Lecture 11: de Rham cohomology in characteristic p: pathological?

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After some heavy work the past few lectures, I think we've earned a break. So here's a little leisurely lecture where we survey some interesting results in de Rham cohomology in characteristic p. There will be no proofs for the moment, though I hope to give proofs in later lectures.

We already said in the previous lecture that de Rham cohomology satisfies Poincaré duality (in the derived sense) for a general smooth proper map $f : X \to S$ of relative dimension $d \in \mathbb{Z}_{\geq 0}$. In particular, it holds in the underived sense over an arbitrary field, irrespective of characteristic:

Theorem 1. *Let $f : X \to Spec(k)$ be a smooth proper variety of dimension d over a field k. Then the top nonvanishing de Rham cohomology group is $H^{2d}dR_{X/k}$, and there is a canonical map $H^{2d}dR_{X/k} \to k$ inducing via the product structure on de Rham cohomology a perfect pairing of finite dimensional k-vector spaces*

$$H^i dR_{X/k} \otimes H^{2d-i} dR_{X/k} \to k$$

for all $0 \leq i \leq 2d$.

This indicates that the de Rham cohomology of smooth proper varieties is "good", even in finite characteristic. Actually, there are other indications of this as well. For example, suppose you have a smooth proper variety $X \to Spec(\mathbb{Q})$ over the rational numbers. We can spread out to a smooth proper map $\mathcal{X} \to Spec(\mathbb{Z}[1/N])$ for some $N \in \mathbb{N}$. Then, as we've mentioned, you get two different integral structures on the topological cohomology of $X(\mathbb{C})$ with complex coefficients, via

$$R\Gamma(X(\mathbb{C}); \mathbb{Z}) \otimes \mathbb{C} \simeq R\Gamma(X(\mathbb{C}); \mathbb{C}) \simeq dR_{X_{\mathbb{C}}/\mathbb{C}} \simeq dR_{\mathcal{X}/\mathbb{Z}[1/N]} \otimes \mathbb{C}$$

where the first isomorphism follows from the homology of $X(\mathbb{C})$ being finite dimensional in each degree, the second isomorphism is Grothendieck's comparison theorem, and the third isomorphism is the base-change property of algebraic de Rham cohomology. When you have an integral structure, you get extra torsion information, which can be investigated by reducing mod p for $(p, N) = 1$. On the topological side, we have

$$R\Gamma(X(\mathbb{C}); \mathbb{Z}) \otimes \mathbb{F}_p \simeq R\Gamma(X(\mathbb{C}); \mathbb{F}_p)$$

where we take the derived tensor product; this is what's responsible for the usual universal coefficient sequence. On the de Rham side, we have

$$dR_{\mathcal{X}/\mathbb{Z}[1/N]} \otimes \mathbb{F}_p \simeq dR_{\mathcal{X}_{\mathbb{F}_p}/\mathbb{F}_p}$$

by the base change property for de Rham cohomology. This again gives a universal coefficient sequence expressing the algebraic de Rham cohomology of a characteristic p variety in terms of torsion information in our "other" integral structure on the topological cohomology of the complex points.

Thus, in a sense, algebraic de Rham cohomology in characteristic p is the natural answer to a question which arises purely in characteristic zero, and it is analogous to the topological cohomology with $\mathbb{F}_p$-coefficients.

Remark 2. *Actually the relevance of characteristic p is even greater than indicated by this discussion. If $f : X \to S$ is an arbitrary smooth proper map, then by base change and limit arguments we reduce the study of de Rham cohomology of f to the case where S is Spec of a finitely generated $\mathbb{Z}$-algebra R. For such R, the maximal ideals are dense in $Spec(R)$, and the residue field at any maximal ideal is a finite field, hence characteristic p for some p. Thus to a large degree the de Rham cohomology in total generality reduces to de Rham cohomology in characteristic p. This is how Deligne-Illusie's proof of Hodge degeneration (in characteristic zero!) proceeds.*

This all speaks to the relevance of de Rham cohomology in characteristic p. But there are some troublesome aspects, even for smooth proper varieties. Indeed, basically all of the three points in Deligne's theorem can fail in general:

1. There are smooth proper varieties over $\mathbb{F}_p$ such that the associated Hodge-de Rham spectral sequence does not degenerate; in fact differentials on the first page can be nonzero.
2. There are smooth proper varieties over $\mathbb{F}_p$ such that Hodge symmetry $h^{p,q} = h^{q,p}$ fails.
3. There are smooth proper families $f : X \to S$ in characteristic p such that the Hodge numbers jump, i.e. are not locally constant on the base. Same for the Betti numbers in algebraic de Rham cohomology.

