# A QUANTITATIVE BURTON-KEANE ESTIMATE UNDER STRONG FKG CONDITION

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ABSTRACT. We consider translationally-invariant percolation models on $\mathbb{Z}^d$ satisfying the finite energy and the FKG properties. We provide explicit upper bounds on the probability of having two distinct clusters going from the endpoints of an edge to distance $n$ (this corresponds to a finite size version of the celebrated Burton-Keane [4] argument proving uniqueness of the infinite-cluster). The proof is based on the generalization of a reverse Poincaré inequality proved in [5]. As a consequence, we obtain upper bounds on the probability of the so-called four-arm event for planar random-cluster models with cluster-weight $q \geq 1$.

## 1. INTRODUCTION

1.1. **The main result.** This article is devoted to deriving a weak reverse Poincaré-type inequality for percolation models satisfying strong association and finite-energy properties, and examining some of its consequences. Let $\Lambda$ be a finite set and consider a percolation model on $\Lambda$, *i.e.* a random binary field $\omega \in \{0,1\}^\Lambda$. The value of the field at $i \in \Lambda$ is denoted by $\omega_i$, and the field on the complementary set $\Lambda \setminus i$ is denoted by $\omega^i$. The law of $\omega$ on $\{0,1\}^\Lambda$ is denoted by $\mathbb{P}$. There is a standard partial order $\prec$ on $\{0,1\}^\Lambda$, and a function $f$ on $\{0,1\}^\Lambda$ is said to be non-decreasing if $f(\omega) \leq f(\psi)$ whenever $\omega \prec \psi$. An event $A \subset \{0,1\}^\Lambda$ is said to be non-decreasing if its indicator function $\mathbb{1}_A$ is.

We will be interested in percolation models satisfying the following two conditions:

**(FE):** *Finite Energy:* There exists $c_{\mathsf{FE}} > 0$ such that, for any $i \in \Lambda$ and $\omega \in \{0,1\}^\Lambda$,

$$\mathbb{P}\left(\omega_i = 1 \,|\, \omega_j, j \neq i\right) \in \left(c_{\mathsf{FE}}, 1 - c_{\mathsf{FE}}\right). \tag{1.1}$$

**(FKG):** *Strong Positive Association:* For any $i \in \Lambda$ and $\xi \prec \psi$ in $\{0,1\}^{\Lambda \setminus \{i\}}$,

$$\mathbb{P}\left(\omega_i = 1 \,|\, \omega_j = \xi_j : j \neq i\right) \leq \mathbb{P}\left(\omega_i = 1 \,|\, \omega_j = \psi_j : j \neq i\right). \tag{1.2}$$

Recall that (1.2) is equivalent to the so-called FKG lattice condition; see [10, Theorem (2.24)].

Before stating the theorem, let us define two more objects. For a configuration $\omega$ and $i \in \Lambda$, define the configurations $\omega^i \times 1$ and $\omega^i \times 0$ obtained from $\omega$ by changing the state $i$ to 1 and 0 respectively. Define

$$\nabla_i f(\omega) = f(\omega^i \times 1) - f(\omega^i \times 0).$$

Also define the set $\mathrm{Piv}_i(A)$ of configurations $\omega$ such that $\omega^i \times 1 \in A$ and $\omega^i \times 0 \notin A$. Equivalently, $\mathrm{Piv}_i(A) = \{\omega \ : \ \nabla_i \mathbb{1}_A(\omega) = 1\}$.

**Theorem 1.1.** *Consider a percolation model on a finite set $\Lambda$ satisfying* (FE) *and* (FKG). *Then, there exists $c_\mathsf{P} = c_\mathsf{P}(c_\mathsf{FE}) > 0$ such that, for any non-decreasing function $f : \{0,1\}^\Lambda \longrightarrow \mathbb{R}$,*

$$\mathrm{Var}\big(f(\omega)\big) \geq c_\mathsf{P} \sum_{i \in \Lambda} \big(\mathbb{E}[\nabla_i f]\big)^2. \tag{1.3}$$

*In particular, for any non-decreasing event $A$*

$$\mathbb{P}(A)\big(1 - \mathbb{P}(A)\big) \geq c_\mathsf{P} \sum_{i \in \Lambda} \mathbb{P}\big(\mathrm{Piv}_i(A)\big)^2. \tag{1.4}$$

We emphasize that the constant $c_\mathsf{P}$ is not depending on the size of $\Lambda$. One may think of this theorem as a weak reverse Poincaré inequality. Indeed, when $\{\omega_i : i \in \Lambda\}$ are independent, the standard discrete Poincaré inequality, see for instance [9], states that

$$\mathbb{P}(A)\big(1 - \mathbb{P}(A)\big) \leq \frac{1}{4} \sum_{i \in \Lambda} \mathbb{P}\big(\mathrm{Piv}_i(A)\big). \tag{1.5}$$

In the independent case, $\mathbb{P}\big(\mathrm{Piv}_i(A)\big) = \mathbb{P}(A|\omega_i = 1) - \mathbb{P}(A|\omega_i = 0) = I_A(i)$ is the so-called *influence* of $i$ on $A$. Let us also mention that some inequalities for influences in models with dependency have been obtained by encoding strongly positively-associated measures in terms of the Lebesgue measure on the hypercube $[0,1]^\Lambda$. Nevertheless, these inequalities bound influences from below; see [10, Theorem (2.28)].

Equation (1.3) was derived in the independent Bernoulli case in [5]. The latter work was one of the motivations for our study of dependent models here.

Our proof of (1.3) hinges on the following simple but apparently new observation, which may be of independent interest. Fix $0 < p < 1$; given a realization $\omega \in \{0,1\}^\Lambda$ of the percolation model, we construct a field $\sigma \in \{0,1\}^\Lambda$ of (conditionally on $\omega$) independent random variables, $\sigma_i$ taking value 1 with probability

$$\frac{p \cdot \omega_i}{\mathbb{P}(\omega_i = 1|\omega^i)},$$

for each $i \in \Lambda$. Then, the distribution of the field $\sigma$, once integrated over $\omega$, enjoys a form of *negative* dependence. Namely, for any $i \in \Lambda$ and for any non-decreasing functions $f : \{0,1\}^{\Lambda \setminus \{i\}} \to \mathbb{R}$ and $g : \{0,1\} \to \mathbb{R}$,

$$\widehat{\mathbb{E}}\big[f(\sigma^i)g(\sigma_i)\big] \leq \widehat{\mathbb{E}}\big[f(\sigma^i)\big]\widehat{\mathbb{E}}\big[g(\sigma_i)\big].$$

This is proved in Theorem 2.1 below, together with additional relevant properties.

## 1.2. **Applications.**

*Some examples of percolation models.* Our applications to percolation models will be mostly dealing with connectivity properties of the graph induced by $\{i \in \Lambda : \omega_i = 1\}$. For simplicity, we will focus on *bond percolation* models – similar results would also hold for so-called *site percolation* models. The set $\Lambda$ is now the edge-set $E_G$ of a finite graph $G = (V_G, E_G)$. The edge $i$ is said to be *open* (resp. *closed*) if $\omega_i = 1$ (resp. $\omega_i = 0$). The configuration $\omega$ can therefore be seen as a subgraph of $G$ with

vertex set $V_G$ and edge set composed of open edges. Two vertices $x$ and $y$ are said to be *connected* if they belong to the same connected component of $\omega$ (we denote the event that $x$ and $y$ are connected by $x \longleftrightarrow y$). Connected components of $\omega$ are called *clusters*.

The most classical example of a percolation model is provided by Bernoulli percolation. This model was introduced by Broadbent and Hammersley in the fifties [3]. In this model, each edge $i$ is open with probability $p$, and closed with probability $1 - p$ independently of the states of the other edges. For general background on Bernoulli percolation, we refer the reader to the books [11, 12].

More generally, the states of edges may not be independent. In such case, we speak of a *dependent* percolation model. Among classical examples, we mention the *random-cluster model* (or *Fortuin-Kasteleyn percolation*) introduced by Fortuin and Kasteleyn in [8]. Let $o(\omega)$ be the number of open edges in $\omega$, $c(\omega)$ be the number of closed edges and $k(\omega)$ be the number of clusters. The probability measure $\phi_{p,q,G}$ of the random-cluster model on a finite graph $G$ with parameters $p \in [0, 1]$ and $q > 0$ is defined by

$$\phi_{p,q,G}(\{\omega\}) := \frac{p^{o(\omega)}(1 - p)^{c(\omega)} q^{k(\omega)}}{Z_{p,q,G}}$$

for every configuration $\omega$ on $G$, where $Z_{p,q,G}$ is a normalizing constant referred to as the *partition function*.

