ a polynomial (or a more general function) whose terms of order $\Phi(x)^{2k}$ are interpreted heuristically as representing k-particle interactions. An example of a quadratic form which is positive, reflection positive, and rotation invariant is given by

$$(\Phi, A\Phi) := \int_{\mathbb{R}^d} \left(K|\nabla\Phi|^2(x) + b|\Phi(x)|^2\right)\, dx. \tag{1.3}$$

The functionals $F(\Phi)$ to which (1.1) is intended to apply include the smeared averages of the form

$$T_f(\Phi) := \int_{\mathbb{R}^d} f(x)\Phi(x)dx \tag{1.4}$$

associated with continuous functions of compact support $f \in C_0(\mathbb{R}^d)$. Then, expectation values of products of such variables take the form

$$\langle \prod_{j=1}^n T_{f_j}(\Phi)\rangle := \int_{(\mathbb{R}^d)^n} dx_1 \dots dx_n\, S_n(x_1, \dots, x_n) \prod_{j=1}^n f(x_j), \tag{1.5}$$

where the functions $S_n(x_1, \dots, x_n)$ are referred to as the *Schwinger functions* of the corresponding euclidean field theory. Equation (1.5) can be regarded as saying that in a distributional sense

$$S_n(x_1, \dots, x_n) = \langle \prod_{j=1}^n \Phi(x_j)\rangle\,. \tag{1.6}$$

A relatively simple class of Euclidean fields are the Gaussian fields, for which H contains the quadratic term only. Gaussian fields (whether reflection positive or not) are alternatively characterized by having their structure determined by just the two-point function, with the $2n$-point Schwinger functions computable through Wick's law:

$$S_{2n}(x_1, \dots, x_{2n}) = \sum_\pi \prod_{j=1}^n S_2(x_{\pi(2j-1)}, x_{\pi(2j)}) \;=: \mathscr{G}_n[S_2](x_1, \dots, x_{2n})\,, \tag{1.7}$$

where π ranges over pairing permutations of $\{1, \dots, 2n\}$. The field theoretical interpretation of (1.7) is the absence of interaction. Due to that, and to their algebraically simple structure, such fields have been referred to as *trivial*.

When interpreting (1.1), one quickly encounters a number of problems. Even in the generally understood case of the Gaussian *free field*, with H consisting of just the quadratic term (1.3), Equation (1.1) is not to be taken literally as the measure is supported by non-differentiable functions for which the integral in the exponential is almost surely divergent. Attempts to incorporate interaction terms to try to define non-trivial quantum field theories lead to additional nasty divergences.

A natural step to tackle next seems to be the incorporation, in addition to quadratic terms, of a term of the form

$$P(\Phi(x)) = \lambda : \Phi^4 : , \tag{1.8}$$

which is the lowest order even non-quadratic term of relevance at the level of the renormalization group analysis, but this inevitably leads to substantial additional problems. Partially successful attempts at their resolution have been the focus of a substantial body of works, through various means such as counter-terms (indicated by the ellipses above), regularizing cutoffs, scale decomposition, renormalization group flows, the theory or regularity structures [31], etc.

A natural approach to the definition of the Φ^4_d functional integral (1.1) is to start by regularizing it with a pair of *ultraviolet* (short distance) and *infrared* (long distance) cutoffs. A lattice version of the ultraviolet cutoff is the restriction of $\Phi(\cdot)$ to the vertices of a graph such as $(a\mathbb{Z})^d$ with $a \ll 1$. The infrared cutoff is implemented by the restriction to finite volumes boxes

$$\Lambda_R \coloneqq [-R, R]^d. \tag{1.9}$$

The cutoffs are then removed, taking $R \nearrow \infty$ and then $a \searrow 0$, allowing the system's parameters to be adjusted in the process, i.e., to depend on (a, R).

The CFT program has yielded non-trivial scalar field theories over $\mathbb{R}^2$ and $\mathbb{R}^3$ [11, 23, 30, 45] (we do not discuss here gauge field theories, cf. [35]). However, the expected progression of constructive results was halted when it was proved that for dimensions $d > 4$ the attempt to construct Φ^4_d with

$$\lim_{|x-y|\to\infty} S_2(x,y) \;=\; 0 \tag{1.10}$$

by the method outlined above (in essence: taking the scaling limit of the lattice models at $\beta \leq \beta_c$) yields only Gaussian fields [1, 19].

Various partial results have indicated that the same may hold true for the critical dimension $d = 4$ (cf. [3, 9, 10, 22, 33]), however a sweeping statement such as proved for $d > 4$ has remained open. In this work we address this case.

More explicitly, with the above cutoffs the probability measure which is indicated by (1.1) takes the form of a statistical-mechanics Gibbs equilibrium state average

$$\langle F(\phi)\rangle = \frac{1}{\text{norm}} \int F(\phi) \exp\left[-H(\phi)\right] \prod_{x\in\Lambda_R} \rho(d\phi_x), \tag{1.11}$$

with a Hamiltonian $H(\phi)$ and an a-priori measure $\rho(d\phi)$ of the form

$$H(\phi) = -\sum_{\{x,y\}\subset\Lambda_R} J_{x,y}\, \phi_x\phi_y\,, \qquad \rho(d\phi_x) = e^{-\lambda\phi_x^4 + b\phi_x^2} d\phi_x\,, \tag{1.12}$$

where $d\phi_x$ is the Lebesgue measure on $\mathbb{R}$ and $J_{x,y}$ is zero for non-nearest neighbour vertices, and $J \geq 0$ otherwise. To keep the notation simple, the basic variables are written here as they appear from the perspective of the lattice, i.e. with $a = 1$, but our attention is focused on the correlations at distances of the order of L, with

$$1 \ll L \ll R. \tag{1.13}$$

A point of fundamental importance is that since the interaction through which the field variables are correlated is local (nearest neighbor on the lattice scale), for the field correlations functions to exhibit non-singular variation on the scales $L \gg 1$, the system's parameters (J, λ, b) need to be very close to the critical manifold, along which the correlation length of the lattice system diverges[1].

Quantities whose joint distribution we track in the scaling limit are based on the collections of random variables of the form

$$T_{f,L}(\phi) := \frac{1}{\sqrt{\Sigma_L}} \sum_{x \in \mathbb{Z}^d} f(x/L)\, \phi_x\,, \tag{1.14}$$

where f ranges over compactly supported continuous functions, whose collection is denoted $C_0(\mathbb{R}^d)$, and Σ_L denotes the variance of the sum of spins over the box of size L, i.e.

$$\Sigma_L := \Big\langle \Big(\sum_{x \in \Lambda_L} \phi_x \Big)^2 \Big\rangle. \tag{1.15}$$

Definition 1.1 *A discrete system as described above, parametrized by* (J, λ, b, R, L)*,* converges in distribution*, in the double limit* $\lim_{L\to\infty} \lim_{R/L\to\infty}$ *(with a possible restriction to a subsequence along which also the other parameters are allowed to vary) if for any finite collection of test functions* $f \in C_0(\mathbb{R}^d)$ *the joint distributions of the random variables* $\{T_{f,L}(\phi)\}$ *converge.*

Through a standard probabilistic construction, the limit can be presented as a random field Φ, to whose weighted averages $T_f(\Phi)$ the above variables converge in distribution. We omit here the detailed discussion of this point[2], but remark that for the models considered here the construction is simplified by i) the exclusion of delta functions $\delta(x)$ and their derivatives from the family of considered test functions, and ii) the uniform local integrability of the rescaled correlation functions (before and at the limit). This important condition is implied in the present case by the *infrared bound* which is presented below in Section 4.3.

Our main result, from the perspective of euclidean field theory, is the following.

Theorem 1.2 (Gaussiannity of Φ^4_4) *For dimension* $d = 4$*, any random field reachable by the above constructions, and satisfying* (1.10)*, is a generalized Gaussian process.*

Let us mention that the precise asymptotic behaviour of scaling limits of lattice models which start from sufficiently small perturbations of the Gaussian free field, i.e. small enough λ, have been obtained through rigorous renormalization techniques [10, 18, 22, 33]. In comparison, our result also covers arbitrarily "hard" ϕ^4 fields, but we do not currently provide comparable analysis of the convergence in terms of the exact scale of the logarithmic corrections and the exact expression for the covariance of the limiting Gaussian field.

Concerning the main statement of gaussianity, let us also note that what from the perspective of constructive field theory may be regarded as disappointment is a positive and constructive result from the perspective of statistical mechanics as the theoreticians' goal there is to understand the critical behaviour in models which lie beyond the reach of exact solutions. Correspondingly, we next turn to a brief introduction of this perspective on the result presented above.

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[1] The scaling limit of a correlation function with exponential decay which on the lattice scale is of a fixed correlation length results in a white noise distribution in the limit.

[2] By the Kolmogorov extension theorem, one may start by selecting sequences of the parameter values so as to establish convergence in distribution for a countable collection of test functions f, which is dense in $C_0(\mathbb{R}^d)$, and then use the uniform local integrability of the rescaled correlation function and of the limiting Schwinger functions, to extend the statement by continuity arguments to all $f \in C_0(\mathbb{R}^d)$. One may then recast the limiting variables as associated with a single random Φ, as in (1.4).

1.2 The statistical mechanics perspective

Statistical mechanics provides a general approach for studying the behaviour of extensive systems of a divergent number of degrees of freedom. Among the theoretically gratifying observations in this field has been the discovery of "universality", meaning here that these key features of phase diagrams and correlations observed on large scales at or near the critical points (including the numerical values of critical exponents), are common across broad classes of systems of rather different microscopic structure. This has accorded relevance to studies of the phase transitions in drastically streamlined mathematical models. The ferromagnetic Ising spin models to which we turn next are among the earliest, and most studied such systems.

An Ising spin model on $\Lambda \subset \mathbb{Z}^d$ has as its basic variables a collection of ± 1 valued variables $\{\sigma_x\}_{x\in\Lambda}$, and a Hamiltonian (the energy function) of the form

$$H_{\Lambda,J,h}(\sigma) := -\sum_{\{x,y\}\subset\Lambda} J_{x,y}\,\sigma_x\sigma_y \;-\; \sum_{x\in\Lambda} h\sigma_x\,. \tag{1.16}$$

Its finite volume Gibbs equilibrium state $\langle\cdot\rangle_{\Lambda,J,h,\beta}$ at *inverse temperature* $\beta \geq 0$ is the probability measure under which the expectation value of any function $F : \{\pm 1\}^\Lambda \to \mathbb{R}$ is given by

$$\langle F\rangle_{\Lambda,J,h,\beta} := \frac{1}{Z(\Lambda,J,h,\beta)} \sum_{\sigma\in\{\pm1\}^\Lambda} F(\sigma)\exp[-\beta H_{\Lambda,J,h}(\sigma)], \tag{1.17}$$

where the normalizing factor $Z(\Lambda,J,h,\beta)$ is the model's *partition function.* Infinite volume Gibbs states on $\mathbb{Z}^d$, which we shall denote by $\langle\cdot\rangle_{J,h,\beta}$, are defined through suitable limits (over sequences $\Lambda_n \nearrow \mathbb{Z}^d$) of the above.

We focus here on the *ferromagnetic* case, with $J_{x,y} \geq 0$ for every x and y, and in particular on the *nearest neighbor* interaction, for which the coupling constants vanish except for nearest neighbor pairs, at which $J_{x,y} = J$. For this reason, we write n.n.f. to refer to these models and we drop the dependency in J in the notation. In dimension $d > 1$, this model undergoes a phase transition at $(\beta,h) = (\beta_c(d),0)$. Since that occurs at zero magnetic field, we restrict our attention to $h = 0$ and will omit h from the notation.

Away from the critical point, the model's truncated correlation functions decay exponentially fast [4, 16] in the lattice distance. This leads to the definition of the *correlation length* $\xi(\beta)$ as:

$$\xi(\beta) := \lim_{n\to\infty} -n/\log\langle\sigma_0;\sigma_{n\mathbf{e}_1}\rangle_\beta. \tag{1.18}$$

where $\mathbf{e}_1$ is the unit vector with first coordinate equal to 1. The correlation length is proven to be finite for any $\beta < \beta_c$ [4], divergent in the limit $\beta \nearrow \beta_c$, and at the critical point itself the decay of the 2-point function slows to a power-law (see [49] and the precise discussion around Corollary 4.4). For this reason, we use the convention $\xi(\beta_c) = +\infty$ in the whole article.

At this point, one may notice the similarity between the Ising model's Gibbs equilibrium distribution (1.17) and the discretized functional integral (1.11). For instance, the Ising spin's a-priori (binary) distribution can be viewed as a limit of the ϕ^4 measure

$$\delta(\sigma^2 = 1) = \lim_{\lambda\to\infty} e^{-\lambda(\phi^2-1)^2} d\phi/\mathrm{Norm}. \tag{1.19}$$

Hence included in Theorem 1.2 is the statement that for $d = 4$ any scaling limit of the critical Ising model is Gaussian. Despite the previous observation, we do not state here the Ising statement as a formal corollary of Theorem 1.2 as the theorem's derivation proceeds through the analysis of the Ising model. That is, while (1.19) allows one to view the Ising model as a special case of a ϕ^4 theory, our analysis proceeds in the opposite direction, as will be explained below.

As it is known, and made explicit in Section 5.3, a bellwether for Gaussian behaviour at large scale distances for models with Ising type interaction is the asymptotic validity of Wick's law at the level of the four-point correlation function [1, 42]. For the Ising model the deviation is expressed in terms of the four-point *Ursell function*

$$U_4^\beta(x,y,z,t) := \langle\sigma_x\sigma_y\sigma_z\sigma_t\rangle_\beta - \big[\langle\sigma_x\sigma_y\rangle_\beta\langle\sigma_z\sigma_t\rangle_\beta + \langle\sigma_x\sigma_z\rangle_\beta\langle\sigma_y\sigma_t\rangle_\beta + \langle\sigma_x\sigma_t\rangle_\beta\langle\sigma_y\sigma_z\rangle_\beta\big]\,.$$

The relevant question then is the ratio of $U_4^\beta(x,y,z,t)$ to the full four-point function $\langle\sigma_x\sigma_y\sigma_z\sigma_t\rangle_\beta$, when the mutual distances between the four points are large, and of comparable order between the pairs. An asymptotic vanishing statement was previously established for dimensions $d > 4$ through a combination of the *tree diagram bound* of [1]:

$$|U_4^\beta(x,y,z,t)| \le 2\sum_{u\in\mathbb{Z}^d}\langle\sigma_u\sigma_x\rangle_\beta\langle\sigma_u\sigma_y\rangle_\beta\langle\sigma_u\sigma_z\rangle_\beta\langle\sigma_u\sigma_t\rangle_\beta \tag{1.20}$$

and the *Infrared Bound* of [21, 23],

$$\langle\sigma_x\sigma_y\rangle_{\beta_c} \le \frac{C}{|x-y|^{d-2}} \tag{1.21}$$

which, by reflection positivity, is valid for n.n.f. Ising models below the correlation length in any dimension $d > 2$ (cf. Section 4.3).

At the heuristic level, the triviality of the scaling limit for $d > 4$ is indicated by the following dimensional analysis. Assume that the two-point function has a power-law decay (below the correlation length). Then, the observation that for every x, y, z, t at mutual distances of order L, the sum in the tree diagram bound (1.20) contributes a factor L^d while the summand has two extra factors which are dominated by $1/L^{d-2}$ (by the Infrared Bound) in comparison to $\langle\sigma_x\sigma_y\sigma_z\sigma_t\rangle_\beta$ suggests that the deviation from the Gaussian behaviour may be of the relative size $O(L^{4-d})$, which for $d > 4$ vanishes in the limit L tends to infinity. This is indeed the essence of the argument which was presented in [1, 19]. However, the above observation is clearly inconclusive for $d = 4$.

The hard step in our analysis of the behaviour at the marginal dimension $d = 4$ is the following multi-scale improvement of the tree diagram bound (1.20).

Theorem 1.3 (Improved tree diagram bound inequality) *For the n.n.f. Ising model in dimension $d = 4$, there exist $c, C > 0$ such that for every $\beta \le \beta_c$, every $L \le \xi(\beta)$ and every $x, y, z, t \in \mathbb{Z}^d$ at a distance larger than L of each other,*

$$|U_4^\beta(x,y,z,t)| \le \frac{C}{B_L(\beta)^c}\sum_{u\in\mathbb{Z}^4}\langle\sigma_u\sigma_x\rangle_\beta\langle\sigma_u\sigma_y\rangle_\beta\langle\sigma_u\sigma_z\rangle_\beta\langle\sigma_u\sigma_t\rangle_\beta, \tag{1.22}$$

where $B_L(\beta)$ is the bubble diagram *truncated at a distance L defined by the formula*

$$B_L(\beta) := \sum_{x\in\Lambda_L}\langle\sigma_0\sigma_x\rangle_\beta^2. \tag{1.23}$$

For a heuristic insight on the implications of this improvement for $d = 4$, one may consider separately the two following scenarios: the two-point function $\langle\sigma_0\sigma_x\rangle_\beta$ may be roughly of the order L^{2-d} (meaning that the Infrared Bound is saturated up to constant), or it may be much smaller. In the first case (which is conjectured to hold when $d = 4$), $B_L(\beta)$ is of order $\log L$, so that the improved tree diagram bound indicates that $|U_4|/S_4$

is of the order $(\log L)^{-c}$, and thus asymptotically negligible. In the second case (which is not the one expected to hold), already the previous heuristic argument involving the tree diagram bound (1.20) suffices.

We derive (1.22) making extensive use of the Ising model's random current representation that we present in Section 2. It enables combinatorial identities through which the non-gaussian deviations from Wick's law can be expressed in terms of intersection probabilities of random clusters which link the specified source points pairwise. We summarized the random current representation and discuss at length the intuitive picture relating the behaviour of these random clusters with the one of random walk traces in Section 2.

To conclude this section, let us mention the following quantitative statement of the gaussianity of the Ising model's scaling limit in dimension $d = 4$. It is expressed in terms of the characteristic function of smeared averages of spins, and makes the logarithmic correction explicit.

Proposition 1.4 *There exist $c, C > 0$ such that for the n.n.f. Ising model on $\mathbb{Z}^4$, every $\beta \le \beta_c$, every $L \le \xi(\beta)$, and every test function $f \in C_0(\mathbb{R}^4)$,*

$$\Big| \Big\langle \exp[z\, T_{f,L}(\sigma) - \tfrac{z^2}{2} \langle T_{f,L}(\sigma)^2 \rangle_\beta] \Big\rangle_\beta - 1 \Big| \ \le\ \frac{C \|f\|_\infty^4 r_f^{12}}{(\log L)^c}\, z^4, \tag{1.24}$$

with $\|f\|_\infty := \max\{|f(x)| : x \in \mathbb{R}^4\}$ and r_f the smallest $r \ge 1$ such that f vanishes outside $[-r, r]^4$.

Since, as it is easy to see (using the Infrared Bound on the left-hand side), for any non-negative continuous function $f \not\equiv 0$ with bounded support,

$$C\, r_f^2 \|f\|_\infty^2 \ge \langle T_{f,L}(\sigma)^2 \rangle_\beta \ge c_f > 0, \tag{1.25}$$

uniformly in $\beta \le \beta_c$ and L, we get that for $L \gg 1$ the distribution of $T_{f,L}(\sigma)$ is approximately Gaussian of variance $\langle T_{f,L}(\sigma)^2 \rangle_\beta$.

1.3 Links between the ϕ^4 and Ising variables

That the two areas of research, field theory and statistical mechanics, are linked through a common underlying mathematical structure has been well recognized in both the physics and mathematics discourse. In one direction, the injection to the constructive program of insights based on probabilistic tools such as the domain Markov property or FKG and other inequalities has led to many simplifying insights (cf. [30]). In the converse direction, tools such as Infrared Bounds [23] and reflection positivity (cf. [20, 21]) had a transformative effect on the rigorous study of critical phenomena.

Specific links between the Φ^4 and Ising random variables also exist in both directions. In one direction, one has the above mentioned (1.19) connecting the Ising model to the lattice version of Φ^4. In the converse direction, Simon-Griffiths [50] observed that the Φ^4 variables can be represented as the asymptotic distribution of a block Ising variable under suitable ferromagnetic coupling, as explained in Section 6. This will enable us to prove Theorem 1.2 using a generalization of the arguments for the Ising model.

More generally, such a deconstruction of a random variable into binary components allows to extend the Ising model's analysis to results valid for systems with a similar Hamiltonian but single spin distributions in the more general Griffiths-Simon class, which is discussed in Section 6.

Organization of the proof: The result proven here is unconditional. However, to better convey the argument's structure, we first establish the claimed result for the scaling limits of critical models ($\beta = \beta_c$) under the auxiliary assumption that the two-point function behaves regularly on all scales, in a sense defined below. We then present an unconditional proof for $\beta \le \beta_c$ in which we add to the above analysis the proof that the two-point function is regular on a sufficiently large collection of distance scales, up to the correlation length $\xi(\beta)$.

Organization of the article: In Section 2 we present the basics of the random current representation, and the intuition based on random walk intersection probabilities. Section 3 contains a conditional proof of the improved tree diagram bound at criticality, derived under a power-law decay assumption on the two-point function. Next, as a preparation for the unconditional proof, in Section 4 we present some relevant properties of Ising model's two-point function. Included there are mostly known but also some new results. Section 5 contains the unconditional proof of our main results for the Ising model. Section 6 is devoted to its extension to systems of real valued variables with the single-spin distribution in the Griffiths-Simon class, which includes the ϕ^4 variables. The appendix contains some auxiliary technical statements that are of independent interest.

2 Random current intersection probabilities

2.1 Definition and switching lemma

Starting with the Ising model, in this section we briefly introduce its random current representation, which allows to express the model's subtle correlation effects in more tangible stochastic geometric terms. The utility of the random current representation is enhanced by the combinatorial symmetry expressed in its *switching lemma*, which enables to structure some of the essential truncated correlations in terms guided by the analysis of the intersection properties of the traces of random walks.

Definition 2.1 *A* current *configuration* $\mathbf{n}$ *on* Λ *is an integer-valued function defined over unordered pairs* $(x, y) \in \Lambda$*. The current's set of* sources *is defined as the set*

$$\partial \mathbf{n} := \{x \in \Lambda : (-1)^{\sum_{y \in \Lambda} \mathbf{n}(x,y)} = -1\}. \tag{2.1}$$

For a given Ising model on Λ*, we associate to a current configuration the* weight

$$w(\mathbf{n}) = w_{\Lambda,J,\beta}(\mathbf{n}) := \prod_{\{x,y\} \subset V} \frac{(\beta J_{x,y})^{\mathbf{n}(x,y)}}{\mathbf{n}(x,y)!}. \tag{2.2}$$

Starting from Taylor's expansion

$$\exp(\beta J_{x,y} \sigma_x \sigma_y) = \sum_{\mathbf{n}(x,y) \ge 0} \frac{(\beta J_{x,y} \sigma_x \sigma_y)^{\mathbf{n}(x,y)}}{\mathbf{n}(x,y)!}, \tag{2.3}$$

one can see that the Ising model's partition function (defined below (1.17)) can be expressed in terms of the corresponding random current:

$$Z(\Lambda, \beta) = 2^{|\Lambda|} \sum_{\mathbf{n}: \partial \mathbf{n} = \varnothing} w(\mathbf{n}). \tag{2.4}$$

Furthermore, the spin-spin correlation functions can be represented as

$$\langle \prod_{x\in A} \sigma_x \rangle_{\Lambda,\beta} = \frac{\sum\limits_{\mathbf{n}:\,\partial\mathbf{n}=A} w(\mathbf{n})}{\sum\limits_{\mathbf{n}:\partial\mathbf{n}=\varnothing} w(\mathbf{n})}\,. \tag{2.5}$$

At this point, it helps to note that any configuration with $\partial\mathbf{n} = \varnothing$, i.e. without sources, can be viewed as the edge count of a multigraph which is decomposable into a union of loops. In contrast, any configuration with $\partial\mathbf{n} = A$, such as the one appearing in the numerator of (2.5), can be viewed as describing the edge count of a multigraph which is decomposable into a collection of loops and of paths connecting pairwise the sources, i.e. sites of A. In particular, a configuration with $\partial\mathbf{n} = \{u, v\}$ can be viewed as giving the "flux numbers" of a family of loops together with a path from u to v. Thus, the random current representation allows to present the spin-spin correlation as the effect on the partition function of a loop system with the addition of a path linking the two sources. In these terms, the spin-spin correlation $\langle\sigma_{x_1}\cdots\sigma_{x_{2n}}\rangle_\beta$ represents the sum of the multiplicative effect of the introduction of n paths pairing the sources.

Connectivity properties of currents play a significant role in our analysis. To express those we shall employ the following terminology and notation.

Definition 2.2 *i) We say that x is* connected *to y (in $\mathbf{n}$), and denote the event by $x \overset{\mathbf{n}}{\leftrightarrow} y$, if there exists a path of vertices $x = u_0, u_1, \ldots, u_k = y$ with $\mathbf{n}(u_i, u_{i+1}) > 0$ for every $0 \le i < k$. We say that x is connected to a set S if it is connected to a vertex in S.*
ii) The cluster *of x, denoted by $\mathbf{C}_\mathbf{n}(x)$, is the set of vertices connected to x in $\mathbf{n}$.*
iii) For a set of vertices B, we denote by $\mathscr{F}_B$ the set of $\mathbf{n}$ satisfying that there exists a sub-current $\mathbf{m} \le \mathbf{n}$ such that $\partial\mathbf{m} = B$.

Some of the most powerful properties of the random current representation are best seen when considering pairs of random currents and using the following lemma.

Lemma 2.3 (Switching lemma) *For any $A, B \subset \Lambda$ and any function F from the set of currents into $\mathbb{R}$,*

$$\sum_{\substack{\mathbf{n}_1:\partial\mathbf{n}_1=A\\ \mathbf{n}_2:\partial\mathbf{n}_2=B}} F(\mathbf{n}_1+\mathbf{n}_2)w(\mathbf{n}_1)w(\mathbf{n}_2) = \sum_{\substack{\mathbf{n}_1:\partial\mathbf{n}_1=A\Delta B\\ \mathbf{n}_2:\partial\mathbf{n}_2=\varnothing}} F(\mathbf{n}_1+\mathbf{n}_2)w(\mathbf{n}_1)w(\mathbf{n}_2)\mathbf{1}_{\mathbf{n}_1+\mathbf{n}_2\in\mathscr{F}_B}. \tag{2.6}$$

where $A\Delta B$ denotes the symmetric difference of sets, $A\Delta B := (A \smallsetminus B) \cup (B \smallsetminus A)$.

The switching lemma appeared as a combinatorial identity in Griffiths-Hurst-Sherman's derivation of the GHS inequality [27]. Its greater potential for the geometrization of the correlation functions was developed in [1], and works which followed. In this paper, we employ two generalizations of this useful identity. In the first, the two currents $\mathbf{n}_1$ and $\mathbf{n}_2$ need not be defined on the same graph (see [5, Lemma 2.2] for details). The second will involve a slightly more general switching statement, which was used in several occasions in the past (see [6, Lemma 2.1] for a statement).

It should be recognized that other stochastic geometric representations of spin correlations and/or interactions can be found, e.g. the Symanzik representation of the ϕ^4 action, and the BFS random walk representation of the correlation functions [11] and that it is conceivable that the strategy could be applied. However we find the random current particularly useful for our purpose.