On the other hand, in a number of relevant examples, these results do hold.

Example 3. *If $f : X \to S$ is a smooth proper map of schemes such that either:*

1. *f has relative dimension one;*
2. *f is a relative abelian variety (see Berthelot-Breen-Messing, "Théorie de Dieudonné cristalline, II", Prop 2.5.2.)*
3. *f is a projective bundle, or more generally a Grassmannian;*

Then all the conclusions of Deligne's theorem hold for f: we have Hodge-de Rham degeneration, Hodge symmetry, and the Hodge cohomology in each bidegree forms a vector bundle on S. Moreover, the Hodge numbers and Betti numbers are all as "expected" from the situation over the complex numbers.

Remark 4. *Given our work so far, it's actually fairly easy to explain point 1. For a relative curve, the E_1 page lives in the box $0 \leq p, q \leq 1$, hence there are only two possible nonzero differentials, and they are exactly the differentials we could prove vanish in complete generality in the previous lecture. Thus we have degeneration. Next we argue that each derived pushforward $f_*\Omega^p$ is split-perfect. We can reduce to S affine, and then the degree zero part of $f_*\mathcal{O}$ splits off because it's finitely generated projective, as seen in the previous lecture. This dually splits the degree one part of $f_*\Omega^1$ off. It follows that $H^1 f_*\mathcal{O}$ is itself a perfect complex (in degree 0) and is dual to $H^0 f_*\Omega^1$. But if a perfect complex and its dual both live in degree zero, then that perfect complex is a locally free sheaf of finite rank, whence the conclusion.*

The fact that the de Rham cohomology of projective spaces is "correct" is important in and of itself, because, as Grothendieck showed in his paper on Chern classes, it implies the existence of a theory of Chern classes for vector bundles in algebraic de Rham cohomology, again over a general base scheme, with properties completely analogous to those of the familiar Chern classes in the topological setting. Even more striking is the following theorem recently proved by Totaro, see his paper "The Hodge theory of classifying stacks":

Theorem 5. *Let R be an arbitrary commutative ring. Define the algebraic de Rham cohomology of the classifying stack BGL_n/R as the limit*

$$\varprojlim_{X \to BGL_n} dR_{X/R}$$

in $D(R)$ as X runs over all smooth R-schemes with a map to BGL_n, i.e. a vector bundle of rank n. Then

$$H^* dR_{BGL_n/R} \simeq R[[c_1, \dots c_n]]$$

with c_i in degree $2i$, just as expected from the topological situation.

The reason this is striking is that the de Rham cohomology of GL_n itself, a smooth affine R-scheme, is "pathological" in various ways in characteristic p, as we'll soon discuss. Yet all the pathology cancels in the limit somehow.

So far we've seen a lot of positives concerning de Rham cohomology in characteristic p, or rather de Rham cohomology independent of the characteristic: Poincaré duality, Chern classes, uniform expected calculations in various standard examples of schemes or stacks. There was only a small negative, namely that Deligne's theorem doesn't always hold, so we can't always make things "underived". But other than Totaro's result, these nice properties were confined to the smooth and proper situation. Now we'll discuss the affine situation, where there are stark differences between characteristic p and characteristic 0.

This is visible even in the simplest example of the affine line.

Example 6. *Let A be a commutative ring. The de Rham complex $\Omega^\bullet_{A[T]/A}$ is the two-term complex*

$$\left(A[T] \overset{d/dT}{\to} A[T] \right),$$

which is isomorphic to

$$A \oplus \bigoplus_{n \geq 1} (A \overset{\cdot n}{\to} A).$$

Thus $\mathbb{A}^1$ is contractible in the eyes of de Rham cohomology if and only if A is a $\mathbb{Q}$-algebra.

