Lecture 10: Deligne's theorem, part 2

Dustin Clausen

June 2, 2021

Let us recall the statement of Deligne's theorem from last time.

Theorem 1. *Let $f : X \to S$ be a smooth and proper map of $\mathbb{Q}$-schemes. Then:*

1. *For all $p \geq 0$ and $q \geq 0$, the quasicoherent sheaf $R^q f_*(\Omega^p) = H_{-q} f_* |\Omega^p|$ on S is locally free of finite rank.*
2. *The local ranks of $R^q f_*(\Omega^p)$ and $R^p f_*(\Omega^q)$ are the same.*
3. *The spectral sequence of quasi-coherent sheaves on S*
$$E_1^{p,q} = R^q f_* \Omega^p \Rightarrow H^{p+q} \mathrm{dR}_{X/S},$$
associated to the Hodge filtration of $\mathrm{dR}_{X/S}$, degenerates at E_1. Thus each $H^n \mathrm{dR}_{X/S}$ has a functorial filtration with associated graded the $R^q f_* \Omega^p$ for $p + q = n$ and in particular is also locally free of finite rank.

The proof is based on the following reduction.

Lemma 2. *Suppose Deligne's theorem is true when S is the Spec of an artinian local $\mathbb{C}$-algebra with residue field $\mathbb{C}$. Then it is true in general.*

Proof. Let's first prove that claim 1 for general S reduces to claim 1 for S the Spec of an artinian local $\mathbb{C}$-algebra with residue field $\mathbb{C}$. In the language of the last lecture, claim 1 says that $f_*|\Omega^p|$ is split-perfect. In what follows we will constantly and without explicit mention use the fact that $f_*|\Omega^p|$ base-changes appropriately, i.e. if f' is a pullback of f via some map $g : S' \to S$, then $g^* f_* |\Omega^p| = f'_* |\Omega^p|$. This is the derived version of the "cohomology and base-change theorem" and is more-or-less a triviality, though an extremely useful one.

Because the property of being split-local is local on S, we can then assume $S = Spec(R)$ is affine. Note that if the claim is true for $X \to S$, then it is also true for any base-change $X' \to S'$. As the data of smooth proper map of schemes to $Spec(R)$ is in the end determined by finitely many elements of R satisfying some polynomial equations, and R can be written as a filtered colimit of finitely generated $\mathbb{Z}$-algebras, we can reduce to the case where R is a finitely generated $\mathbb{Z}$-algebra, in particular noetherian.

Then it follows from the main reduction in the previous lecture that we can reduce to the case where R is an artinian local ring with residue field k of characteristic 0, which can be taken to be a residue field of R, hence finitely generated as a field. By Cohen's theorem, we can further give ourselves a splitting $k \to R$ of the map $R \to k$. Now we arbitrarily embed $k \to \mathbb{C}$ and take the faithfully flat base change of $X \to Spec(R) \to Spec(k)$ along it. As split-perfectness can be checked after faithfully flat base-change, we effectuate the reduction as desired.

Now suppose 2 is true for $S = Spec(\mathbb{C})$ and 1 is true in general. Then by the same argument 2 is true for $Spec(k)$ for any finitely generated field of characteristic zero k, hence for any field of characteristic zero k. Since ranks are detected on base-change to residue fields, the field case implies the general case.

Finally, suppose 3 is true for S the Spec of an artinian local $\mathbb{C}$-algebra with residue field $\mathbb{C}$, and 1 is true in general. The claim is that all differentials vanish on all pages; we can prove this by induction on the page number, and therefore we can assume that our differential is a natural map from a locally free sheaf of the form $R^q f_* \Omega^p$ to another such locally free sheaf of the same form. Moreover these locally free sheaves are compatible under base change by 1. Thus we can reduce to the finitely generated hence noetherian case, then to the local case, then to the complete local case, then to the artinian case, then to artinian over $\mathbb{C}$ just as previously. $\square$

