Lecture 12: Equivariant de Rham cohomology: set-up and definitions

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We're working towards proving Poincaré duality for de Rham cohomology over a general base scheme. But the proof is not easy, and it requires some preliminaries. In particular, and this may be slightly surprising, at a key point we will make use of *equivariant* de Rham cohomology. The theory of equivariant de Rham cohomology also gives a useful way of treating the cohomology of projective bundles, which leads to Chern classes and Thom classes, and these will also play a role in our proof of Poincaré duality.

Moreover, we would like to produce counterexamples to the three claims of Deligne's theorem in characteristic p. Following a paper of Antieau-Bhatt-Matthew, "Counterexamples to HKR in characteristic p" this can be done by first finding the counterexamples in equivariant de Rham cohomology, then using an approximation argument to see that they also manifest in ordinary de Rham cohomology.

So it is well worth our time to make this detour into equivariant de Rham cohomology!

What is the idea of equivariant de Rham cohomology? First, *equivariant* means equivariant with respect to some group action, so we should talk about groups. Suppose S is a fixed scheme, our "base scheme", and suppose $G \to S$ is an *S-group scheme*. This means that:

1. For all $T \to S$, the set $Hom_S(T, G) = G(T)$ is equipped with a group structure, functorially in the S-scheme T.

2. Or, equivalently by the Yoneda lemma, we equip G with a single "multiplication map" of S-schemes
$$m : G \times_S G \to G$$
satisfying the group axioms on T-valued points for all S-schemes T. (Alternatively, these axioms can also be expressed diagramatically, for example associativity is the equality of two maps $G \times_S G \times_S G \to G$.)

3. If S and G are affine, this is also equivalent, by duality, to promoting the coordinate $\mathcal{O}(S)$-algebra $\mathcal{O}(G)$ to a *Hopf algebra* in $\mathcal{O}(S)$-modules, but we won't use this perspective.

Example 1. *The following examples of group schemes will play a role for us. All except the last are defined over an arbitrary base scheme S; actually they are defined over the universal base $S = Spec(\mathbb{Z})$, and their version over S is obtained by pullback. The last example is only defined over an $\mathbb{F}_p$-scheme, and is base-changed from its universal version over $Spec(\mathbb{F}_p)$.*

1. *For any $n \geq 1$, define the group scheme GL_n by having its values on an affine scheme $T = Spec(B)$ be*
$$GL_n(B) = \{\text{invertible } n \times n \text{ matrices with coefficients in } B\}$$
with group structure of matrix multiplication. For a general scheme T we have $GL_n(T) = Aut(\mathcal{O}_T^{\oplus n})$. This is an affine scheme with coordinate ring $\mathbb{Z}[x_{ij} \mid 1 \leq i, j \leq n][det^{-1}]$.

2. *In particular, we have $GL_1 = \mathbb{G}_m = Spec(\mathbb{Z}[x, x^{-1}])$, the* multiplicative group scheme *which represents the functor sending a commutative ring B to the multiplicative group $B^\times$ of units in B. This is an abelian group scheme, so the "multiplication by n map" (which is raising to the n^{th} power in this case) is a homomorphism, and by taking the kernel we get another group scheme*

$$\mu_n := ker(\mathbb{G}_m \xrightarrow{n} \mathbb{G}_m).$$

This is also affine, with coordinate ring $\mathbb{Z}[x]/(x^n - 1)$.

3. *There is also the additive group scheme $\mathbb{G}_a$ such that $\mathbb{G}_a(B) = B$ with group law of addition; the coordinate ring over $\mathbb{Z}$ is $\mathbb{Z}[x]$.*

4. *If our base S has characteristic p, then raising to the p^{th} power is a homomorphism $F : \mathbb{G}_a \to \mathbb{G}_a$, and its kernel is denoted*

$$\alpha_p := ker(F : \mathbb{G}_a \to \mathbb{G}_a).$$

Over $\mathbb{F}_p$ we have $\alpha_p = Spec(\mathbb{F}_p[x]/x^p)$.