Even more, if $R = k$ is a field of characteristic p, then the de Rham cohomology groups of $\mathbb{A}^1$ are infinite-dimensional. In contrast:

Proposition 7. *If $f: X \to Spec(k)$ is a smooth scheme over a characteristic zero field k, then $H^i dR_{X/k}$ is finite-dimensional for all i.*

Proof. By limiting techniques and base-change, we can reduce to where k is a finitely generated field, then to where $k = \mathbb{C}$. Then by Grothendieck's comparison, we reduce to showing that the topological cohomology of $X(\mathbb{C})$ is finite-dimensional. But indeed we can make a finite CW-complex (indeed a compact manifold with boundary) which is homotopy equivalent to $X(\mathbb{C})$, by deleting a tubular neighborhood of the boundary divisor in a smooth normal crossings compactification. (There's also an alternate argument available using the de Rham complex with log poles on a normal crossings compactification, as in the exercises to the lecture on Grothendieck's theorem.) $\square$

This looks like a serious pathology: the affine line is not contractible, and its cohomology, as well as the cohomology of any non-proper smooth variety, is infinite dimensional. But actually, we'll see, when we discuss derived de Rham cohomology, that far from being pathological, the situation with $\mathbb{A}^1$ is as good as it could possibly be, and from a certain perspective it's actually the characteristic zero situation, where the affine line is contractible, that is pathological.

Namely, it turns out that in characteristic p, *the de Rham cohomology of an arbitrary smooth variety is formally determined by the de Rham cohomology of* $\mathbb{A}^1$, and this is the key to a simple understanding of many properties of de Rham cohomology. Of course, for this to work, the de Rham cohomology of $\mathbb{A}^1$ has to be nonzero, and even quite large.

One example of something which will be simple to explain from that perspective is the existence of *crystalline cohomology*, which is another indication that de Rham cohomology in characteristic p is a worthwhile and non-pathological theory. We state it in the simplest case, with ground field $\mathbb{F}_p$.

Theorem 8. *Let p be a prime, and let X be a smooth variety over $\mathbb{F}_p$. There is a functorially attached* crystalline cohomology

$$cr_{X/\mathbb{Z}_p} \in D(\mathbb{Z}_p),$$

a derived p-complete object with the following properties:

1. *There is a natural identification $cr_{X/\mathbb{Z}_p} \otimes \mathbb{F}_p \simeq dR_{X/\mathbb{F}_p}$;*
2. *If X arises as the special fiber of a smooth map $\mathcal{X} \to \mathbb{Z}_p$, there is an induced natural identification*

   $$cr_{X/\mathbb{Z}_p} \simeq (dR_{\mathcal{X}/\mathbb{Z}_p})_{\widehat{p}}$$

   (one must derived p-complete the right hand side; though this doesn't do anything when $\mathcal{X} \to \mathbb{Z}_p$ is proper).
3. *When X is proper, $cr_{X/\mathbb{Z}_p}$ is a perfect complex and there is Poincaré duality.*

The way to read this is that the p-complete de Rham cohomology of a smooth $\mathbb{Z}_p$-algebra really only depends on the special fiber. This should be compared with the result we proved in the complex-analytic setting in our discussion of Deligne's theorem, that the de Rham cohomology over an artinian local $\mathbb{C}$-algebra only depends on the special fiber. But here it is even more impressive because $\mathbb{Z}_p$ is not an $\mathbb{F}_p$-algebra, and moreover we have torsion information.

In both contexts, this crystalline nature applies only to de Rham cohomology iteslf, not to the Hodge filtration, and in fact, the variation of the Hodge filtration is an extremely important and fine invariant of the deformation $\mathcal{X}$ of X, which for example for elliptic curves uniquely recovers the deformation.

Exercise 9. *Let R be a commutative ring. Directly calculate the de Rham cohomology of $\mathbb{P}^1_R \to Spec(R)$ by using Mayer-Vietoris on the standard affine covering by two copies of $\mathbb{A}^1$ with intersection $\mathbb{G}_m$.*

Exercise 10. *Let R be a commutative ring. If $P \in Perf(R)$ with $P, P^\vee \in D(R)_{\geq 0}$, then P is a finitely generated projective module living in degree 0.*