The random-cluster models satisfy (FE) for $q > 0$ and (FKG) for any $q \geq 1$. For this reason, random-cluster models are good examples of models satisfying our two assumptions, but they are not the only ones. The uniform spanning tree ($\mathbb{P}$ is simply the uniform measure on trees containing every vertices of $G$) is a typical example of a model not satisfying (FE).

Our applications give upper bounds on probabilities of two-leg type events In two dimensions because of dual connections, the usual name would be four-arm type events, namely probabilities of having two long disjoint clusters attached to two vertices of a given edge.

In order to deduce such estimates for individual bonds from (1.3) or (1.4), we need to assume some form of translation invariance.

*First application.* To give a simple illustration of how Theorem 1.1 might be put to work, let us mention the following result. Consider the $d$-dimensional torus $\mathbb{T}_n^{(d)}$ of size $2n + 1$ and denote by $\tilde{A}_2^e(n)$ the event that the edge $e$ is pivotal for the existence of an open circuit of non-trivial homotopy in $\mathbb{T}_n^{(d)}$.

**Proposition 1.1.** *Let $d \geq 2$, there exists $c_{\tilde{A}_2} = c_{\tilde{A}_2}(c_{FE}, d) > 0$ such that, for every $n \geq 1$ and any edge $e$ of $\mathbb{T}_n^{(d)}$,*

$$\mathbb{P}[\tilde{A}_2^e(n)] \leq \frac{c_{\tilde{A}_2}}{n^{d/2}},$$

*where $\mathbb{P}$ is the law of an arbitrary translation invariant percolation model on $\mathbb{T}_n^{(d)}$ satisfying (FE) and (FKG).*

Note that $\tilde{A}_2^e(n)$ is basically the event that there are two disjoint clusters emanating from the end-points of $e$ and going to distance $n$, with some additional topological requirement on the macroscopic structure of these clusters (among these requirements, they should join into a cluster of $\mathbb{T}_n^{(d)}$). This additional condition is not so nice, and it does not directly apply to models on $\mathbb{Z}^d$. We would like to replace this by the event that there are two disjoint clusters going from the end-points of *some fixed bond $e$* to distance $n$. Let $A_2^e(n)$ be the event that there are two disjoint clusters going from the endpoints of the edge $e$ to distance $n$.

In the next two applications we explain two ways of deriving upper bounds on $\mathbb{P}\left(A_2^e(n)\right)$.

*A quantitative Burton-Keane argument.* Our second application is an extension of the results of [5] to arbitrary bond percolation models $\mathbb{P}$ on $\mathbb{Z}^d$ which satisfy (FE), (FKG) and are invariant under translations:

(TI)**:** The measure $\mathbb{P}$ is invariant under shift $\tau_x : \{0,1\}^{\mathbb{Z}^d} \to \{0,1\}^{\mathbb{Z}^d}$ defined by

$$\tau_x(\omega)_{(u,v)} = \omega_{(u+x,v+x)} \quad \forall u, v \in \mathbb{Z}^d.$$

**Theorem 1.2.** *Consider a percolation model on $\mathbb{Z}^d$ satisfying* (FE), (FKG) *and* (TI). *Then there exists $c_{\mathsf{BK}} > 0$ such that, for any edge $e$,*

$$\mathbb{P}[A_2^e(n)] \leq \frac{c_{\mathsf{BK}}}{(\log n)^{d/2}}. \tag{1.6}$$

As we have already mentioned a quantitative Burton-Keane argument leading to (1.6) for Bernoulli percolation-type models was developed in [5].

*Spanning clusters and four-arm event for critical planar random-cluster models with $q \geq 1$.* Proposition 1.1 and Theorem 1.2 are based on (1.4). Nevertheless, (1.3) provides us with additional degrees of freedom: one can try various model-dependent monotone functions $f$.

Let $\mathbb{P}$ be a bond percolation measure on $\mathbb{Z}^d$. Consider the boxes $\Lambda_k = [-k, \ldots, k]^d$ and the annuli $A_{m,n} = \Lambda_n \setminus \Lambda_m$ for $0 < m < n$. Let $N = N_{m,n}$ be the number of distinct clusters of $\partial \Lambda_n$ *crossing* $A_{m,n}$ in the *restriction* of the percolation configuration to the bonds of $\Lambda_n$. In the sequel we shall use $\eta$ for a percolation configuration on $\mathbb{Z}^d \setminus \Lambda_m$, $\omega$ for a percolation configuration on $\Lambda_m$, and $\eta \times \omega$ for the configuration obtained by merging the two previous configurations. For a given $\eta$ the function $\omega \mapsto N(\eta \times \omega) \overset{\Delta}{=} N^\eta(\omega)$ is decreasing. Hence, (1.3) implies that

$$\mathbb{V}\mathrm{ar}\left(N^\eta(\omega)\big|\eta\right) \geq c_{\mathsf{P}} \sum_{e \in \mathcal{E}_{\Lambda_m}} \left(\mathbb{E}\left(\nabla_e N^\eta(\omega)\big|\eta\right)\right)^2. \tag{1.7}$$

Above $\mathcal{E}_{\Lambda_m}$ is the set of nearest neighbor bonds of $\Lambda_m$. Note that, for any $e \in \mathcal{E}_{\Lambda_m}$,

$$-\nabla_e N_{m,n}^\eta(\omega) \geq \mathbb{1}_{A_2^e(2n)}(\eta \times \omega).$$

Therefore, we infer from (1.7) the following corollary.

**Corollary 1.1.** *Consider a percolation model on $\mathbb{Z}^d$ satisfying* (FE), (FKG) *and* (TI). *Then, for any edge $e$,*

$$\mathbb{P}\left(A_2^e(2n)\right) \le \sqrt{\frac{1}{c_\mathsf{P}(2m)^d}\mathbb{E}\left(\mathbb{V}\mathrm{ar}\left(N_{m,n}^\eta(\omega)\big|\eta\right)\right)} \le \sqrt{\frac{1}{c_\mathsf{P}(2m)^d}\mathbb{V}\mathrm{ar}\left(N_{m,n}\right)}, \qquad (1.8)$$

*for any $0 < m < n$.*

Of course (1.8) is useful only when one is able to control the number of crossing clusters of $A_{m.n}$, specifically $\mathbb{E}\left(\mathbb{V}\mathrm{ar}\left(N_{m,n}^\eta(\omega)\big|\eta\right)\right)$. This requires work – a trivial upper bound of order $m^{2(d-1)}$ gives nothing even in two dimensions. Settling this in any dimension would be a feat even in the case of Bernoulli percolation, see [1]. For the moment it is not clear to us that a nice closed form bound can be obtained in the full generality suggested by (FE), (FKG) and (TI), even if one requires ergodicity instead of just translation invariance.

However, using very recent results of [6, 7] for the random-cluster model on $\mathbb{Z}^2$, the distribution of the number of crossing clusters can be controlled, and the upper bound (1.8) implies the following refinement of Theorem 1.2, which is of the same order as the bound of Proposition 1.1:

**Theorem 1.3.** *Let $d = 2$, $q \in [1, 4]$, there exists $c_{\mathsf{A}_2} = c_{\mathsf{A}_2}(p, q) > 0$ such that for any edge $e$ and every $n \ge 1$,*

$$\phi_{p,q,\mathbb{Z}^2}[A_2^e(n)] \le \frac{c_{\mathsf{A}_2}}{n},$$

*where $\phi_{p,q,\mathbb{Z}^2}$ is the unique infinite volume random-cluster measure with edge-weight $p$ and cluster-weight $q$ (see Section 3.3 for a precise definition).*

The proof is easy whenever $p \ne p_c$. In the critical case $p = p_c$ the proof is based on Russo-Seymour-Welsh (RSW) bounds obtained in [6, 7]. We give two arguments: one is based on the implied mixing properties of $\phi_{p,q,\mathbb{Z}^2}$ and on subsequent reduction to Proposition 1.1. The second directly relies on RSW bounds to check that one can fix $\epsilon > 0$ such that $\left\{\phi_{p,q,\mathbb{Z}^2}\left(N_{\epsilon n,n}^2\right)\right\}$ is a bounded sequence. Then (1.7) applies.

Note that for $q > 4$, the phase transition is expected to be discontinuous (see discussion in Section 3.3) and the probability of $A_2(n)$ should decay exponentially fast at every $p$.

## 2. Proof of Theorem 1.1

We shall prove Theorem 1.1 with $c_\mathsf{P} = \frac{c_{\mathsf{FE}}^3}{(2-c_{\mathsf{FE}})^2}$.