Next we need not just a group, but something for the group to act on. Suppose $X \to S$ is an S-scheme. A (left) *G-action* on X is the data of, for every S-scheme T, an action of the group $G(T)$ on $X(T)$ which is functorial in T; or equivalently by Yoneda, the data of a map $a : G \times_S X \to X$ satisfying the axioms of a group action on T-points for all T.

Example 2. *The following examples of actions of group schemes will play a role for us.*

1. *For any G and X, we have the* trivial action *of G on X, where $G(T)$ acts by the identity on $X(T)$ for all T. The most important action is (!) the trivial G-action on the terminal S-scheme S.*

2. *For any homomorphism $H \to G$ of group schemes, we get an action of H on G by left translation.*

3. *If $E \to X$ is a vector bundle over an S-scheme, so $E = Spec_X(Sym(\mathcal{E}^\vee))$ where $\mathcal{E}$ is a locally free sheaf of finite rank on X, we get an action of $\mathbb{G}_m$ on E given by scalar multiplication coming from the description of T-valued points*

$$E(T) = \{(f, s) \mid f : T \to X, s \in \Gamma(T; f^*\mathcal{E}).\}$$

We can also restrict this action to the complement of the zero section $E \setminus 0$ and we still get a $\mathbb{G}_m$-action.

Now, if we have a group scheme G acting on X, all over our implicit base S, then we want to define an associated *equivariant de Rham cohomology*

$$dR_{X/S;G} \in D(S)$$

in the derived quasicoherent ∞-category of S. This should furthermore promote to a filtered object of $D(S)$ via some version of the Hodge filtration.

The main desiderata of this theory are as follows:

1. It is functorial in the scheme X with G-action;

2. If G acts freely on X with quotient $X' = G\backslash X$,[1] then we should have

$$dR_{X/S;G} \simeq dR_{X'/S}.$$

In the end, we will essentially define $dR_{X/S;G}$ by a formal reduction to the case of free actions. But for motivation and orientation we will start from a more basic perspective, and only a posteriori see that it amounts to formally reducing to the case of free actions. This is the perspective of modifying the notion of quotient in such a manner that it behaves like the quotient by a free action, even when the action is not free.

1 Warmup: quotients by G-actions in sets

We start with a question:

Question 3. *What's so nice about free actions, anyway?*

Let's discuss this question in the category of sets. We can then transport it to more general settings by the Yoneda embedding. A free G-action on a set X exhibits a certain nice regularity: intuitively speaking, all the orbits are of the "same size" (precisely, they are all bijective with G, and canonically so up to translation), and this size is independent not just of the point $x \in X$ but even of the set X and the free G-action on it as well. In this way there's a uniformity to free G-actions: they all resemble one another.

In fact, this uniformity can be extended to arbitrary G-actions as well, but we have to redefine what we mean by the orbits. Or rather, instead of talking about the orbits, i.e. the fibers of the map to the quotient set $X \to G\backslash X$, we should talk about the fibers of the map to the quotient *groupoid* $X \to G\backslash\backslash X$.

We can say what this groupoid quotient is both abstractly and concretely. Abstractly, a G-set X is encoded by a functor

$$X : BG \to Sets,$$

where BG is the category with one object having automorphism group G, and the quotient $G\backslash X$ is the colimit of this functor X. But we can view $Sets$ as a full subcategory of the $(2,1)$-category $Groupoids$ and take the colimit there instead, and this gives $G\backslash\backslash X$. We could of course go all the way to $Anima$ and take the colimit there for the "ultimate" quotient, but we would just get the same groupoid $G\backslash\backslash X$ anyway, so we might as well stop at $Groupoids$.

Concretely, $G\backslash\backslash X$ can be modeled as the "action category" of G on X: the object set is X, and the Hom-set from x to y is the set of $g \in G$ such that $gx = y$.