2.2 Representation of Ursell's four-point function

The switching lemma enables one to rewrite spin-spin correlation ratios in terms of probabilities of events expressed in terms of the random currents. The first of these is the relation

$$\frac{\langle\sigma_A\rangle_{\Lambda,\beta}\langle\sigma_B\rangle_{\Lambda,\beta}}{\langle\sigma_A\sigma_B\rangle_{\Lambda,\beta}} := \mathbf{P}^{A\Delta B,\varnothing}_{\Lambda,\beta}[\mathbf{n}_1+\mathbf{n}_2 \in \mathscr{F}_B], \tag{2.7}$$

for which we denote by $\mathbf{P}^A_{\Lambda,\beta}(\mathbf{n})$ the probability distribution on random currents constrained by the source condition $\partial\mathbf{n} = A$, or more explicitly

$$\mathbf{P}^A_{\Lambda,\beta}(\mathbf{n}) := \frac{2^{|\Lambda|}w(\mathbf{n})}{\langle\prod_{x\in A}\sigma_x\rangle_{\Lambda,\beta}\, Z(\Lambda,\beta)}\mathbb{I}[\partial\mathbf{n} = A], \tag{2.8}$$

and by $\mathbf{P}^{A_1,\dots,A_i}_{\Lambda,\beta}$ we denote the law of an independent family of currents $(\mathbf{n}_1,\dots,\mathbf{n}_i)$

$$\mathbf{P}^{A_1,\dots,A_i}_{\Lambda,\beta} := \mathbf{P}^{A_1}_{\Lambda,\beta}\otimes\cdots\otimes\mathbf{P}^{A_i}_{\Lambda,\beta}. \tag{2.9}$$

For two-point sets we may write $A = xy$ instead of $\{x,y\}$.

As we will also work with the infinite volume Gibbs measures, let us note that random currents and the switching lemma admit a generalization to infinite volume[3]. Existing continuity results [5] permit to extend (2.7) to the infinite volume, expressed in terms of the weak limits of the random current measures $\mathbf{P}^A_{\Lambda_n,\beta}$ and $\mathbf{P}^{A_1,\dots,A_i}_{\Lambda_n,\beta}$, in the limit $\Lambda_n \nearrow \mathbb{Z}^d$. The limiting statement is similar to (2.7) but without the finite volume subscript Λ:

$$\frac{\langle\sigma_A\rangle_{\beta}\langle\sigma_B\rangle_{\beta}}{\langle\sigma_A\sigma_B\rangle_{\beta}} = \mathbf{P}^{A\Delta B,\varnothing}_{\beta}[\mathbf{n}_1+\mathbf{n}_2 \in \mathscr{F}_B]. \tag{2.10}$$

Combining (2.10) for the different values of the product of spin-spin correlations leads to

$$U_4^\beta(x,y,z,t) = -2\langle\sigma_x\sigma_y\rangle_\beta\langle\sigma_z\sigma_t\rangle_\beta\, \mathbf{P}^{xy,zt}_\beta[\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(x)\cap\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(z)\neq\varnothing]. \tag{2.11}$$

This equality is of fundamental importance to the question discussed here. It was the basis of the analysis of [1], and is the starting point for our discussion.

By (2.11), the relative magnitude of the deviation of the four-point function $\langle\sigma_x\sigma_y\sigma_z\sigma_t\rangle_\beta$ from the Gaussian law (i.e. the discrepancy in Wick's formula) is bounded in terms of intersection properties of the two clusters that link the indicated sources pairwise:

$$\frac{|U_4^\beta(x,y,z,t)|}{\langle\sigma_x\sigma_y\sigma_z\sigma_t\rangle_\beta} \;\le\; 2\mathbf{P}^{xy,zt}_\beta[\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(x)\cap\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(z)\neq\varnothing]. \tag{2.12}$$

The random sets $\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(x)$ and $\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(z)$ are not independently distributed. However (2.12) can be further simplified through a monotonicity property of random currents. As proved in [1], and recalled here in the Appendix, the probability of an intersection can only increase upon the two sets' replacement by a pair of independently distributed clusters defined through the addition of two sourceless currents:

$$\mathbf{P}^{xy,zt}_\beta[\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(x)\cap\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(z)\neq\varnothing] \le \mathbf{P}^{xy,zt,\varnothing,\varnothing}_\beta[\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_3}(x)\cap\mathbf{C}_{\mathbf{n}_2+\mathbf{n}_4}(z)\neq\varnothing]. \tag{2.13}$$

[3] The extension of the switching lemma to $\mathbb{Z}^d$ is straightforward for $\beta\le\beta_c$ since then $\mathbf{n}_1+\mathbf{n}_2$ does not contain infinite paths of positive currents, almost surely under $\mathbf{P}^{A,B}_\beta$. For $\beta<\beta_c$ this is implied by the discussion of [1] for $\beta<\beta_c$, and for $\beta=\beta_c$ it follows from the continuity result of [5] for $\beta=\beta_c$.

This leads to the simpler upper bound in which the two random sets are independent:

$$|U_4^\beta(x,y,z,t)| \le 2\langle\sigma_x\sigma_y\rangle_\beta\langle\sigma_z\sigma_t\rangle_\beta \mathbf{P}_\beta^{xy,zt,\varnothing,\varnothing}[\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_3}(x)\cap\mathbf{C}_{\mathbf{n}_2+\mathbf{n}_4}(z)\neq\varnothing]. \tag{2.14}$$

Bounding the intersection probability by the expected number of intersection sites and applying the switching lemma leads directly to the tree diagram bound (1.20). However, as was explained above, to tackle the marginal dimension $d=4$ one needs to improve on that.

While $\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_3}(x)$ and $\mathbf{C}_{\mathbf{n}_2+\mathbf{n}_4}(z)$ are bulkier and exhibit less independence than simple random walks linking the sources $\{x,y\}$ and $\{z,t\}$, the analogy is of help in guiding the intuition towards useful estimate strategies. In particular, it is classical that in dimension $d=4$ the probability that the traces of two random walks starting at distance L of each other intersect, tends to 0 (as $1/\log L$, see [2, (2.8)] and [37]), but nevertheless the expected number of points of intersection remains of order $\Omega(1)$. The discrepancy is explained by the fact that although the intersections occur rarely, the conditional expectation of the number of intersection sites, conditioned on there being at least one, diverges logarithmically in L. The thrust of our analysis will be to establish similar behaviour in the system considered here. More explicitly, we will prove that the conditional expectation of the clusters' intersection size, conditioned on it being non-empty, grows at least as $(\log L)^c$.

The analysis of clusters' intersection properties is more difficult than that of the paths of simple random walks for at least two reasons:

- Missing information on the two-point function: Most analyses of intersection properties of random walks involve estimates on the Green function. In our system its role is to some extent taken by the two-point spin-spin correlation function. However, unlike the former case we do not a priori know the two-point function's exact order of magnitude (though a good one-sided inequality is provided by the Infrared Bound). This raises a difficulty that we address by studying the regularity properties of the two-point function in Section 4.

- The lack of a simple Markov property: in one way or another, the analysis of intersections for random walks involves the random walk's Markov property. Among its other applications, the walk's renewal property facilitates de-correlating the walks' behaviour at different places. In comparison, the random current clusters exhibit only a multidimensional domain Markov property. One of the main contributions of this paper will be to show a mixing property of random currents which will enable us to bypass the difficulty raised by the lack of a renewal property.

We expect that both the regularity estimates and the mixing properties established here are of independent interest, and may be of help in studies of the model also in three dimensions.

3 A conditional improvement of the tree diagram bound for $\beta=\beta_c$

To better convey the strategy by which the tree diagram bound is improved, we start with a *conditional proof* of (1.22) for the Ising model on $\mathbb{Z}^4$ at criticality (i.e. when $\beta=\beta_c$), under the following assumption on the model's two-point function. The removal of this assumption will raise substantial problems which are presented in the sections that follow. Below, $|\cdot|$ denotes the infinite-norm

$$|x| := \max\{|x_i|, 1\le i\le d\}. \tag{3.1}$$

Assumption 3.1 (Power-law decay) *There exist η and $c, C \in (0,\infty)$ such that for every $x \in \mathbb{Z}^d$,*

$$\frac{c}{|x|^{d-2+\eta}} \le \langle \sigma_0 \sigma_x \rangle_{\beta_c} \le \frac{C}{|x|^{d-2+\eta}}. \tag{3.2}$$

The Infrared Bound (4.44) guarantees that $\eta \ge 0$ in any dimension $d > 2$. Note that if $\eta > 0$ for $d = 4$, then $B_L(\beta_c)$ is bounded uniformly in L in which case the tree diagram bound implies the improved one. Thus, under this assumption the case requiring attention is just $\eta = 0$ (which is the generally expected value).

3.1 Intersection clusters

Our starting point is (2.14) in which $U_4^{\beta_c}$ is bounded by the probability of intersection of two independently distributed clusters $\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_3}(x)$ and $\mathbf{C}_{\mathbf{n}_2+\mathbf{n}_4}(z)$, of which $\mathbf{n}_1$ and $\mathbf{n}_2$ include paths linking pairwise widely separated sources, $\partial\mathbf{n}_1 = \{x,y\}$ and $\partial\mathbf{n}_2 = \{z,t\}$. Introduce the notation

$$\mathscr{T} := \mathbf{C}_{\mathbf{n}_1+\mathbf{n}_3}(x) \cap \mathbf{C}_{\mathbf{n}_2+\mathbf{n}_4}(z), \tag{3.3}$$

and let $|\mathscr{T}|$ be the set's cardinality. The tree diagram bound corresponds to the first moment estimate:

$$\mathbf{P}_{\beta_c}^{xy,zt,\varnothing,\varnothing}[|\mathscr{T}| > 0] \le \mathbf{E}_{\beta_c}^{xy,zt,\varnothing,\varnothing}[|\mathscr{T}|], \tag{3.4}$$

in which the intersection probability is bounded by the intersection set's expected size.

Although the set $\mathscr{T}$ is less tractable than the intersection of a pair of Markovian random walks, their intuitive example provides a useful guide. The intersection of the traces of two simple random walks in dimension $d = 4$ has a Cantor-set like structure. Guided by this analogy, and taking advantage of the switching lemma, we show that conditioned on the event that u belongs to $\mathscr{T}$, the intersection $|\mathscr{T}|$ is typically very large. This is in line with our expectation that the vertices in the intersection set occur in large (disconnected) clusters, causing the expected size of $|\mathscr{T}|$ to be much larger than the probability of it being non-zero.

Below and in the rest of this article, we introduce the annulus of sizes $k \le n$ and the boundary of a box as follows:

$$\mathrm{Ann}(k,n) := \Lambda_n \setminus \Lambda_{k-1} \quad \text{and} \quad \partial\Lambda_n := \mathrm{Ann}(n,n). \tag{3.5}$$

In the proof, we apply the following deterministic covering lemma, which links the number of points in a set $\mathscr{X} \subset \mathbb{Z}^d$ with the number of concentric annuli of the form $u + \mathrm{Ann}(\ell_k, \ell_{k+1})$, with $u \in \mathscr{X}$, which it takes to cover $\mathscr{X}$. To state it we denote, for any (possibly finite) increasing sequence of lengths $\mathscr{L} = (\ell_k)$, every $u \in \mathbb{Z}^d$, and every integer K,

$$\mathbf{M}_u(\mathscr{X}; \mathscr{L}, K) = \mathrm{card}\{k \le K : \mathscr{X} \cap [u + \mathrm{Ann}(\ell_k, \ell_{k+1})] \neq \varnothing\}. \tag{3.6}$$

Lemma 3.2 *(Annular covering) In the above notation, for any sequence $\mathscr{L} = (\ell_k)$ with $\ell_1 \ge 1$ and $\ell_{k+1} \ge 2\ell_k$,*

$$\text{if } |\mathscr{X}| < 2^r, \text{ then there exists a site } u \in \mathscr{X} \text{ for which } \mathbf{M}_u(\mathscr{X}; \mathscr{L}, K) < 5r. \tag{3.7}$$

Proof We prove the following stronger statement: For every set $\mathscr{X}$ containing the origin and every K, if $|\mathscr{X}\cap\Lambda_{\ell_K}|<2^r$, then there exists $u\in\mathscr{X}\cap\Lambda_{\ell_K}$ with $M_u(\mathscr{X};\mathscr{L},K)<5r$.

The assertion is obviously true for $r=1$ as one can pick u to be the origin. Next, consider the case of $r>1$ assuming the statement holds for all smaller values. If the intersection of $\mathscr{X}$ and $\Lambda_{\ell_{K-1}}$ is reduced to the origin, then $M_0(\mathscr{X};\mathscr{L},K)\le 2$ (only the annuli $\mathrm{Ann}(\ell_l,\ell_{l+1})$ with l equal to $K-1$ or K can intersect $\mathscr{X}$) as required so we now assume that this is not the case. Consider $0\le k\le K-2$ maximal such that there exists $u\in\mathscr{X}$ with $\ell_k<|u|\le\ell_{k+1}$.

Since $\mathscr{X}\cap\Lambda_{\ell_{k-1}}$ and $\mathscr{X}\cap(u+\Lambda_{\ell_{k-1}})$ are disjoint (we use that $\ell_k\ge 2\ell_{k-1}$), one of the two sets has cardinality strictly smaller than 2^{r-1}. Assume first that it is $\mathscr{X}\cap\Lambda_{\ell_{k-1}}$. The induction hypothesis implies the existence of $v\in\mathscr{X}\cap\Lambda_{\ell_{k-1}}$ such that

$$\mathbf{M}_v(\mathscr{X};\mathscr{L},k-1)<5(r-1). \tag{3.8}$$

By our choice of k, every site in $\mathscr{X}$ is either in $\Lambda_{\ell_{k+1}}$ or outside of $\Lambda_{\ell_{K-1}}$. This implies that only the annuli $\mathrm{Ann}(\ell_l,\ell_{l+1})$ with l equal to k, $k+1$, $K-2$, $K-1$ or K can intersect $\mathscr{X}$, so that

$$\mathbf{M}_v(\mathscr{X};\mathscr{L},K)\le\mathbf{M}_v(\mathscr{X};\mathscr{L},k-1)+5<5r\,. \tag{3.9}$$

If it is $\mathscr{X}\cap(u+\Lambda_{\ell_{k-1}})$ which has small cardinality, simply translate the set by u and apply the same reasoning. The distance between the vertex v obtained by the procedure and 0 is at most $\ell_{k-1}+\ell_k\le\ell_K$, so that the claim follows in this case as well. □

In the following conditional statement, we denote by $\mathscr{L}_\alpha$ a sequence of integers defined recursively so that $\ell_{k+1}=\ell_k^\alpha$ with a specified $\alpha>1$ and ℓ_0 a large enough integer.

Proposition 3.3 (Conditional intersection-clustering bound) *Under the assumption that the Ising model on $\mathbb{Z}^4$ satisfies* (3.2) *with $\eta=0$ and restricting to $\alpha>3^8$: there exist $\ell_0=\ell_0(\alpha)$ and $\delta=\delta(\alpha)>0$ such that for every $K>2$ and every $u,x,y,z,t\in\mathbb{Z}^4$ with mutual distance between x,y,z,t larger than $2\ell_K$,*

$$\mathbf{P}_{\beta_c}^{ux,uz,uy,ut}[\mathbf{M}_u(\mathscr{T};\mathscr{L}_\alpha,K)<\delta K]\le 2^{-\delta K}. \tag{3.10}$$

Before deriving this estimate, which is proven in the next section, let us show how it leads to the improved tree diagram bound.

Proof of Theorem 1.3 under the assumption (3.2). As the discussion is limited here to $\beta=\beta_c$, we omit it from the notation. If $\eta>0$ the bubble diagram is finite and hence the desired statement is already contained in the tree diagram bound (1.20). Focus then on the case $\eta=0$, for which the bubble diagram diverges logarithmically. Fix $\alpha>3^8$ and let ℓ_0 and δ be given by Proposition 3.3. Since x,y,z,t are at mutual distances at least L, there exists $c=c(\alpha)>0$ such that one may pick

$$K=K(L)\ge c\log\log L \tag{3.11}$$

in such a way that $L\ge 2\ell_K$.

Using Lemma 3.2, then the switching lemma, and finally Proposition 3.3, we get

$$\begin{aligned}\mathbf{P}^{xy,zt,\varnothing,\varnothing}[0<|\mathscr{T}|<2^{\delta K/5}]&\le\sum_{u\in\mathbb{Z}^4}\mathbf{P}^{xy,zt,\varnothing,\varnothing}[u\in\mathscr{T},\mathbf{M}_u(\mathscr{T};\mathscr{L}_\alpha,K)<\delta K]\\&=\sum_{u\in\mathbb{Z}^4}\frac{\langle\sigma_u\sigma_x\rangle\langle\sigma_u\sigma_y\rangle\langle\sigma_u\sigma_z\rangle\langle\sigma_u\sigma_t\rangle}{\langle\sigma_x\sigma_y\rangle\langle\sigma_z\sigma_t\rangle}\mathbf{P}^{ux,uz,uy,ut}[\mathbf{M}_u(\mathscr{T};\mathscr{L}_\alpha,K)<\delta K]\\&\le 2^{-\delta K}\sum_{u\in\mathbb{Z}^4}\frac{\langle\sigma_u\sigma_x\rangle\langle\sigma_u\sigma_y\rangle\langle\sigma_u\sigma_z\rangle\langle\sigma_u\sigma_t\rangle}{\langle\sigma_x\sigma_y\rangle\langle\sigma_z\sigma_t\rangle}.\end{aligned} \tag{3.12}$$

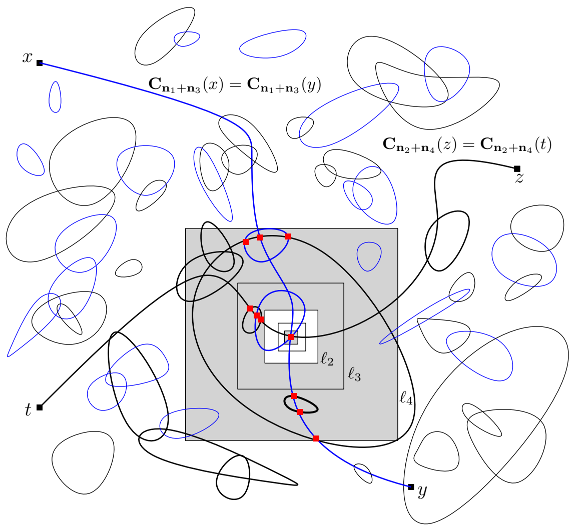

Figure 1: The two (duplicated)-currents $\mathbf{n}_1+\mathbf{n}_3$ and $\mathbf{n}_2+\mathbf{n}_4$ in blue and black respectively. The clusters of x (or equivalently y) in $\mathbf{n}_1+\mathbf{n}_3$ and z (or equivalently t) in $\mathbf{n}_2+\mathbf{n}_4$ are depicted in bold. The red vertices are the elements of the intersection $\mathscr{T}$. We illustrated the annuli around one element, denoted u, of $\mathscr{T}$ and draw them in gray when an intersection occurs. Here, we therefore have $M_u(\mathscr{T};\mathscr{L},5) = 3$ since three annuli contain an intersection.

For the larger values of $|\mathscr{T}|$, the Markov inequality and the switching lemma give

$$\begin{aligned}\mathbf{P}^{xy,zt,\varnothing,\varnothing}[|\mathscr{T}| \geq 2^{\delta K/5}] &\leq 2^{-\delta K/5}\mathbf{E}^{xy,zt,\varnothing,\varnothing}[|\mathscr{T}|] \\ &= 2^{-\delta K/5}\sum_{u\in\mathbb{Z}^4}\frac{\langle\sigma_u\sigma_x\rangle\langle\sigma_u\sigma_y\rangle\langle\sigma_u\sigma_z\rangle\langle\sigma_u\sigma_t\rangle}{\langle\sigma_x\sigma_y\rangle\langle\sigma_z\sigma_t\rangle}. \qquad (3.13)\end{aligned}$$

Adding (3.12) and (3.13) gives an improved tree diagram bound which, in view of (3.11) and of the logarithmic divergence of $B_L(\beta_c)$ implied by $\eta = 0$, yields (1.22). □

3.2 Derivation of the conditional intersection-clustering bound (Proposition 3.3)

The intuition underlying the conditional intersection-clustering bound and the choice of ℓ_k is guided by the aforementioned example of simple random walks. In dimension 4, the traces of two independent random walks starting at the origin intersect in an annulus of the form $\mathrm{Ann}(n, n^\alpha)$ with probability at least $c(\alpha) > 0$ uniformly in n. Since the paths traced by these random walks within different annuli are roughly independent, one may expect the number of annuli among the K first ones in which the paths intersect to be, with large probability, of the order of δK.

However, in the case considered here, the clusters of u in $\mathbf{n}_1+\mathbf{n}_3$ and $\mathbf{n}_2+\mathbf{n}_4$ do not have the renewal structure of Markovian random walks. We shall compensate for that in two steps:

(i) *reformulate the intersection property*

(ii) *derive an asymptotic mixing statement.*

For the first step, let I_k be the event (with I standing for intersection) that there exist unique clusters of $\mathrm{Ann}(\ell_k, \ell_{k+1})$ in $\mathbf{n}_1+\mathbf{n}_3$ and $\mathbf{n}_2+\mathbf{n}_4$ crossing the annulus from the inner boundary to the outer boundary and that these two clusters are intersecting. Lemma 3.4 presents the statement that the probability that the event occurs and that these clusters intersect, is bounded away from 0 uniformly in k.

Note that the annuli $\mathrm{Ann}(\ell_k,\ell_{k+1})$ are wide enough so that sourceless currents will typically have no radial crossing, and when such crossings are forced by the placement of sources (for instance when one source, is at the common center of a family of nested annuli and the other at a distant site outside), in each annulus there will most likely be only one crossing cluster. It then follows that all the crossing clusters of $\mathbf{n}_1+\mathbf{n}_3$ belong to the $\mathbf{n}_1+\mathbf{n}_3$ cluster of the sources, and a similar property holds for the crossing clusters of $\mathbf{n}_2+\mathbf{n}_4$.

For the second step, we prove that events observed within sufficiently separated annuli are roughly independent. The exact assertion is presented below in Proposition 3.6 and will be the crux of the whole paper.

Following is the first of these two statements.

Lemma 3.4 (Conditional intersection-clustering property) *Assume* (3.2) *holds for the Ising model on $\mathbb{Z}^4$ with $\eta=0$. For $\alpha>3^4$, there exist $\ell_0=\ell_0(\alpha)$ and $c=c(\alpha,\ell_0)>0$ such that for every $x,z\notin\Lambda_{2\ell_{k+1}}$,*

$$\mathbf{P}^{0x,0z,\varnothing,\varnothing}_{\beta_c}[I_k]\geq c. \tag{3.14}$$

The main ingredient in the proof is a second moment method on the number of intersections in $\mathrm{Ann}(\ell_k,\ell_{k+1})$ of the clusters of the origin in $\mathbf{n}_1+\mathbf{n}_3$ and $\mathbf{n}_2+\mathbf{n}_4$. A second part of the proof is devoted to the uniqueness of the clusters crossing the annulus. This makes the event under consideration measurable in terms of the currents within just the specified annulus, allowing us to apply the mixing property for the proof of Proposition 3.3, which follows further below.

Proof Drop β_c from the notation. Fix $\alpha>3^4$ and set $\varepsilon>0$ so that $\alpha>(1+\varepsilon)(3+\varepsilon)^4$. The constants c_i below depend on ε only. Introduce the intermediary integers $n\leq m\leq M\leq N$ satisfying

$$n\geq \ell_k^{3+\varepsilon}\ ,\ m\geq n^{3+\varepsilon}\ ,\ M\geq m^{1+\varepsilon}\ ,\ N\geq M^{3+\varepsilon}\ ,\ \ell_{k+1}\geq N^{3+\varepsilon}. \tag{3.15}$$

We start by proving that $\mathscr{M}:=\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_3}(0)\cap\mathbf{C}_{\mathbf{n}_2+\mathbf{n}_4}(0)\cap\mathrm{Ann}(m,M)$ is non-empty with positive probability by applying a second-moment method on $|\mathscr{M}|$. Namely, the switching lemma (more precisely (A.10)) and (3.2) imply that

$$\begin{aligned}\mathbf{E}^{0x,0z,\varnothing,\varnothing}[|\mathscr{M}|]&=\sum_{v\in\mathrm{Ann}(m,M)}\mathbf{P}^{0x,\varnothing}[v\overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow}0]\mathbf{P}^{0z,\varnothing}[v\overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow}0]\\&=\sum_{v\in\mathrm{Ann}(m,M)}\frac{\langle\sigma_0\sigma_v\rangle\langle\sigma_v\sigma_x\rangle}{\langle\sigma_0\sigma_x\rangle}\frac{\langle\sigma_0\sigma_v\rangle\langle\sigma_v\sigma_z\rangle}{\langle\sigma_0\sigma_z\rangle}\\&\geq c_1(B_M-B_{m-1})\ \geq\ c_2\log(M/m)\,.\end{aligned}\tag{3.16}$$

On the other hand, we find that

$$\mathbf{E}^{0x,0z,\varnothing,\varnothing}[|\mathscr{M}|^2]=\sum_{v,w\in\mathrm{Ann}(m,M)}\mathbf{P}^{0x,\varnothing}[v,w\overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow}0]\mathbf{P}^{0z,\varnothing}[v,w\overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow}0]. \tag{3.17}$$

Now, by a delicate application of the switching lemma and a monotonicity argument we have the following inequality (stated and proven as Proposition A.3 in the Appendix),

$$\mathbf{P}^{0x,\varnothing}[v,w\overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow}0]\leq\frac{\langle\sigma_0\sigma_v\rangle\langle\sigma_v\sigma_w\rangle\langle\sigma_w\sigma_x\rangle}{\langle\sigma_0\sigma_x\rangle}+\frac{\langle\sigma_0\sigma_w\rangle\langle\sigma_w\sigma_v\rangle\langle\sigma_v\sigma_x\rangle}{\langle\sigma_0\sigma_x\rangle}\,. \tag{3.18}$$

Together with (3.2), this gives

$$\mathbf{E}^{0x,0z,\varnothing,\varnothing}[|\mathscr{M}|^2] \ \le C_3(B_M - B_{m-1})\, B_{2M} \le C_4(\log M)^2. \tag{3.19}$$

The second moment (or Cauchy-Schwarz) inequality, and the bound $M \ge m^{1+\varepsilon}$ thus imply

$$\mathbf{P}^{0x,0z,\varnothing,\varnothing}[\mathscr{M} \ne \varnothing] \ge \frac{\mathbf{E}^{0x,0z,\varnothing,\varnothing}[|\mathscr{M}|]^2}{\mathbf{E}^{0x,0z,\varnothing,\varnothing}[|\mathscr{M}|^2]} \ge c_5 > 0. \tag{3.20}$$

At this stage, one may feel that the main point of the lemma was established: we showed that with uniformly positive probability the clusters of 0 in $\mathbf{n}_1 + \mathbf{n}_3$ and $\mathbf{n}_2 + \mathbf{n}_4$ intersect in $\mathrm{Ann}(m, M)$. However, to conclude the argument we need to establish the uniqueness, with large probability, of the crossing cluster in $\mathbf{n}_1 + \mathbf{n}_3$ (the same then holds true for $\mathbf{n}_2 + \mathbf{n}_4$). This part of the proof is slightly more technical and may be omitted in a first reading. It is here that we shall need α to be large enough.