From now on in this section, we fix a finite set $\Lambda$. Consider a percolation model on $\Omega := \{0,1\}^\Lambda$ satisfying (FE) and (FKG) and let $\mathbb{P}$ be the law of the random configuration $\omega$. Furthermore, for $I \subset \Lambda$, we define $\omega_I = \{\omega_i : i \in I\}$ and $\omega^I = \{\omega_i : i \notin I\}$. To keep notation compatible, we set $\omega_i$ and $\omega^i$ when $I = \{i\}$.

Recall that $\omega^i \times 1$ and $\omega^i \times 0$ denote the configurations obtained from $\omega$ by setting the value of $\omega_i$ to 1 and 0 respectively.

*To lighten the notation, we write $p = c_{\mathsf{FE}}/2$ for the rest of this section.*

2.1. **A representation of fields satisfying** (FE) **and** (FKG)**.** In order to prove Theorem 1.1 we introduce an auxiliary Bernoulli filed $\sigma \in \{0,1\}^\Lambda$ and utilize the projection method of [5] with respect to $\sigma$-algebras generated by this auxiliary field. The efficiency of such approach hinges on the fact that $\sigma_i$-s happen to be negatively correlated in the sense specified in P3 of Theorem 2.1 below.

**Definition 1.** *Consider a probability space $(\widehat{\Omega}, \widehat{\mathbb{P}})$ containing $(\Omega, \mathbb{P})$ and an additional field $\sigma \in \{0,1\}^\Lambda$ which, conditionally on $\omega \in \Omega$, has independent entries satisfying for every $i \in \Lambda$,*

$$\widehat{\mathbb{P}}(\sigma_i = 1 | \omega) = \frac{p \cdot \omega_i}{\mathbb{P}(\omega_i = 1 | \omega^i)}. \tag{2.1}$$

We claim that $\sigma$ enjoys the following set of properties:

**Theorem 2.1.** *Let $\omega$ be a field satisfying* (FE) *and* (FKG)*. Then,*

- P1 *For each $i \in \Lambda$, if $\sigma_i = 1$, then $\omega_i = 1$.*
- P2 *For any $i \in \Lambda$ and any non-decreasing function $f : \Omega \to \mathbb{R}$, the conditional expectation $\widehat{\mathbb{E}}\left(f(\omega) \, | \, \sigma_i\right)$ is also non-decreasing.*
- P3 *For any $i \in \Lambda$ and for any non-decreasing functions $f : \{0,1\}^{\Lambda \setminus \{i\}} \to \mathbb{R}$ and $g : \{0,1\} \to \mathbb{R}$,*

$$\widehat{\mathbb{E}}\left[f(\sigma^i)g(\sigma_i)\right] \le \widehat{\mathbb{E}}\left[f(\sigma^i)\right]\widehat{\mathbb{E}}\left[g(\sigma_i)\right]. \tag{2.2}$$

- P4 *The family $\{\sigma_i - p : i \in \Lambda\}$ is free in $\widehat{\Omega}$ (seen as a vector space).*

Property P3 is providing a weak form of negative association (note that it does not directly imply negative association is the classical sense).

*Proof. Property* P1. The first property follows directly from the definition of $\sigma$.

*Property* P2. Let us first prove that for each $i \in \Lambda$, $\sigma_i$ is a Bernoulli random variable of parameter $p$. and $\sigma_i$ is independent of $\omega^i$. This follows from (2.1) and the following computation

$$\widehat{\mathbb{P}}\left(\sigma_i = 1; \omega^i\right) = \widehat{\mathbb{P}}\left(\sigma_i = 1; \omega_i = 1; \omega^i\right)$$
$$= \frac{p}{\mathbb{P}\left(\omega_i = 1 \, | \, \omega^i\right)} \, \mathbb{P}\left(\omega_i = 1; \omega^i\right) = p \, \mathbb{P}\left(\omega^i\right).$$

Hence $\sigma_i$ is indeed a Bernoulli random variable of parameter $p$ (also $\sigma_i$ and $\omega^i$ are independent). Now, let us simplify the notation by setting $f_i(\sigma_i) := \widehat{\mathbb{E}}\left(f(\omega) \, | \, \sigma_i\right)$.

The previous claim used in the second equality implies that

$$(1-p)(f_i(1) - f_i(0)) := (1-p)\left[\frac{\widehat{\mathbb{E}}\left(f(\omega)\mathbb{1}_{\sigma_i=1}\right)}{\widehat{\mathbb{P}}(\sigma_i = 1)} - \frac{\widehat{\mathbb{E}}\left(f(\omega)\mathbb{1}_{\sigma_i=0}\right)}{\widehat{\mathbb{P}}(\sigma_i = 0)}\right]$$

$$= (\tfrac{1}{p} - 1)\widehat{\mathbb{E}}\left(f(\omega)\mathbb{1}_{\sigma_i=1}\right) - \widehat{\mathbb{E}}\left(f(\omega)\mathbb{1}_{\sigma_i=0}\right)$$

$$\overset{(\mathsf{P1})}{=} \tfrac{1}{p}\widehat{\mathbb{E}}\left(f(\omega)\mathbb{1}_{\omega_i=1}\mathbb{1}_{\sigma_i=1}\right) - \mathbb{E}\left(f(\omega)\right)$$

$$\overset{(2.1)}{=} \tfrac{1}{p}\mathbb{E}\left(f(\omega)\mathbb{1}_{\omega_i=1}\frac{p}{\mathbb{P}\left(\omega_i = 1 \mid \omega^i\right)}\right) - \mathbb{E}\left(f(\omega)\right)$$

$$= \mathbb{E}\left(f(\omega^i \times 1)\right) - \mathbb{E}\left(f(\omega)\right) \overset{(\mathsf{FKG})}{\geq} 0. \tag{2.3}$$

*Property* P3. We wish to prove the negative association formula (2.2). Since $g$ is a non-decreasing function of only one site, we only need to treat the case $g = \mathrm{id}$ (any non-decreasing function of one site is of the form $\alpha\,\mathrm{id} + \beta$ with $\alpha \geq 0$). For $\omega_I \in \{0,1\}^I$ and $\omega^I \in \{0,1\}^{\Lambda\setminus I}$, let $\omega_I \times \omega^I$ be the configuration in $\Omega$ coinciding with $\omega_I$ on $I$ and $\omega^I$ on $\Lambda \setminus I$.

**Claim.** *For any subset $I \subset \Lambda$ the following happens: If $\omega_I \succ \widetilde{\omega}_I$, then, for any $\omega^I$ and for any non-decreasing function $f : \{0,1\}^{\Lambda\setminus I} \to \mathbb{R}$,*

$$\widehat{\mathbb{E}}\left(f(\sigma^I) \mid \omega_I \times \omega^I\right) \leq \widehat{\mathbb{E}}\left(f(\sigma^I) \mid \widetilde{\omega}_I \times \omega^I\right). \tag{2.4}$$

*Proof of the Claim.* Under the conditional measure $\widehat{\mathbb{P}}\left(\cdot \mid \omega\right)$, the sequence $\sigma$ is simply a collection of independent Bernoulli random variables with probabilities of success specified by (2.1). By (1.2),

$$\frac{p}{\mathbb{P}\left(\omega_i = 1 \middle| (\omega_I \times \omega^I)^i\right)} \leq \frac{p}{\mathbb{P}\left(\omega_i = 1 \middle| (\widetilde{\omega}_I \times \omega^I)^i\right)},$$

for any $i \notin I$ and $\omega_I \geq \widetilde{\omega}_I$, a fact which implies that the random variable $\sigma_I$ conditioned on $\omega_I \times \omega^I$ is stochastically dominated by the random variable $\sigma_I$ conditioned on $\widetilde{\omega}_I \times \omega^I$. The claim follows by definition of stochastic domination. $\qquad\square$

In particular, the claim yields that if $f$ is a non-decreasing function of $\sigma^i$, then, for any $i$ and any $\omega^i$,

$$\widehat{\mathbb{E}}\left(f(\sigma^i) \mid \omega^i \times 1\right) \leq \widehat{\mathbb{E}}\left(f(\sigma^i) \mid \omega^i\right). \tag{2.5}$$