From now on, we therefore assume $S = Spec(R)$ with R an artinian local $\mathbb{C}$-algebra with residue field $\mathbb{C}$, or equivalently a finite dimensional local $\mathbb{C}$-algebra. We will now explain how to reduce even further to the case $R = \mathbb{C}$. This is based on the following theorem, itself an instance of an important general principle in de Rham cohomology:

Theorem 3. *Let $f : X \to Spec(R)$ be a smooth proper map where R is an artinian local $\mathbb{C}$-algebra with residue field $\mathbb{C}$. Then there is a canonical isomorphism in $D(R)$*

$$dR_{X/R} \simeq dR_{X_{\mathbb{C}}/\mathbb{C}} \otimes_{\mathbb{C}} R,$$

where $X_{\mathbb{C}}$ is the base-change of X to the residue field.

In other words, de Rham cohomology does not vary infinitessimally. Note that this is very much not true for $dR_{X/R}$ viewed as a filtered object; we already saw in our discussion of elliptic curves over the complex numbers that variation of the Hodge filtration on H^1 exactly corresponds to the variation in the moduli of elliptic curves.

Note that the statement is purely algebraic. Indeed, the analogous result holds for any field of characteristic zero replacing $\mathbb{C}$. You can read about that, for example, in Grothendieck's letter to Tate on crystals. But we'll restrict to $\mathbb{C}$ and produce the isomorphism via an analytic/topological construction.

Proof. We'll produce an isomorphism

$$dR_{X/R} \simeq R\Gamma(X^{top}; R)$$

where $X^{top} = X_{\mathbb{C}}(\mathbb{C})$. Then the theorem will follow by comparing with the isomorphism

$$dR_{X_{\mathbb{C}}/\mathbb{C}} \simeq R\Gamma(X^{top}; \mathbb{C})$$

coming from Grothendieck's comparison with analytic de Rham cohomology and the Poincare Lemma.

Actually, the first isomorphism can be proved by following essentially the same argument as gives the second isomorphism, but this requires a little bit of the theory of analytifications of finite type $\mathbb{C}$-schemes. If

$$A = \mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$$

is a finitely generated $\mathbb{C}$-algebra, then the associated complex analytic space $Spec(A)^{an}$, as a topological space, is the common zero locus

$$Z(f_1, \ldots, f_m) \subset \mathbb{C}^n$$

with the subspace topology, where $\mathbb{C}^n$ has its standard complex topology (basis given by open balls, or open polydisks). It is equipped with the sheaf of rings $\mathcal{O}_{Spec(A)^{an}}$, whose sections on the intersection of a poldisk with $Z(f_1, \dots, f_m)$ is the quotient of the ring of holomorphic functions on that polydisk by the ideal generated by the restrictions of the functions $f_1, \dots, f_n$. This construction is functorial in $Spec(A)$ and globalizes to define the analytificiation of any finite type $\mathbb{C}$-scheme X. If X is smooth over $\mathbb{C}$, one checks using the implicit function theorem that this is the same as the associated complex manifold with its structure sheaf of holomorphic functions.

This applies in particular to our $Spec(R)$ and X, so we get an induced map of complex analytic spaces

$$f^{an} : X^{an} \to Spec(R)^{an}.$$

In fact, $Spec(R)^{an} = Spec(R)$ as locally ringed spaces: they are both just a point with sheaf of rings R.

Moreover, there is a notion of smooth map of analytic spaces: they locally look like projection off a polydisk. There is an associated relative de Rham complex in the analytic setting. The analytification of a smooth map of finite type schemes is a smooth map of analytic spaces, and there is a comparison map between the relative de Rham complexes. Both the proof of Grothendieck's theorem comparing analytic and algebraic de Rham cohomology, and the proof of the Poincare lemma, go through just fine in the context of our map $f : X \to Spec(R)$ and its analytification. Note that the underlying topological space of X^{an} is just X^{top}; indeed the underlying topological space is not sensitive to nilpotents, but on the other hand $X_{red} = X_{\mathbb{C}}$ as $\mathbb{C}$-schemes. Thus this sketch explains why the desired isomorphism $dR_{X/R} \simeq R\Gamma(X^{top}; R)$ holds. □