To prove the uniqueness of crossings, we employ the notion of the current's *backbone*[4], on which more can be found in [1, 4, 14, 15, 17]. If the event $\{\mathscr{M} \ne \varnothing\}$ occurs but not I_k, then one of the following four events must occur (see e.g. Fig. 2):

$F_1 :=$ the backbone $\Gamma(\mathbf{n}_1)$ of $\mathbf{n}_1$ does two successive crossings of $\mathrm{Ann}(\ell_k, n)$;
$F_2 :=$ $\mathbf{n}_1 + \mathbf{n}_2$ contains a cluster crossing $\mathrm{Ann}(n, m) \smallsetminus \Gamma(\mathbf{n}_1)$;
$F_3 :=$ $\mathbf{n}_1 + \mathbf{n}_2$ contains a cluster crossing $\mathrm{Ann}(M, N) \smallsetminus \Gamma(\mathbf{n}_1)$;
$F_4 :=$ the backbone $\Gamma(\mathbf{n}_1)$ of $\mathbf{n}_1$ does two successive crossings of $\mathrm{Ann}(N, \ell_{k+1})$.

We bound the probabilities of these events separately. For F_1 to occur, the backbone $\Gamma(\mathbf{n}_1)$ must do a zigzag: to go from 0 to a vertex $v \in \partial\Lambda_n$, then to a vertex $w \in \partial\Lambda_{\ell_k}$, and finally to x. The *chain rule* for backbones (see e.g. [4]) combined with the assumed condition (3.2), jointly imply that

$$\mathbf{P}^{0x,\varnothing}[F_1] \le \sum_{\substack{v\in\partial\Lambda_n\\ w\in\partial\Lambda_{\ell_k}}} \frac{\langle\sigma_0\sigma_v\rangle\langle\sigma_v\sigma_w\rangle\langle\sigma_w\sigma_x\rangle}{\langle\sigma_0\sigma_x\rangle} \le C_6 n^3 \ell_k^3 n^{-4} \le C_7 \ell_k^{-\varepsilon}. \tag{3.21}$$

To bound the probability of F_2, condition on $\Gamma(\mathbf{n}_1)$. The remaining current in $\mathbf{n}_1$ is a sourceless current with depleted coupling constants (see [4, 14, 15] for details on this type of reasoning). The probability that some $v \in \partial\Lambda_n$ and $w \in \partial\Lambda_m$ are connected in $\mathbb{Z}^4 \smallsetminus \Gamma(\mathbf{n}_1)$ to each other can then be bounded by $\langle\sigma_v\sigma_w\rangle\langle\sigma_v\sigma_w\rangle'$ where the $\langle\cdot\rangle'$ denotes an Ising measure with depleted coupling constants (the depletion depends on $\Gamma(\mathbf{n}_1)$ and the switching lemma concerns one current with depletion and one without; we refer to [5] for the statement and proof of the switching lemma in this context, and some applications). At the risk of repeating ourselves, we refer to [4] for an illustration of this line of reasoning. The Griffiths inequality [25] implies that this probability is bounded by $\langle\sigma_v\sigma_w\rangle^2$, which together with (3.2), immediately leads to the following sequence of inequalities:

$$\mathbf{P}^{0x,\varnothing}[F_2] \le \sum_{\substack{v\in\partial\Lambda_n\\ w\in\partial\Lambda_m}} \langle\sigma_v\sigma_w\rangle^2 \le C_8 n^{-\varepsilon}. \tag{3.22}$$

The event F_4 is bounded similarly to F_1, and F_3 similarly to F_2. For $\ell_0 = \ell_0(\varepsilon)$ large enough the sum of the four probabilities does not exceed half of the constant c_5 in (3.20), and the main statement follows. □

[4]We mentioned that a current $\mathbf{n}$ with sources x and y can be seen as the superposition of one path from x to y and loops. The backbone $\Gamma(\mathbf{n})$ is an appropriate choice of such a path induced by an ordering of the edges. Again, we refrain ourselves from providing more details here and refer to the relevant literature for details on this notion.

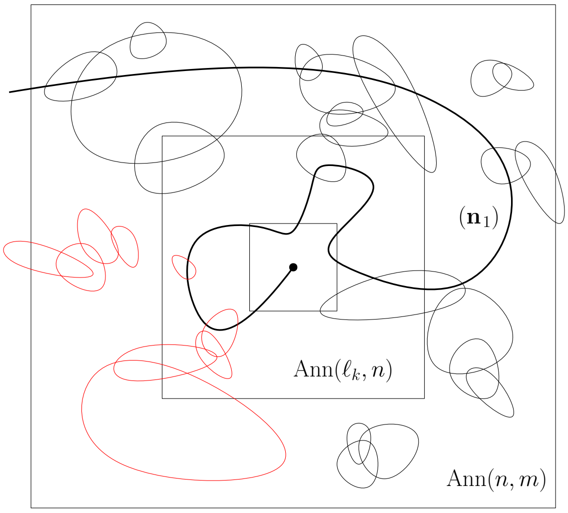

Figure 2: In this picture, $\Gamma(\mathbf{n}_1)$ does only one crossing of $\mathrm{Ann}(\ell_k, n)$, and $\mathbf{n}_1 + \mathbf{n}_2 - \Gamma(\mathbf{n}_1)$ does not cross $\mathrm{Ann}(n, m) \setminus \Gamma(\mathbf{n}_1)$. This prevents the fact that the cluster in red, made of loops in $\mathbf{n}_1 + \mathbf{n}_2 - \Gamma(\mathbf{n}_1)$ would connect an excursion of $\Gamma(\mathbf{n}_1)$ outside of Λ_{ℓ_k} but not reaching $\partial\Lambda_n$ to $\partial\Lambda_m$ (which would potentially create an additional cluster crossing $\mathrm{Ann}(\ell_k, m)$).

Remark 3.5 *The condition $\alpha > 3^4$ is used in the second part of the proof, where we need the exponent connecting the inner and outer radii of annuli to be strictly larger than 3. We did not try to improve on this exponent.*

The second of the above described statements is one of the main innovations of this paper. It concerns a mixing property, which in Section 5.1 will be stated under a more general form and derived unconditionally for every $d \geq 4$.

Proposition 3.6 (Conditional mixing property) *Assume that* (3.2) *holds for the Ising model on $\mathbb{Z}^4$ with $\eta = 0$. For $\alpha > 3^8$, there exists $C > 0$ such that for every $n^\alpha \leq N$, every $x \notin \Lambda_N$, and every events E and F depending on the restriction of $\mathbf{n}$ to edges within Λ_n and outside of Λ_N respectively,*

$$|\mathbf{P}^{0x}_{\beta_c}[E \cap F] - \mathbf{P}^{0x}_{\beta_c}[E]\mathbf{P}^{0x}_{\beta_c}[F]| \leq \frac{C}{\sqrt{\log(N/n)}}. \tag{3.23}$$

The heart of the proof will be the use of a (random) resolution of identity $\mathbf{N}$, meaning a random variable which is concentrated around 1, given by a weighted sum on the cluster of 0 in $\mathbf{n}_1 + \mathbf{n}_2$, where $\partial\mathbf{n}_1 = \{0, x\}$ and $\partial\mathbf{n}_2 = \varnothing$, which will enable us to write

$$\mathbf{P}^{0x}[E \cap F] \approx \mathbf{E}^{0x,\varnothing}[\mathbf{N}\mathbb{I}(\mathbf{n}_1 \in E \cap F)]. \tag{3.24}$$

Since $\mathbf{N}$ will be a certain convex combination of the random variables $\mathbb{I}[y \overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow} 0]/\langle\sigma_0\sigma_y\rangle$, the term on the right will be a convex sum of $\mathbf{P}^{0x,\varnothing}$-probabilities of the events $\{y \overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow} 0, \mathbf{n}_1 \in E \cap F\}$. For each fixed y, we will use the switching principle to transform the sources $\{0, x\}$ and $\varnothing$ of $\mathbf{n}_1$ and $\mathbf{n}_2$ into $\{0, y\}$ and $\{y, x\}$, exchanging at the same time the roles of $\mathbf{n}_1$ and $\mathbf{n}_2$ inside Λ_n without changing anything outside Λ_N. This useful operation has a nice byproduct: the event $\mathbf{n}_1 \in F$ becomes $\mathbf{n}_2 \in F$ which is independent of $\mathbf{n}_1 \in E$. Deducing the mixing from there will be a matter of elementary algebraic manipulations.

The error term will be (almost entirely) due to how concentrated around 1 $\mathbf{N}$ is. In order to prove this fact, we will implement a refine second moment method in which we estimate the expectation sharply and the second moment of $\mathbf{N}$. The proof will require some regularity assumptions on the gradient of the two-point function: for every $x \in \mathbb{Z}^d$,

$$|\nabla_x \langle \sigma_0 \sigma_x \rangle| \le \frac{C}{|x|} \langle \sigma_0 \sigma_x \rangle, \tag{3.25}$$

which follows from (3.2) by an argument that we choose to postpone to Section 4.4 (after the required technology has been introduced).

Proof Let us recall that we are discussing here $\beta = \beta_c$, omitting the symbol from the notation. Fix $\alpha > 3^4$ (the power 4 instead of 8 suffices at this stage) and choose $\varepsilon > 0$ so that $\alpha > (1+\varepsilon)(9+\varepsilon)^2$. Below, the constants C_i are independent of β and $n^\alpha = N \le \xi(\beta)$ (we may assume equality between N and n^α without loss of generality). Introduce two intermediary integers $m \le M$ satisfying that

$$m \ge n^{9+\varepsilon} \ , \ M \ge m^{1+\varepsilon} \ , \ N \ge M^{9+\varepsilon} \tag{3.26}$$

as well as the notation $n_k = 2^k m$ for $k \ge 1$. Set K such that $n_{K+1} \le M < n_{K+2}$. The key to our proof will be the random variable

$$\mathbf{N} := \frac{1}{K} \sum_{k=1}^{K} \frac{1}{\alpha_k} \sum_{y \in \mathrm{Ann}(n_k, n_{k+1})} \mathbb{I}[y \overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow} 0] \quad \text{where} \quad \alpha_k := \sum_{y \in \mathrm{Ann}(n_k, n_{k+1})} \langle \sigma_0 \sigma_y \rangle. \tag{3.27}$$

Combining the regularity assumptions (3.25) and (3.2) with Proposition A.3 (the precise computation is presented in Section 5.2), we find

$$\mathbf{E}^{0x,\varnothing}[\mathbf{N}] = \frac{1}{K} \sum_{k=1}^{K} \frac{1}{\alpha_k} \sum_{y \in \mathrm{Ann}(n_k, n_{k+1})} \frac{\langle \sigma_0 \sigma_y \rangle \langle \sigma_y \sigma_x \rangle}{\langle \sigma_0 \sigma_x \rangle} \ge 1 - \frac{C_1}{K}, \tag{3.28}$$

$$\mathbf{E}^{0x,\varnothing}[\mathbf{N}^2] \le \frac{1}{K^2} \sum_{k,\ell=1}^{K} \frac{1}{\alpha_k \alpha_\ell} \sum_{\substack{y \in \mathrm{Ann}(n_k, n_{k+1}) \\ z \in \mathrm{Ann}(n_\ell, n_{\ell+1})}} \frac{\langle \sigma_0 \sigma_y \rangle \langle \sigma_y \sigma_z \rangle \langle \sigma_z \sigma_x \rangle + \langle \sigma_0 \sigma_z \rangle \langle \sigma_z \sigma_y \rangle \langle \sigma_y \sigma_x \rangle}{\langle \sigma_0 \sigma_x \rangle} \le 1 + \frac{C_2}{K}. \tag{3.29}$$

The Cauchy-Schwarz inequality and the fact that $\mathbf{P}^{0x}[E \cap F] = \mathbf{P}^{0x,\varnothing}[\mathbf{n}_1 \in E \cap F]$ thus imply that

$$|\mathbf{P}^{0x,\varnothing}[\mathbf{n}_1 \in E \cap F] - \mathbf{E}^{0x,\varnothing}[\mathbf{N} \mathbb{I}_{\mathbf{n}_1 \in E \cap F}]| \le \sqrt{\mathbf{E}^{0x,\varnothing}[(\mathbf{N}-1)^2]} \le \frac{C_3}{\sqrt{K}}. \tag{3.30}$$

Now, fix $y \in \mathrm{Ann}(m, M)$ and let $G(y)$ be the event (depending on $\mathbf{n}_1 + \mathbf{n}_2$ only) that there exists $\mathbf{k} \le \mathbf{n}_1 + \mathbf{n}_2$ such that $\mathbf{k} = 0$ on Λ_n, $\mathbf{k} = \mathbf{n}_1 + \mathbf{n}_2$ outside Λ_N, and $\partial \mathbf{k} = \{x, y\}$. We find that

$$\mathbf{P}^{0x,\varnothing}[\mathbf{n}_1 \in E \cap F, y \overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow} 0, G(y)] = \frac{\langle \sigma_0 \sigma_y \rangle \langle \sigma_y \sigma_x \rangle}{\langle \sigma_0 \sigma_x \rangle} \mathbf{P}^{0y,yx}[\mathbf{n}_1 \in E, \mathbf{n}_2 \in F, G(y)], \tag{3.31}$$

where we use the following reasoning: for $\mathbf{m} \in G(y)$, consider the multi-graph $\mathscr{M}$ obtained by duplicating every edge of the graph into $\mathbf{m}(x,y)$ edges. If $G(y)$ occurs, the existence of $\mathbf{k}$ guarantees the existence of a subgraph $\mathscr{K} \subset \mathscr{M}$ with $\partial \mathscr{K} = \{x, y\}$ containing no edge with endpoints in Λ_n and all those of $\mathscr{M}$ with endpoints outside Λ_N, so that the generalized switching principle formulated in [6, Lemma 2.1] implies that

$$\begin{aligned}\sum_{\mathscr{T}\leq\mathscr{M}:\partial\mathscr{T}=\{0,x\}}\mathbb{I}[\mathscr{T}\in E\cap F]&=\sum_{\mathscr{T}\leq\mathscr{M}:\partial\mathscr{T}\Delta\mathscr{K}=\{0,x\}}\mathbb{I}[\mathscr{T}\Delta\mathscr{K}\in E\cap F]\\&=\sum_{\mathscr{T}\leq\mathscr{M}:\partial\mathscr{T}=\{0,y\}}\mathbb{I}[\mathscr{T}\in E,\mathscr{M}\smallsetminus\mathscr{T}\in F],\end{aligned}\tag{3.32}$$

where we allow ourselves the latitude of calling E and F the events defined for multi-graphs corresponding to the events E and F for currents. One gets (3.31) when rephrasing this equality in terms of weighted currents (exactly like in standard proofs of the switching principle, see e.g. [1] or [6] for a closely related reasoning).

Observe now that forgetting about $G(y)$ on the right-hand side of (3.31) gives

$$\mathbf{P}^{0y,yx}[\mathbf{n}_1\in E,\mathbf{n}_2\in F]=\mathbf{P}^{0y}[E]\mathbf{P}^{yx}[F].\tag{3.33}$$

Furthermore, since $x\notin\Lambda_N$ and $y\in\Lambda_m$, (3.25) implies that

$$\Big|\frac{\langle\sigma_0\sigma_y\rangle\langle\sigma_y\sigma_x\rangle}{\langle\sigma_0\sigma_x\rangle}-\langle\sigma_0\sigma_y\rangle\Big|\leq\frac{C_4m}{N}.\tag{3.34}$$

Last but not least, we can bound (from below) $\mathbf{P}^{0x,\varnothing}[G(y)]$ and $\mathbf{P}^{0y,yx}[G(y)]$ as follows. We only briefly describe the argument since we will present it in full details in Section 5.2. The event $G(y)$ clearly contains the event that $\mathrm{Ann}(M,N)$ is not crossed by a cluster in $\mathbf{n}_1$, and $\mathrm{Ann}(n,m)$ is not crossed by a cluster in $\mathbf{n}_2$, since in such case $\mathbf{k}$ can be defined as the sum of $\mathbf{n}_1$ restricted to the clusters intersecting Λ_N^c (this current has no sources) and $\mathbf{n}_2$ restricted to the clusters intersecting Λ_m^c (this current has sources x and y). Now, we can bound the probability of $\mathbf{n}_1$ crossing $\mathrm{Ann}(M,N)$ in the same spirit as we bounded the probabilities for F_1 and F_3 in the previous proof by splitting $\mathrm{Ann}(M,N)$ in two annuli $\mathrm{Ann}(\sqrt{MN},N)$ and $\mathrm{Ann}(M,\sqrt{MN})$, then estimating the probability that the backbone of $\mathbf{n}_1$ crosses the inner annulus more than once, and then the probability that the remaining current (which is sourceless) crosses the outer annulus. Doing the same for the probability that a cluster of $\mathbf{n}_2$ crosses $\mathrm{Ann}(n,m)$, we find that

$$\frac{\langle\sigma_0\sigma_x\rangle}{\langle\sigma_0\sigma_y\rangle\langle\sigma_y\sigma_x\rangle}\mathbf{P}^{0x,\varnothing}[G(y),y\overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow}0]=\mathbf{P}^{0y,yx}[G(y)]\geq1-\frac{C_5}{n^{\varepsilon}}\geq1-C_6\big(\frac{n}{N}\big)^{\varepsilon/(\alpha-1)}.\tag{3.35}$$

Note that we use that $M\geq N^{9+\varepsilon}$ in this part of the proof.

Overall, the value of K and (3.30)–(3.35) put together imply

$$|\mathbf{P}^{0x}[E\cap F]-\sum_{y\in\mathrm{Ann}(m,M)}\delta(y)\mathbf{P}^{0y,\varnothing}[E]\mathbf{P}^{yx,\varnothing}[F]|\leq\frac{C_7}{\sqrt{\log(N/n)}},\tag{3.36}$$

with $\delta(y)=\langle\sigma_0\sigma_y\rangle/(K\alpha_{k(y)})$ where $k(y)$ is such that $y\in\mathrm{Ann}(n_{k(y)},n_{k(y)+1})$.

The end of the proof is now a matter of elementary algebraic manipulations. Applying this inequality twice (once with x and once with x') for F being the full set, we obtain that for every $x,x'\notin\Lambda_N$ and every event E which is depending on Λ_n only,

$$|\mathbf{P}^{0x}[E]-\mathbf{P}^{0x'}[E]|\leq\frac{2C_7}{\sqrt{\log(N/n)}}.\tag{3.37}$$

Now, assume the stronger assumption that $\alpha>3^8$ and fix $m=\lfloor\sqrt{Nn}\rfloor$. Applying

- (3.36) for m and N, the full event and F,
- then (3.37) for n, m and E (note that $m \geq n^{3^4}$),
- and (3.36) for m and N, E and F,

gives that for every $x \notin \Lambda_N$,

$$\begin{aligned}|\mathbf{P}^{0x}[E\cap F]-\mathbf{P}^{0x}[E]\mathbf{P}^{0x}[F]| &\leq |\mathbf{P}^{0x}[E\cap F]-\mathbf{P}^{0x}[E]\sum_y \delta(y)\mathbf{P}^{yx}[F]|+\frac{C_7}{\sqrt{\log(N/n)}}\\ &\leq |\mathbf{P}^{0x}[E\cap F]-\sum_y \delta(y)\mathbf{P}^{0y}[E]\mathbf{P}^{yx}[F]|+\frac{3C_7}{\sqrt{\log(N/n)}}\\ &\leq \frac{4C_7}{\sqrt{\log(N/n)}}. \end{aligned}\tag{3.38}$$

□

Using Lemma 3.4 and Proposition 3.6, we may now establish the clustering of intersections, under (3.2).

Proof of Proposition 3.3 In view of the translation invariance of the claimed statement, we take u to be the origin. Since x and y are at a distance larger than $2\ell_K$ of each other, one of them is at a distance (larger than or equal to) ℓ_K of u. Without loss of generality we take that to be x, and make a similar assumption about z.

Let $\mathscr{S}_K$ denote the set of subsets of $\{1,\dots,K-2\}$ containing *even* integers only and fix $S\in\mathscr{S}_K$. Let A_S be the event that no I_k occurs for $k\in S$. If s denotes the maximal element of S, the mixing property Proposition 3.6 used with $n=\ell_{s-1}$ and $N=\ell_s$ gives

$$\mathbf{P}^{0x,0z,\varnothing,\varnothing}[A_S]\leq \mathbf{P}^{0x,0z,\varnothing,\varnothing}[I_s^c]\mathbf{P}^{0x,0z,\varnothing,\varnothing}[A_{S\setminus\{s\}}]+\frac{C}{\sqrt{\log \ell_{s-1}}}.\tag{3.39}$$

To be precise and honest, we use a multi-current version, with four currents, of the mixing property. We will state and prove this property in Sections 5.1 and 5.2 and ignore this additional difficulty for now. Also, it is here that the stronger restriction $\alpha>3^8$ is used, along with the choice of $\ell_0=\ell_0(\alpha)$, to enable the mixing. Note that we used that the event I_s is expressed in terms of just the restriction of the currents $\mathbf{n}_1,\dots,\mathbf{n}_4$ to $\mathrm{Ann}(\ell_s,\ell_{s+1})$.

Now, the intersection property Lemma 3.4 and an elementary bound on ℓ_{s-1} gives the existence of $c_0>0$ small enough that

$$\mathbf{P}^{0x,0z,\varnothing,\varnothing}[A_S]\leq(1-2c_0)\mathbf{P}^{0x,0z,\varnothing,\varnothing}[A_{S\setminus\{s\}}]+c_0(1-c_0)^{|S|-1}.\tag{3.40}$$

An induction gives immediately that for every $S\in\mathscr{S}_K$,

$$\mathbf{P}^{0x,0z,\varnothing,\varnothing}[A_S]\leq(1-c_0)^{|S|}.\tag{3.41}$$

Let $B_S\subset A_S$ be the event that the clusters of 0 in $\mathbf{n}_1+\mathbf{n}_3$ and $\mathbf{n}_2+\mathbf{n}_4$ do not intersect in any of the annuli $\mathrm{Ann}(\ell_s,\ell_{s+1})$ for $s\in S$. Thanks to Corollary A.2, the probability of B_S increases when removing sources, so that

$$\mathbf{P}^{0x,0z,0y,0t}[B_S]\leq\mathbf{P}^{0x,0z,\varnothing,\varnothing}[B_S]\leq\mathbf{P}^{0x,0z,\varnothing,\varnothing}[A_S]\leq(1-c_0)^{|S|}.\tag{3.42}$$

To conclude, observe that if $\mathbf{M}_0(\mathscr{T};\mathscr{L}_\alpha,K)\leq\delta K$, then there must exist a set $S\in\mathscr{S}_K$ of cardinality at least $(\frac{1}{2}-\delta)K$ such that B_S occurs. We deduce that

$$\begin{aligned}\mathbf{P}^{0x,0z,0y,0t}[\mathbf{M}_0(\mathscr{T};\mathscr{L}_\alpha,K)<\delta K] &\leq \sum_{S\in\mathscr{S}_K:|S|\geq(1/2-\delta)K}\mathbf{P}^{0x,0z,0y,0t}[B_S]\\ &\leq\binom{K/2}{\delta K}(1-c_0)^{(1/2-\delta)K},\end{aligned}\tag{3.43}$$

which implies the claim by appropriately choosing the value of δ. $\square$

4 Weak regularity of the two-point function

In this section, we focus on the Ising model's two-point function and present some old and new observations. The main goal of the section is the proof of the abundance, below the correlation length, of *regular scales* at which the two-point function has properties similar to those it would have under the power-law decay assumption (3.2) of Section 3. Regular scales play a key role in the unconditional proof of Theorem 1.3. In order to obtain the main auxiliary result, stated in Section 4.5, we present here three properties of the two-point function. These may respectively be referred to as *monotonicity* (Section 4.1), a *sliding-scale spatial Infrared Bound* (Section 4.3), and a *gradient estimate* (Section 4.4).

Throughout this section, we set $|J| = \sum_y J_{0,y}$ and

$$S_\beta(x) := \langle \sigma_0 \sigma_x \rangle_\beta. \tag{4.1}$$

The coordinates of an element x of $\mathbb{R}^d$ will be written as $x_1, \dots, x_d$, and $|x|$ taken to denote its infinite norm. We write $\mathbf{e}_i$ for the unit vector in the i-th direction.

Remark 4.1 *The results presented here apply to systems with real valued variables of arbitrary distribution (including variables in the Griffiths-Simon class which is of particular interest for us, see Section 6), and ferromagnetic Reflection-Positive pair couplings of specified properties. We refrain from discussing these generalizations and, unless specified otherwise, focus on the n.n.f. Ising model.*

4.1 Messager-Miracle-Sole monotonicity for the two-point function

The Messager-Miracle-Solé (MMS) inequality [34, 41, 46] states that for models with n.n.f. interactions (and more generally *reflection positive* interactions) in a region Λ endowed with reflection symmetry, the correlation function $\langle \prod_{x\in A} \sigma_x \prod_{x\in B} \sigma_x \rangle_{\Lambda,\beta}$ at sets of sites A and B which are on the same side of a reflection plane, can only decrease when B is replaced by its reflected image, $\mathscr{R}(B)$, i.e.

$$\Big\langle \prod_{x\in A} \sigma_x \prod_{x\in B} \sigma_x \Big\rangle_{\Lambda,\beta} \ \geq\ \Big\langle \prod_{x\in A} \sigma_x \prod_{x\in \mathscr{R}(B)} \sigma_x \Big\rangle_{\Lambda,\beta}. \tag{4.2}$$

In the infinite volume limit on $\mathbb{Z}^d$, this principle can be invoked for reflections with respect to

- hyperplanes passing through vertices or mid-edges, i.e. reflections changing only one coordinate x_i, which is sent to $L - x_i$ for some fixed $L \in \frac{1}{2}\mathbb{Z}$,
- "diagonal" hyperplanes, i.e. reflections changing only two coordinates x_i and x_j, which are sent to $x_j \pm L$ and $x_i \mp L$ respectively, for some $L \in \mathbb{Z}$.

In particular, this implies the following useful comparison principle (see Fig. **??** for an illustration).

Proposition 4.2 (MMS monotonicity) *For the n.n.f. Ising model on $\mathbb{Z}^d$ with $d \geq 1$, along the principal axes two point function is monotone decreasing in $|x|$, and for any $x = (x_1, ... x_d) \in \mathbb{Z}^d$*

$$S_\beta((\|x\|_\infty, 0_\perp) \geq S_\beta(x) \geq S_\beta((\|x\|_1, 0_\perp) \tag{4.3}$$

where $\|x\|_1 = \sum_{j=1}^d |x_j|$, $\|x\|_\infty \max_j |x_j|$, and $0_\perp$ is the null vector in a $\mathbb{Z}^{d-1}$.

The above carries the useful implication that for any $x, y \in \mathbb{Z}^d$ with $\|y\|_\infty \geq d\,\|x\|_\infty$

$$S_\beta(x) \geq S_\beta(y) \tag{4.4}$$

(Since $\|x\|_1 \leq d\,\|x\|_\infty \leq \|y\|_\infty$, by (4.3) the two quantities are on the correspondingly opposite sides of $S_\beta((\|x\|_1, 0_\perp))$.)

The above is useful in extracting the implications of the following well-known principle (of which an early analog can be found in Hammersley's analysis of percolation [32]).