As a result, we infer:

$$\widehat{\mathbb{E}}(f(\sigma^i)\sigma_i) = \widehat{\mathbb{E}}(f(\sigma^i)\mathbb{1}_{\sigma_i=1}\mathbb{1}_{\omega_i=1})$$

$$\overset{(2.1)}{=} \sum_{\omega^i} p\,\widehat{\mathbb{E}}\left(f(\sigma^i) \mid \omega^i \times 1\right)\mathbb{P}\left(\omega^i\right)$$

$$\overset{(2.5)}{\leq} p\sum_{\omega^i}\widehat{\mathbb{E}}\left(f(\sigma^i) \mid \omega^i\right)\mathbb{P}\left(\omega^i\right)$$

$$= p\,\widehat{\mathbb{E}}[f(\sigma^i)] \overset{(\mathsf{P1})}{=} \widehat{\mathbb{E}}[\sigma_i]\,\widehat{\mathbb{E}}[f(\sigma^i)].$$

*Property* P4. Assume that $X = \sum_{i \in \Lambda} \lambda_i (\sigma_i - p) = 0$. We deduce that

$$0 = \widehat{\mathbb{E}}[X^2] = \widehat{\mathbb{E}}[(X - \widehat{\mathbb{E}}[X|\omega])^2] + \widehat{\mathbb{E}}[\widehat{\mathbb{E}}[X|\omega]^2] \geq \widehat{\mathbb{E}}[(X - \widehat{\mathbb{E}}[X|\omega])^2].$$

Now,

$$X - \widehat{\mathbb{E}}[X|\omega] = \sum_{i \in \Lambda} \lambda_i (\sigma_i - \widehat{\mathbb{E}}[\sigma_i|\omega]).$$

and conditionally on $\omega$, the random variables $\sigma_i - \widehat{\mathbb{E}}[\sigma_i|\omega]$ are independent random variables of mean 0. We deduce that

$$\begin{aligned}
\widehat{\mathbb{E}}[(X - \widehat{\mathbb{E}}[X|\omega])^2] &= \widehat{\mathbb{E}}\big[\widehat{\mathbb{E}}[(X - \widehat{\mathbb{E}}[X|\omega])^2|\omega]\big] \\
&= \widehat{\mathbb{E}}\Big[\widehat{\mathbb{E}}\big[\big(\sum_{i \in \Lambda} \lambda_i (\sigma_i - \widehat{\mathbb{E}}[\sigma_i|\omega])\big)^2 \mid \omega\big]\Big] \\
&= \widehat{\mathbb{E}}\Big[\sum_{i \in \Lambda} \lambda_i^2 \widehat{\mathbb{E}}\big[(\sigma_i - \widehat{\mathbb{E}}[\sigma_i|\omega])^2|\omega\big]\Big] \\
&= \sum_{i \in \Lambda} \lambda_i^2 \widehat{\mathbb{E}}[(\sigma_i - \widehat{\mathbb{E}}[\sigma_i|\omega])^2].
\end{aligned}$$

Now, $\widehat{\mathbb{E}}[\sigma_i|\omega] \leq 1/2$ by (2.1) and (1.1), and therefore

$$\widehat{\mathbb{E}}[(\sigma_i - \widehat{\mathbb{E}}[\sigma_i|\omega])^2] = \mathbb{E}\big[\widehat{\mathbb{E}}[\sigma_i|\omega](1 - \widehat{\mathbb{E}}[\sigma_i|\omega])\big] \geq \tfrac{1}{2}\mathbb{E}[\widehat{\mathbb{E}}[\sigma_i|\omega]] = \tfrac{p}{2} > 0 \qquad (2.6)$$

and therefore $\lambda_i = 0$ for every $i \in \Lambda$. $\qquad\square$

2.2. **Proof of Theorem 1.1.** Before proceeding with the proof of the theorem, let us formulate and prove the following elementary lemma.

**Lemma 2.1.** *Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a finite dimensional Hilbert space, and let $\{f_1, \ldots, f_n\}$ be a normalized basis of $\mathcal{H}$, such that $\langle f_i, f_j \rangle \leq 0$ for every pair $i \neq j$. Then, for any $f = \sum_{i=1}^n \lambda_i f_i$ such that $\langle f, f_i \rangle \geq 0$ for every $1 \leq i \leq n$, we have that $\lambda_i \geq \langle f, f_i \rangle$ for every $1 \leq i \leq n$.*

Note that when the basis $\{f_1, \ldots, f_n\}$ is orthogonal, we find that $\lambda_i = \langle f, f_i \rangle$ for every $1 \leq i \leq n$.

*Proof.* Lemma 2.1 has a transparent geometric meaning: Consider a positive (and by assumption obtuse) cone $\mathcal{C} = \mathcal{C}(f_1, \ldots, f_n)$ spanned by vectors $f_i$. Then, $\langle f, f_i \rangle \geq 0$ for every $i$ implies that $f \in \mathcal{C}$, that is all the coefficients $\lambda_j$ in the decomposition $f = \sum_j \lambda_j f_j$ are non-negative. Taking scalar product with normalized vectors $f_i$:

$$\langle f, f_i \rangle = \lambda_i + \sum_{j \neq i} \lambda_j \langle f_j, f_i \rangle \leq \lambda_i,$$

one then gets the conclusion.

Perhaps the simplest way to prove the above positive cone property is by induction on the cardinality of the basis. The two dimensional case is straightforward. Let us now consider the basis $\{f_1, \ldots, f_{n+1}\}$. For $i \geq 2$, let

$$F_i = \frac{f_i - \langle f_i, f_1 \rangle f_1}{1 - \langle f_1, f_i \rangle^2} \qquad (2.7)$$

be the normalized projection of $f_i$ on $\mathrm{vec}(f_1)^\perp$. Also write $f = \langle f, f_1 \rangle f_1 + F$ where $F \in \mathrm{vec}(F_2, \ldots, F_{n+1})$. We wish to apply the induction hypothesis with $F$ and $F_2, \ldots, F_{n+1}$. For that, simply observe that for $i \neq j$,

$$\langle F_i, F_j \rangle = \frac{\langle f_i, f_j \rangle - \langle f_i, f_1 \rangle \langle f_j, f_1 \rangle}{(1 - \langle f_1, f_i \rangle^2)(1 - \langle f_1, f_j \rangle^2)} \leq 0$$

(we used that $\langle f_i, f_j \rangle \leq 0$ for $i \neq j$) and

$$\langle F, F_i \rangle = \langle f, F_i \rangle = \frac{\langle f, f_i \rangle - \langle f_i, f_1 \rangle \langle f, f_1 \rangle}{1 - \langle f_i, f_1 \rangle^2} \geq \langle f, f_i \rangle \geq 0 \tag{2.8}$$

where the first equality is due to the orthogonality of $f_1$ and $F_i$, and the first inequality to the fact that $\langle f_i, f_1 \rangle \leq 0$ and $\langle f, f_1 \rangle \geq 0$. Therefore, by the induction assumption $F$ lies in the cone $\mathcal{C}(F_2, \ldots, F_{n+1})$. By (2.7) all $F_i$-s lie in $\mathcal{C}(f_1, \ldots, f_{n+1})$, and hence $F \in \mathcal{C}(f_1, \ldots, f_{n+1})$ as well. Since the coefficient $\langle f, f_1 \rangle$ in $f = \langle f, f_1 \rangle f_1 + F$ is non-negative, we are done. $\square$

*Proof of Theorem 1.1.* Consider a field $\omega \in \{0, 1\}^\Lambda$ satisfying (FE) and (FKG). We keep the notations from the previous section. In particular, on the probability space $(\widehat{\Omega}, \widehat{\mathbb{P}})$, we associate the field $\sigma$ to $\omega$.

Let $f$ be a square integrable non-decreasing function of $\omega$. As before, we set $f_i(\sigma_i) := \widehat{\mathbb{E}}[f(\omega) | \sigma_i]$. Without loss of generality, we assume that $\mathbb{E}[f(\omega)] = 0$, and therefore $\widehat{\mathbb{E}}[f_i(\sigma_i)] = 0$ for every $i \in \Lambda$.