How to articulate that now all the orbits have the same size, independent of the action? We should again consider the fibers of the map $X \to G\backslash\backslash X$, but now "fiber" means fiber product of the form $* \times_{G\backslash\backslash X} X$, calculated in $Groupoids$. Fixing a lift $x_0 : * \to X$ of the chosen basepoint in $G\backslash\backslash X$, this identifies with the groupoid of pairs $(x \in X, g \in G)$ with $gx_0 = x$. Obviously the x is redundant data, and this groupoid identifies with just the set G. This bijection is again canonical up to translations, corresponding to the choice of lift x_0. So taken in the appropriate groupoid-theoretic sense, we see that all the fibers of $X \to G\backslash\backslash X$ look the same irrespective of the action, and look like the ordinary fibers in the case of a free action.

[1] Quotients, even by free actions, need not exist in the category of schemes; one should really pass to algebraic spaces. But when we discuss things more precisely this subtlety won't play a role.

Actually, there is an even better way of expressing this uniformity: there is a *universal* quotient by a G-action, one from which every quotient is deduced by pullback. This is the case $X = *$ of the terminal set, where the quotient $G\backslash\backslash *$ is the groupoid BG.

Lemma 4. *Let G be a group and X be a set with G-action. The commutative square*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ G\backslash\backslash X \arrow[r] & BG, \end{tikzcd}$$

deduced from the G-equivariant map $X \to *$, is a pullback square.

In fact, the story is even nicer: for a fixed groupoid Y, giving a map of groupoids $Y \to G\backslash\backslash *$ is the same, via pullback of $* \to G\backslash\backslash *$, as giving a groupoid Y' with G-action together with an identification $G\backslash\backslash Y' \simeq Y$. Specializing to the case where Y is a set, we can come full circle:

Lemma 5. *Let Y be a set.*

1. *The groupoid of maps $Y \to BG$ is equivalent to the groupoid of free G-sets Y' together with an isomorphism $Y \simeq G\backslash Y'$.*

2. *More generally, for a G-set X, the groupoid of maps $Y \to G\backslash\backslash X$ identifies with the groupoid of free G-sets Y' together with an isomorphism $Y \simeq G\backslash Y'$ and a G-map $Y' \to X$.*

This shows the sense in which the general quotient $G\backslash\backslash X$ essentially reduces to the case of free actions. Note that the groupoid $G\backslash\backslash X$ is determined by data of the groupoid of maps $Y \to G\backslash\backslash X$ from sets Y; in fact already just $Y = *$ suffices. In particular we actually get a new description of $G\backslash\backslash X$: it is the groupoid of pairs (S, f) where S is a G-torsor (nonempty transitive G-set) and $f : S \to X$ is a G-map. This is in fact a much better description than the original description as an action category. Why? Because now the morphisms in the category are "the obvious ones" given the nature of the objects. This is just like as for our favorite categories: sets, abelian groups, etc., etc., and it contrasts with the description of the action category, where our object set was X and this tells us nothing about what the morphisms should be.

2 Back to schemes

Now we should feel like we understand how to take the quotient by a group action in the groupoid sense, in the world of sets. What about the world of S-schemes? Suppose given an S-group scheme G with an action on an S-scheme X. One can proceed rather formally and define a presheaf of groupoids on S-schemes by

$$(G\backslash\backslash X)(T) = G(T)\backslash\backslash X(T).$$

This naive definition of $G\backslash\backslash X$ as a presheaf would actually be sufficient for our purposes, but it does have a fairly substantial drawback, at least from an aesthetic perspective. Namely, if G acts freely on X with quotient $X' = G\backslash X$, then while there is a natural map of presheaves on S-schemes

$$G\backslash\backslash X \to X',$$

this map is not an isomorphism in general. Indeed, if it were an isomorphism, then we would learn that the quotient map $X \to X'$ has a section. But then acting by G on this section would given an isomorphism of schemes with G-action $G \times X' \simeq X$ where G acts by translations purely on the first factor in the source. However, not all free actions are of this trivial form, for example point 3 of Example 2 is a nontrivial free action when E has rank bigger than one, and the quotient is indeed a scheme: the projective bundle.