Lemma 4.3 *For every ferromagnetic Ising model on $\mathbb{Z}^d$ $(d \geq 1)$ with coupling constants that are invariant under translations, every finite $0 \in \Lambda \subset \mathbb{Z}^d$ and every $y \notin \Lambda$,*

$$S_\beta(y) \leq \sum_{\substack{u \in \Lambda \\ v \notin \Lambda}} S_\beta(u)\,\beta J_{u,v}\,S_\beta(y - v). \tag{4.5}$$

This statement is a mild extension of Simon's inequality which was originally formulated for the n.n.f. Ising models [49]. Being spin-dimension balanced, it is valid also for the more general Simon-Griffiths class of variables discussed in Section 6 below (and more general interactions)[5].

Corollary 4.4 (Lower bound on S_β) *For the n.n.f. Ising model on $\mathbb{Z}^d$ $(d \geq 1)$, there exists $c = c(d) > 0$ such that for every $\beta \leq \beta_c$ and $x \in \mathbb{Z}^d$,*

$$S_\beta(x) \geq \frac{c}{\beta|J|\,|x|^{d-1}} \exp\Big(-\frac{d|x| + 1}{\xi(\beta)}\Big). \tag{4.6}$$

Proof Let us introduce

$$Y_\beta(\Lambda) := \sum_{\substack{u \in \Lambda \\ v \notin \Lambda}} S_\beta(u)\,\beta J_{u,v}. \tag{4.7}$$

Applying (4.5) with Λ_L and $y = n\mathbf{e}_1$, and iterating it $\lfloor \frac{n}{L+1} \rfloor$ times (i.e. as many as possible without reducing the last factor to a distance shorter than L), we get

$$\beta|J| S_\beta(n\mathbf{e}_1) \leq Y_\beta(\Lambda_L)^{\lfloor \frac{n}{L+1} \rfloor}. \tag{4.8}$$

Since $\lim_n S_\beta(n\mathbf{e}_1)^{1/n} = e^{-1/\xi(\beta)}$, we deduce that

$$e^{-1/\xi(\beta)} \leq Y_\beta(\Lambda_L)^{\frac{1}{L+1}}. \tag{4.9}$$

On the other hand, by (4.4), for each $x \in \mathbb{Z}^d$, $S_\beta(u) \leq S_\beta(x)$ for all $u \in \partial\Lambda_{d|x|}$, and hence

$$\frac{Y_\beta(\Lambda_{d|x|})}{|\partial\Lambda_{d|x|}|} \leq \beta|J| S_\beta(x). \tag{4.10}$$

The substitution of (4.9) in (4.10) yields the claimed lower bound (4.6). $\square$

[5] The factor $S_\beta(u)$ in (4.5) can also be replaced by the finite volume expectation $\langle\sigma_0\sigma_u\rangle_\Lambda$, as in Lieb's improvement of Simon's inequality [39]. Both versions have an easy proof through a simple application of the switching lemma, in its mildly improved form.

4.2 The two-point function's Fourier transform

In view of the model's translation invariance, it is natural to expect the Fourier transform to be of use. It indeed appears at different levels. For the infinite system on $\mathbb{Z}^d$ at $\beta < \beta_c$ one learns useful information by considering the Fourier transform of the two-point function

$$\widehat{S}_\beta(p) := \sum_{x \in \mathbb{Z}^d} e^{ip\cdot x} S_\beta(x). \tag{4.11}$$

where p ranges over $[-\pi, \pi]^d$.

The above sum is absolutely convergent for $\beta < \beta_c$, though not at β_c. However for any β the finite-volume model in $\Lambda_L = (-L, L]^d$ may be considered in terms of the spin-Fourier modes, defined as

$$\widehat{\sigma}_\beta(p) := \frac{1}{\sqrt{(2L)^d}} \sum_{x \in (-L,L]^d} e^{ip\cdot x} \sigma_x. \tag{4.12}$$

with p ranging over $\Lambda_L^* := [-\pi, \pi]^d \cap \frac{\pi}{L}\mathbb{Z}^d$.

The above *spin-wave modes* are especially relevant in case the Hamiltonian is taken with the periodic boundary conditions, under which sites $x, y \in \Lambda_L$ are neighbors if either $\|x - y\| = 1$ or $|y_i - x_i| = 2L - 1$ for some $i \in \{1, \dots, d\}$. With these boundary conditions, the model is invariant under cyclic shifts, and its Hamiltonian decomposes into a sum of single-mode contributions:

$$H_{\Lambda_R}(\sigma) = \sum_{p \in \Lambda_L^*} \mathscr{E}(p) \, |\widehat{\sigma}_\beta(p)|^2, \tag{4.13}$$

where

$$\mathscr{E}(p) := 2 \sum_{j=1}^{d} [1 - \cos(p_j)] = 4 \sum_{j=1}^{d} \sin^2(p_j/2) \,. \tag{4.14}$$

Among the various relations in which the Fourier transform plays a useful role, the following statements will be of relevance for our discussion.

i) The finite volume version of the two point function's Fourier transform (4.11) reemerges in its dual role, as the second moment of the spin-wave mode:

$$\widehat{S}_\beta^{(R)}(p) := \sum_{x \in \Lambda_L} e^{ip\cdot x} \langle \sigma_0 \sigma_x \rangle_{\Lambda_L, \beta} = \langle |\widehat{\sigma}_\beta(p)|^2 \rangle_{\Lambda_L, \beta} \geq 0 \,. \tag{4.15}$$

ii) The gaussian domination (aka *infrared*) bound

$$\mathscr{E}(p) \widehat{S}_\beta(p) \leq \frac{1}{2|J|\beta} \,. \tag{4.16}$$

The bound may remind one of equipartition law, and in this form it appeals to the physics intuition. Alas, it has so far been proven only for reflection positive interactions [20, 21].

iii) The Parseval-Plancherel identity sum rule

$$\langle \sigma_0^2 \rangle_{\Lambda_L, \beta} = \frac{1}{|\Lambda_L|} \sum_{p \in \Lambda_L^*} \widehat{S}_\beta^{(R)}(p) \,. \tag{4.17}$$

As was pointed out in [21], the combination of (4.17) with (4.16) yields a (then novel) way to prove the existence of spontaneous magnetization for β high enough.

More explicitly, in (4.17) one may note that the Infrared Bound (4.16) does not provide any direct control on the $p = 0$ term, since $\mathscr{E}(0) = 0$. And, in fact, the hallmark of the low temperature phase ($\beta > \beta_c$) is that this single value of the summand attains macroscopic size:

$$\widehat{S}^{(R)}_{\beta}(0) \approx |\Lambda_L|\, M(\beta)^2 \tag{4.18}$$

with $M(\beta)$ the Ising model's spontaneous magnetization.

For the disordered regime, where $M(\beta) = 0$ the sum rule combined with the Infrared Bound implies that for every $\beta < \beta_c$,

$$\langle \sigma_0^2 \rangle_\beta = \int_{[-\pi,\pi]^d} \widehat{S}_\beta(p) dp \;\leq\; \int_{[-\pi,\pi]^d} \frac{dp}{2|J|\beta\, \mathscr{E}(p)} \; . \tag{4.19}$$

Since $\mathscr{E}(p)$ vanishes only at $p = 0$ and there at the rate $\mathscr{E}(p) \sim |p|^2$, the integral is convergent for $d > 2$ and one gets

$$\langle \sigma_0^2 \rangle_{\beta_c} \;\leq\; \frac{C_d}{2|J|\beta_c} \tag{4.20}$$

with $C_d < \infty$ for $d > 2$. This bound will be used in our discussion of the more general single spin distribution (including the ϕ^4 type measures) in Section 6.

We shall also use the following statement on the relation between the finite volume and the infinite volume states.

Proposition 4.5 *For the Ising model on $\mathbb{Z}^d$ ($d \geq 1$) with translation invariant finite range interactions, for any $\beta < \beta_c$:*

1. *the system has only one infinite volume Gibbs equilibrium state.*
2. *the correction functions of that state satisfy, for any finite $A \subset \mathbb{Z}^d$, and any sequence of finite volumes $V_n \subset \mathbb{Z}^d$ which asymptotically cover any finite region,*

$$\Big\langle \prod_{x \in A} \sigma_x \Big\rangle_\beta \;=\; \lim_{V_n \to \mathbb{Z}^d} \Big\langle \prod_{x \in A} \sigma_x \Big\rangle^{(b.c.)}_{V_n,\beta} \tag{4.21}$$

   with $\langle - \rangle^{(b.c.)}_{V_n,\beta}$ denoting the correlation function under boundary conditions which may include either cross-boundary spin couplings (e.g. periodic), or arbitrary specified values of $\sigma_{\partial V_n}$.

3. *with the finite volumes taken as the rectangular domains Λ_L, also the Fourier Transform functions converge, i.e. for any $p \in [-\pi,\pi]^d$, and sequence as in* (4.21)

$$\widehat{S}_\beta(p) \;=\; \lim_{n \to \infty} \sum_{x \in V_n} e^{ip\cdot x} \langle \sigma_0 \sigma_x \rangle^{(b.c.)}_{V_n,\beta} \tag{4.22}$$

The statement follows by standard arguments which we omit here. The main ingredients are the exponential decay of correlations, which at any $\beta < \beta_c$ are exponentially bounded, uniformly in the volume, and the FKG inequality. The first two points hold also for $\beta = \beta_c$ [5]. However not the last, (4.22), since at the critical temperature the correlation function is not summable.

We shall employ the freedom which Proposition 4.5 provides in establishing the different monotonicity properties of $S(p)$ in p.

4.3 The spectral representation and a sliding-scale Infrared Bound

We next present a Fourier transform counterpart (though one derived by different means) of the Messager-Miracle-Sole monotonicity stated in Section 4.1, and use it for a sliding-scale extension of the infrared bound (4.16). The results, which include both old [23, 51] and new observations, are based on the relation of the two-point function with the transfer matrix, and the positivity of the latter.

The transfer matrix has been the source of many insights on the structure of statistical mechanical systems with finite range interactions. Its appearance can be seen in Ising's study of one dimensional systems, for which it permits a simple proof of the absence of phase transition. Also, in higher dimensions it has played an essential role in many important developments [21, 23, 47], some of which rely on positivity properties. Here we shall use the following consequences of its spectral representation for the two-point function.

Proposition 4.6 (Spectral Representation) *For the n.n.f. Ising model on $\mathbb{Z}^d$ $(d \geq 1)$, at $\beta < \beta_c$, for every bounded function $v : \mathbb{Z}^{d-1} \to \mathbb{C}$, there exists a positive measure $\mu_{v,\beta}$ of finite mass*

$$\int_{1/\xi(\beta)}^{\infty} d\mu_{v,\beta}(a) = \sum_{x_\perp, y_\perp \in \mathbb{Z}^{d-1}} v_{x_\perp} \overline{v_{y_\perp}} S_\beta((0, y_\perp - x_\perp)) \tag{4.23}$$

such that for every $n \in \mathbb{Z}$,

$$\sum_{x_\perp, y_\perp \in \mathbb{Z}^{d-1}} v_{x_\perp} \overline{v_{y_\perp}} S_\beta((n, x_\perp - y_\perp)) = \int_{1/\xi(\beta)}^{\infty} e^{-a|n|} d\mu_{v,\beta}(a)\,. \tag{4.24}$$

and for any $p = (p_1, p_\perp) \in [-\pi, \pi]^d$

$$\widehat{S}_\beta(p) = \int_0^\infty \frac{e^a - e^{-a}}{\mathscr{E}_1(p_1) + (e^{a/2} - e^{-a/2})^2} d\mu_{v_{p_\perp},\beta}(a), \tag{4.25}$$

with $\mathscr{E}_1(k) := 2[1 - \cos(k)] = 4\sin^2(k/2)$.

Although the spectral representation is quite well-known (cf. [23] and references therein) for completeness of the presentation we include the derivation of (A.20) in the Appendix. Eq. (4.25) then follows by applying (4.25) to the function $v_{p_\perp}(x_\perp) := e^{ip_\perp \cdot x_\perp}$ and taking the Fourier transform in the first coordinate. It is of relevance to note that the proof of the spectral representation is developed first in the context of finite volumes, with boundary conditions which are periodic in the directions of the principal axes, and then taking the infinity volume limit.

Of particular interest for us are the following implications of (4.25) (the first was noted and applied in [23]).

Proposition 4.7 *For the n.n.f. Ising model on $\mathbb{Z}^d$ $(d \geq 1)$, at any $\beta < \beta_c$:*

1. *$\widehat{S}_\beta(p_1, p_2, \dots, p_d)$ is monotone decreasing in each $|p_j|$, over $[-\pi, \pi]$,*
2. *$\mathscr{E}_1(p_1)\widehat{S}_\beta(p)$ and $|p_1|^2 \widehat{S}_\beta(p_1)$ are monotone increasing in $|p_1|$,*
3. *the function*

$$\widehat{S}_\beta^{(mod)}(p) := \widehat{S}_\beta(p) + \widehat{S}_\beta(p + \pi(1, 1, 0, \dots, 0)) \tag{4.26}$$

is monotone decreasing in δ along the line of constant $\{p_3, \dots, p_d\}$ and

$$(p_1, p_2) \;=\; (|p_1 - p_2| + \delta, |p_1 - p_2| - \delta)\,, \quad \delta \in [0, |p_{1,2}|]\,. \tag{4.27}$$

and the above remains true under any permutation of the indices.

The correction in (4.26) is insignificant in the regime where $\widehat{S}_\beta(p)$ is large. That is so since $|\widehat{S}_\beta(p+\pi(1,1,0,...,0))| \le C/\beta$ uniformly for $|p| \le \pi/2$. (The main term diverges in the limit $\beta \nearrow \beta_c$ and $p \to 0$.)

Proof The first two statements are implied by the combination of (4.25) with the observation that each of the following functions is monotone in $k \in [0,\pi]$: $k \mapsto \mathscr{E}_1(k)$, $k \mapsto k^2/\mathscr{E}_1(k)$, and for each $a \ge 0$, $k \mapsto \mathscr{E}_1(k)/(\mathscr{E}_1(k) + (e^{a/2} - e^{-a/2})^2)$.

In general terms, the third statement is based on the application of the transfer matrix in the diagonal direction, in each step increasing by 1 the value of $x_1 + x_2$. The transfer is by a pair of alternating application of, T and T^*, mapping back and forth between the space of spin configurations on the subgraphs of Λ_L defined by the parity of $x_1 + x_2$. However, a situation similar to the previously discussed case is recovered by considering the transfer by TT^* which maps from one even plane to the next even plane.

More explicitly: to produce the spectral representation one may start by considering a partially rotated rectangular region, whose main axes are associated with the coordinate system $(x_1+x_2, x_1-x_2, x_3, \dots, x_d)$. The finite-volume Hamiltonian is taken with the correspondingly modified periodic boundary conditions which produce cyclicity in these directions. As stated in (4.22), for $\beta < \beta_c$ the change does not affect the two-point function's infinite volume limit (although the correlation function is modified at distances approaching the box's shortest dimension, and beyond it).

For the rotated system the argument by which monotonicity was proven above for the Cartesian directions, applies to two point function's restriction to the even sub-lattice, since for that one has the single and positive transfer matrix TT^*. This function's Fourier transform is proportional to $\widehat{S}^{(mod)}$, of (4.26), since

$$\begin{aligned} \widehat{S}^{(mod)}_\beta(p) &= \sum_{x\in\Lambda_L} e^{ip\cdot x} S_\beta(0,x) \sum_{k=0,1} e^{i\pi(x_1+x_2)k} \\ &= 2 \sum_{\substack{x\in\Lambda_L \\ x_1+x_2 \text{ even}}} e^{ip\cdot x} S_\beta(0,x). \end{aligned} \tag{4.28}$$

Thus, the third monotonicity statement follows by a direct adaptation of the proof of the first one. □

Corollary 4.8 *The two-point function of the n.n.f. Ising model on $\mathbb{Z}^d$ $(d \ge 1)$ at any $\beta < \beta_c$ satisfies, for all $p \in [-\pi/2, \pi/2]^d$,*

$$\widehat{S}_\beta(\|p\|_\infty, 0_\perp) \;\ge\; \widehat{S}_\beta(p) \;\ge\; \widehat{S}_\beta(\|p\|_1, 0_\perp) - \frac{C}{\beta}, \tag{4.29}$$

with C which depends only on the dimension.

Proof The inequality follows from Proposition 4.7 through the monotonicity lines used for (4.3). □

The previous bound combined with the second statement in Proposition 4.7 yield an interesting consequence for the behaviour of the *susceptibility* truncated at a distance ℓ, which we define as

$$\chi_L(\beta) := \sum_{x\in\Lambda_L} S_\beta(x). \tag{4.30}$$

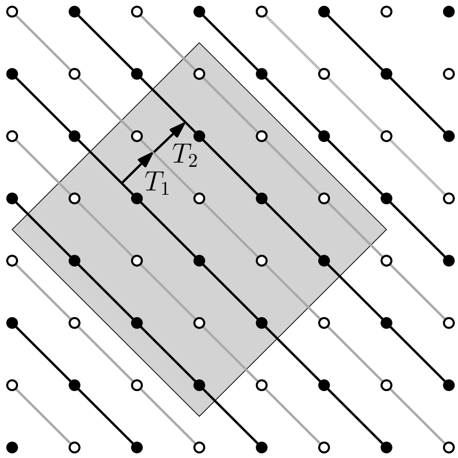

Figure 3: The split of Z^2 into even and odd sub-lattices and their stratification into intertwined diagonal hyperplanes. The partition function of $(-L, L]^2$ with rotated-periodic boundary condition can be written as $Z_{2L} = \mathrm{tr}(T_2 T_1)^L$, with T_j a pair of conjugate mapping between the Hilbert spaces of the even and the odd hyperplanes. The product $T_2 T_1 = T_1^* T_1$ provides the even subgraph's transfer matrix in one of this graph's principal directions.

Theorem 4.9 (Sliding-scale Infrared Bound) *For any n.n.f. model on $\mathbb{Z}^d$ $(d > 2)$, there exists a constant $C = C(d) > 0$ such that for every $\beta \le \beta_c$ and $L \ge \ell \ge 1$,*

$$\frac{\chi_L(\beta)}{L^2} \le \frac{C}{\beta} \frac{\chi_\ell(\beta)}{\ell^2}. \tag{4.31}$$

The case $\ell = 1$ is in essence similar to the Infrared Bound (4.16), as is explained below, so that (4.31) may be viewed as a *sliding-scale* version of this inequality. One may also note that (4.31) is a sharp improvement (replacing the exponent d by 2) on the more naive application of the Messager-Miracle-Sole inequality giving that for every $L \ge \ell \ge 1$,

$$\frac{\chi_L(\beta)}{L^d} \le \frac{\chi_\ell(\beta)}{\ell^d}. \tag{4.32}$$

Proof Let us first note that it suffices to prove the claim for all $\beta < \beta_c$, with a uniform constant C. Its extension to the critical point can be deduced from the continuity

$$S_\beta(x) \;=\; \lim_{\beta \to \beta_c} S_{\beta_c}(x) \tag{4.33}$$

(which follows from the main result of [5]). This observation allows us to apply the monotonicity results discussed above.

Below, the constants C_i are to be understood as dependend on d only. Consider the smeared version of $\chi_L(\beta)$ defined by

$$\widetilde{\chi}_L(\beta) := \sum_{x \in \mathbb{Z}^d} e^{-(\|x\|_2/L)^2} S_\beta(x). \tag{4.34}$$

with $\|p\|_2^2 = \sum_{i=1}^d p_i^2$. The MMS monotonicity statement (4.4) implies that

$$e^{-d} \chi_L(\beta) \le \widetilde{\chi}_L(\beta) \le C_1 \chi_L(\beta) \tag{4.35}$$

for every L, so that it suffices to prove that for every $L \geq \ell \geq 1$,

$$\frac{\widetilde{\chi}_L(\beta)}{L^2} \leq C_2 \frac{\widetilde{\chi}_\ell(\beta)}{\ell^2}. \tag{4.36}$$

We will work in Fourier space, and use the identity

$$\widetilde{\chi}_L(\beta) = L^d \int_{[-\pi,\pi]^d} e^{-\|p\|_2^2 L^2}\, \widehat{S}_\beta(p) dp\,. \tag{4.37}$$

Now, let

$$A := \{p \in [-\tfrac{\pi\ell}{L}, \tfrac{\pi\ell}{L}]^d : |p_1| = \|p\|_\infty\}. \tag{4.38}$$

Using the symmetries of $\widehat{S}_\beta$ and the decay of Corollary 4.8, we find that

$$\int_{[-\pi,\pi]^d} e^{-\|p\|_2^2 L^2}\, \widehat{S}_\beta(p) dp \leq (d + C_3 e^{-\ell^2}) \int_A e^{-\|p\|_2^2 L^2}\, \widehat{S}_\beta(p) dp. \tag{4.39}$$

Since $|p_1| = \|p\|_\infty$ for $p \in A$ and $\|p\|_\infty \geq \|p\|_1/d$, the second property of Proposition 4.7 and Corollary 4.8 give that

$$\widehat{S}_\beta(p) \leq \widehat{S}_\beta(\|p\|_\infty, 0_\perp) \leq (d\tfrac{L}{\ell})^2 \widehat{S}_\beta(\tfrac{L}{\ell}\|p\|_1, 0_\perp) \leq (d\tfrac{L}{\ell})^2 (\widehat{S}_\beta(\tfrac{L}{\ell}p) + C/\beta). \tag{4.40}$$

Using this inequality and making the change of variable $p \mapsto q = \frac{L}{\ell}p$ gives

$$\int_A \exp[-\|p\|_2^2 L^2] \widehat{S}_\beta(p) dp \leq C_4\, (\tfrac{\ell}{L})^{d-2} \Big(\int_{[-\pi,\pi]^d} \exp[-\|q\|_2^2 \ell^2] \widehat{S}_\beta(q) dq + C_5/\beta \Big), \tag{4.41}$$

which after plugging in (4.39) and taking the Fourier transform implies that

$$\widetilde{\chi}_L(\beta) \leq C_6 (\tfrac{\ell}{L})^{d-2} (\widetilde{\chi}_\ell(\beta) + C_5/\beta). \tag{4.42}$$

The inequality (4.36) follows from the fact that $\widetilde{\chi}_\ell(\beta) \geq 1$, so that the constant C_5/β can be removed by changing C_6 into a larger constant C_7/β. $\square$

Inequality (4.4) and then the sliding-scale Infrared Bound with $L = |x|$ and $\ell = 1$ (4.31) implies that for every $x \in \mathbb{Z}^d$,

$$S_\beta(x) \;\leq\; \frac{C_1}{|x|^d} \sum_{y \in \mathrm{Ann}(d|x|, 2d|x|)} S_\beta(y) \;\leq\; \frac{C_1}{|x|^d} \chi_{2d|x|}(\beta) \;\leq\; \frac{C_2 \langle \sigma_0^2 \rangle_\beta}{|x|^{d-2}}. \tag{4.43}$$

The factor $\langle \sigma_0^2 \rangle_\beta$ in the upper bound may seem pointless for the Ising model where it is simply equal to 1, but it becomes very important when studying unbounded spins, as in Section 6, where it is essential for a dimensionless improved tree diagram bound.

It may be noted that the combination of (4.43) with (4.19) leads to the more standard formulation [20, 21] of the Infrared Bound in x-space:

$$S_\beta(x) \leq \frac{C}{\beta |J|\, |x|^{d-2}}. \tag{4.44}$$

4.4 Regularity of the gradient of the two-point function

Proposition 4.10 (gradient estimate) *For the n.n.f. Ising model, there exists $C = C(d) > 0$ such that for every $\beta \le \beta_c$, every $x \in \mathbb{Z}^d$ and every $1 \le i \le d$,*

$$|S_\beta(x \pm \mathbf{e}_i) - S_\beta(x)| \le \frac{F(|x|)}{|x|} S_\beta(x), \tag{4.45}$$

where

$$F(n) := C \frac{S_\beta(dn\mathbf{e}_1)}{S_\beta(n\mathbf{e}_1)} \log\Big(\frac{2S_\beta(\frac{n}{2}\mathbf{e}_1)}{S_\beta(n\mathbf{e}_1)}\Big). \tag{4.46}$$

The previous proposition is particularly interesting when $S_\beta(2dn\mathbf{e}_1) \ge c_0 S_\beta(\frac{n}{2}\mathbf{e}_1)$, in which case we obtain the existence of a constant $C_0 = C_0(c_0, d) > 0$ such that for every $x \in \partial\Lambda_n$ and $1 \le i \le d$,

$$|S_\beta(x \pm \mathbf{e}_i) - S_\beta(x)| \le \frac{C_0}{|x|} S_\beta(x). \tag{4.47}$$

Proof Without loss of generality, we may assume that $x = (|x|, x_\perp)$. We first assume that $i = 1$. Introduce the three sequences $u_n := S_\beta(n\mathbf{e}_1)$, $v_n := S_\beta((n, x_\perp))$ and $w_n := u_n + v_n$. The spectral representation applied to the function v being the sum of the Dirac functions at $0_\perp$ and $x_\perp$ implies the existence of a finite measure $\mu_{x_\perp,\beta}$ such that

$$w_n = \int_0^\infty e^{-na} d\mu_{x_\perp,\beta}(a). \tag{4.48}$$

Cauchy-Schwarz gives $w_n^2 \le w_{n-1}w_{n+1}$, which when iterated between n and $n/2$ (assume n is even, the odd case is similar) leads to

$$\frac{w_{n+1}}{w_n} \ge \Big(\frac{w_n}{w_{n/2}}\Big)^{2/n} \ge 1 - \tfrac{2}{n} \log\Big(\frac{w_{n/2}}{w_n}\Big). \tag{4.49}$$

We now use that $u_{n/2} \ge v_{n/2}$, $u_n \ge v_n$, and $u_n \ge u_{n+1}$ which are all consequences of the Messager-Miracle-Sole inequality. Together with trivial algebraic manipulations, we get

$$v_{n+1} \ge v_n - \frac{4\log(2u_{n/2}/u_n)}{n} u_n. \tag{4.50}$$

The bound we are seeking corresponds to $n = |x|$.

To get the result for $i \ne 1$, use the Messager-Miracle-Sole inequality applied twice to get that

$$|S_\beta(x \pm \mathbf{e}_i) - S_\beta(x)| \le S_\beta(x - d\mathbf{e}_1) - S_\beta(x + d\mathbf{e}_1), \tag{4.51}$$

and then refer to the previous case to conclude (to be perfectly honest one obtains the result for $n = |x| - d$, but the proof can be easily adapted to get the result for $n = |x|$). □

Remark 4.11 *When $x = n\mathbf{e}_1$ and $i = 1$, running through the lines of the previous proof shows that one can take $F(n) = 2\log(S_\beta(n\mathbf{e}_1)/S_\beta(\frac{n}{2}\mathbf{e}_1))$ which is bounded by $(2+o(1))\log n$ thanks to the lower bound (4.6) and the Infrared Bound (4.44). We therefore get that for every $n \le \xi(\beta)$,*

$$S_\beta(n\mathbf{e}_1) - S_\beta((n+1)\mathbf{e}_1) \le (2 + o(1)) \frac{\log n}{n} S_\beta(n\mathbf{e}_1). \tag{4.52}$$

It would be of interest to remove the $\log n$ factor, as this would enable a proof that $S_\beta(n\mathbf{e}_1)$ does not drop too fast between different scales.

4.5 Regular scales

Using the dyadic distance scales, we shall now introduce the notion of *regular scales*, which in essence means that on the given scale the two-point function has the properties which in the conditional proof of Section 3, were available under the assumption (3.2).