Let $I$ be the set of indices $i$ so that $\mathbb{E}[f_i(\sigma_i)^2] > 0$. Consider $V = \mathrm{vec}(f_i(\sigma_i) : i \in I)$ and write

$$f = \sum_{i \in I} \gamma_i f_i(\sigma_i) + f^\perp,$$

where $f^\perp \in V^\perp$. Properties P2 and P3 of Theorem 2.1 show that $\langle f_i(\sigma_i), f_j(\sigma_j) \rangle \leq 0$ for every $i \neq j$ in $I$. Furthermore,

$$\langle f(\omega), f_i(\sigma_i) \rangle = \widehat{\mathbb{E}}[f(\omega) f_i(\sigma_i)] = \widehat{\mathbb{E}}[f_i(\sigma_i)^2] \geq 0.$$

(The last equality is due to the definition of the conditional expectation.) Last but not least, the family $\{f_i(\sigma_i) : i \in I\}$ forms a basis of $V$. Indeed, since $\widehat{\mathbb{E}}[f_i(\sigma_i)] = 0$ and $\widehat{\mathbb{E}}[\sigma_i] = p$ by P1 of Theorem 2.1, we deduce that

$$f_i(\sigma_i) = (f_i(1) - f_i(0))(\sigma_i - p). \tag{2.9}$$

Now, $f_i(1) - f_i(0) > 0$ for $i \in I$, and the family $\{\sigma_i - p : i \in I\}$ is free thanks to P4. We are therefore in position to apply the previous lemma with $f_i(\sigma_i) = f_i(\sigma_i) / \|f_i(\sigma_i)\|$ in order to obtain that

$$\gamma_i \geq \frac{\langle f(\omega), f_i(\sigma_i) \rangle}{\|f_i(\sigma_i)\|^2} = \frac{\widehat{\mathbb{E}}[f_i(\sigma_i)^2]}{\|f_i(\sigma_i)\|^2} = 1. \tag{2.10}$$

We deduce that

$$\widehat{\mathbb{E}}[f(\omega)^2] \geq \widehat{\mathbb{E}}\Big[\Big(\sum_{i \in I} \gamma_i f_i(\sigma_i)\Big)^2\Big]$$

$$= \widehat{\mathbb{E}}\Big[\Big(\sum_{i \in I} \gamma_i (f_i(\sigma_i) - \widehat{\mathbb{E}}[f_i(\sigma_i)|\omega])\Big)^2\Big] + \widehat{\mathbb{E}}\Big[\Big(\sum_{i \in I} \gamma_i \widehat{\mathbb{E}}[f_i(\sigma_i)|\omega]\Big)^2\Big]$$

$$\geq \widehat{\mathbb{E}}\Big[\Big(\sum_{i \in I} \gamma_i (f_i(\sigma_i) - \widehat{\mathbb{E}}[f_i(\sigma_i)|\omega])\Big)^2\Big]$$

$$= \sum_{i \in I} \gamma_i^2 \widehat{\mathbb{E}}[(f_i(\sigma_i) - \widehat{\mathbb{E}}[f_i(\sigma_i)|\omega])^2]$$

$$\overset{(2.10)}{\geq} \sum_{i \in I} \widehat{\mathbb{E}}[(f_i(\sigma_i) - \widehat{\mathbb{E}}[f_i(\sigma_i)|\omega])^2]$$

$$= \sum_{i \in \Lambda} \widehat{\mathbb{E}}[(f_i(\sigma_i) - \widehat{\mathbb{E}}[f_i(\sigma_i)|\omega])^2],$$

where the second equality is due to the fact that conditionally on $\omega$, the variables $\{f_i(\sigma_i) - \widehat{\mathbb{E}}[f_i(\sigma_i)|\omega] : i \in I\}$ are orthogonal (since the $\sigma_i - \widehat{\mathbb{E}}[\sigma_i|\omega]$ are), and the last equality to the observation that for $i \notin I$,

$$0 \leq \widehat{\mathbb{E}}[(f_i(\sigma_i) - \widehat{\mathbb{E}}[f_i(\sigma_i)|\omega])^2] \leq \widehat{\mathbb{E}}[f_i(\sigma_i)^2] = 0.$$

We are now ready to conclude. Similarly to (2.9), we find that (remember that we chose $p = c_{\mathsf{FE}}/2$)

$$\widehat{\mathbb{E}}\big[(f_i(\sigma_i) - \widehat{\mathbb{E}}[f_i(\sigma_i)|\omega])^2\big] = (f_i(1) - f_i(0))^2 \widehat{\mathbb{E}}\big[(\sigma_i - \widehat{\mathbb{E}}[\sigma_i|\omega])^2\big]$$

$$\overset{(2.6)}{\geq} \tfrac{p}{2}(f_i(1) - f_i(0))^2$$

$$\overset{(2.3)}{=} \frac{p}{2(1-p)^2}\big(\mathbb{E}[f(\omega^i \times 1)] - \mathbb{E}[f(\omega)]\big)^2$$

$$= \frac{p}{2(1-p)^2}\big(\mathbb{E}[(f(\omega^i \times 1) - f(\omega^i \times 0))\mathbb{1}_{\omega_i=0}]\big)^2$$

$$\overset{(1.1)}{\geq} \frac{2p^3}{(1-p)^2}\big(\mathbb{E}[f(\omega^i \times 1) - f(\omega^i \times 0)]\big)^2. \qquad (2.11)$$

Overall, we find that

$$\mathbb{Var}(f(\omega)) = \widehat{\mathbb{E}}[f(\omega)^2] \geq \frac{2p^3}{(1-p)^2} \sum_{i \in \Lambda} \big(\mathbb{E}[f(\omega^i \times 1) - f(\omega^i \times 0)]\big)^2.$$

$$\square$$

**Remark 1.** *Note that up to (2.11), the proof relied only on the lower bound $\mathbb{P}(\omega_i = 1 \,|\, \omega^i) \geq c_{\mathsf{FE}} = 2p$ in (1.1).*

## 3. Applications

### 3.1. **Proof of Proposition 1.1.**

*Proof of Proposition 1.1.* Let $E_n$ be the event that there exists an open circuit with non trivial homotopy in $\mathbb{T}_n$. Theorem 1.1 implies that

$$\sum_{i \in E_{\mathbb{T}_n}} \mathbb{P}[\mathrm{Piv}_i(E_n)]^2 \leq \frac{1}{c_{\mathsf{P}}} \mathbb{P}(E_n)(1 - \mathbb{P}(E_n)) \leq \frac{1}{c_{\mathsf{P}}}.$$

Note that edges are of two types: the "vertical" and "horizontal" edges. By shift invariance of $\mathbb{P}$ and of the event $E_n$ we therefore obtain, for an horizontal edge $e$ (the same reasoning can be applied to vertical edges),

$$|\mathbb{T}_n| \cdot \mathbb{P}[\tilde{A}_2^e(n)]^2 \leq \sum_{i \in E_{\mathbb{T}_n}} \mathbb{P}[\mathrm{Piv}_i(E_n)]^2 \leq \frac{1}{c_{\mathsf{P}}}.$$

$\square$

### 3.2. **Quantitative Burton-Keane argument.**

Recall that we are working with (nearest neighbor) bond percolation models. In this section, let $\Lambda_k(x) = x + [-k, k]^d$ and $\Lambda_k = \Lambda_k(0)$. For $x \in \Lambda_n$, the set $\mathcal{C}_n(x)$ is the connected component of $x$ in the restriction of the percolation configuration to the edges with at least one end-point in $\Lambda_n$.

For $x \in \Lambda_n$ let $\mathrm{Trif}_n(x)$ be the event that:

  a) There are *exactly* three open bonds incident to $x$.
  b) $\mathcal{C}_n(x) \setminus x$ is a disjoint union of *exactly* three connected clusters, and each of these three clusters is connected to $\partial \Lambda_{n+1} = \Lambda_{n+1} \setminus \Lambda_n$.

Recall the following classical fact [4]:

**Lemma 3.1.** *Consider a percolation model on $\Lambda_n$. Then, for any $\omega$,*

$$\sum_{x \in \Lambda_n} \mathbb{1}_{\mathrm{Trif}_n(x)}(\omega) \leq |\partial \Lambda_{n+1}|. \tag{3.1}$$

Note that this lemma has the following useful consequence: if $\mathbb{P}$ is in fact a translation invariant measure on the whole plane or on a torus, it implies that

$$\mathbb{P}[\mathrm{Trif}_{2n}(0)] \leq \frac{|\partial \Lambda_{n+1}|}{|\Lambda_n|}. \tag{3.2}$$

Let $k \leq n$. Define the event $\mathrm{CoarseTrif}_{k,n}$ that there are at least three distinct clusters in the annulus $A_{k,n} := \Lambda_n \setminus \Lambda_k$ connecting the inner to the outer boundaries of $A_{k,n}$. In other words, there are at least three distinct crossing clusters of $A_{k,n}$.