Now consider the Hodge-de Rham spectral sequence for f

$$E_1^{p,q} = H^q(X; \Omega^p) \Rightarrow H^{p+q} dR_{X/R}.$$

This is a convergent spectral sequence of finitely generated R-modules. It follows that for all $n \geq 0$,

$$\sum_{p+q=n} l_R(H^q(X; \Omega^p)) \geq l_R(H^n dR_{X/R}) = l_R(R) \cdot dim_{\mathbb{C}} H^n dR_{X_{\mathbb{C}}/\mathbb{C}},$$

where the equality follows from the theorem we just proved. On the other hand, we know from the previous lecture and the base-change property of Hodge cohomology that

$$l_R(H^q(X; \Omega^p)) \leq l_R(R) \cdot dim_{\mathbb{C}}(H^q(X_{\mathbb{C}}; \Omega^p))$$

with equality for all q if and only if $Rf_*\Omega^p$ is split-perfect. Summing over $p+q=n$ and using Hodge-de Rham geneneration over $\mathbb{C}$ we get

$$\sum_{p+q=n} l_R(R^q f_* \Omega^p) \leq l_R(R) \cdot dim_{\mathbb{C}} H^n dR_{X_{\mathbb{C}}/\mathbb{C}}$$

with equality for all n if and only if each $Rf_*\Omega^p$ is split-perfect.

But we had exactly the opposite inequality above. It follows that $Rf_*\Omega^p$ is split perfect, and we have equality above. Then since length is additive in short exact sequences and length is zero if and only if the module is zero, we also deduce degeneration over R.

To summarize, we have reduced proving all three points to the case $S = Spec(\mathbb{C})$. The first claim is vacuous over a field, so again using the comparison of algebraic and analytic de Rham cohomology it suffices to prove is the following:

Theorem 4. *Let $M = X^{an}$ be the compact complex manifold associated to a smooth proper $\mathbb{C}$-scheme . Then Hodge degeneration and Hodge decomposition hold for M, i.e.:*

1. *The Hodge-de Rham spectral sequence*

$$H^q(M;\Omega^p) \Rightarrow H^{p+q}(M;\mathbb{C})$$

degenerates;

2. *The filtration $F^{\geq p}H^n(M;\mathbb{C})$ associated to this spectral sequence is opposed to its complex-conjugate, i.e.*

$$F^{\geq p}H^n(M;\mathbb{C}) \oplus \overline{F^{\geq n-p+1}H^n(M;\mathbb{C})} = H^n(M;\mathbb{C})$$

for all $0 \leq p \leq n$.

We have seen that this theorem holds for M if M is a compact complex Kähler manifold. But unfortunately, our $M = X^{an}$ need not be Kähler! Actually, Moishezon in his paper "On n-dimensional compact varieties with n algebraically independent meromorphic functions" proved a theorem which implies that if X is a smooth and proper $\mathbb{C}$-scheme, then X^{an} is Kähler if and only if X is *projective*, which is generally a stronger condition than just being proper.

However, a smooth proper $\mathbb{C}$-scheme is not too far from a smooth projective $\mathbb{C}$-scheme. Combining Chow's lemma with Hironaka's resolution theorem, we find that there is a birational map

$$f : Y \to X$$

with Y smooth and projective. We will use this in combination with the following.

Lemma 5. *Let $f : N \to M$ be a birational map between compact complex manifolds. Then for all $p, q \geq 0$, the pullback map*

$$f^* : H^q(M;\Omega^p) \to H^q(N;\Omega^p)$$

is injective.

Proof. Breaking M up into its components, we can assume M connected, of dimension d, say. Then N is also connected of dimension d, as the map is birational.

By Serre duality, we reduce to proving that $f^* : H^d(M;\Omega^d) \to H^d(N;\Omega^d)$ is injective. Indeed, by Serre duality $H^q(M;\Omega^p)$ is in perfect pairing with $H^{d-q}(M;\Omega^{d-p})$ via the natural product to $H^d(M;\Omega^d) = \mathbb{C}$. If $c \in H^q(M;\Omega^p)$ has $f^*c = 0$, then in particular

$$f^*(cc') = f^*(c)f^*(c') = 0$$

for all $c' \in H^{d-q}(M;\Omega^{d-p})$. If f^* is injective on $H^d(M;\Omega^d)$ we deduce $cc' = 0$ for all c', whence $c = 0$ by duality.