Definition 4.12 *Fix $c, C > 0$. An annular region* $\mathrm{Ann}(n/2, 4n)$ *is said to be* regular *if the following four properties are satisfied:*

P1 for every $x, y \in \mathrm{Ann}(n/2, 4n)$, $S_\beta(y) \le C S_\beta(x)$;

P2 for every $x, y \in \mathrm{Ann}(n/2, 4n)$, $|S_\beta(x) - S_\beta(y)| \le \frac{C|x-y|}{|x|} S_\beta(x)$;

P3 for every $x \in \Lambda_n$ and $y \notin \Lambda_{Cn}$, $S_\beta(y) \le \frac{1}{2} S_\beta(x)$;

P4 $\chi_{2n}(\beta) \ge (1+c)\chi_n(\beta)$.

A scale k is said to be regular *if the above holds for $n = 2^k$, and a vertex $x \in \mathbb{Z}^d$ will be said to be in a regular scale if it belongs to an annulus with the above properties.*

One may note that P1 follows trivially from P2 but we still choose to state the two properties independently (the proof would work with weaker versions of P2 so one can imagine cases where the notion of regular scale could be used with a different version of P2 not implying P1).

Under the power-law assumption (3.2) of Section 3 every scale is regular at criticality. However, for now we do not have an unconditional proof of that. For an unconditional proof of our main results, this gap will be addressed through the following statement, which is the main result of this section.

Theorem 4.13 (abundance of regular scales) *Fix $d > 2$ and $\alpha > 2$. There exist $c = c(d) > 0$ and $C = C(d) > 0$ such that for every $n^\alpha \le N \le \xi(\beta)$, there are at least $c\log_2(N/n)$ regular scales k with $n \le 2^k \le N$.*

Proof The lower bound (4.4) for S_β and the Infrared Bound (4.44) imply that

$$\chi_N(\beta) \ge c_0 N \ge c_0 (N/n)^{(\alpha-2)/(\alpha-1)} n^2 \ge c_1 (N/n)^{(\alpha-2)/(\alpha-1)} \chi_n(\beta). \tag{4.53}$$

Using the sliding-scale Infrared Bound (4.32), there exist $r, c_2 > 0$ (independent of n, N) such that there are at least $c_2 \log_2(N/n)$ scales $m = 2^k$ between n and N such that

$$\chi_{rm}(\beta) \ge \chi_{4dm}(\beta) + \chi_m(\beta). \tag{4.54}$$

Let us verify that the different properties of regular scales are satisfied for such an m. Applying (4.4) in the first inequality, the assumption (4.54) in the second, and (4.4) in the third, one has

$$|\mathrm{Ann}(4dm, rm)| S_\beta(4dm\mathbf{e}_1) \ge \chi_{rm}(\beta) - \chi_{4dm}(\beta) \ge \chi_m(\beta) \ge |\Lambda_{m/(4d)}| S_\beta(\tfrac{1}{4}m\mathbf{e}_1). \tag{4.55}$$

This implies that $S_\beta(4dm\mathbf{e}_1) \ge c_0 S_\beta(\frac{1}{4}m\mathbf{e}_1)$, which immediately gives P1 by (4.4) for S_β and P2 by the gradient estimate given by Proposition 4.10. Also, the fact that $S_\beta(x) \ge S_\beta(4dm\mathbf{e}_1) \ge \frac{c_3}{m^d}\chi_m(\beta)$ for every $x \in \mathrm{Ann}(m, 2m)$ implies P4. To prove P3, observe that for every R, the previous displayed inequality together with the sliding-scale Infrared Bound (4.31) give that for every $y \notin \Lambda_{dRm}$ and $x \in \Lambda_m$,

$$|\Lambda_{Rm}| S_\beta(y) \le \chi_{Rm}(\beta) \le C_4 R^2 \chi_m(\beta) \le C_5 R^2 m^d S_\beta(x), \tag{4.56}$$

which implies the claim for C and c respectively large and small enough using here the assumption that $d > 2$. □

5 Unconditional proofs of the Ising's results

In this section, we prove our results for every $\beta \leq \beta_c$ without making the power-law assumption of Section 3. We emphasize that unlike the introductory discussion of that section, the proofs given below are unconditional. The discussion is also not restricted to the critical point itself and covers more general approaches of the scaling limits, from the side $\beta \leq \beta_c$ (hence the correlation length will be mentioned in several places). However, at this stage the discussion is still restricted to the n.n.f. Ising model.

5.1 Unconditional proofs of the intersection-clustering bound and Theorem 1.3 for the Ising model

The notation remains as in Section 3. The endgame in this section will be the unconditional proof of the intersection-clustering bound that we restate below in the right level of generality. The main modification is that the sequence $\mathscr{L}$ of integers ℓ_k will be chosen dynamically, adjusting it to the behaviour of the two-point function. More precisely, recall the definition of the bubble diagram $B_L(\beta)$ truncated at a distance L. Fix $D \gg 1$ and define recursively a (possibly finite) sequence $\mathscr{L}$ of integers $\ell_k = \ell_k(\beta, D)$ by the formula $\ell_0 = 0$ and

$$\ell_{k+1} = \inf\{\ell : B_\ell(\beta) \geq D B_{\ell_k}(\beta)\}. \tag{5.1}$$

By the Infrared Bound (4.44), $B_L - B_\ell \leq C_0 \log(L/\ell)$ (in dimension $d = 4$) from which it is a simple exercise to deduce that under the above definition

$$D^k \leq B_{\ell_k}(\beta) \leq C D^k \tag{5.2}$$

for every k and some large constant C independent of k.

Proposition 5.1 (clustering bound) *For $d = 4$ and D large enough, there exists $\delta = \delta(D) > 0$ such that for every $\beta \leq \beta_c$, every $K > 3$ with $\ell_K \leq \xi(\beta)$, and every $u, x, y, z, t \in \mathbb{Z}^4$ with mutual distances between x, y, z, t larger than $2\ell_K$,*

$$\mathbf{P}_\beta^{ux,uz,uy,ut}[\mathbf{M}_u(\mathscr{T}; \mathscr{L}, K) < \delta K] \leq 2^{-\delta K}. \tag{5.3}$$

Before proving this proposition, let us explain how it implies the improved tree diagram bound.

Proof of Theorem 1.3 Choose D large enough that the previous proposition holds true. We follow the same lines as in Section 3.1, simply noting that since $B_{\ell_k}(\beta) \leq C D^k$, we may choose $K \geq c \log B_L(\beta)$ with $2\ell_K \leq L$, where c is independent of L and β, so that (3.13) implies the improved tree diagram bound inequality. □

The main modification we need for an unconditional proof of the intersection-clustering bound lies in the derivation of the intersection and mixing properties. The former is similar to Lemma 3.4, but restricted to sources that lie in regular scales. We restate it here in a slightly modified form.

Lemma 5.2 (Intersection property) *Fix $d = 4$. There exists $c > 0$ such that for every $\beta \leq \beta_c$, every k, and every $y \notin \Lambda_{2\ell_{k+1}}$ in a regular scale,*

$$\mathbf{P}_\beta^{0y,0y,\varnothing,\varnothing}[I_k] \geq c. \tag{5.4}$$

Proof Restricting our attention to the case of y belonging to a regular scale enables us to use properties P1 and P2 of the regularity assumption on the scale. With this additional assumption, we follow the same proof as the one of the conditional version (Lemma 3.4). Introduce the intermediary integers $n \le m \le M \le N$ satisfying

$$\ell_k^4 \ge n \ge \ell_k^{3+\varepsilon} \ , \ n^4 \ge m \ge n^{3+\varepsilon} \ , \ M^4 \ge N \ge M^{3+\varepsilon} \ , \ N^4 \ge \ell_{k+1} \ge N^{3+\varepsilon}. \tag{5.5}$$

For the second moment method on $\mathscr{M}$, the first and second moments take the following forms

$$\mathbf{E}_\beta^{0y,0y,\varnothing,\varnothing}[|\mathscr{M}|] \ge c_1(B_M(\beta) - B_{m-1}(\beta)) \ge c_2 B_{\ell_{k+1}}(\beta), \tag{5.6}$$

$$\mathbf{E}_\beta^{0y,0y,\varnothing,\varnothing}[|\mathscr{M}|^2] \le c_3(B_M(\beta) - B_{m-1}(\beta))B_{2M}(\beta) \le c_3 B_{\ell_{k+1}}(\beta)^2, \tag{5.7}$$

where in the second inequality of the first line, we used that D is large enough and Lemma 5.3 below to get that

$$B_M(\beta) \ge \frac{B_{\ell_{k+1}}(\beta)}{1+15C} \quad \text{and} \quad B_{m-1}(\beta) \le (1+15C)B_{\ell_k}(\beta) \le \frac{1+15C}{D} B_{\ell_{k+1}}(\beta).$$

For the bound on the probabilities of the events $F_1, \dots, F_4$ defined as in Section 3.2, recall that the vertices x and z there are in our case both equal to y that belongs to a regular scale. Using Property 2 of the regularity of scales, the bounds in (3.21) and (3.22) follow readily from the Infrared Bound (4.44). □

In the previous proof, we used the following statement.

Lemma 5.3 *For $d = 4$, there exists $C > 0$ such that for every $\beta \le \beta_c$ and every $\ell \le L \le \xi(\beta)$,*

$$B_L(\beta) \le \Big(1 + C\frac{\log(L/\ell)}{\log \ell}\Big) B_\ell(\beta). \tag{5.8}$$

Proof For every $n \le N$ for which $n = 2^k$ with k regular, we have that (recall the definition of $\chi_n(\beta)$ from the previous section)

$$\begin{aligned} B_{2N}(\beta) - B_N(\beta) &\le C_0 N^{-4} \chi_{N/d}(\beta)^2 \\ &\le C_1 n^{-4} \chi_n(\beta)^2 \\ &\le C_2 n^{-4} (\chi_{2n}(\beta) - \chi_n(\beta))^2 \\ &\le C_3 (B_{2n}(\beta) - B_n(\beta)), \end{aligned} \tag{5.9}$$

where in the first inequality we used (4.4), in the second the sliding-scaled Infrared Bound (4.31), in the third Property P4 of the regularity of n, and in the last Cauchy-Schwarz.

Now, there are $\log_2(L/\ell)$ scales between ℓ and L, and at least $\frac{1}{C}\log_2 \ell$ regular scales between 1 and ℓ by abundance of regular scales (Theorem 4.13). Since the sums of squared correlations on any of the former contribute less to $B_L(\beta) - B_\ell(\beta)$ that any of the latter to $B_\ell(\beta)$, we deduce that

$$B_L(\beta) \le \Big(1 + C\frac{\log_2(L/\ell)}{\log_2 \ell}\Big) B_\ell(\beta). \tag{5.10}$$

□

Next comes the unconditional mixing property.

Theorem 5.4 (random currents' mixing property) *For $d \geq 4$, there exist $\alpha, c > 0$ such that for every $t \leq s$, every $\beta \leq \beta_c$, every $n^\alpha \leq N \leq \xi(\beta)$, every $x_i \in \Lambda_n$ and $y_i \notin \Lambda_N$ $(i \leq t)$, and every events E and F depending on the restriction of $(\mathbf{n}_1, \ldots, \mathbf{n}_s)$ to edges within Λ_n and outside of Λ_N respectively,*

$$\left|\mathbf{P}_\beta^{x_1y_1,\ldots,x_ty_t,\varnothing,\ldots,\varnothing}[E \cap F] - \mathbf{P}_\beta^{x_1y_1,\ldots,x_ty_t,\varnothing,\ldots,\varnothing}[E]\mathbf{P}_\beta^{x_1y_1,\ldots,x_ty_t,\varnothing,\ldots,\varnothing}[F]\right| \leq s(\log \tfrac{N}{n})^{-c}. \tag{5.11}$$

Furthermore, for every $x_1', \ldots, x_t' \in \Lambda_n$ and $y_1', \ldots, y_t' \notin \Lambda_N$, we have that

$$\left|\mathbf{P}_\beta^{x_1y_1,\ldots,x_ty_t,\varnothing,\ldots,\varnothing}[E] - \mathbf{P}_\beta^{x_1y_1',\ldots,x_ty_t',\varnothing,\ldots,\varnothing}[E]\right| \leq s(\log \tfrac{N}{n})^{-c}, \tag{5.12}$$

$$\left|\mathbf{P}_\beta^{x_1y_1,\ldots,x_ty_t,\varnothing,\ldots,\varnothing}[F] - \mathbf{P}_\beta^{x_1'y_1,\ldots,x_t'y_t,\varnothing,\ldots,\varnothing}[F]\right| \leq s(\log \tfrac{N}{n})^{-c}. \tag{5.13}$$

We postpone the proof to Section 5.2 below. Before showing how Theorem 5.4 is used in the proof of the improved tree diagram bound, let us make an interlude and comment on this statement.

Discussion The relation (5.11) is an assertion of approximate independence between events at far distances, and (5.12)–(5.13) expresses a degree of independence of the probability of an event from the precise placement of the sources when these are far from the event in question. This result should be of interest on its own, and possibly have other applications, since mixing properties efficiently replace independence in statistical mechanics.

The main difficulty of the theorem concerns currents with a source inside Λ_n and a source outside Λ_N (i.e. the first t ones). In this case, the currents are constrained to have a path linking the two, and that may be a conduit for information, and correlation, between Λ_n and the exterior of Λ_N. To appreciate the point it may be of help to compare the situation with Bernoulli percolation: there the mixing property without sources is a triviality (by the variables' independence); while an analogue of the mixing property with sources x and y would concern Bernoulli percolation conditioned on having a path from x to y. Proving convergence at criticality, for x set as the origin and y tending to infinity, of these conditioned measures is a notoriously hard problem. It would in particular imply the existence of the so-called Incipient Infinite Cluster (IIC), and the definition of the IIC was justified in 2D [36] and in high dimension [53], but it is still open in dimensions $3 \leq d \leq 10$. When the number of sources is even inside Λ_n, things become much simpler and one may in fact prove a quantitative ratio weak mixing using mixing properties for (sub)-critical random-cluster measures with cluster-weight 2 provided by [5].

Theorem 5.4 has an extension to three dimensions using [5], but there it becomes non-quantitative (the careful reader will notice that the condition $d > 3$ is coming from the exponent appearing in the proof of (5.36) in Lemma 5.7 in the next section). More precisely, one may prove that in dimension $d = 3$, for every n, s and ε, there exists a constant N sufficiently large that the previous theorem holds with an error ε instead of $s(\log \frac{N}{n})^{-c}$. This has a particularly interesting application: one may construct the IIC in dimension $d = 3$ for this model, since the random-cluster model with cluster weight $q = 2$ conditioned on having a path from x to y can be obtained as the random current model with sources x and y together with an additional independent sprinkling (see [6]). This represents a non-trivial result for critical 3D Ising. More generally, we believe that the previous mixing result may be a key tool in the rigorous description of the critical behaviour of the Ising model in three dimensions.

This concludes the interlude, and we return now to the proof of the intersection-clustering bound.

Proof of Proposition 5.1 We follow the same argument as in the proof of the conditional version (Proposition 3.3) and borrow the notation from the corresponding proof at the end of Section 3.2. We fix $\alpha > 2$ large enough that the mixing property Theorem 5.4 holds true. Using Lemma 5.3, we may choose $D = D(\alpha)$ such that $\ell_{k+1} \geq \ell_k^\alpha$.

The proof is exactly identical to the proof of Proposition 3.3, with the exception of the bound on $\mathbf{P}^{0x,0z,\varnothing,\varnothing}[A_S]$ and the fact that we restrict ourselves to subsets S of even integers in $\{1,\dots,K-3\}$. In order to obtain this result, first observe that since we assumed $\ell_K \leq \xi(\beta)$, by Theorem 4.13 there exists $y \in \mathrm{Ann}(\ell_{K-1},\ell_K)$ in a regular scale. Since the event A_S depends on the currents inside $\Lambda_{\ell_{K-2}}$ (since S does not contain integers strictly larger than $K-3$), and that $\ell_{K-1} \geq \ell_{K-2}^\alpha$, the mixing property (Theorem 5.4) shows that

$$\mathbf{P}^{0x,0z,\varnothing,\varnothing}[A_S] \leq \mathbf{P}^{0y,0y,\varnothing,\varnothing}[A_S] + \frac{C}{\sqrt{\log \ell_{K-1}}} \leq \mathbf{P}^{0y,0y,\varnothing,\varnothing}[A_S] + 2^{-\delta K}. \tag{5.14}$$

To derive the first bound on the right-hand side, we apply the mixing property repeatedly (Theorem 5.4) and the intersection property (Lemma 5.2) exactly as in the conditional proof. For the second inequality, we lower bound ℓ_{K-1} using $B_{\ell_{K-1}}(\beta) \geq D^{K-1}$ and the Infrared Bound (4.44). □

5.2 The mixing property: proof of Theorem 5.4

As we saw, the mixing property is in the core of the proof of our main result. The strategy of the proof was explained in Section 3.2 when we proved mixing for one current under the power-law assumption. In this section we again define a random variable $\mathbf{N}$ which is approximately 1 and is a weighted sum over (t-tuple of) vertices connected to the origin. The main difficulty will come from the fact that since we do not fully control the spin-spin correlations, we will need to define $\mathbf{N}$ in a smarter fashion. Also, whereas in Section 3.2 we treated the case of a single current ($s = 1$), here we generalize to multiple currents.

Fix $\beta \leq \beta_c$ and drop it from the notation. Also fix $s \geq t \geq 1$ and $n^\alpha \leq N \leq \xi(\beta)$. Below, constants c_i and C_i are independent of the choices of s,t,β,n,N satisfying the properties above. We introduce the integers m and M such that $m/n = (N/n)^{1/3}$ and $N/M = (N/n)^{1/3}$ (we omit the details of the rounding operation).

For $\mathbf{x} = (x_1,\dots,x_t)$ and $\mathbf{y} = (y_1,\dots,y_t)$, we will use the following shortcut notation

$$\mathbf{P}^{\mathbf{xy}} := \mathbf{P}^{x_1y_1,\dots,x_ty_t,\varnothing,\dots,\varnothing} \quad \text{and} \quad \mathbf{P}^{\mathbf{xy}} \otimes \mathbf{P}^{\varnothing}, \tag{5.15}$$

where the second measure is the law of the random variable $(\mathbf{n}_1,\dots,\mathbf{n}_s,\mathbf{n}'_1,\dots,\mathbf{n}'_s)$, where $(\mathbf{n}'_1,\dots,\mathbf{n}'_s)$ is an independent family of sourceless currents.

To define $\mathbf{N}$, first introduce for every vertex $y \notin \Lambda_{2dm}$, the set (see Fig. 4)

$$\mathbb{A}_y(m) := \big\{u \in \mathrm{Ann}(m,2m) : \forall x \in \Lambda_{m/d}, \langle\sigma_x\sigma_y\rangle \leq \big(1 + \tfrac{C|x-u|}{|y|}\big)\langle\sigma_u\sigma_y\rangle\big\}, \tag{5.16}$$

where C is given by the definition of good scales.

Remark 5.5 *When y is in a regular scale, then $\mathbb{A}_y(m)$ is equal to* $\mathrm{Ann}(m,2m)$ *by Property P2 of regular scales. The reason why we consider $\mathbb{A}_y(m)$ instead of the full annulus* $\mathrm{Ann}(m,2m)$ *is technical: since y will not a priori be assumed to belong to a regular scale (in fact $|y|$ may be much larger than $\xi(\beta)$ when $\beta < \beta_c$), we will use (for* (5.26) *and* (5.37) *below) the inequality between $\langle\sigma_x\sigma_y\rangle$ and $\langle\sigma_u\sigma_y\rangle$ in several bounds. Now, if $y_1 = |y|$, then*

$$\mathbb{A}_y(m) \supset \{z \in \mathbb{Z}^d : m \leq z_1 \leq 2m \text{ and } 0 \leq z_j \leq m/d \text{ for } j > 1\} \tag{5.17}$$

as the Messager-Miracle-Sole inequality implies[6] *that* $\langle\sigma_z\sigma_y\rangle \geq \langle\sigma_x\sigma_y\rangle$ *for every* $x \in \Lambda_{m/d}$.

From now on, fix a set $\mathscr{K}$ of regular scales k with $m \leq 2^k \leq M/2$ satisfying that distinct $k, k' \in \mathscr{K}$ are differing by a multiplicative factor at least C (where the constant C is given by Theorem 4.13). We further assume that $|\mathscr{K}| \geq c_1 \log(N/n)$, where c_1 is sufficiently small. The existence of $\mathscr{K}$ is guaranteed by the definition of m and M and the abundance of regular scales given by Theorem 4.13.

Define $\mathbf{N} := \prod_{i=1}^t \mathbf{N}_i$, where

$$\mathbf{N}_i := \frac{1}{|\mathscr{K}|}\sum_{k\in\mathscr{K}} \frac{1}{A_{x_i,y_i}(2^k)} \sum_{u\in\mathbb{A}_{y_i}(2^k)} \mathbb{I}[u \overset{\mathbf{n}_i+\mathbf{n}'_i}{\longleftrightarrow} x_i], \tag{5.20}$$

where $a_{x,y}(u) := \langle\sigma_x\sigma_u\rangle\langle\sigma_u\sigma_y\rangle/\langle\sigma_x\sigma_y\rangle$ and $A_{x,y}(m) := \sum_{u\in\mathbb{A}_y(m)} a_{x,y}(u)$. The first step of the proof is the following concentration inequality.

Proposition 5.6 (Concentration of N) *For every* $\alpha > 2$, *there exists* $C_0 = C_0(\alpha, t) > 0$ *such that for every* n *large enough and* $n^\alpha \leq N \leq \xi(\beta)$,

$$\mathbf{E}^{\mathbf{xy},\varnothing}[(\mathbf{N}-1)^2] \leq \frac{C_0}{\log(N/n)}. \tag{5.21}$$

Proof We shall apply the telescopic formula

$$\mathbf{N} - 1 \equiv \prod_{i=1}^t \mathbf{N}_i - 1 = \sum_{i=1}^t (\mathbf{N}_i - 1)\prod_{j>i}\mathbf{N}_j$$

with the last product interpreted as 1 for $i = t$. Hence, by Cauchy-Schwarz inequality and the currents' independence,

$$\mathbf{E}^{\mathbf{xy},\varnothing}[(\mathbf{N}-1)^2] \leq t\sum_{i=1}^t \mathbf{E}^{\mathbf{xy},\varnothing}[(\mathbf{N}_i-1)^2]\prod_{j>i}\mathbf{E}^{\mathbf{xy},\varnothing}[\mathbf{N}_j^2]. \tag{5.22}$$

It therefore suffices to show that there exists a constant $C_1 > 0$ such that for every $i \leq t$,

$$\mathbf{E}^{\mathbf{xy},\varnothing}[(\mathbf{N}_i-1)^2] \leq \frac{C_1}{\log(N/n)}. \tag{5.23}$$

To lighten the notation, and since the random variable $\mathbf{N}_i$ depends only on $\mathbf{n}_i$ and $\mathbf{n}'_i$, we omit the index in $x_i, y_i, \mathbf{n}_i, \mathbf{n}'_i$ and write instead just $x, y, \mathbf{n}, \mathbf{n}'$. We keep the index in $\mathbf{N}_i$ to avoid confusion with $\mathbf{N}$ which is the product of these random variables.

The proof of (5.23) is also based on a computation of the first and second moments of $\mathbf{N}_i$. For the first moment, the switching lemma and the definition of $\mathbf{N}_i$ imply that

[6] The claim follows directly from the inequality $S_\beta(y) \leq S_\beta(x)$ for every x, y such that $x_1 \geq 0$ and $y_1 \geq x_1 + \sum_{j>1}|y_j - x_j|$. In order to prove this inequality, define, for $0 \leq i \leq d$,

$$v^{(i)} := (x_1 + \sum_{j=i}^d |y_j - x_j|, x_2, \ldots, x_i, y_{i+1}, \ldots, y_d). \tag{5.18}$$

Successive applications of the Messager-Miracle-Sole inequality with respect to the sum or the difference (depending on whether x_i is positive or negative) of the first and i-th coordinates implies that

$$S_\beta(y) \leq S_\beta(v^{(1)}) \leq S_\beta(v^{(2)}) \leq \cdots \leq S_\beta(v^{(d)}) = S_\beta(x). \tag{5.19}$$

$\mathbf{E}^{xy,\varnothing}[\mathbf{N}_i] = 1$. From the lower bound on $|\mathscr{K}|$, to bound the second moment it therefore suffices to show that

$$\mathbf{E}^{xy,\varnothing}[\mathbf{N}_i^2] \le 1 + \frac{C_2}{|\mathscr{K}|}, \tag{5.24}$$

which follows from the inequality, for every $\ell \ge k$ in $\mathscr{K}$,

$$\sum_{\substack{u\in\mathbb{A}_y(2^k)\\ v\in\mathbb{A}_y(2^\ell)}} \mathbf{P}^{xy,\varnothing}[u, v \overset{\mathbf{n}+\mathbf{n}'}{\longleftrightarrow} x] \le A_{x,y}(2^k)A_{x,y}(2^\ell)(1 + C_3 2^{-(\ell-k)}). \tag{5.25}$$

Case $\ell > k$. We find by (A.11) that

$$\mathbf{P}^{xy,\varnothing}[u, v \overset{\mathbf{n}+\mathbf{n}'}{\longleftrightarrow} x] \le a_{x,y}(u)a_{x,y}(v)\left(\frac{\langle\sigma_x\sigma_y\rangle}{\langle\sigma_u\sigma_y\rangle}\frac{\langle\sigma_u\sigma_v\rangle}{\langle\sigma_x\sigma_v\rangle} + \frac{\langle\sigma_x\sigma_y\rangle}{\langle\sigma_v\sigma_y\rangle}\frac{\langle\sigma_u\sigma_v\rangle}{\langle\sigma_x\sigma_u\rangle}\right). \tag{5.26}$$

Now, since $u \in \mathbb{A}_y(2^k)$, $\langle\sigma_x\sigma_y\rangle \le (1+C\frac{|u-x|}{|y|})\langle\sigma_u\sigma_y\rangle$. Furthermore, since ℓ is a regular scale, Property P2 of regular scales implies that $\langle\sigma_u\sigma_v\rangle \le (1 + C\frac{|u-x|}{|v|})\langle\sigma_x\sigma_v\rangle$. We deduce that

$$\frac{\langle\sigma_x\sigma_y\rangle}{\langle\sigma_u\sigma_y\rangle}\frac{\langle\sigma_u\sigma_v\rangle}{\langle\sigma_x\sigma_v\rangle} \le 1 + C_0 2^{-(\ell-k)}. \tag{5.27}$$

Similarly, since $v \in \mathbb{A}_y(2^\ell)$, $\langle\sigma_x\sigma_y\rangle \le (1 + C\frac{|v-x|}{|y|})\langle\sigma_v\sigma_y\rangle$. Property P3 for the $\ell - k$ regular scales in $\mathscr{K}$ between k and ℓ implies that

$$\frac{\langle\sigma_x\sigma_y\rangle}{\langle\sigma_v\sigma_y\rangle}\frac{\langle\sigma_u\sigma_v\rangle}{\langle\sigma_x\sigma_u\rangle} \le C_1 2^{-(\ell-k)}. \tag{5.28}$$

Plugging (5.27)–(5.28) into (5.26) and summing over $u \in \mathbb{A}_y(2^k)$ and $v \in \mathbb{A}_y(2^\ell)$ gives (5.25).