**Corollary 3.1.** *Consider a percolation model on $\mathbb{Z}^d$ satisfying* (FE), (FKG) *and* (TI). *There exists $c_1 = c_1(c_{\mathsf{FE}}) > 0$ such that, for any $0 \leq k \leq n$,*

$$\mathbb{P}[\mathrm{CoarseTrif}_{k,n}] \leq \frac{\exp(c_1 k)}{n}.$$

*Proof.* By conditioning on the clusters in $A_{k,n}$, one may easily check that

$$\mathbb{P}[\mathrm{Trif}_n(0)|\mathrm{CoarseTrif}_{k,n}] \geq c_{\mathsf{FE}}{}^{6dk}.$$

(There may be some problems if the three clusters reach $\partial\Lambda_k$ near the corner, yet such cases can be treated separately.) The result follows readily from Lemma 3.1. □

*Proof of Theorem 1.2.* Set $\varepsilon := 1/(2c_1)$ and let $k = \lfloor \varepsilon \log n \rfloor$. Let $E_n$ be the event that there are exactly two clusters in $A_{k,n}$ from the inner to the outer boundaries. On $E_n$, let $\mathcal{C}$ be the set of vertices of $A_{k,n}$ connected to the boundary of $\Lambda_n$ by an open path. Since there are only two distinct clusters connecting the inside and outside boundaries of $A_{k,n}$, the vertices of $\mathcal{C} \cap \Lambda_k$ can be divided into two subsets $E_1 := E_1(\mathcal{C})$ and $E_2 := E_2(\mathcal{C})$ depending on which clusters they belong to.

For every possible realization $C$ of $\mathcal{C}$ so that $\{\mathcal{C} = C\} \subset E_n$, define $\mathrm{Cross}(C)$ to be the event that $E_1$ and $E_2$ are connected by an open path inside $\Lambda_k$. Theorem 1.1 applied to $\mathbb{P}(\cdot|\mathcal{C} = C)$ and $\mathrm{Cross}(C)$ gives

$$\sum_{e \in E_{\Lambda_k}} \mathbb{P}[\mathrm{Piv}_e(\mathrm{Cross}(C))|\mathcal{C} = C]^2 \leq \tfrac{1}{4},$$

which by Cauchy-Schwarz implies

$$\sum_{e \in E_{\Lambda_k}} \mathbb{P}[\mathrm{Piv}_e(\mathrm{Cross}(C))|\mathcal{C} = C] \leq \tfrac{1}{2}\sqrt{|E_{\Lambda_k}|}.$$

For an edge $e'$, let $A_2(n, e')$ be the event that the two end-points of $e'$ are connected to the boundary of $\Lambda_n$ by two disjoint clusters. By definition of $C$, we have that $\mathrm{Piv}_{e'}(\mathrm{Cross}(C)) \cap \{\mathcal{C} = C\} = A_2(n, e') \cap \{\mathcal{C} = C\}$ which, by summing over all possible $C$, implies that

$$\sum_{e' \in E_{\Lambda_k}} \mathbb{P}[A_2(n, e'), E_n] \leq \tfrac{1}{2}\sqrt{|E_{\Lambda_k}|}\,\mathbb{P}(E_n) \leq \tfrac{1}{2}\sqrt{|E_{\Lambda_k}|} = \sqrt{|\Lambda_k|/2}.$$

It only remains to see that Corollary 3.1 implies

$$\mathbb{P}[A_2(n, e'), \mathrm{CoarseTrif}_{k,n}] \leq \mathbb{P}[\mathrm{CoarseTrif}_{k,n}] = \frac{\exp(c_1\varepsilon \log n)}{n} = \frac{1}{\sqrt{n}}.$$

At the end, we find that

$$\frac{1}{|\Lambda_k|} \sum_{e' \in E_{\Lambda_k}} \mathbb{P}[A_2(n, e')] \leq \frac{1}{\sqrt{2|\Lambda_k|}} + \frac{1}{\sqrt{n}}.$$

Consider for a moment that the edge $e$ involved in $A_2^e(2n)$ is horizontal. Since $A_2^e(2n)$ is included in a translate of $A_2(n, e')$ for any horizontal edge $e' \in E_{\Lambda_k}$, we conclude that

$$\mathbb{P}[A_2^e(2n)] \leq \frac{1}{|\Lambda_k|} \sum_{e' \in E_{\Lambda_k}} \mathbb{P}[A_2(n, e')] \leq \frac{1}{\sqrt{2|\Lambda_k|}} + \frac{1}{\sqrt{n}},$$

which implies the proposition with $c_{\mathsf{BK}} = c_{\mathsf{BK}}(c_{\mathsf{FE}}) > 0$ small enough thanks to our choice for $k$. □

3.3. **Proof of Theorem 1.3.** Let us recall some additional details on the random-cluster model. First, let us introduce random-cluster measures with boundary conditions. Fix a finite graph $G$. *Boundary conditions* $\xi$ are given by a partition $P_1 \sqcup \cdots \sqcup P_k$ of $\partial G$. Two vertices are *wired in* $\xi$ if they belong to the same $P_i$. The graph obtained from the configuration $\omega$ by identifying the wired vertices together is denoted by $\omega^\xi$. Let $k(\omega^\xi)$ be the number of connected components of the graph $\omega^\xi$. The probability measure $\mathbb{P}^\xi_{p,q,G}$ of the random-cluster model on $G$ with *edge-weight* $p \in [0,1]$ and *cluster-weight* $q > 0$ and boundary conditions $\xi$ is defined by

$$\mathbb{P}^\xi_{p,q,G}(\{\omega\}) := \frac{p^{o(\omega)}(1-p)^{c(\omega)}q^{k(\omega^\xi)}}{Z^\xi_{p,q,G}} \tag{3.3}$$

for every configuration $\omega$ on $G$. The constant $Z^\xi_{p,q,G}$ is a normalizing constant, referred to as the *partition function*, defined in such a way that the sum over all configurations equals 1. For $q \geq 1$, infinite-volume random-cluster measures can be defined by weak limits of random-cluster measures on larger and larger boxes.

Recall that the planar random-cluster model possesses a dual model on the dual graph $(\mathbb{Z}^2)^\star$. The configuration $\omega^\star \in \{0,1\}^{E_{(\mathbb{Z}^2)^\star}}$ is defined as follows: each dual-edge $e^\star \in (\mathbb{Z}^2)^\star$ is dual-open in $\omega^\star$ if and only if the edge of $\mathbb{Z}^2$ passing through its middle (there is a unique such edge) is closed in $\omega$. If the law of $\omega$ is $\mathbb{P}^\xi_{p,q,G}$, then the law of the dual model is $\mathbb{P}^{\xi^\star}_{p^\star,q,G^\star}$ for some dual boundary conditions. We will only use that the dual boundary conditions of wired boundary conditions are the free ones and vice-versa.

On $\mathbb{Z}^2$, the random-cluster model undergoes a phase transition at some parameter $p_c(q)$ satisfying for every infinite-volume random-cluster model $\mathbb{P}_{p,q,\mathbb{Z}^2}$ with parameters $p$ and $q$,

$$\mathbb{P}_{p,q,\mathbb{Z}^2}[0 \longleftrightarrow \infty] = \begin{cases} \theta(p,q) > 0 & \text{if } p > p_c(q), \\ 0 & \text{if } p < p_c(q). \end{cases}$$

The critical point of the planar random-cluster model on $\mathbb{Z}^2$ is known to correspond to the self-dual point of the model, i.e. $p_c(q) = \sqrt{q}/(1+\sqrt{q})$ [2]. Also, for $q \in [1,4]$, the behavior at criticality is known to be the following (see [7]): there is a unique infinite-volume measure and, for any numbers $1 < a < b \leq \infty$ there exists $c_{\mathsf{RSW}} = c_{\mathsf{RSW}}(a,b) > 0$ such that, for all $n \geq 1$ and any boundary conditions $\xi$,

$$c_{\mathsf{RSW}} \leq \mathbb{P}^\xi_{p_c,q,\widehat{R}_n}\left[\mathrm{Cross}(R_n)\right] \leq 1 - c_{\mathsf{RSW}}, \tag{3.4}$$

where $R_n = R_n[a] := [-an, an] \times [-n, n]$, $\widehat{R}_n = \widehat{R}_n[b] := [-bn, bn] \times [-2n, 2n]$, and $\mathrm{Cross}(R_n)$ is the event that the left and right sides of $R_n$ are connected by an open path in $R_n$.

*Proof of Theorem 1.3.* Fix $q \in [1,4]$. Note that for $p < p_c(q)$, there exists $c = c(p,q) > 0$ such that, for any edge $e$ and every $n \geq 1$,

$$\mathbb{P}_{p,q,\mathbb{Z}^2}[A_2^e(n)] \leq \mathbb{P}_{p,q,\mathbb{Z}^2}[0 \longleftrightarrow \partial\Lambda_n] \leq e^{-c(p,q)n},$$

thanks to exponential decay of correlations, see [2] one more time. Similarly, when $p > p_c(q)$, for any edge $e$ and every $n \geq 1$,

$$\mathbb{P}_{p,q,\mathbb{Z}^2}[A_2^e(n)] = \mathbb{P}_{p^\star,q,(\mathbb{Z}^2)^\star}[A_2(n)] \leq \mathbb{P}_{p^\star,q,(\mathbb{Z}^2)^\star}[u \overset{\star}{\longleftrightarrow} \partial\Lambda_n^\star] \leq e^{-c(p^\star,q)n}.$$

The only interesting case is therefore the critical point $p = p_c$.