But we saw in the lecture on compact complex manifolds that $H^d(-;\Omega^d) = H^{2d}(-;\mathbb{C})$ via the Hodge de Rham spectral sequence. Thus we reduce to showing that f, as a map of real manifolds, induces an isomorphism on top cohomology. This follows from degree theory, as by assumption there is a point $p \in M$ and an open neighborhood of p over which f is an isomorphism. □

Now to finish the proof of Deligne's theorem it suffices to show the following.

Proposition 6. *Let $f : N \to M$ be a birational map of compact complex manifolds. If Hodge degeneration and decomposition hold for N, then they hold for M.*

Proof. The Hodge-de Rham spectral sequence for M

$$H^q(M;\Omega^p) \Rightarrow H^{p+q}(M;\mathbb{C})$$

maps by f^* to the Hodge-de Rham spectral sequence for N. By the lemma, f^* is injective on the first page. We deduce inductively that all differentials are zero and it's injective on all pages. In particular, we get degeneration.

Moreover, since f^* is injective and commutes with both complex conjugation and the Hodge filtration, we deduce

$$F^{\geq p}H^n(M;\mathbb{C}) \cap \overline{F^{\geq n-p+1}H^n(M;\mathbb{C})} = 0$$

from the analogous statement for N. With $h^{p,q} = dim H^q(M;\Omega^p)$, we deduce

$$\sum_{i\geq p} h^{i,n-i} + \sum_{i>n-p} h^{i,n-i} \leq \sum_{0\leq i\leq n} h^{i,n-i},$$

or

$$\sum_{i\geq p} h^{i,n-i} \leq \sum_{i\leq n-p} h^{i,n-i}.$$

On the other hand, using Serre duality this implies the opposite inequality in the complementary degree $2d-n$. Thus we have equality, which gives the Hodge decomposition theorem. □

By now, there are some purely algebraic proofs of the degeneration part of Deligne's theorem. One, due to Faltings, uses the p-adic numbers instead of the complex numbers. Another, by Deligne-Illusie, uses reduction to characteristic p. But I'm not aware of any purely algebraic proofs of the Hodge symmetry $h^{p,q} = h^{q,p}$, nor am I aware of any proof of degeneration which just works in the general algebraic context without reduction to a situation where there's extra structure, e.g. Frobenius in characteristic p.

The last thing I'd like to discuss in this lecture is the consequence for Poincaré duality. We start by recalling Grothendieck-Serre duality as we stated it previously: if $f : X \to S$ is a proper smooth map of schemes, then the right adjoint $f^!$ to the pushforward $f_* : D(X) \to D(S)$ satisfies

$$f^!M \simeq f^!\mathcal{O} \otimes f^*M$$

via a natural map from right to left which one can construct a priori using the projection formula for f_*, and moreover

$$f^!\mathcal{O} \simeq \Omega^d[d]$$

naturally with respect to pullback, where d stands for the relative dimension, a locally constant $\mathbb{Z}$-valued function on X.

As we showed, this implies that f_* sends perfect complexes to perfect complexes. For that deduction we used the locally-valid characterization of perfect complexes as the compact objects. On the other hand, the perfect complexes are also exactly the dualizable objects, and so it's reasonable to ask how pushforward interacts with duality. Such an interaction also follows from Grothendieck-Serre duality.

Proposition 7. *Let $f : X \to S$ be a proper smooth map of schemes of relative dimension $d : X \to \mathbb{Z}_{\geq 0}$, and let $P \in Perf(X)$. Then there is a natural identification*

$$(f_* P)^\vee \simeq f_*(P^\vee \otimes \Omega^d[d]),$$

where natural means both functorial and natural with respect to pullback along maps $S' \to S$.