Case $\ell = k$. Assume that $\langle\sigma_u\sigma_y\rangle \le \langle\sigma_v\sigma_y\rangle$. Use (A.11) to write

$$\mathbf{P}^{xy,\varnothing}[u, v \overset{\mathbf{n}+\mathbf{n}'}{\longleftrightarrow} x] \le \langle\sigma_v\sigma_u\rangle\left(\frac{\langle\sigma_x\sigma_u\rangle}{\langle\sigma_x\sigma_v\rangle} + \frac{\langle\sigma_u\sigma_y\rangle}{\langle\sigma_v\sigma_y\rangle}\right)a_{x,y}(v). \tag{5.29}$$

By Property P1 of regular scales, the first term under parenthesis is bounded by a constant. The second one is bounded by 1 by assumption. Now, for each $v \in \mathbb{A}_y(2^k)$,

$$\begin{aligned}\sum_{u\in\mathbb{A}_y(2^k):\langle\sigma_u\sigma_y\rangle\le\langle\sigma_v\sigma_y\rangle} \langle\sigma_v\sigma_u\rangle &\le \chi_{2^{k+1}}(\beta) \le C_2(\chi_{2^{k+1}}(\beta) - \chi_{2^k}(\beta))\\ &\le C_3 \sum_{u\in\mathbb{A}_y(2^k)} \langle\sigma_0\sigma_u\rangle\\ &\le C_4 \sum_{u\in\mathbb{A}_y(2^k)} \frac{\langle\sigma_x\sigma_u\rangle\langle\sigma_u\sigma_y\rangle}{\langle\sigma_x\sigma_y\rangle} = C_4 A_{x,y}(2^k),\end{aligned}$$

where the first inequality is trivial, the second one is true by Property P4, the third by Remark 5.5 (when y is regular then it is a direct consequence of P4, and when it is not one can use (5.17) and the Messager-Miracle-Sole inequality), and the fourth inequality follows from Property P1 of regular scales (to replace $\langle\sigma_0\sigma_u\rangle$ by $\langle\sigma_x\sigma_u\rangle$) and the fact that since $u \in \mathbb{A}_y(2^k)$, $\langle\sigma_x\sigma_y\rangle \le (1 + C|u - x|/|y|)\langle\sigma_u\sigma_y\rangle \le C_5\langle\sigma_u\sigma_y\rangle$.

We deduce that

$$\sum_{u,v\in\mathbb{A}_y(2^k)} \mathbf{P}^{xy,\varnothing}[u,v \overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow} x] \le 2 \sum_{\substack{u,v\in\mathbb{A}_y(2^k)\\ \langle\sigma_u\sigma_y\rangle\le\langle\sigma_v\sigma_y\rangle}} \mathbf{P}^{xy,\varnothing}[u,v \overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow} x] \le C_6 A_{x,y}(2^k)^2. \quad (5.30)$$

□

For a proof of Theorem 5.4 we fix $\alpha > 2$ (which will be taken large enough later). Applying the Cauchy-Schwarz inequality gives

$$|\mathbf{P}^{\mathbf{xy}}[E\cap F] - \mathbf{E}^{\mathbf{xy},\varnothing}[\mathbf{N}\mathbb{I}_{(\mathbf{n}_1,\dots,\mathbf{n}_s)\in E\cap F}]| \le \sqrt{\mathbf{E}^{\mathbf{xy},\varnothing}[(\mathbf{N}-1)^2]} \le \frac{C_1}{\sqrt{\log(N/n)}}. \quad (5.31)$$

Now, for $\mathbf{u} = (u_1,\dots,u_t)$ with $u_i \in \mathrm{Ann}(m,M)$ for every i, let $G(u_1,\dots,u_t)$ be the event that for every $i \le s$, there exists $\mathbf{k}_i \le \mathbf{n}_i + \mathbf{n}'_i$ such that $\mathbf{k}_i = 0$ on Λ_n, $\mathbf{k}_i = \mathbf{n}_i + \mathbf{n}'_i$ outside Λ_N, and $\partial\mathbf{k}_i$ is equal to $\{u_i, y_i\}$ if $i \le t$ and $\varnothing$ if $t < i \le s$. The switching principle implies as in Section 4.3 that

$$\begin{aligned}\mathbf{P}^{\mathbf{xy},\varnothing}&[(\mathbf{n}_1,\dots,\mathbf{n}_s)\in E\cap F, u_i \overset{\mathbf{n}_i+\mathbf{n}'_i}{\longleftrightarrow} x_i \text{ for } i\le t, G(u_1,\dots,u_t)]\\ &= \Big(\prod_{i=1}^t a_{x_i,y_i}(u_i)\Big)\mathbf{P}^{\mathbf{xu},\mathbf{uy}}[(\mathbf{n}_1,\dots,\mathbf{n}_s)\in E, (\mathbf{n}'_1,\dots,\mathbf{n}'_s)\in F, G(u_1,\dots,u_t)]. \quad (5.32)\end{aligned}$$

Also, as before, we have the trivial identity

$$\mathbf{P}^{\mathbf{xu},\mathbf{uy}}[(\mathbf{n}_1,\dots,\mathbf{n}_s)\in E, (\mathbf{n}'_1,\dots,\mathbf{n}'_s)\in F] = \mathbf{P}^{\mathbf{xu}}[E]\mathbf{P}^{\mathbf{uy}}[F]. \quad (5.33)$$

We now pause the argument to establish the following lemma.

Lemma 5.7 *For $d \ge 4$, there exists $\varepsilon > 0$ and $\alpha_0 = \alpha_0(\varepsilon) > 0$ large enough such that for every $n^{\alpha_0} \le N \le \xi(\beta)$ and for every $\mathbf{u}$ with $u_i \in \mathbb{A}_{y_i}(2^{k_i})$ for some k_i with $m \le 2^{k_i} \le M/2$ for every $1 \le i \le t$,*

$$\begin{aligned}\Big(\prod_{i=1}^t a_{x_i,y_i}(u_i)\Big)^{-1}\mathbf{P}^{\mathbf{xy},\varnothing}[u_i \overset{\mathbf{n}_i+\mathbf{n}'_i}{\longleftrightarrow} x_i, \forall i\le t, G(u_1,\dots,u_t)^c] &= \mathbf{P}^{\mathbf{xu},\mathbf{uy}}[G(u_1,\dots,u_t)^c]\\ &\le s\big(\tfrac{n}{N}\big)^{\varepsilon}. \quad (5.34)\end{aligned}$$

Proof Fix $\varepsilon > 0$ sufficiently small (we will see below how small it should be). The first identity follows from the switching lemma so we focus on the second one. Let G_i be the event that the current $\mathbf{k}_i$ exists. This event clearly contains (see Fig. 4) the event that $\mathrm{Ann}(M,N)$ is not crossed by a cluster in $\mathbf{n}_i$, and $\mathrm{Ann}(n,m)$ is not crossed by a cluster in $\mathbf{n}'_i$, since in such case $\mathbf{k}_i$ can be defined as the sum of $\mathbf{n}_i$ restricted to the clusters intersecting Λ_N^c (this current has no sources) and $\mathbf{n}'_i$ restricted to the clusters intersecting Λ_m^c (this current has sources u_i and y_i). We focus on the probability of this event for $i \le t$, the case $t < i \le s$ being even simpler since there are no sources.

We bound the probability of $\mathbf{n}_i$ crossing $\mathrm{Ann}(M,N)$ by splitting $\mathrm{Ann}(M,N)$ in two annuli $\mathrm{Ann}(M,R)$ and $\mathrm{Ann}(R,N)$ with $R = \sqrt{MN}$, then estimating the probability that the backbone of $\mathbf{n}_i$ crosses the inner annulus, and then the probability that the remaining current crosses the outer annulus. More precisely, the chain rule for backbones [4] gives that for $\alpha_0 = \alpha_0(\varepsilon) > 0$ large enough and $N \ge n^{\alpha_0}$,

$$\mathbf{P}^{\mathbf{xy}}[\Gamma(\mathbf{n}_i) \text{ crosses } \mathrm{Ann}(M,R)] \le \sum_{v\in\partial\Lambda_R} \frac{\langle\sigma_{x_i}\sigma_v\rangle\langle\sigma_v\sigma_{u_i}\rangle}{\langle\sigma_{x_i}\sigma_{u_i}\rangle} \le C_2 R^3\frac{M^3}{R^4} \le (n/N)^{\varepsilon}, \quad (5.35)$$

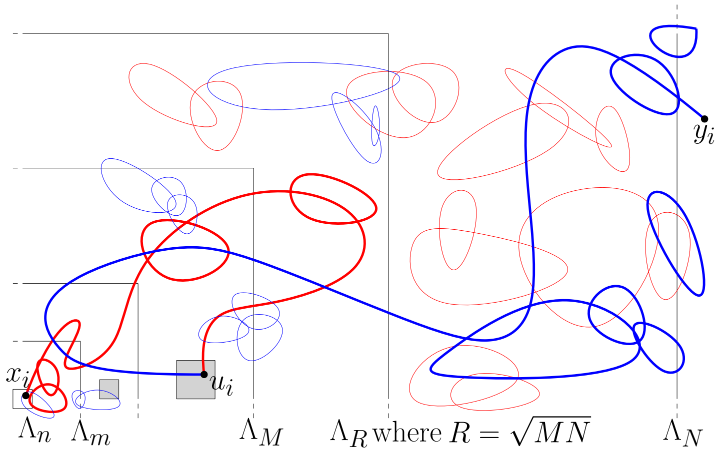

Figure 4: The currents $\mathbf{n}_i$ (red) and $\mathbf{n}'_i$ (blue). Since the sources of $\mathbf{n}_i$, i.e. x_i and u_i, are both in Λ_M, a reasoning similar to the proof of uniqueness in the intersection property (first control the backbone, proving that it does not cross the annulus $\mathrm{Ann}(M,R)$, and then the remaining sourceless current) enables us to conclude that the probability that the current contains a crossing of $\mathrm{Ann}(M,N)$ is small. Similarly, since the sources u_i and y_i of $\mathbf{n}'_i$ lie both outside of Λ_m, we can prove that the probability that $\mathbf{n}'_i$ crosses $\mathrm{Ann}(n,m)$ is small. An extra care is needed for establishing the latter since y is not assumed to be regular. To circumvent this problem, we consider only intersection sites u_i in one of the boxes $\mathbb{A}_k(y_i)$, which are depicted here in gray.

where the lower bound (4.6) to bound the denominator and the Infrared Bound (4.44) for the numerator. Then, observe that the remaining current $\mathbf{n}_i \setminus \Gamma(\mathbf{n}_i)$ is sourceless. Adding an additional sourceless current and using the switching lemma and Griffiths inequality [25] (very much like in the bound on F_2 in the proof of Lemma (3.22)) gives

$$\mathbf{P}^{\mathbf{x}\mathbf{y}}[\mathbf{n}_i \setminus \Gamma(\mathbf{n}_i) \text{ crosses } \mathrm{Ann}(R,N)] \le \sum_{\substack{v\in\partial\Lambda_R\\ w\in\partial\Lambda_N}} \langle\sigma_v\sigma_w\rangle^2 \le C_3 R^3 N^3 (R/N)^4 \le (n/N)^{\varepsilon}, \quad (5.36)$$

where we used the Infrared-Bound 4.44, and in the last one the definition of R and the fact that $N \ge n^{\alpha_0}$ for α_0 large enough.

When dealing with the probability of $\mathbf{n}'_i$ crossing $\mathrm{Ann}(n,m)$, fix $r = \sqrt{nm}$ and apply the same reasoning with the annuli $\mathrm{Ann}(n,r)$ and $\mathrm{Ann}(r,m)$. The equivalent of (5.36) is the same as before, but one must be more careful about the bound on the probability of the event dealing with the backbone:

$$\mathbf{P}^{\mathbf{x}\mathbf{y}}[\Gamma(\mathbf{n}'_i) \text{ crosses } \mathrm{Ann}(r,m)] \le C_4 \sum_{v\in\partial\Lambda_r} \frac{\langle\sigma_{u_i}\sigma_v\rangle\langle\sigma_v\sigma_{y_i}\rangle}{\langle\sigma_{u_i}\sigma_{y_i}\rangle} \le C_5 r^3/m^2 \le (n/N)^{\varepsilon}, \quad (5.37)$$

where we used the Infrared Bound (4.44) and our assumption that u_i belongs to one of the $\mathbb{A}_k(y_i)$ (to show that $\langle\sigma_v\sigma_{y_i}\rangle \le C_4\langle\sigma_{u_i}\sigma_{y_i}\rangle$). □

Invoking the above lemma we now return to the proof of Theorem 5.4.

Introduce the coefficients $\delta(\mathbf{u},\mathbf{x},\mathbf{y})$ equal to

$$\delta(\mathbf{u},\mathbf{x},\mathbf{y}) := \prod_{i=1}^{t} \frac{a_{x_i,y_i}(u_i)}{|\mathscr{K}| A_{x_i,y_i}(2^{k_i})} \quad (5.38)$$

for $\mathbf{u}$ such that for every $i \le t$, $u_i \in \mathbb{A}_{y_i}(2^{k_i})$ for some k_i, and equal to 0 for other $\mathbf{u}$. Gathering (5.31)–(5.33) as well as Lemma 5.7, and observing that the sum on $(u_1,\dots,u_t)$ of $\delta(\mathbf{u},\mathbf{x},\mathbf{y})$ is 1, we obtain that

$$|\mathbf{P}^{\mathbf{xy}}[E\cap F]-\sum_{\mathbf{u}}\delta(\mathbf{u},\mathbf{x},\mathbf{y})\mathbf{P}^{\mathbf{xu}}[E]\mathbf{P}^{\mathbf{uy}}[F]|\le\frac{C_5 s}{\sqrt{\log(N/n)}}+2C_6 s(n/N)^{\varepsilon}\le\frac{C_7 s}{\sqrt{\log(N/n)}},\tag{5.39}$$

provided that $N\ge n^{\alpha_0}$ where α_0 is given by the previous lemma.

To conclude the proof is now a matter of elementary algebraic manipulations. We begin by proving (5.12) when all the y_i, y_i' for $i\le t$ belong to regular scales (not necessarily the same ones). In this case, apply twice (once for $\mathbf{y}$ and once for $\mathbf{y}'$) the previous inequality for our event E and the event on the outside being the full event to find

$$|\mathbf{P}^{\mathbf{xy}}[E]-\mathbf{P}^{\mathbf{xy}'}[E]|\le\Big|\sum_{\mathbf{u}}(\delta(\mathbf{u},\mathbf{x},\mathbf{y})-\delta(\mathbf{u},\mathbf{x},\mathbf{y}'))\mathbf{P}^{\mathbf{xu}}[E]\Big|+\frac{2C_7 s}{\sqrt{\log(N/n)}}.\tag{5.40}$$

Since all the y_i, y_i' are in regular scales, Remark 5.5 implies that $\mathbb{A}_{y_i}(2^{k_i})=\mathbb{A}_{y_i'}(2^{k_i})=\mathrm{Ann}(2^{k_i},2^{k_i+1})$. Furthermore, Property 2 of regular scales implies[7] that

$$|\delta(\mathbf{u},\mathbf{x},\mathbf{y})-\delta(\mathbf{u},\mathbf{x},\mathbf{y}')|\le C_8 s\tfrac{M}{N}\delta(\mathbf{u},\mathbf{x},\mathbf{y})\le C_9 s(n/N)^{1/3}\delta(\mathbf{u},\mathbf{x},\mathbf{y}).\tag{5.41}$$

Therefore, (5.12) follows readily (with a large constant C_{10}) in this case. The same argument works for the second identity (5.13).

Now, assume further that $N\ge n^{3\alpha_0}$. Consider $\mathbf{z}=(z_1,\dots,z_t)$ with z_i in a regular scale for every $i\le t$ and $z_i\in\mathrm{Ann}(m,M)$. We also pick $\mathbf{y}$ on which we do not assume anything. The fact that the $\delta(\mathbf{u},\mathbf{x},\mathbf{y})$ sum to 1 implies that

$$\begin{aligned}|\mathbf{P}^{\mathbf{xy}}[E]-\mathbf{P}^{\mathbf{xz}}[E]|&=\Big|\mathbf{P}^{\mathbf{xy}}[E]-\sum_{\mathbf{u}}\delta(\mathbf{u},\mathbf{x},\mathbf{y})\mathbf{P}^{\mathbf{xz}}[E]\Big|\\&\le\Big|\mathbf{P}^{\mathbf{xy}}[E]-\sum_{\mathbf{u}}\delta(\mathbf{u},\mathbf{x},\mathbf{y})\mathbf{P}^{\mathbf{xu}}[E]\Big|+\frac{C_6 s}{\sqrt{\log(m/n)}}\le\frac{C_7 s}{\sqrt{\log(N/n)}},\end{aligned}\tag{5.42}$$

where in the second line we have used (5.40) for $\mathbf{u}$ and $\mathbf{z}$ for n and m which is justified since $m/n=(N/n)^{1/3}\ge n^{\alpha_0}$.

Finally, we repeat the reasoning above. For $N\ge n^{9\alpha_0}$, the proof of (5.11) is obtained by applying (5.39) for E, F, N and n, the previous inequality for E, m and n (which is justified since $m/n=(N/n)^{1/3}\ge n^{3\alpha_0}$), and then again (5.39) for F, N and n. This concludes the proof with $\alpha=9\alpha_0$.

5.3 Proof of Proposition 1.4

As mentioned in the introduction, Newman [42] showed, using the model's Lee-Yang property, that gaussianity of the limit is implied by the asymptotic vanishing of the scaling

[7] Note that in this case $\delta(\mathbf{u},\mathbf{x},\mathbf{y})$ and $\delta(\mathbf{u},\mathbf{x},\mathbf{y}')$ are both close to

$$\delta'(\mathbf{u},\mathbf{x}):=\prod_{i\le t}\frac{\langle\sigma_{x_i}\sigma_{u_i}\rangle}{|\mathscr{K}|\sum_{v_i\in\mathrm{Ann}(2^{k_i},2^{k_i+1})}\langle\sigma_{x_i}\sigma_{v_i}\rangle}.$$

limit of $|U_4^\beta|$. A more quantitative proof is available through the inequality, valid for every $n \geq 2$ and derived using the switching lemma in [1, Proposition 12.1],

$$\begin{aligned}
0 &\leq \mathscr{G}_n[S_\beta](x_1, \ldots, x_{2n}) - S_\beta(x_1, \ldots, x_{2n}) \\
&\leq -\frac{3}{2} \sum_{1 \leq i < j < k < l \leq 2n} S_\beta(x_1, \ldots, \cancel{x_i}, \ldots, \cancel{x_j}, \ldots, \cancel{x_k}, \ldots, \cancel{x_l}, \ldots, x_{2n})\, U_4^\beta(x_i, x_j, x_k, x_l).
\end{aligned} \tag{5.43}$$

Combined with the improved tree diagram bound (1.22), this inequality has the following consequence: for a continuous function f which vanishes outside $[-r, r]^4$, and for $\beta \leq \beta_c$ and $L \leq \xi(\beta)$,

$$\Big|\langle T_{f,L}(\sigma)^{2n}\rangle_\beta - \tfrac{(2n)!}{2^n n!}\langle T_{f,L}(\sigma)^2\rangle_\beta^n\Big| \;\leq\; \tfrac{3}{2}(2n)^4 \langle T_{f,L}(\sigma)^{2n-4}\rangle_\beta \, \|f\|_\infty^4 \, S(L, r, \beta), \tag{5.44}$$

where

$$S(L, r, \beta) \;:=\; \sum_{\substack{x \in \mathbb{Z}^d \\ x_1, \ldots, x_4 \in \Lambda_{rL}}} 2\frac{\langle \sigma_x \sigma_{x_1}\rangle_\beta \langle \sigma_x \sigma_{x_2}\rangle_\beta \langle \sigma_x \sigma_{x_3}\rangle_\beta \langle \sigma_x \sigma_{x_4}\rangle_\beta}{\Sigma_L(\beta)^2 \cdot B_{L(x_1, \ldots, x_4)}(\beta)^c}, \tag{5.45}$$

where $L(x_1, \ldots, x_4)$ is the minimal distance between the x_i. Also note that by flip symmetry we find that $\langle T_{f,L}(\sigma)^{2n+1}\rangle_\beta = 0$. Multiplying (5.44) by $\frac{z^{2n}}{(2n)!}$ and summing over n, we obtain

$$\Big|\langle \exp[z\, T_{f,L}(\sigma)]\rangle_\beta - \exp[\tfrac{z^2}{2}\langle T_{f,L}(\sigma)^2\rangle_\beta]\Big| \;\leq\; \exp[\tfrac{z^2}{2}\langle T_{f,L}(\sigma)^2\rangle_\beta] C_1 z^4 \|f\|_\infty^4 S(L, r, \beta). \tag{5.46}$$

Proposition 1.4 therefore follows from the bound

$$S(L, r, \beta) \leq C_2 r^{12}\Big(\frac{\log\log L}{\log L}\Big)^c \tag{5.47}$$

that we prove next. Note that (5.46) is easy to obtain under the power-law assumption (3.2). Without this assumption, the derivation requires a few lines of (not particularly interesting) computations to address the fact that we do not a priori know that $B_L(\beta)$ grows logarithmically. Roughly speaking, the argument below uses the idea that in the case in which $B_L(\beta)$ is small, then $\Sigma_L(\beta)$ is much smaller than L^2, so that both combine in such a way that the previous bound is always valid for $c > 0$ sufficiently small. We now provide the proof of (5.47). Fix $0 < a < 1$ (any choice would do) and split the sum into four sums

$$S(L, r, \beta) = \underbrace{\sum_{\substack{x \in \Lambda_{drL} \\ x_1, \ldots, x_4 \in \Lambda_{rL} \\ L(x_1, \ldots, x_4) \geq L^a}} (\cdots)}_{(1)} + \underbrace{\sum_{\substack{x \notin \Lambda_{drL} \\ x_1, \ldots, x_4 \in \Lambda_{rL} \\ L(x_1, \ldots, x_4) \geq L^a}} (\cdots)}_{(2)} + \underbrace{\sum_{\substack{x \in \Lambda_{drL} \\ x_1, \ldots, x_4 \in \Lambda_{rL} \\ L(x_1, \ldots, x_4) < L^a}} (\cdots)}_{(3)} + \underbrace{\sum_{\substack{x \notin \Lambda_{drL} \\ x_1, \ldots, x_4 \in \Lambda_{rL} \\ L(x_1, \ldots, x_4) < L^a}} (\cdots)}_{(4)}. \tag{5.48}$$

Bound on (1) We focus on this term and give more details since it is in fact the main contributor. By Lemma 5.3,

$$B_{L(x_1, \ldots, x_4)}(\beta) \geq \tfrac{1}{C_3} B_L(\beta). \tag{5.49}$$

Summing over the sites in Λ_{rL}, we get that

$$(1) \leq C_3 \frac{|\Lambda_{rL}| \chi_{2rL}(\beta)^4}{\Sigma_L(\beta)^2 B_L(\beta)^c} \leq C_4 r^{12}\Big(\frac{\chi_L(\beta)^2}{L^4 B_L(\beta)}\Big)^c, \tag{5.50}$$

where in the second inequality we used the sliding-scale Infrared Bound (4.31) to bound $\chi_{2drL}(\beta)$ in terms of $\chi_L(\beta) \le C_5 L^{-4}\Sigma_L(\beta)$ and the Infrared Bound (4.44) to write

$$\chi_L(\beta) \le C_6 L^2. \tag{5.51}$$

Using Cauchy-Schwarz in the first inequality below, (5.51) to bound the first term in the middle and Lemma 5.3 for the second one, we find that

$$\frac{\chi_L(\beta)^2}{L^4 B_L(\beta)} \le 2\frac{\chi_{L/\log L}(\beta)^2}{L^4} + C_7 \frac{B_L(\beta) - B_{L/\log L}(\beta)}{B_L(\beta)} \le C_8 \frac{\log\log L}{\log L}. \tag{5.52}$$

Plugging this estimate in (5.50) gives

$$(1) \le C_9 r^{12} \Big(\frac{\log\log L}{\log L}\Big)^c. \tag{5.53}$$

Bound on (2) Combine (4.4) and the sliding-scale Infrared Bound (4.31) to get that for $i = 1, \ldots, 4$,

$$\langle \sigma_x \sigma_{x_i} \rangle_\beta \le \frac{C_{10}}{|x|^4} \chi_{|x|/d}(\beta) \le \frac{C_{11}}{L^2 |x|^2} \chi_L(\beta). \tag{5.54}$$

Summing over the sites gives the same bound as in (5.50) so that the reasoning in (1) gives

$$(2) \le C_{12} r^{12} \Big(\frac{\log\log L}{\log L}\Big)^c. \tag{5.55}$$

Bound on (3) This term is much smaller than the previous two due to the constraint that two sites must be close to each other. In fact, we will not even need the improved part of the tree diagram bound and will simply use that $B_{L(x_1,\ldots,x_4)}(\beta) \ge 1$. Then, we use the Infrared Bound (4.44) to bound the terms $\langle \sigma_x \sigma_{x_i} \rangle_\beta$ and $\langle \sigma_x \sigma_{x_j} \rangle_\beta$ for which x_i and x_j are at a distance exactly $L(x_1, \ldots, x_4)$. Summing over the other two sites x_k and x_l gives a contribution bounded by $\chi_{2rL}(\beta)^2 \le C_{13} L^{-8} r^4 \Sigma_L(\beta)^2$ by the sliding-scale Infrared Bound (4.31). Summing over x and then x_i and x_j gives that

$$(3) \le \frac{C_{14} r^4 \log(Lr)}{L^{4-4a}}. \tag{5.56}$$

Bound on (4) This sum is even simpler to bound than (3). Again, we simply use $B_{L(x_1,\ldots,x_4)}(\beta) \ge 1$, bound two of the terms $\langle \sigma_x \sigma_{x_i} \rangle_\beta$ using (5.54), and the other two using the Infrared Bound (4.44). Summing over the vertices and using the constraint that two of the sites must be close to each other gives the bound

$$(4) \le \frac{C_{15} r^8}{L^{4-4a}}. \tag{5.57}$$

In conclusion, all the sums (1)–(4) are sufficiently small (recall that by definition $r \ge 1$) and the claim is derived.

Remark 5.8 *For $\beta < \beta_c$, applying* (5.45) *with $r = 1$ and $L = \xi(\beta)$ gives the following bound on the* renormalized coupling constant *$g(\beta)$:*

$$g(\beta) := \frac{1}{\xi(\beta)^4 \chi(\beta)^2} \sum_{x,y,z \in \mathbb{Z}^d} |U_4^\beta(0,x,y,z)| \le \log\big(\tfrac{1}{|\beta - \beta_c|}\big)^{-c}, \tag{5.58}$$

where we used that

$$\xi(\beta)^2 \geq c_0 \chi_{\xi(\beta)}(\beta) \geq c_1 \chi(\beta) \geq c_2/(\beta_c - \beta)$$

(the inequalities follow respectively from the Infrared Bound, a classical bound obtained first by Sokal [51], and the mean-field lower bound on $\chi(\beta)$ [1]). In field theory this quantity is often referred to as the (dimensionless) renormalized coupling constant. In [33] it was proved that for lattice ϕ^4_4 measures of small enough λ it converges to 0 at the rate $1/\log(\frac{1}{|\beta-\beta_c|})$. Such behaviour is expected to be true, in dimension $d = 4$, also for the n.n.f. Ising model.