*Proof using mixing properties and Proposition 1.1.* Recall that, by Proposition 1.1, we already know that, for any edge $e'$ of $\mathbb{T}_n^{(2)}$ and every $n \geq 1$,

$$\mathbb{P}_{p_c,q,\mathbb{T}_n^{(2)}}[\tilde{A}_2^{e'}(n)] \leq \frac{c_{\tilde{A}_2}}{n},$$

where $\mathbb{P}_{p_c,q,\mathbb{T}_n^{(2)}}$ is the random-cluster measure on $\mathbb{T}_n^{(2)}$. We therefore only need to show that there exists $C > 0$ such that

$$\mathbb{P}_{p_c,q,\mathbb{Z}^2}[A_2^e(n)] \leq C\,\mathbb{P}_{p_c,q,\mathbb{T}_n^{(2)}}[\tilde{A}_2^{e'}(n)].$$

Embed $\mathbb{T}_n^{(2)}$ into $\mathbb{Z}^2$ in such a way that the vertex set is $\Lambda_n := [-n,n]^2$ and $e$ is an edge having 0 as an endpoint. First, we wish to highlight that (3.4) (more precisely the mixing result [6, Theorem 5.19]) classically implies the existence of $c_1 > 0$ so that for every boundary conditions $\xi$ and $n \geq 1$, We should add a reference... H. I guess [6, Theorem 5.19] is the only one here

$$\mathbb{P}_{p_c,q,\mathbb{Z}^2}[A_2^e(n/2)] \leq c_1\mathbb{P}_{p_c,q,\Lambda_n}^{\xi}[A_2^e(n/2)].$$

In particular, this is also true for so-called *periodic boundary conditions*, so that

$$\mathbb{P}_{p_c,q,\mathbb{Z}^2}[A_2^e(n/2)] \leq c_1\mathbb{P}_{p_c,q,\mathbb{T}_n^{(2)}}[A_4^e(n/2)]. \tag{3.5}$$

Now, introduce the event $A_2^{\text{sep}}(n/2)$ that there exist two open paths $\gamma, \tilde{\gamma}$ and two dual-open dual-paths $\gamma^\star$ and $\tilde{\gamma}^\star$, originating from the endpoints of $e$ and $e^*$ respectively, satisfying

- the endpoints (on the boundary of $\Lambda_{n/2}$ and $\Lambda_{n/2}^\star$ respectively) $x$, $\tilde{x}$, $x^\star$ and $\tilde{x}^\star$ of the paths are at distance larger or equal to $\frac{n}{10}$ from each others.
- $x$ and $\tilde{x}$ are connected to $\partial\Lambda_{\frac{3n}{5}}$ in $x + \Lambda_{\frac{n}{10}}$ and $\tilde{x} + \Lambda_{\frac{n}{10}}$,
- $x^\star$ and $\tilde{x}^\star$ are connected to $\partial\Lambda_{\frac{3n}{5}}^\star$ in $x^\star + \Lambda_{\frac{n}{10}}$ and $\tilde{x}^\star + \Lambda_{\frac{n}{10}}$.

Classically (see [13, 6] for a treatment in the case of Bernoulli percolation and Ising respectively, the same proof applies), (3.4) implies that there exists $c_2 > 0$ so that, for any $n \geq 1$,

$$\mathbb{P}_{p_c,q,\mathbb{T}_n^{(2)}}[A_2^e(n/2)] \leq c_2\mathbb{P}_{p_c,q,\mathbb{T}_n^{(2)}}[A_2^{\text{sep}}(n/2)]. \tag{3.6}$$

It remains to see that there exists $c_3 > 0$ so that, for any $n \geq 1$,

$$\mathbb{P}_{p_c,q,\mathbb{T}_n^{(2)}}[A_2^{\text{sep}}(n/2)] \leq c_3\mathbb{P}_{p_c,q,\mathbb{T}_n^{(2)}}[\tilde{A}_2^e(n)].$$

In order to do so, mimic the classical argument to prove quasi-multiplicativity of arm-probabilities for Bernoulli percolation (see [13] again and Figure 1).

We only sketch the proof. Condition on $A_2^{\text{sep}}(n/2)$. Consider a thin area $S$ of width $\frac{n}{10}$ going from $x + \Lambda_{\frac{n}{10}}$ to $\tilde{x} + \Lambda_{\frac{n}{10}}$ outside $\Lambda_{\frac{n}{2}}$, an a thin dual area $S^\star$ of width $\frac{n}{10}$ going from $x^\star + \Lambda_{\frac{n}{10}}$ to $\tilde{x}^\star + \Lambda_{\frac{n}{10}}$ outside $\Lambda_{\frac{n}{2}}$ so that these two areas do not
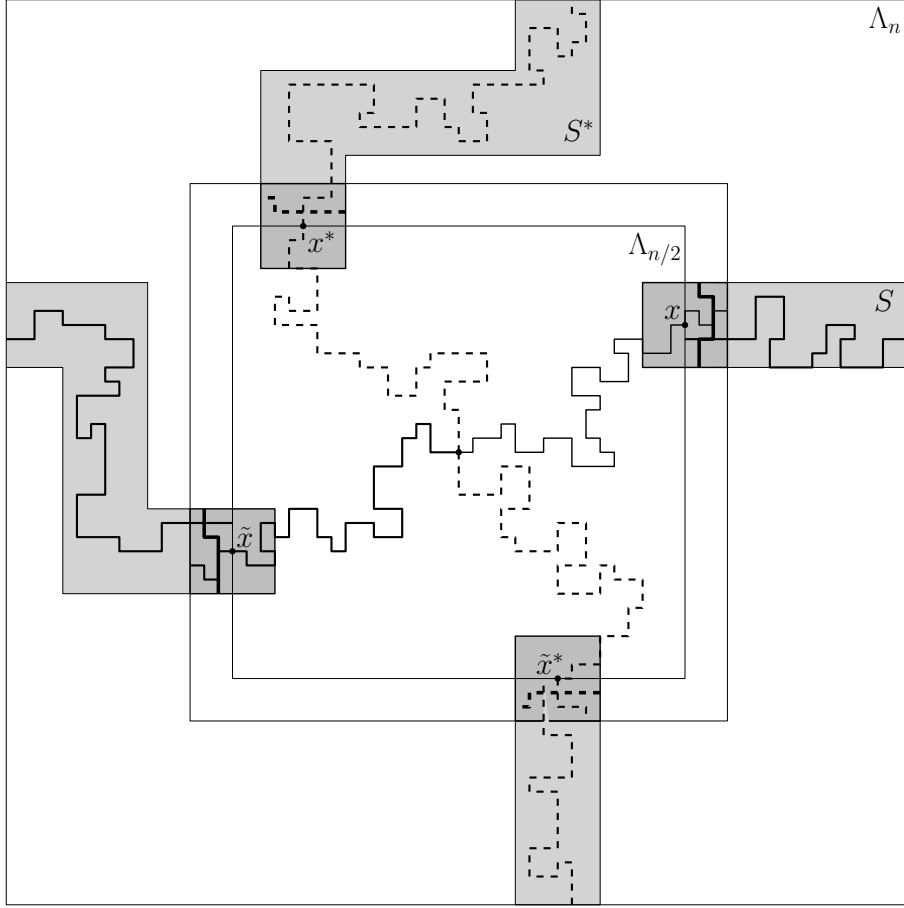
FIGURE 1. The event $A_2^{\mathrm{sep}}(n/2)$ together with the extension of the four paths using the sets $S$ and $S^*$. Estimates on crossing probabilities available from [7] show that these extensions cost a mutiplicative constant (not depending on $n$).

intersect. Now, (3.4) implies that there exist a primal path in $S$ connecting the two paths $\gamma$ and $\tilde{\gamma}$, and a dual path in $S^\star$ connecting $\gamma^\star$ and $\tilde{\gamma}^\star$ with probability $c_4 > 0$. But if this occurs, the event $\tilde{A}_2^e(n)$ is satisfied, so that

$$\mathbb{P}_{p_c,q,\mathbb{T}_n^{(2)}}[\tilde{A}_4^e(n)] \geq c_4 \mathbb{P}_{p_c,q,\mathbb{T}_n^{(2)}}[A_2^{\mathrm{sep}}(n/2)].$$

It only remains to invoke (3.5) and (3.6) to conclude.                    □

*Proof using bounds on* $\mathbb{E}_{p_c,q,\mathbb{Z}^2}\left(N_{m,n}^2\right)$ *and* (1.8). We shall check that there exists $c_{\mathsf{N}} = c_{\mathsf{N}}(c_{\mathsf{RSW}}) < \infty$, such that, uniformly in $m$,

$$\mathbb{E}_{p_c,q,\mathbb{Z}^2}\left(N_{m,5m}^2\right) \leq c_{\mathsf{N}}. \tag{3.7}$$

A substitution into (1.8) yields the claim.