Thus we have the dissatisfying situation where there are two different candidates for the "good quotient". But it's easy enough to modify the definition to make them collapse, at least in most practical situations. We just have to sheafify with respect to the appropriate topology. We will assume G is a flat quasicompact S-group scheme, which is therefore automatically faithfully flat because of the identity section, and use the fpqc (=faithfully flat quasicompact, in French) topology. I will refer to the Stacks Project https://stacks.math.columbia.edu/tag/03NV for the definition, basic properties, and basic examples of fpqc covers and the fpqc topology on S-schemes. Probably the most important result on the fpqc topology is the following, "descent for derived quasi-coherent sheaves".

Theorem 6. *The presheaf of ∞-categories $X \mapsto D(X)$ on schemes satisfies descent for the fpqc topology.*

Remark 7. *The basic idea of the proof is that equivalences in $D(-)$ can be detected on pullback to any fpqc cover. This is not difficult to show by looking on homology groups to reduce to the situation in ordinary algebra. Why does that help? Given a cover, we can split it by pullback along an fpqc cover (itself, essentially), and thus attempt to reduce the descent question to the case where the cover is split. But when the cover is split the covering sieve is trivial and descent is thereby automatic. For the actual proof, in fact of a generalization giving flat hyperdescent, you can see Lurie's DAG VII.*

This descent for $D(-)$ is very robust and propagates to all sorts of other descent results. For example, all of the following are also fpqc sheaves:

Example 8.

1. *The presheaf $X \mapsto QCoh(X)$, in other words the full subcategory of $D(X)$ consisting of those objects sitting in degree zero. (Use that flat base-changes preserve homology group sheaves.)*
2. *The presheaves $X \mapsto Perf(X)$ and $X \mapsto Vect(X)$. (Pass to dualizable objects in respectively $D(-)$ and $QCoh(-)$).*
3. *The presheaf $X \mapsto \{$relatively affine X-schemes $Y \to X\}$. (Pass to commutative algebra objects in $QCoh(-)$.)*
4. *The presheaf $X \mapsto \{$relatively affine X-schemes $Y \to X$ satisfying property $P\}$, where P is any property of maps of schemes which is* fpqc*-local.*[2]
5. *The previous four examples were all sheaves of categories or ∞-categories. Here is a sheaf of sets. For a scheme S and S-scheme X, the presheaf of sets $T \mapsto Hom_S(T, X)$ on S-schemes is an fpqc sheaf. (Reduce to the affine case, then look at hom sets in example 3.)*
6. *For any $M \in D(S)$, the presheaf $(f : T \to S) \mapsto f_*(f^*M)$ is an fpqc sheaf on S-schemes with values in $D(S)$. This follows by looking at mapping spaces in the above theorem giving descent for $D(-)$.*

[2] At this link https://stacks.math.columbia.edu/tag/02YJ you can find a long list of properties of morphisms of schemes which are fpqc-local. This includes some of our favorite examples: smooth, etale, and flat.

Results of this type are referred to as "faithfully flat descent".

Definition 9. *Let S be a scheme, $G \to S$ a quasicompact flat S-group scheme acting on an S-scheme $X \to S$. We define the* stack quotient $G\backslash\backslash X$ *to be the fpqc sheaf of groupoids on S-schemes given by sheafifying the presheaf*

$$T \mapsto G(T)\backslash\backslash X(T).$$

As in the case of sets, the study of these general quotients in essence reduces to the case where $X = *$, where $*$ means the terminal S-scheme S, namely there is a pullback diagram of fpqc sheaves of groupoids

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ G\backslash\backslash X \arrow[r] & BG. \end{tikzcd}$$

This follows from the discussion in the setting of groupoids, because sheafification commutes with finite limits, for example pullbacks. The most pertinent question, therefore, is how to more concretely describe the functor on S-schemes represented by BG? Here is an essentially tautological axiomatic approach.