6 Generalization to models in the Griffiths-Simon class

6.1 Definition of the models

While the random current properties are rooted in the binary structure of the Ising spins, the results extend to a broader family of statistic-mechanical Gibbs equilibrium states of systems of variables $\tau = (\tau_x)$ defined over $\Lambda \subset \mathbb{Z}^d$, with an Ising-like Hamiltonian

$$H_{\Lambda,J}(\tau) = -\sum_{\{x,y\}\subset\Lambda} J_{x,y}\,\tau_x\tau_y \tag{6.1}$$

and an a-priori site measure ρ of the form

$$\rho(\tau) := \sum_{\underline{\sigma}\in\{\pm1\}^N} \exp[\sum_{i,j=1}^{n} K_{i,j}\sigma_i\sigma_j] \times \delta(\tau - \sum_{j=1}^{N} Q_j\sigma_j)\,, \tag{6.2}$$

where δ denotes the Dirac function.

In discussions of the Gibbs equilibrium states of such systems, i.e. measures of the form $e^{-\beta H(\tau)}\prod_x \rho(\tau_x)/\mathrm{Norm}$, the variables τ_x can be viewed as local weighted averages of an array of Ising *constituent spins* attached to the vertices (x,j) of a decorated version $\Lambda\times\{1,\ldots,N\}$ of the original graph:

$$\tau_x := \sum_{j=1}^{n} Q_j\sigma_{x,j}\,. \tag{6.3}$$

This device was introduced by Griffiths [26] as a tool for the extension of inequalities which were initially established for Ising spin models to similar systems of more general spins, of which a number of interesting examples were presented in [26]. It was subsequently pointed out by Simon and Griffiths [29] that upon taking weak limits this class can be extended to include also the ϕ^4 variables described in Section 1.1. This general family of models is referred to below as the Griffiths-Simon (GS) class. Simon and Griffiths' observation was employed in the proof of the "no-go" theorem for $d > 4$ of [1]. It will also play a similar role here, allowing us to extend the results proven above to a similarly broader class of variables.

While this is not relevant for all the properties discussed above, we restrict our attention here to nearest-neighbor ferromagnetic (n.n.f.) models and omit J from the notation.

An important observation is that Reflection-Positivity remains true regardless of the distribution of the single-site variables τ. In particular, the n.n.f. models satisfy this condition so that the results of Section 4 extend mutatis mutandis to ρ in the GS class. Note however that ρ can have unbounded support, so that to be of relevance the relations of interest need to be expressed in spin-dimension balanced forms, a fact on which we were

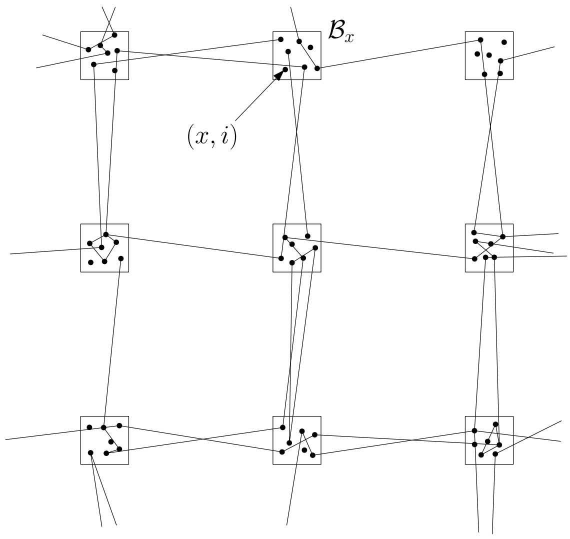

Figure 5: The decorated graph, in which the sites $x \in \Lambda$ of a graph of interest are replaced by "blocks" $\mathscr{B}_x$ of sites indexed as (x, i). The Ising "constituent spins" $\sigma_{x,j}$ are coupled pairwise through intra-block couplings $K_{i,j}$ and inter-block couplings $Q_i J_{x,y} Q_j$. The depicted lines indicate a possible realization of the corresponding random current.

careful in Section 4. Many of the basic diagrammatic bounds which are available for the Ising model extend to the GS class essentially by linearity, and then to the GS class by continuity. In particular, the regularity estimate (4.45) extends verbatim.

When discussing systems of GS single site variables we denote by $\langle \cdot \rangle_{\rho,\beta}$ the model's infinite volume Gibbs measure, $\beta_c(\rho)$ the corresponding critical inverse temperature, $\xi(\rho,\beta)$ the correlation length, $U_4^{\rho,\beta}$ the 4-point Ursell function of the τ variables, and $B_L(\rho,\beta)$ the bubble diagram truncated at a distance L.

6.2 An improved tree diagram bound for models in the GS class

For bounds which are not homogeneous in the spin dimension, one needs to pay attention to the fact that τ is neither dimensionless nor bounded, and prepare the extension by reformulating the Ising relations in a spin-dimensionless form.

For example, the basic tree diagram bound (1.20) has four Ising spins on the left side and four pairs on the right. An extension of the inequality to GS models can be reached by site-splitting the terms in which an Ising spin is repeated, using the inequality

$$\langle \sigma_x \sigma_u \rangle_{\rho,\beta} \langle \sigma_u \sigma_y \rangle_{\rho,\beta} \leq \sum_{u'} \langle \sigma_x \sigma_u \rangle_{\rho,\beta} \, \beta J_{u,u'} \langle \sigma_{u'} \sigma_y \rangle_{\rho,\beta} \tag{6.4}$$

(which has a simple proof by means of the switching lemma)[8]. The resulting diagrammatic bounds may at first glance appear as slightly more complicated than the one for the Ising case, but it has the advantage of being dimensionally balanced. That is a required condition for a bound to hold uniformly throughout the GS class of models. Additional consideration is needed for the factors by which the tree diagram bound of [1] is improved here. Taking care of that we get the following extension of the result, which also covers the ϕ^4 lattice models.

[8]An alternative method for reducing a diagrammatic expression's spin-dimension is to divide by $\langle \sigma_u^2 \rangle_0$. Both methods are of use, and may be compared through (4.20).

Theorem 6.1 (Improved tree diagram bound for the GS class) *There exist $C, c > 0$ such that for every n.n.f. model in the GS class on $\mathbb{Z}^4$, every $\beta \le \beta_c(\rho)$, $L \le \xi(\rho,\beta)$ and every $x, y, z, t \in \mathbb{Z}^d$ at distances larger than L of each other,*

$$|U_4^{\rho,\beta}(x,y,z,t)| \le C\Big(\frac{B_0(\rho,\beta)}{B_L(\rho,\beta)}\Big)^c \sum_u \sum_{u',u''} \langle \tau_x \tau_u\rangle_{\rho,\beta}\, \beta J_{u,u'}\, \langle \tau_{u'}\tau_y\rangle_{\rho,\beta} \langle \tau_z \tau_u\rangle_{\rho,\beta}\, \beta J_{u,u''}\, \langle \tau_{u''}\tau_z\rangle_{\rho,\beta}. \tag{6.5}$$

Before diving into the proof, note that the improved tree diagram bound implies, as it did for the Ising model, the following quantitative bound on the convergence to gaussian of the scaling limit of the τ field in four dimensional models with variables in the GS class.

Proposition 6.2 *There exist two constants $c, C > 0$ such that for every n.n.f. model in the GS class on $\mathbb{Z}^4$, every $\beta \le \beta_c(\rho)$, $L \le \xi(\rho,\beta)$, every continuous function $f : \mathbb{R}^4 \to \mathbb{R}$ with bounded support and every $z \in \mathbb{R}$,*

$$\Big|\Big\langle \exp[z\, T_{f,L}(\tau) - \tfrac{z^2}{2}\langle T_{f,L}(\tau)^2\rangle_{\rho,\beta}]\Big\rangle_{\rho,\beta} - 1\Big| \;\le\; \frac{C\|f\|_\infty^4 r_f^{12}}{(\log L)^c}\, z^4. \tag{6.6}$$

We now return to the proof of the improved tree diagram bound, following the path outlined above. The GS class of variables is naturally divided into two kinds. The core consists of those that directly falls under the definition (6.2). The rest can be obtained as weak limits of the former. Since the constants in (6.5) are uniform, it suffices to prove the result for the former to get it for the latter. We therefore focus on site-measures ρ satisfying the condition (6.2), which can directly be represented as Ising measures on a graph where every vertex is replaced by blocks, as explained in the previous section. In this case, we identify $\langle\cdot\rangle_{\rho,\beta}$ with the Ising measure, and τ_x with the proper average of Ising's variables. With this identification, we can harvest all the nice inequalities that are given by Ising's theory. In particular, we can use the random current representation.

More explicitly, to generalize the argument used in the Ising's proof, we introduce the measure $\mathbf{P}^{xy}$ defined on the graph $\mathbb{Z}^d \times \{1,\dots,N\}$ in two steps:

- first, sample two integers $1 \le i, j \le N$ with probability $Q_i Q_j \langle \sigma_{x,i}\sigma_{y,j}\rangle_{\rho,\beta} / \langle \tau_x \tau_y\rangle_{\rho,\beta}$,
- second, sample a current according to the measure $\mathbf{P}_{\rho,\beta}^{\{(x,i),(y,j)\}}$ corresponding to the random current representation of the Ising model $\langle\cdot\rangle_{\rho,\beta}$.

The interpretation of this object is that of a random current with two *random sources* $(x,i) \in \mathscr{B}_x$ and $(y,j) \in \mathscr{B}_y$. Also note that the superscript xy will unequivocally denote this type of measures (we will avoid using measures with deterministic sources in this section to prevent confusion) and $\mathbf{P}_{\rho,\beta}^{\varnothing}$ which have no sources.

Remark 6.3 *The interest of the measure $\mathbf{P}_{\rho,\beta}^{xy}$ over measures with deterministic sets of sources comes from the fact that the probability that the cluster of the sources intersects a set of the form $\mathscr{B}_u$ can be bounded in terms of correlations of the variables $\tau_x, x \in \mathbb{Z}^d$ (see Proposition A.8).*

Proof of Theorem 6.1 As mentioned above, every ρ in the GS class is a weak limit of measures of the form (6.2). We therefore focus on such measures.

Exactly like in the case of the Ising model, the core of the proof of Theorem 6.1 will be the proof of the intersection-clustering property that we now state and whose proof is postponed after the proof of the theorem. Define $\ell_0 = 0$ and $\ell_k = \ell_k(\rho,\beta)$ using the same definition (using $B_L(\rho,\beta)$ this time) as in (5.1). Let $\mathscr{T}_u$ be the set of vertices $v \in \mathbb{Z}^d$ such that $\mathscr{B}_v$ is connected in $\mathbf{n}_1 + \mathbf{n}_3$ to a box $\mathscr{B}_{u'}$ and in $\mathbf{n}_2 + \mathbf{n}_4$ to a box $\mathscr{B}_{u''}$, with u' and u'' at graph distance at most 2 of u. Note that $\mathscr{T}_u$ is now a function of u and that it is defined in terms of "coarse intersections", i.e. lattice sites v such that both clusters intersect $\mathscr{B}_v$ (but do not necessarily intersect each other).

Proposition 6.4 (intersection-clustering bound for the GS class) *For $d = 4$ and D large enough, there exists $\delta = \delta(D)$ such that for every model with ρ satisfying (6.2), every $\beta \le \beta_c(\rho)$, every K such that $\ell_K \le \xi(\rho,\beta)$ and every $u, u', u'', x, y, z, t \in \mathbb{Z}^d$ with u' and u'' neighbors of u and x, y, z, t at mutual distances larger than $2\ell_K$,*

$$\mathbf{P}^{ux,uz,u'y,u''t}_{\rho,\beta}[\mathbf{M}_u(\mathscr{T}_u;\mathscr{L},K) \le \delta K] \le 2^{-\delta K}. \tag{6.7}$$

We now keep going with the proof of the theorem. Express $U_4^{\rho,\beta}$ in terms of intersection properties of currents by summing (2.14) over vertices of $\mathscr{B}_x,\dots,\mathscr{B}_z$:

$$|U_4^{\rho,\beta}(x,y,z,t)| \le 2\langle\tau_x\tau_y\rangle_{\rho,\beta}\langle\tau_z\tau_t\rangle_{\rho,\beta}\mathbf{P}^{xy,zt,\varnothing,\varnothing}_{\rho,\beta}[\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_3}(\partial\mathbf{n}_1)\cap\mathbf{C}_{\mathbf{n}_2+\mathbf{n}_4}(\partial\mathbf{n}_2) \ne \varnothing], \tag{6.8}$$

where $\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_3}(\partial\mathbf{n}_1)$ and $\mathbf{C}_{\mathbf{n}_2+\mathbf{n}_4}(\partial\mathbf{n}_2)$ refer to the clusters in $\mathbf{n}_1+\mathbf{n}_3$ and $\mathbf{n}_2+\mathbf{n}_4$ of the sources in $\partial\mathbf{n}_1$ and $\partial\mathbf{n}_2$ respectively (we introduce this notation since the sources are not deterministic anymore).

Define $K \ge c\log[B_L(\rho,\beta)/B_0(\rho,\beta)]$ as in the Ising case. We now implement the same reasoning as for the Ising model, with the twist that we consider coarse intersections. If $\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_3}(\partial\mathbf{n}_1)$ and $\mathbf{C}_{\mathbf{n}_2+\mathbf{n}_4}(\partial\mathbf{n}_2)$ intersect, then

- either the number of $u \in \mathbb{Z}^d$ such that $\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_3}(\partial\mathbf{n}_1)$ and $\mathbf{C}_{\mathbf{n}_2+\mathbf{n}_4}(\partial\mathbf{n}_2)$ intersect $\mathscr{B}_u$ is larger than or equal to $2^{\delta K/5}$,
- or there exists $u \in \mathbb{Z}^d$ such that $\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_3}(\partial\mathbf{n}_1)$ and $\mathbf{C}_{\mathbf{n}_2+\mathbf{n}_4}(\partial\mathbf{n}_2)$ intersect $\mathscr{B}_u$, and $\mathbf{M}_u(\mathscr{T}_u;\mathscr{L},K) < \delta K$.

Using the Markov inequality and (A.34) on the first line, and Lemma A.7 in the second one, we find (drop ρ and β from notation)

$$\begin{aligned}|U_4(x,y,z,t)| \le{}& 2^{-\delta K/5}\sum_{u,u',u''\in\mathbb{Z}^d}\langle\tau_x\tau_u\rangle\,\beta J_{u,u'}\langle\tau_{u'}\tau_y\rangle\langle\tau_z\tau_u\rangle\,\beta J_{u,u''}\,\langle\tau_{u''}\tau_t\rangle\\ &+\sum_{u,u',u''\in\mathbb{Z}^d}\langle\tau_x\tau_u\rangle\,\beta J_{u,u'}\,\langle\tau_{u'}\tau_y\rangle\langle\tau_z\tau_u\rangle\,\beta J_{u,u''}\,\langle\tau_{u''}\tau_t\rangle\mathbf{P}^{xu,zu,u'y,u''t}[\mathbf{M}_u(\mathscr{T}_u;\mathscr{L},K) < \delta K].\end{aligned} \tag{6.9}$$

It may be noted that the definition of $\mathbf{M}_u(\mathscr{T}_u;\mathscr{L},K)$ in terms of coarse intersections instead of true intersections is crucial to use Lemma A.7 (like that the event does not depend on the specific sources in each block). Also, the fact that $\mathscr{T}_u$ was defined in terms of clusters intersecting boxes at graph distance at most 2 of u enables us to ignore the tedious additional of δ_{ab} in Lemma A.7. The intersection-clustering bound (Proposition 6.4) concludes the proof. □

We now need to prove Proposition 6.4. The proof itself is exactly the same as for Proposition 5.1 (the monotonicity property of (A.2) is not impacted), except for the proofs of the mixing and intersection properties (i.e. statements corresponding to Lemma 5.2 and Theorem 5.4 respectively). Below, we briefly detail the statements and proofs of these results. Let $I_k(0)$ be the event that there exists $v \in \mathrm{Ann}(\ell_k,\ell_{k+1})$ such that $\mathscr{B}_v$ is connected in $\mathbf{n}_1+\mathbf{n}_3$ and to $\mathbf{n}_2+\mathbf{n}_4$ to the union of the boxes $\mathscr{B}_w$ with w at a distance at most 2 of 0.

Lemma 6.5 (intersection property for the GS class) *There exists $c > 0$ such that for every ρ satisfying* (6.2)*, every $\beta \le \beta_c(\rho)$, every k, every neighbour $0'$ of the origin, and every $y \notin \Lambda_{2\ell_{k+1}}$ in a regular scale,*

$$\mathbf{P}^{0y,0'y,\varnothing,\varnothing}_{\rho,\beta}[I_k(0)] \ge c. \tag{6.10}$$

Proof Reuse the notions included in the proofs of the intersection property in previous sections. Let

$$\mathscr{M} := \sum_{v\in \mathrm{Ann}(m,M)} \sum_{i,i'=1}^{n} Q_i^2 \mathbb{I}[\partial \mathbf{n}_1 \overset{\mathbf{n}_1+\mathbf{n}_3}{\longleftrightarrow} (v,i)]\, Q_{i'}^2 \mathbb{I}[\partial \mathbf{n}_1 \overset{\mathbf{n}_1+\mathbf{n}_3}{\longleftrightarrow} (v,i')]. \tag{6.11}$$

A computation similar to before gives

$$\mathbf{E}^{0y,0'y,\varnothing,\varnothing}_{\rho,\beta}[|\mathscr{M}|] \ge c_1(B_M(\rho,\beta) - B_{m-1}(\rho,\beta)) \tag{6.12}$$

$$\mathbf{E}^{0y,0'y,\varnothing,\varnothing}_{\rho,\beta}[|\mathscr{M}|^2] \le C_2 B_{\ell_{k+1}}(\rho,\beta)^2. \tag{6.13}$$

Now, in the first line we use the same reasoning as in (**??**). We include it for completeness to see where the division by $B_0(\rho,\beta)$ enters into the game (it is the only place it does). The Infrared Bound (4.43) (note that $\langle \tau_0^2\rangle_{\rho,\beta} = B_0(\rho,\beta)$) implies that

$$B_M(\rho,\beta) - B_{m-1}(\rho,\beta) \ge B_{\ell_{k+1}}(\rho,\beta) - B_{\ell_k}(\rho,\beta) - C_3 B_0(\rho,\beta) \ge (1 - \tfrac{1+C_3}{D}) B_{\ell_{k+1}}(\rho,\beta). \tag{6.14}$$

Cauchy-Schwarz therefore implies the fact that $\mathscr{M} \ne \varnothing$ with positive probability, which implies in particular the existence of a vertex $v \in \mathrm{Ann}(m,M)$ which is connected in $\mathbf{n}_1 + \mathbf{n}_3$ to $\mathscr{B}_0$ and in $\mathbf{n}_2 + \mathbf{n}_4$ to $\mathscr{B}_0'$.

The second part of the proof bounding the probabilities of $F_1, \dots, F_4$ follows by the same proof as for the Ising model. More precisely, for F_1, the chain rule for backbones [4] and a decomposition on the first edge of the backbone with one endpoint in (a block of a vertex in) Λ_{n-1} and the other (in a block of a vertex) in $\partial\Lambda_n$, and then the first edge after this between an endpoint (in a block of a vertex) outside Λ_{ℓ_k} and one in (a block of a vertex in) Λ_{ℓ_k} implies that

$$\mathbf{P}^{0y,\varnothing}_{\rho,\beta}[F_1] \le \sum_{\substack{v\in\partial\Lambda_n \\ w\in\partial\Lambda_{\ell_k} \\ v',w'\in\mathbb{Z}^d}} \frac{\langle \tau_0\tau_v\rangle_{\rho,\beta}\, \beta J_{v',v}\, \langle \tau_v \tau_{w'}\rangle_{\rho,\beta}\, \beta J_{w',w}\, \langle \tau_w\tau_y\rangle_{\rho,\beta}}{\langle \tau_0\tau_y\rangle_{\rho,\beta}} \le C_3 n^3 \ell_k^3 n^{-4} \le C_4 \ell_k^{-\varepsilon}. \tag{6.15}$$

This inequality uses Property P2 of regular scales, the lower bound (4.6) on the two-point function, and the Infrared Bound (4.44). For F_3, the same reasoning as for Ising, with Proposition A.8 replacing the switching lemma, leads to

$$\mathbf{P}^{0x,\varnothing}_{\rho,\beta}[F_3] \le \sum_{\substack{v\in\partial\Lambda_n \\ w\in\partial\Lambda_m}} \mathbf{P}^{\varnothing,\varnothing}_{\rho,\beta}[\mathscr{B}_v \overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow} \mathscr{B}_w] \le \sum_{\substack{v\in\partial\Lambda_n \\ w\in\partial\Lambda_m \\ v',w'\in\mathbb{Z}^d}} \langle \tau_v\tau_w\rangle\, \beta J_{w,w'}\, \langle \tau_{w'}\tau_{v'}\rangle\, \beta J_{v',v} \le C_5 \ell_k^{-\varepsilon}, \tag{6.16}$$

where in the last line we used again the Infrared Bound (4.44). $\square$

We now turn to the proof of the mixing property for the measures $\mathbf{P}^{xy}_{\beta}$, which is the exact replica of the Ising statement.

Theorem 6.6 (mixing of random currents for the GS class) *For $d \geq 4$, there exist $\alpha, c > 0$ such that for every ρ satisfying (6.2), every $t \leq s$, every $\beta \leq \beta_c(\rho)$, every $n^\alpha \leq N \leq \xi(\rho,\beta)$, every $x_i \in \Lambda_n$ and $y_i \notin \Lambda_N$ for every $i \leq t$, and every events E and F depending on the restriction of $(\mathbf{n}_1, \ldots, \mathbf{n}_s)$ to edges within Λ_n and outside of Λ_N respectively,*

$$\left|\mathbf{P}_{\rho,\beta}^{x_1y_1,\ldots,x_ty_t,\varnothing,\ldots,\varnothing}[E \cap F] - \mathbf{P}_{\rho,\beta}^{x_1y_1,\ldots,x_ty_t,\varnothing,\ldots,\varnothing}[E]\mathbf{P}_{\rho,\beta}^{x_1y_1,\ldots,x_ty_t,\varnothing,\ldots,\varnothing}[F]\right| \leq s(\log \tfrac{N}{n})^{-c}. \tag{6.17}$$

Furthermore, for every $x_1', \ldots, x_t' \in \Lambda_n$ and $y_1', \ldots, y_t' \notin \Lambda_N$,

$$\left|\mathbf{P}_{\rho,\beta}^{x_1y_1,\ldots,x_ty_t,\varnothing,\ldots,\varnothing}[E] - \mathbf{P}_{\rho,\beta}^{x_1y_1',\ldots,x_ty_t',\varnothing,\ldots,\varnothing}[E]\right| \leq s(\log \tfrac{N}{n})^{-c}, \tag{6.18}$$

$$\left|\mathbf{P}_{\rho,\beta}^{x_1y_1,\ldots,x_ty_t,\varnothing,\ldots,\varnothing}[F] - \mathbf{P}_{\rho,\beta}^{x_1'y_1,\ldots,x_t'y_t,\varnothing,\ldots,\varnothing}[F]\right| \leq s(\log \tfrac{N}{n})^{-c}. \tag{6.19}$$

Proof The beginning is the same as for the Ising model, until the definition of the variable $\mathbf{N}_i$ which becomes

$$\mathbf{N}_i := \frac{1}{|\mathcal{K}|} \sum_{k \in \mathcal{K}} \frac{1}{A_{x_i,y_i}(k)} \sum_{u \in \mathbb{A}_k(y_i)} \sum_{j=1}^{N} Q_j^2 \, \mathbb{I}[(u,j) \overset{\mathbf{n}_i + \mathbf{n}_i'}{\longleftrightarrow} \partial \mathbf{n}_i], \tag{6.20}$$

where $a_{x,y}(u) := \langle \tau_x \tau_u \rangle \langle \tau_u \tau_y \rangle / \langle \tau_x \tau_y \rangle$ and $A_{x,y}(k) := \sum_{u \in \mathbb{A}_k(y_i)} a_{x,y}(u)$. The proof of the concentration inequality follows the same lines as in the Ising case. Indeed, the choice of the weight Q_j^2 enables to rewrite the moments of the random variables $\mathbf{N}_i$ in terms of the correlations of the random variables $(\tau_z : z \in \mathbb{Z}^d)$. The rest of the proof is exactly the same, with trivial changes. For instance, in the proof of Lemma 5.7, one must be careful to derive bounds on probabilities involving $\beta|J|$. This is easily doable using Proposition A.8 exactly like in the previous proof. □

A Appendix

A.1 Partial monotonicity statements

An inconvenient feature of the random current representation is the lack of an FKG-type monotonicity, as the one valid for the Fortuin-Kasteleyn random cluster models (cf. [28]). The addition of a pair of sources may enhance the configuration, e.g. forcing a long line where such were rare, but in some situations it may facilitate a split in a connecting line, thereby reducing the current's connectivity properties. Nevertheless, some monotonicity properties can still be found, and are used in our analysis.

In this section, we set σ_A for the product of the spins in A and write $\mathbf{C}_\mathbf{n}(S) = \cup_{x \in S} \mathbf{C}_\mathbf{n}(x)$.

Lemma A.1 *Let A, B, S be subsets of Λ and F a non-negative function defined over pairs of currents, which is determined by just the values of $(\mathbf{n}_1, \mathbf{n}_2)$ along the edges touching the connected cluster $\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(S)$ and such that $F(\mathbf{n}_1, \mathbf{n}_2) = 0$ whenever that cluster intersects B and $(\partial \mathbf{n}_1, \partial \mathbf{n}_2) = (A, B)$. Then*

$$\mathbf{E}_{\Lambda,\beta}^{A,B}[F(\mathbf{n}_1,\mathbf{n}_2)] = \mathbf{E}_{\Lambda,\beta}^{A,\varnothing}[F(\mathbf{n}_1,\mathbf{n}_2)\frac{\langle \sigma_B \rangle_{\Lambda \smallsetminus \mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(S),\beta}}{\langle \sigma_B \rangle_{\Lambda,\beta}}] \leq \mathbf{E}_{\Lambda,\beta}^{A,\varnothing}[F(\mathbf{n}_1,\mathbf{n}_2)]. \tag{A.1}$$

Proof The second inequality is a trivial application of Griffiths' inequality [25]. The first one is proven by a fairly straightforward manipulation involving currents that we now present. We drop β from the notation. Fix $T \subset \Lambda$ not intersection B and choose F given by

$$F(\mathbf{n}_1, \mathbf{n}_2) := \mathbb{I}[\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(S) = T]\,\mathbb{I}[\mathbf{n}_1 = \mathbf{n}]\,\mathbb{I}[\mathbf{n}_2 = \mathbf{m} \text{ on } T] \tag{A.2}$$

for $\mathbf{n}$ and $\mathbf{m}$ currents on Λ and T respectively. For such a choice of function, we find that

$$\begin{aligned}
\langle\sigma_A\rangle_\Lambda\langle\sigma_B\rangle_\Lambda\,\mathbf{E}_\Lambda^{A,B}[F(\mathbf{n}_1,\mathbf{n}_2)] &= \frac{4^{|\Lambda|}}{Z(\Lambda,\beta)^2}\sum_{\mathbf{n}_1:\partial\mathbf{n}_1=A}\ \sum_{\mathbf{n}_2:\partial\mathbf{n}_2=B} F(\mathbf{n}_1,\mathbf{n}_2)w(\mathbf{n}_1)w(\mathbf{n}_2)\\
&= \frac{4^{|\Lambda|}w(\mathbf{n})w(\mathbf{m})}{Z(\Lambda,\beta)^2}\sum_{\mathbf{n}_2':\partial\mathbf{n}_2'=B} w(\mathbf{n}_2)\\
&= \frac{4^{|\Lambda|}w(\mathbf{n})w(\mathbf{m})}{Z(\Lambda,\beta)^2}\langle\sigma_B\rangle_{\Lambda\smallsetminus T}\sum_{\mathbf{n}_2':\partial\mathbf{n}_2'=\varnothing} w(\mathbf{n}_2)\\
&= \langle\sigma_A\rangle_\Lambda\langle\sigma_B\rangle_{\Lambda\smallsetminus T}\,\mathbf{E}_\Lambda^{A,\varnothing}[F(\mathbf{n}_1,\mathbf{n}_2)],
\end{aligned} \tag{A.3}$$

where $\mathbf{n}_2'$ is referring to a current on $\Lambda\smallsetminus T$. In the second line, we used that for $F(\mathbf{n}_1,\mathbf{n}_2)$ to be non-zero, $\mathbf{n}_1$ must be equal to $\mathbf{n}$ and $\mathbf{n}_2$ be decomposed into the current $\mathbf{m}$ on T and a current $\mathbf{n}_2'$ outside T (also, $\mathbf{n}_2(x,y)$ is equal to zero for every $x\in T$ and $y\notin T$). In the last line, we skipped the steps corresponding to going backward line to line to end up with $\mathbf{E}_\Lambda^{A,\varnothing}[F(\mathbf{n}_1,\mathbf{n}_2)]$.