Proof. Let $Q \in Perf(S)$. Then

$$\begin{aligned} Hom(Q, (f_*P)^\vee) &= Hom(f_*P, Q^\vee) \\ &= Hom(P, f^!(Q^\vee)) \\ &= Hom(P, f^*(Q^\vee) \otimes \Omega^d[d]) \\ &= Hom(P, (f^*Q)^\vee \otimes \Omega^d[d]) \\ &= Hom(f^*Q, P^\vee \otimes \Omega^d[d])) \\ &= Hom(Q, f_*(P^\vee \otimes \Omega^d[d])), \end{aligned}$$

whence the result by the Yoneda lemma. □

Remark 8. *A duality datum can always be encoded by the corresponding perfect pairing. Usually perfect pairings are taken to have values in the unit $\mathcal{O}$, but we can just as well take them to have values in an invertible object, for example $\Omega^d[d]$. Then, an equivalent way of phrasing the above lemma is the following: given a perfect pairing*

$$P \otimes P' \to \Omega^d[d]$$

in $Perf(X)$, the composition

$$f_*P \otimes f_*P' \to f_*(P \otimes P') \to f_*\Omega^d[d] \to \mathcal{O}$$

is a perfect pairing in $Perf(S)$, where the last map is the counit for the adjunction $(f_*, f^!)$ and the first map comes from the fact that f_* is right adjoint to the symmetric monoidal functor f^*.

With this we can state Poincaré duality for Hodge cohomology.

Theorem 9. *Let $f : X \to S$ be a smooth proper map of schemes. The for any locally constant functions $p, p' : X \to \mathbb{Z}_{\geq 0}$ with $p + p' = d$ the relative dimension, the multiplication map*

$$\Omega^p[p] \otimes \Omega^{p'}[p'] \to \Omega^d[d]$$

is a perfect pairing in $Perf(X)$, and hence the induced

$$f_*\Omega^p[p] \otimes f_*\Omega^{p'}[p'] \to f_*\Omega^d[d] \to \mathcal{O}$$

is a perfect pairing in $Perf(S)$.

Proof. Locally, Ω^1 is free of rank d, and we can check that we have the claimed perfect pairing in $Perf(X)$ using the standard basis for the exterior powers. The second follows from the first and relative Serre duality, as explained in the previous remark. □

This is a perfect pairing in $Perf(S)$. In general, perfect pairings in $Perf(S)$ do not pass to homology groups: indeed we have seen that the homology groups of a perfect complex need not even be perfect i.e. dualizable. But if we have a split-perfect complex P and a duality pairing

$$P \otimes P' \to \mathcal{O},$$

then on homology we get maps

$$H_i P \otimes H_{-i} P' \to H_0 \mathcal{O} = \mathcal{O}$$

which are themselves perfect pairings, because we can choose a splitting $P \simeq \oplus_i H_i(P)[i]$ and use this to a get a dual splitting $P^\vee \simeq \oplus_i H_i(P)^\vee[-i]$ which is implemented by the above pairing. Thus, in the setting of Hodge cohomology we deduce:

Corollary 10. *Let $f : X \to S$ be a smooth proper map of schemes, and let $p, p' : X \to \mathbb{Z}$ be locally constant functions with $p + p' = d$, the relative dimension. Suppose that $f_*\Omega^p$ is split-perfect, where f_* stands for derived pushforward. (For example, this holds if S is a $\mathbb{Q}$-scheme or if $S = Spec(k)$ for a field k.)*

Then for all integers q, q' with $q + q' = 0$, the composition

$$H^q f_*\Omega^p[p] \otimes H^{q'} f_*\Omega^{p'}[p'] \to H^0 f_*\Omega^d[d] \to \mathcal{O}$$

is a perfect pairing of finite locally free sheaves on S.