Lemma 10. *Let S be a scheme and G a flat quasicompact S-group scheme. Suppose given an fqpc sheaf of groupoids $\mathcal{X}$ on S-shemes. Giving an isomorphism $\mathcal{X} \simeq BG$ is equivalent to giving:*

1. *a basepoint $x : * \to \mathcal{X}$;*

2. *a group sheaf isomorphism $G \simeq \pi_1(\mathcal{X}, x)$*

such that the sheafified $\pi_0\mathcal{X}$ is $*$, or equivalently every object is locally isomorphic to x.

In other words, BG classifies any kind of object where any two objects are locally isomorphic, and there is a standard global model for that object which has automorphism group G.

Example 11. *Take $G = GL_n$ and let $\mathcal{X}$ be defined by setting $\mathcal{X}(T)$ to be the groupoid of sheaves of $\mathcal{O}_T$-modules which are locally free of rank n (in the Zariski topology). Then this is not just a Zariski sheaf but an fpqc sheaf by faithfully flat descent. Take the basepoint to be $x(T) = \mathcal{O}_T^{\oplus n}$. This has automorphism group GL_n, and any two objects in $\mathcal{X}(T)$ are locally isomorphic to x by definition, hence we conclude that*

$$\mathcal{X} = BGL_n.$$

However, just as in the discussion of quotients of sets, there is a preferred description of BG. If we have a map $T \to BG$, we can pull back $* \to BG$ and get a *principal G-bundle* over T in the sense of the following definition.

Definition 12. *Let S be a scheme, $G \to S$ a flat quasicompact S-group scheme, and T an S-scheme. A principal G-bundle over T is a T-scheme $\widetilde{T} \to T$ with G-action (covering the trivial action on T) which is fpqc locally on T a* trivial *principal G-bundle.*

A trivial principal G-bundle over T is one of the form $G \times T \to T$, the projection map, with G acting by translation purely on the first factor.

This is the in-families version of a G-torsor. It is also the most basic situation in which you have a free G-action on a scheme where the quotient is also a scheme, and covers all reasonable examples.

Lemma 13. *1. Pulling back $* \to BG$ along a map $T \to BG$ gives an identification of $BG(T)$ with the groupoid of principal G-bundles over T.*

2. More generally, pulling back $X \to G\backslash\backslash X$ gives an identification of $(G\backslash\backslash X)(T)$ with the groupoid of pairs consisting of a principal G-bundle over T and a G-equivariant map from its total space to X.

By definition, a principal G-bundle is fpqc locally trivial. But in some examples a smaller topology suffices to trivialize a torsor. For example it follows from the alternate description of BGL_n given above that in that case, every prinicipal GL_n-torsor is Zariski-locally trivial. Here is another general class of examples. The method of argument is maybe more important than the result.

Proposition 14. *Suppose G is a smooth S-group scheme. Then every principal G-bundle is étale-locally trivial.*

Proof. Trivializing a principal G-bundle is the same thing as giving a section. Every map acquires a section after pullback along itself. Thus every principal G-bundle is trivialized on pullback along itself. But now, smoothness is preserved by pullback and is fpqc local. It follows that an principal G-bundle is smooth. Now it suffices to note that every smooth surjective map has étale local sections. This follows directly from the definition of smooth in terms of being étale over affine space, since affine space has a section. □

Okay, now we can pretend we understand these so-called *stack quotients* $G\backslash\backslash X$. We would now like to define equivariant de Rham cohomology as the de Rham cohomology of this quotient stack. We extend de Rham cohomology to stacks in a naive way.

Definition 15. *Suppose given a functor $\mathcal{F} : Sch_{/S}^{op} \to \mathcal{C}$ where $\mathcal{C}$ is a presentable ∞-category. We define*

$$\mathcal{F}(G\backslash\backslash X) = \varprojlim_{T \to G\backslash\backslash X} \mathcal{F}(T).$$

In particular, this defines the de Rham cohomology of $G\backslash\backslash X$, as a filtered object in $D(S)$:

$$dR_{G\backslash\backslash X/S} = \varprojlim_{T \to G\backslash\backslash X} dR_{T/S}.$$

This is a limit over a rather large diagram category. (In fact, it's so large that it's not even a set, so this definition isn't even well-defined in general. Note that we didn't bound the cardinality of the S-schemes occurring. This is just a technical nuisance however. It doesn't matter in practice and I won't get into it.) But if $\mathcal{F}$ has nice descent properties then we can use a smaller diagram instead.