The proof follows readily for every function F satisfying the assumptions of the lemma. Also, we obtain the result on $\mathbb{Z}^d$ by letting Λ tend to $\mathbb{Z}^d$. □

An interesting application of the lemma is the following pair of *disentangling bounds*. The first inequality appeared in [1, Proposition 5.2], the second is new.

Corollary A.2 *For every $\beta > 0$, every four vertices $x, y, z, t \in \mathbb{Z}^d$ and every set $S \subset \mathbb{Z}^d$,*

$$\mathbf{P}_\beta^{xy,zt}[\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(x)\cap\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(z)\neq\varnothing] \le \mathbf{P}_\beta^{xy,\varnothing,zt}[\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(x)\cap\mathbf{C}_{\mathbf{n}_3}(z)\neq\varnothing], \tag{A.4}$$

$$\mathbf{P}_\beta^{0x,0z,\varnothing,\varnothing}[\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_3}(0)\cap\mathbf{C}_{\mathbf{n}_2+\mathbf{n}_4}(0)\cap S\neq\varnothing] \le \mathbf{P}_\beta^{0x,0z,0y,0t}[\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_3}(0)\cap\mathbf{C}_{\mathbf{n}_2+\mathbf{n}_4}(0)\cap S\neq\varnothing]. \tag{A.5}$$

Proof Fix $\beta > 0$, Λ finite (the claim will then follow by letting Λ tend to $\mathbb{Z}^d$) and drop β from the notation. For the first identity, introduce the random variable

$$\mathbf{C} = \mathbf{C}(\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3) := \mathbf{C}_{\mathbf{n}_3}(\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(x)). \tag{A.6}$$

Lemma A.1 applied in the first and third lines, Griffiths' inequality [25], and the trivial inclusion $\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(x)\subset\mathbf{C}$ in the second, give

$$\begin{aligned}
\mathbf{P}_\Lambda^{xy,zt}[\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(x)\cap\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(z)=\varnothing] &= \mathbf{E}_\Lambda^{xy,\varnothing}\Big[\mathbb{I}[z,t\notin\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(x)]\frac{\langle\sigma_z\sigma_t\rangle_{\Lambda\smallsetminus\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(x)}}{\langle\sigma_z\sigma_t\rangle_\Lambda}\Big]\\
&\ge \mathbf{E}_\Lambda^{xy,\varnothing,\varnothing}\Big[\mathbb{I}[z,t\notin\mathbf{C}]\frac{\langle\sigma_z\sigma_t\rangle_{\Lambda\smallsetminus\mathbf{C}}}{\langle\sigma_z\sigma_t\rangle_\Lambda}\Big]\\
&= \mathbf{P}_\Lambda^{xy,\varnothing,zt}[z,t\notin\mathbf{C}],
\end{aligned} \tag{A.7}$$

which gives the first inequality.

The second identity requires two successive applications of Lemma A.1. First, conditioning on $\mathbf{n}_2 + \mathbf{n}_4$, the proposition applied to $\mathbf{S} := \mathbf{C}_{\mathbf{n_2}+\mathbf{n_4}}(0) \cap S$ gives

$$\mathbf{P}_\Lambda^{0x,0z,\varnothing,0t}[\mathbf{C}_{\mathbf{n_1}+\mathbf{n_3}}(0) \cap \mathbf{C}_{\mathbf{n_2}+\mathbf{n_4}}(0) \cap S \neq \varnothing] \leq \mathbf{P}_\Lambda^{0x,0z,0y,0t}[\mathbf{C}_{\mathbf{n_1}+\mathbf{n_3}}(0) \cap \mathbf{C}_{\mathbf{n_2}+\mathbf{n_4}}(0) \cap S \neq \varnothing]. \tag{A.8}$$

Similarly, conditioning on $\mathbf{n}_1 + \mathbf{n}_3$, the proposition applied to $\mathbf{S}' := \mathbf{C}_{\mathbf{n_1}+\mathbf{n_3}}(0) \cap S$ gives

$$\mathbf{P}_\Lambda^{0x,0z,\varnothing,\varnothing}[\mathbf{C}_{\mathbf{n_1}+\mathbf{n_3}}(0) \cap \mathbf{C}_{\mathbf{n_2}+\mathbf{n_4}}(0) \cap S \neq \varnothing] \leq \mathbf{P}_\Lambda^{0x,0z,\varnothing,0t}[\mathbf{C}_{\mathbf{n_1}+\mathbf{n_3}}(0) \cap \mathbf{C}_{\mathbf{n_2}+\mathbf{n_4}}(0) \cap S \neq \varnothing], \tag{A.9}$$

thus concluding the proof. □

A.2 Multi-point connectivity probabilities

The following two relations facilitate the derivation of estimates guided by the random walk analogy.

Proposition A.3 *For every $x, u, v \in \mathbb{Z}^d$, we have that*

$$\mathbf{P}_\beta^{0x,\varnothing}[u \overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow} 0] = \frac{\langle\sigma_0\sigma_u\rangle_\beta\langle\sigma_u\sigma_x\rangle_\beta}{\langle\sigma_0\sigma_x\rangle_\beta}, \tag{A.10}$$

$$\mathbf{P}_\beta^{0x,\varnothing}[u, v \overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow} 0] \leq \frac{\langle\sigma_0\sigma_v\rangle_\beta\langle\sigma_v\sigma_u\rangle_\beta\langle\sigma_u\sigma_x\rangle_\beta}{\langle\sigma_0\sigma_x\rangle_\beta} + \frac{\langle\sigma_0\sigma_u\rangle_\beta\langle\sigma_u\sigma_v\rangle_\beta\langle\sigma_v\sigma_x\rangle_\beta}{\langle\sigma_0\sigma_x\rangle_\beta}. \tag{A.11}$$

The equality (A.10) is a direct consequence of the switching lemma and has been used several times in the past. The inequality (A.11) is an important new addition, which is proven below. Its structure suggests a more general k-step random walk type bound, but the present proof does not extend to $k > 2$. In particular, if a k-step bound could be proven for every k, it would improve the concentration estimate for $\mathbf{N}$ in the proof of mixing from an inverse logarithmic bound to a small polynomial one, which would translate into a similar bound for the mixing property which may be very useful for the study of the critical regime. Note that this would not improve the log correction in our result since the intersection property also requires the ℓ_k to grow fast.

Proof Fix $\beta > 0$ and drop it from the notation. We work with finite Λ and then take the limit as Λ tends to $\mathbb{Z}^d$. In the whole proof, $\longleftrightarrow$ denotes the connection in $\mathbf{n}_1 + \mathbf{n}_2$, and $\not\longleftrightarrow$ denotes the absence of connection. As mentioned above, (A.10) follows readily from the switching lemma. To prove (A.11), use the switching lemma to find

$$\mathbf{P}_\Lambda^{0x,\varnothing}[u, v \longleftrightarrow 0] = \frac{\langle\sigma_0\sigma_u\rangle_\Lambda\langle\sigma_u\sigma_x\rangle_\Lambda}{\langle\sigma_0\sigma_x\rangle_\Lambda}\mathbf{P}_\Lambda^{0u,ux}[v \longleftrightarrow u]. \tag{A.12}$$

Then, our goal is to show that

$$\mathbf{P}_\Lambda^{0u,ux}[v \longleftrightarrow u] \leq \mathbf{P}_\Lambda^{0u,\varnothing}[v \longleftrightarrow u] + \mathbf{P}_\Lambda^{\varnothing,ux}[v \longleftrightarrow u] - \mathbf{P}_\Lambda^{\varnothing,\varnothing}[v \longleftrightarrow u] \tag{A.13}$$

which implies (A.11) readily using (A.10). In order to show (A.13), set $\mathbf{C} = \mathbf{C}_{\mathbf{n_1}+\mathbf{n_2}}(v)$ and apply Lemma A.1 to $F(\mathbf{n}_1, \mathbf{n}_2) := \mathbb{I}[u \not\longleftrightarrow v]$ to obtain

$$\mathbf{P}_\Lambda^{0u,ux}[u \not\longleftrightarrow v] = \mathbf{E}_\Lambda^{0u,\varnothing}\Big[\mathbb{I}[u \not\longleftrightarrow v]\frac{\langle\sigma_0\sigma_y\rangle_{\Lambda\setminus\mathbf{C}}}{\langle\sigma_0\sigma_y\rangle_\Lambda}\Big]. \tag{A.14}$$

Next, apply Lemma A.1 to

$$F(\mathbf{n}_1, \mathbf{n}_2) := \mathbb{I}[u \not\leftrightarrow v]\Big(1 - \frac{\langle \sigma_0 \sigma_x \rangle_{\Lambda \setminus \mathbf{C}}}{\langle \sigma_0 \sigma_x \rangle_\Lambda}\Big) \geq 0 \tag{A.15}$$

(the inequality is due to Griffiths' inequality [25]) to obtain (A.13) thanks to the following inequalities

$$\begin{aligned}\mathbf{P}_\Lambda^{0x,\varnothing}[u \not\leftrightarrow v] - \mathbf{P}_\Lambda^{0x,0y}[u \not\leftrightarrow v] = \mathbf{E}_\Lambda^{0x,\varnothing}[F(\mathbf{n}_1, \mathbf{n}_2)] = \mathbf{E}_\Lambda^{\varnothing,\varnothing}\Big[F(\mathbf{n}_1, \mathbf{n}_2) \frac{\langle \sigma_0 \sigma_x \rangle_{\Lambda \setminus \mathbf{C}}}{\langle \sigma_0 \sigma_x \rangle_\Lambda}\Big] \\ \leq \mathbf{E}_\Lambda^{\varnothing,\varnothing}[F(\mathbf{n}_1, \mathbf{n}_2)] = \mathbf{P}_\Lambda^{\varnothing,\varnothing}[u \not\leftrightarrow v] - \mathbf{P}_\Lambda^{\varnothing,0y}[u \not\leftrightarrow v].\end{aligned} \tag{A.16}$$

□

Remark A.4 *Griffiths' inequality [25] plugged in* (A.14) *gives*

$$\mathbf{P}^{0u,ux}[v \longleftrightarrow u] \geq \mathbf{P}^{0u,\varnothing}[v \longleftrightarrow u]. \tag{A.17}$$

Remark A.5 *The inequalities* (A.13) *and* (A.17) *can be extended to every set* $S \subset \mathbb{Z}^d$ *and every two vertices* $x, y \in \mathbb{Z}^d$*:*

$$\mathbf{P}_\beta^{0x,\varnothing}[0 \overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow} S] \leq \mathbf{P}_\beta^{0x,0y}[0 \overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow} S] \leq \mathbf{P}_\beta^{0x,\varnothing}[0 \overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow} S] + \mathbf{P}_\beta^{\varnothing,0y}[0 \overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow} S] - \mathbf{P}_\beta^{\varnothing,\varnothing}[0 \overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow} S]. \tag{A.18}$$

A.3 The spectral representation

In Section 4.3 we make use of a spectral representation of the correlation function. Thought the statement is well known, cf. [23] and references therein, for completeness of the presentation following is its derivation.

Proposition A.6 (Spectral Representation) *For the n.n.f. Ising model on* $\mathbb{Z}^d$ *(*$d \geq 1$*), at* $\beta < \beta_c$*, for every bounded function* $v : \mathbb{Z}^{d-1} \to \mathbb{C}$*, there exists a positive measure* $\mu_{v,\beta}$ *of finite mass*

$$\int_{1/\xi(\beta)}^\infty d\mu_{v,\beta}(a) = \sum_{x_\perp, y_\perp \in \mathbb{Z}^{d-1}} v_{x_\perp} \overline{v_{y_\perp}} S_\beta((0, y_\perp - x_\perp)) \tag{A.19}$$

such that for every $n \in \mathbb{Z}$*,*

$$\sum_{x_\perp, y_\perp \in \mathbb{Z}^{d-1}} v_{x_\perp} \overline{v_{y_\perp}} S_\beta((n, x_\perp - y_\perp)) = \int_{1/\xi(\beta)}^\infty e^{-a|n|} d\mu_{v,\beta}(a). \tag{A.20}$$

In particular, with $v = \delta_\perp$ (the Kronecker delta function over $\mathbb{Z}^{d-1}$), this yields the following spectral representation for the correlation function along a principal axis

$$S_{\infty,\ell}((n, 0_\perp)) = \int_{1/\xi(\ell,\beta)}^\infty e^{-an} d\mu_{\delta_\perp,\beta,\ell}(a), \tag{A.21}$$

where $\mu_{\delta_\perp,\beta,\ell}$ is a probability measure.

Proof It is convenient to first derive the corresponding statements for finite volume versions of the model, in tubular domains with periodic boundary conditions $\mathbb{T}(m,\ell) := (\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/\ell\mathbb{Z})^{d-1}$ (with the convention $(\mathbb{Z}/\infty\mathbb{Z}) = \mathbb{Z}$). Let $S_{m,\ell}(x) := \langle \tau_0 \tau_x \rangle_{\mathbb{T}(m,\ell)}$.

Consider an $\mathbb{C}$-vector space $\mathscr{V}_\ell$ with basis $\mathscr{B}$ consisting of elements $|\tau\rangle$ indexed by the spin configurations τ on the transversal hyperplane $(\mathbb{Z}/\ell\mathbb{Z})^{d-1}$. On $\mathscr{V}_\ell$, introduce the *transfer matrix* T_ℓ whose matrix elements in this basis are

$$T_\ell(\tau,\tau') := \exp\Big[\beta \sum_{\substack{\{x,y\}\subset(\mathbb{Z}/\ell\mathbb{Z})^{d-1}\\ x,y \text{ neighbours}}} (\tau_x\tau_y + \tau'_x\tau'_y) + \beta \sum_{x\in(\mathbb{Z}/\ell\mathbb{Z})^{d-1}} \tau_x\tau'_x\Big]. \tag{A.22}$$

Also, consider the family of linear operators Ψ_v on $\mathscr{V}$, depending linearly on functions $v : (\mathbb{Z}/\ell\mathbb{Z})^{d-1} \to \mathbb{C}$, which in the basis $\mathscr{B}$ are diagonal, acting as

$$\Psi_v|\tau\rangle := \Big(\sum_{x\in(\mathbb{Z}/\ell\mathbb{Z})^{d-1}} v(x)\tau_x\Big)|\tau\rangle. \tag{A.23}$$

In this notation, one obtains the following representation for the correlation function of spins at sites which we write as $(n, x_\perp) \in \mathbb{T}_{m,\ell}$:

$$\sum_{x_\perp,y_\perp\in(\mathbb{Z}/\ell\mathbb{Z})^{d-1}} \overline{v_{y_\perp}} v_{x_\perp} S_{m,\ell}((n, y_\perp - x_\perp)) = \frac{\mathrm{Trace}(T_\ell^{m-n}\overline{\Psi}_v T_\ell^n \Psi_v^{(\ell)})}{\mathrm{Trace}(T_\ell^m)}. \tag{A.24}$$

Now, the n.n.f. model's transfer matrix T_ℓ is positive definite. Indeed, the symmetry follows directly from the definition. Positivity with respect to reflections by symmetry hyperplanes passing through mid-edges follows from the elementary criterion of [21] (see also [13]). Being a positive definite matrix, T_ℓ is diagonalizable. The spectral representation of T_ℓ enables us to write

$$\sum_{x_\perp,y_\perp\in\mathbb{Z}^{d-1}} v_{x_\perp}\overline{v_{y_\perp}} S_{m,\ell}((n, x_\perp - y_\perp)) = \frac{\sum_{\lambda_1,\lambda_2\in\mathrm{Spec}(T_\ell)} {\lambda_1}^{m-n}{\lambda_2}^n \langle e_{\lambda_1}|\overline{\Psi}_v|e_{\lambda_2}\rangle\langle e_{\lambda_2}|\Psi_v|e_{\lambda_1}\rangle}{\sum_{\lambda\in\mathrm{Spec}(T_\ell)} \lambda^m}, \tag{A.25}$$

where $\{|e_\lambda\rangle\}$ is an orthonormal basis of eigenvectors of T_ℓ. Since the matrix elements of T_ℓ are all strictly positive, the maximal eigenvalue $\lambda_{\max}$ is non-degenerate. Hence, in the limit $m \to \infty$ only the terms with $\lambda_1 = \lambda_{\max}$ and $\lambda = \lambda_{\max}$ are of relevance, and one is left with the single sum:

$$\sum_{x_\perp,y_\perp\in\mathbb{Z}^{d-1}} v_{x_\perp}\overline{v_{y_\perp}} S_{\infty,\ell}((n, x_\perp - y_\perp)) = \sum_{\lambda\in\mathrm{Spec}(P)} (\tfrac{\lambda}{\lambda_{\max}})^n \langle e_{\lambda_{\max}}|\overline{T}|e_\lambda\rangle\langle e_\lambda|T|e_{\lambda_{\max}}\rangle. \tag{A.26}$$

Writing $\lambda/\lambda_{\max} = e^{-a}$ directly yields

$$\sum_{x_\perp,y_\perp\in\mathbb{Z}^{d-1}} v_{x_\perp}\overline{v_{y_\perp}} S_{\infty,\ell}((n, x_\perp - y_\perp)) = \int_{1/\xi(\ell,\beta)}^\infty e^{-an} d\mu_{v,\beta,\ell}(a), \tag{A.27}$$

where $\mu_{v,\beta,\ell}$ is a discrete finite measure (whose support starts at $\xi(\ell,\beta)$ which is the inverse rate of decay in x of $S_{\infty,\ell}(x)$). The statement follows by taking the limit of this relation with $\ell \to \infty$. For any $\beta < \beta_c$ the existence of the limit can be justified through standard arguments based on the fact that the correlations decay uniformly in ℓ at an exponential rate $1/\xi(\beta)(> 0)$. The limiting measure (which will no longer be discrete) is of support bounded away from 0, starting at the inverse correlation length. □

A.4 Intersection properties for random current representation of models in the GS class

We start with the version of the switching lemma that we will use. Below, δ_{uv} denotes the current equal to 1 on the edge uv and 0 otherwise.

Lemma A.7 (coarse switching lemma) *Let S, T be two disjoint sets of vertices. For every event E depending on the sum of two currents and every $x \neq y$,*

$$\mathbf{P}_\beta^{xy,\varnothing}[x \overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow} S, \mathbf{n}_1+\mathbf{n}_2 \in E] \le \beta \sum_{a\in S, b\notin S} J_{a,b} \frac{\langle\sigma_x\sigma_a\rangle\langle\sigma_b\sigma_y\rangle}{\langle\sigma_x\sigma_y\rangle} \mathbf{P}_\beta^{xa,by}[\mathbf{n}_1+\mathbf{n}_2+\delta_{ab} \in E], \tag{A.28}$$

$$\mathbf{P}_\beta^{\varnothing,\varnothing}[S \overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow} T, \mathbf{n}_1+\mathbf{n}_2 \in E] \le \beta^2 \sum_{\substack{a\in S, b\notin S\\ s\in T, t\notin T}} J_{a,b}J_{s,t}\langle\sigma_a\sigma_s\rangle\langle\sigma_t\sigma_b\rangle \mathbf{P}_\beta^{as,tb}[\mathbf{n}_1+\mathbf{n}_2+\delta_{ab}+\delta_{st} \in E]. \tag{A.29}$$

Proof We start with the first inequality. Fix Λ finite. By multiplying by the quantity $\langle\sigma_x\sigma_y\rangle_\beta 4^{-|\Lambda|}Z(\Lambda,J,\beta)^2$, and then making the change of variable $\mathbf{m} = \mathbf{n}_1+\mathbf{n}_2$, $\mathbf{n}_2 = \mathbf{n}$, we find that

$$\begin{aligned}(1) := &\sum_{\substack{\partial\mathbf{n}_1=\{x,y\}\\ \partial\mathbf{n}_2=\varnothing}} w_\beta(\mathbf{n}_1)w_\beta(\mathbf{n}_2)\mathbb{I}[\mathbf{n}_1+\mathbf{n}_2\in E]\mathbb{I}[x \overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow} S]\\ = &\sum_{\partial\mathbf{m}=\{x,y\}} w_\beta(\mathbf{m})\mathbb{I}[\mathbf{m}\in E]\mathbb{I}[x \overset{\mathbf{m}}{\longleftrightarrow} S] \sum_{\substack{\mathbf{n}\le\mathbf{m}\\ \partial\mathbf{n}=\varnothing}} \binom{\mathbf{m}}{\mathbf{n}}\\ = &\, 2^{-|\Lambda|} \sum_{\partial\mathbf{m}=\{x,y\}} w_{2\beta}(\mathbf{m})\mathbb{I}[\mathbf{m}\in E]\mathbb{I}[x \overset{\mathbf{m}}{\longleftrightarrow} S]2^{k(\mathbf{m})},\end{aligned} \tag{A.30}$$

where in the last line we used that the number of even subgraphs of the multi-graph $\mathscr{M}$ (see for instance definition in [6]) associated with $\mathbf{m}$ is given by $2^{|\mathbf{m}|+k(\mathbf{m})-|\Lambda|}$, where $|\mathbf{m}|$ means the total sum of $\mathbf{m}$, and $k(\mathbf{m})$ is the number of connected components. Now, observe that

$$w_{2\beta}(\mathbf{m})\mathbb{I}[x \overset{\mathbf{m}}{\longleftrightarrow} S]2^{k(\mathbf{m})} \le \sum_{a\notin S, b\in S} \beta J_{a,b} w_{2\beta}(\mathbf{m}-\delta_{ab})\mathbb{I}[x \overset{\mathbf{m}-\delta_{ab}}{\longleftrightarrow} b]\mathbb{I}[\mathbf{m}_{ab}\ge 1]2^{k(\mathbf{m}-\delta_{ab})}. \tag{A.31}$$

Indeed, we are necessarily in one of the following cases: consider the edges ab with $a \in S$, $b \notin S$, and a connected to y in $\mathbf{m}-\delta_{ab}$. Assume that

- there is an edge ab as above with $\mathbf{m}_{ab} \ge 2$, in such case $k(\mathbf{m}-\delta_{ab}) = k(\mathbf{m})$ and $w_{2\beta}(\mathbf{m}) = \frac{2\beta J_{ab}}{\mathbf{m}_{ab}} w_{2\beta}(\mathbf{m}-\delta_{ab}) \le \beta J_{a,b} w_{2\beta}(\mathbf{m}-\delta_{ab})$;
- there is a loop in the cluster of x in $\mathbf{m}$ which is intersecting the edge-boundary of S, in such case there are two edges ab satisfying the property above, with $k(\mathbf{m}-\delta_{ab}) = k(\mathbf{m})$ and $w_{2\beta}(\mathbf{m}) \le 2\beta J_{a,b} w_{2\beta}(\mathbf{m}-\delta_{ab})$;
- otherwise, there is only one edge ab with $\mathbf{m}_{ab} = 1$, in such case $k(\mathbf{m}-\delta_{ab}) = k(\mathbf{m})+1$ and $w_{2\beta}(\mathbf{m}) \le 2\beta J_{a,b} w_{2\beta}(\mathbf{m}-\delta_{ab})$.

Injecting the last displayed inequality in the first one, and then making the change of variable $\mathbf{m}' = \mathbf{m} - \delta_{uv}$, we find that

$$\begin{aligned}
(1) &\le \sum_{a\notin S, b\in S} \beta J_{a,b} \sum_{\partial \mathbf{m}'=\{x,y,a,b\}} w_{2\beta}(\mathbf{m}') 2^{k(\mathbf{m}')} \mathbb{I}[x \overset{\mathbf{m}'}{\longleftrightarrow} a] \mathbb{I}[\mathbf{m}' + \delta_{ab} \in E] \\
&= \sum_{a\notin S, b\in S} \beta J_{a,b} \sum_{\substack{\partial \mathbf{n}_1=\{x,a,b,y\} \\ \partial \mathbf{n}_2=\varnothing}} w_\beta(\mathbf{n}_1) w_\beta(\mathbf{n}_2) \mathbb{I}[x \overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow} a] \mathbb{I}[\mathbf{n}_1 + \mathbf{n}_2 + \delta_{ab} \in E] \\
&= \sum_{a\notin S, b\in S} \beta J_{a,b} \sum_{\substack{\partial \mathbf{n}_1=\{b,y\} \\ \partial \mathbf{n}_2=\{x,a\}}} w_\beta(\mathbf{n}_1) w_\beta(\mathbf{n}_2) \mathbb{I}[\mathbf{n}_1 + \mathbf{n}_2 + \delta_{ab} \in E], \qquad \text{(A.32)}
\end{aligned}$$

where in the last line we used the switching lemma. Dividing by $\langle \sigma_x \sigma_y \rangle_{\Lambda,\beta} 4^{-|\Lambda|} Z(\Lambda, J, \beta)^2$ and letting Λ tend to the full lattice implies the first claim.

The second claim follows from the same reasoning using pairs of edges (ab, st) with $a \in S$, $b \notin S$, $s \in T$ and $t \notin T$ such that a is connected to s in $\mathbf{n}_1 + \mathbf{n}_2$. □

We deduce the following pair of diagrammatic bounds on the connectivity probabilities.

Proposition A.8 *For every distinct $x, y, u, v \in \mathbb{Z}^d$*

$$\mathbf{P}^{\varnothing,\varnothing}_{\rho,\beta}[\mathscr{B}_x \overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow} \mathscr{B}_y] \le \sum_{x',y'\in\mathbb{Z}^d} \langle \tau_x \tau_y \rangle \beta J_{y,y'} \langle \tau_{y'} \tau_{x'} \rangle \beta J_{x',x}\,, \qquad \text{(A.33)}$$

$$\mathbf{P}^{xy,\varnothing}_{\rho,\beta}[\partial \mathbf{n}_1 \overset{\mathbf{n}_1+\mathbf{n}_2}{\longleftrightarrow} \mathscr{B}_u] \le \sum_{u'\in\mathbb{Z}^d} \frac{\langle \tau_x \tau_u \rangle \beta J_{u,u'} \langle \tau_{u'} \tau_x \rangle}{\langle \tau_x \tau_y \rangle}. \qquad \text{(A.34)}$$

Proof For the first one, sum (A.29) for E being the full event and vertices in $\mathscr{B}_x$ and $\mathscr{B}_y$, and use (A.10). For the second one, do the same with (A.28) instead. □

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