Remark 11. *From this corollary we can deduce that for an* arbitrary *smooth proper map $f : X \to S$ of relative dimension d, the quasi-coherent sheaves $R^q f_*\Omega^p$ on S, which make up the E_1 page of the Hodge-de Rham spectral sequence, vanish outside the box $0 \leq p, q \leq d$. Indeed the bound on p is clear because Ω^p itself is zero outside the range $0 \leq p \leq d$. As for the bound on q, we use that the bottom homology group of a perfect complex is finitely presented and base-changes appropriately (in the underived sense) to reduce to the case of spec of a field. Then the previous corollary applies and hence the conclusion $q \leq d$ follows by duality from the trivial conclusion $q \geq 0$ which is a general feature of cohomology.*

Remark 12. *Even without any split-perfect hypothesis, a small part of the corollary nonethless holds for an arbitrary smooth proper map $f : X \to S$ of relative dimension d. Namely, the induced pairing*

$$R^0 f_*\mathcal{O} \otimes R^d f_*\Omega^d \to R^d f_*\Omega^d \to \mathcal{O}$$

is a perfect pairing of locally finite free $\mathcal{O}$-modules. Indeed, we can work locally and assume S is affine; in fact we can even work flat-locally. Recall the Stein factorization, valid for an arbitrary proper map $f : X \to S$, which is

$$X \to S' \to S$$

such that the first map has connected geometric fibers and the second map is given by $S' = Spec(R^0 f_*\mathcal{O})$. The Stein factorization is preserved under flat base-change. When f is smooth one shows that S' is a finite etale S-algebra, see Stacks Project tag 0E0D. It follows that after a finite etale base change we can assume X splits up into a finite disjoint union of smooth proper S-schemes with geometrically connected fibers, and in that case $R^0 f_*\mathcal{O} = \mathcal{O}$.

Thus, we need to show that if f has geometrically connected fibers, then $R^d f_*\Omega^d \to \mathcal{O}$ is an isomorphism. Again we recall that the top cohomology is preserved under arbitrary underived basechange. Thus by use of Nakayama's lemma we can reduce to the case where S is a field. But then we have the required duality in all degrees by the previous corollary.

Now, what about de Rham cohomology? This has a filtration where the associated graded identifies with Hodge cohomology, so we can hope for a duality on de Rham cohomology which, on associated gradeds, recovers the above duality on Hodge cohomology.

More precisely, let $f : X \to S$ be smooth and consider the de Rham complex $\Omega^\bullet$ of quasi-coherent sheaves on X. The product structure on the de Rham complex is encoded in a map of chain complexes

$$\Omega^\bullet \otimes \Omega^\bullet \to \Omega^\bullet.$$

We can equivalently encode this in a map of filtered objects in $D(X)$

$$|\Omega^\bullet| \otimes |\Omega^\bullet| \to |\Omega^\bullet|,$$

where the tensor product on filtered objects, formally defined by Day convolution on the functor category $Fun((\mathbb{Z}, \leq), D(X))$, is such that this amounts to compatible maps

$$F^{\geq p}|\Omega^\bullet| \otimes F^{\geq p'}|\Omega^\bullet| \to F^{\geq p+p'}|\Omega^\bullet|$$

for all $p, q' \geq 0$. On associated graded we recover the product structure

$$\Omega^p[-p] \otimes \Omega^{p'}[-p'] \to \Omega^{p+p'}[-p-p']$$

used above (up to shift).

But now we can also take f_* and get a product structure

$$dR_{X/S} \otimes dR_{X/S} \to dR_{X/S}$$

of filtered objects in $D(S)$, again recovering the product structure on Hodge cohomology by taking associated gradeds. Using this, we can see that promoting Poincare dualtity to de Rham cohomology is equivalent to making sure there is a correct fundamental class.

Proposition 13. *Let $f : X \to S$ be a smooth proper map of schemes of relative dimension d. The following are equivalent:*