Consider the annulus $A_{m,5m}$ and four rectangles

$$S_{m,\mathsf{U}} = [-5m, 5m] \times [m, 5m] \quad S_{m,\mathsf{R}} = [-5m, -m] \times [-5m, 5m]$$
$$S_{m,\mathsf{L}} = [m, 5m] \times [-5m, 5m] \quad S_{m,\mathsf{D}} = [-5m, 5m] \times [-5m, -m]. \tag{3.8}$$

For $* \in \{\mathsf{U}, \mathsf{R}, \mathsf{L}, \mathsf{D}\}$, let $N_{m,*}$ be the number of distinct short side crossing clusters of $S_{m*}$. For instance $N_{m,\mathsf{U}}$ is the number of distinct clusters which connect $[-5m, 5m] \times \{m\}$ to $[-5m, 5m] \times \{5m\}$ in the restriction of the percolation configuration to the rectangle $S_{m,\mathsf{U}}$. Clearly,

$$N_{m,5m} \leq \sum_{* \in \{\mathsf{U}, \mathsf{R}, \mathsf{L}, \mathsf{D}\}} N_{m,*},$$

and, by symmetry, it remains to give an upper bound on $\mathbb{E}_{p_c, q, \mathbb{Z}^2}\left(N_{m,\mathsf{U}}^2\right)$.

**Lemma 3.2.** *The RSW bound* (3.4) *implies:*

$$\mathbb{P}_{p_c, q, \mathbb{Z}^2}\left(N_{m,\mathsf{U}} \geq k\right) \leq \left(1 - c_{\mathsf{RSW}}(5, \infty)\right)^{k-1} \tag{3.9}$$

*uniformly in $k > 1$ and $m$.*

*Proof.* Let us introduce the events $\mathcal{R}_k = \{N_{m,\mathsf{U}} \geq k\}$. We claim that

$$\mathbb{P}_{p_c, q, \mathbb{Z}^2}\left(\mathcal{R}_k \big| \mathcal{R}_{k-1}\right) \leq 1 - c_{\mathsf{RSW}}(5, \infty) \tag{3.10}$$

uniformly in $m$ and $k > 1$. Indeed, distinct crossing clusters which show up in any percolation configuration from $\mathcal{R}_{k-1}$ are naturally ordered from left to right. There are at least $(k-1)$ such clusters. The following somewhat standard construction which we sketch below is depicted on Figure 2:
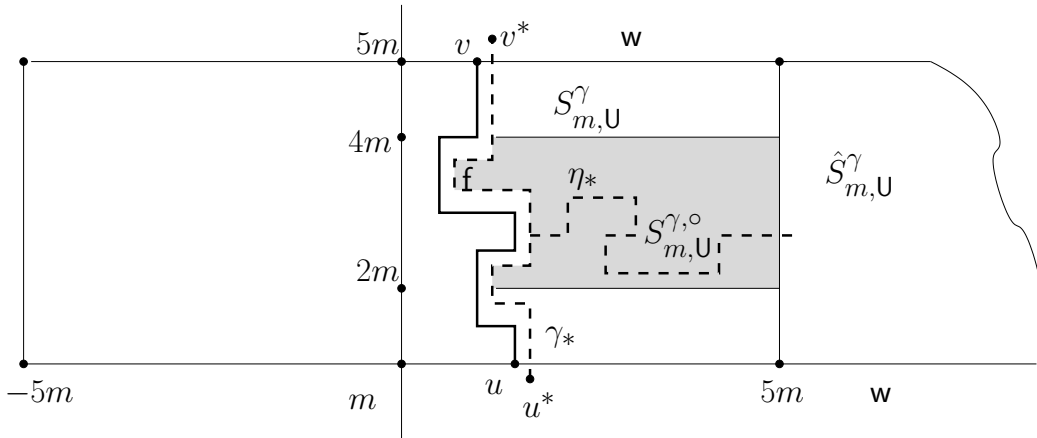


FIGURE 2. The dual path $\gamma_* = \gamma_*^{u,v}$. Event $\mathcal{D}^\circ$: a dual path $\eta_*$ crosses from left to right the middle section $S_{m,\mathsf{U}}^{\gamma,\circ}$ of $S_{m,\mathsf{U}}^\gamma$ and, as such, rules out the occurrence of the event $\mathcal{R}_k$.
The boundary conditions (for the direct model) on the semi-infinite strip $\hat{S}_{m,\mathsf{U}}^\gamma$ are $\mathsf{w}$ on the upper and lower parts, and $\mathsf{f}$ on $\gamma_*$.

Consider the disjoint decomposition $\mathcal{R}_{k-1} = \cup \mathcal{R}_{k-1}^{u,v}$, where $u$ (respectively $v$) is the rightmost vertex of the $(k-1)$-th crossing cluster on the bottom (respectively

top) side of $S_{m,\mathsf{U}}$. The event $\mathcal{R}_{k-1}^{u,v}$ implies that there is the left-most dual crossing $\gamma_*^{u,v}$ from $u^*$ to $v^*$, where $u^* = u + \frac{1}{2}(1,-1)$ and $v^* = v + \frac{1}{2}(1,1)$. Consider the remaining part, denoted by $S_{m,\mathsf{U}}^{\gamma}$, of the rectangle $S_{m,\mathsf{U}}$ to the right of $\gamma_*^{u,v}$. Let $S_{m,\mathsf{U}}^{\gamma,\circ}$ be the middle section of $S_{m,\mathsf{U}}^{\gamma}$, i.e. $S_{m,\mathsf{U}}^{\gamma,\circ} := S_{m,\mathsf{U}}^{\gamma} \cap (\mathbb{Z} \times [2m, 4m])$. Finally consider the infinite strip extension $\hat{S}_{m,\mathsf{U}}^{\gamma}$ to the right of $S_{m,\mathsf{U}}^{\gamma}$.

Let $\mathcal{D}^{\circ}$ be the event that there is a left to right dual crossing of $S_{m,\mathsf{U}}^{\gamma,\circ}$. By the FKG property of the random-cluster model,

$$\mathbb{P}_{p_c,q,\mathbb{Z}^2}\left(\mathcal{R}_k \big| \mathcal{R}_{k-1}\right) \leq 1 - \min_{\gamma} \mathbb{P}_{p_c,q,\hat{S}_{m,\mathsf{U}}^{\gamma}}^{\mathsf{w,f}}\left(\mathcal{D}^{\circ}\right),$$

where the boundary conditions are *direct* boundary conditions on the semi-infinite strip $\hat{S}_{m,\mathsf{U}}^{\gamma}$: wired on upper and lower parts and free on $\gamma$. Note that the model is self-dual at criticality. Hence, for any possible realization of $\gamma$,

$$\mathbb{P}_{p_c,q,\hat{S}_{m,\mathsf{U}}^{\gamma}}^{\mathsf{w,f}}\left(\mathcal{D}^{\circ}\right) \geq \mathbb{P}_{p_c,q,\hat{R}_m[\infty]}^{\mathsf{f}}\left(\mathrm{Cross}(R_m[5])\right),$$

and (3.4) applies. $\square$

**Remark 2.** *Let us highlight the fact that SLE predictions, see [6, Section 13.3.2], suggest that $\mathbb{P}_{p,q,\mathbb{Z}^2}[A_4^e(n)] = n^{-\xi_{1010}+o(1)}$, where*

$$\xi_{1010} = \frac{3\sigma^2 + 10\sigma + 3}{4(1+\sigma)} \quad \text{with} \quad \sigma = \frac{2}{\pi}\arcsin(\sqrt{q}/2).$$

*This implies that $\mathbb{P}_{p,q,\mathbb{Z}^2}[A_4^e(n)] \gg \frac{1}{n}$ for $q < 2\arcsin[\pi\frac{2-\sqrt{3}}{\sqrt{3}}] \approx 0.459$. This illustrates the fact that the claim of Theorem 1.1 can fail to hold when the condition (FKG) is dropped.*

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