1. *The map $H^d f_* \Omega^{d-1} \to H^d f_* \Omega^d$ induced by the de Rham differential is zero.*
2. *The map $f_* \Omega^d[-d] = F^{\geq d} dR_{X/S} \to dR_{X/S}$ from the top nontrivial filtered piece of de Rham cohomology induces an isomorphism on H^{2d}.*
3. *The $-2d$ shift of the counit map $f_* \Omega^d[d] \to \mathcal{O}$ factors through to a map $dR_{X/S} \to \mathcal{O}[-2d]$.*
4. *There exists a perfect pairing of filtered objects*

   $$dR_{X/S} \otimes dR_{X/S} \to \mathcal{O}[-2d]\{d\},$$

   where $\mathcal{O}[-2d]\{d\}$ stands for the filtered object whose only nonzero associated graded is in degree d and equals $\mathcal{O}[-2d]$; such a perfect pairing amounts to compatible perfect pairings of perfect complexes on S

   $$F^{\geq p} dR_{X/S} \otimes \left(dR_{X/S} / F^{\geq p'+1} dR_{X/S} \right) \to \mathcal{O}[-2d]$$

   for all $p, p' \in \mathbb{Z}$ with $p + p' = d$, and we moreover ask that the induced perfect pairing on associated gradeds identify with our previous perfect pairing on Hodge cohomology.

Proof. By degree reasons, the first differential $H^d f_* \Omega^{d-1} \to H^d f_* \Omega^d$ is the only possible nonzero differential in the Hodge-de Rham spectral sequence which enters $H^d f_* \Omega^d$. Thus this differential is zero if and only if the edge map $H^d f_* \Omega^d \to H^{2d} dR_{X/S}$ is an isomorphism, which is exactly condition 2. Hence 1 and 2 are equivalent.

By Remark 11, $H^{2d} dR_{X/S}$ is the bottom homology group of $dR_{X/S}$, so there's a canonical truncation map $dR_{X/S} \to H^{2d} dR_{X/S}[-2d]$ which shows that condition 2 implies condition 3. Now, condition 3 is equivalent to saying that we have a map of filtered objects

$$dR_{X/S} \to \mathcal{O}[-2d]\{d\}.$$

Thus, assuming condition 3, we can define a pairing of filtered objects by using the product structure:

$$dR_{X/S} \otimes dR_{X/S} \to dR_{X/S} \to \mathcal{O}[-2d]\{d\}.$$

Inducting up the filtration shows that we can check it's a perfect pairing on associated graded, but then we exactly see the perfect pairing on Hodge cohomology, whence condition 4. Finally, assume condition 4. It follows that there is an induced pairing on the the Hodge-de Rham spectral sequence. On the E_1-page, recall from Remark 12 that the $(0,0)$ spot and the (d,d)-spot are in perfect duality. It follows that to show condition 1, it suffices to show the dual claim, that the map $H^0 f_* \mathcal{O} \to H^0 f_* \Omega^1$ is zero. This follows from the Stein factorization as discussed in Remark 12, because by functoriality we reduce to checking the analogous claim for S' which is finite etale over S and hence has $\Omega^1 = 0$. □

If these conditions hold, we'll say that $f : X \to S$ satisfies Poincare duality for de Rham cohomology. Actually, we'll see later that all smooth proper maps of relative dimension d satisfy Poincare duality for de Rham cohomology, but the argument is not at all easy in spite of the apparently simplicity of condition 1.

Note that this is a derived duality, and just as in the discussion of Hodge cohomology, it doesn't necessarily pass to a duality on the cohomology groups themselves. However in the situation of Deligne's theorem this holds.

Corollary 14. *Suppose $f : X \to S$ is a smooth proper map of relative dimension d which satisfies Poincare duality for de Rham cohomology. Suppose that each entry on each page of the Hodge-de Rham spectral sequence is finite locally free, e.g. if S has characteristic zero or S is Spec of a field. Then the induced pairing on the Hodge-de Rham spectral sequence is a perfect pairing in all degrees on all pages, as is the induced pairing*

$$H^i dR_{X/S} \otimes H^{2d-i} dR_{X/S} \to H^{2d} dR_{X/S} \to \mathcal{O}$$

on de Rham cohomology quasi-coherent sheaves, and this pairing also respects the Hodge filtration in that the right annihilator of $F^{\geq p} H^i$ identifies with $F^{\geq d-p+1} H^{2d-i}$.

Proof. The pairing is perfect on the E_1 page by Corollary 10, and we can inductively split all occurring filtrations to deduce the claims by induction on the page number. □