Definition 16. *1. An* fpqc atlas *for $G\backslash\backslash X$ is a collection of maps $\{T_i \to G\backslash\backslash X\}_{i\in I}$ from schemes such that for any map $T \to G\backslash\backslash X$ from a scheme, each pullback $T_i \times_{G\backslash\backslash X} T$ is represented by a scheme, and together they form an $fpqc$ cover of T.*

2. A smooth atlas *for $G\backslash\backslash X$ is a collection of maps $\{T_i \to G\backslash\backslash X\}_{i\in I}$ from schemes such that for any map $T \to G\backslash\backslash X$ from a scheme, each pullback $T_i \times_{G\backslash\backslash X} T$ is represented by a scheme, and together they give an $fpqc$ cover of T where moreover all the constituent maps are smooth.*

Example 17. *1. Since G itself is fpqc over S by assumption, it follows that the single map $X \to G\backslash\backslash X$ constitutes an fpqc cover of $G\backslash\backslash X$.*

2. If G is smooth, then the single map $X \to G\backslash\backslash X$ is even a smooth atlas.

3. Somewhat surprisingly, even if G is not smooth there can be smooth atlases for $G\backslash\backslash X$. Actually a general result of Artin says that if there's a finitely presented flat atlas, then there's a smooth atlas. In particular if G is finitely presented and flat there's always a smooth atlas. For example, for any $n \geq 1$ the map $\mathbb{G}_m \to B\mu_n$ classifying the principal μ_n-bundle $\mathbb{G}_m \overset{z\mapsto z^n}{\to} \mathbb{G}_m$ is a smooth atlas for $B\mu_n$ over any base scheme, but μ_p is not smooth in characteristic p (its coordinate ring is not reduced).

Then we have the following, showing how to calculate $\mathcal{F}$ in terms of an atlas:

Lemma 18. *1. Suppose that $\mathcal{F} : Sch^{op}_{/S} \to \mathcal{C}$ is a sheaf for the fpqc topology. Then $\mathcal{F}(G\backslash\backslash X)$ identifies with the limit over the Cech nerve of any fpqc atlas.*

2. Suppose that $\mathcal{F} : Sch^{op}_{/S} \to \mathcal{C}$ is a sheaf for the étale topology. Then $\mathcal{F}(G\backslash\backslash X)$ identifies with the limit over the Cech nerve of any smooth atlas.

The proof is very formal and doesn't require much. The only thing to keep in mind for part 2 is that étale descent implies smooth descent, because smooth surjective maps have étale local sections. Now, here are two relevant examples:

Example 19. *The functor $\mathcal{F} = D(-)$ of derived quasi-coherent sheaf ∞-category satisfies fpqc descent. Thus*

$$D(G\backslash\backslash X) = \varprojlim_{[n]\in\Delta} D(G^n \times X),$$

intuitively the G-equivariant quasicoherent sheaves on X.

For the sheaf of ordinary categories $\mathcal{F} = QCoh(-)$, this reduces to: an object in $QCoh(G\backslash\backslash X)$ is an object $M \in D(X)$ together with an isomorphism $a^*M \simeq p^*M$ satisfying a cocycle condition, where $a, p : G \times X \to X$ are the action and projection maps.

Example 20. *The functor $\mathcal{F} = dR_{-/S}$ with values in filtered objets in $D(S)$ satisfies étale descent. This follows from derived fpqc descent for quasi-coherent sheaves because on the étale site of any scheme the sheaf of $\mathcal{O}$-modules Ω^p is quasi-coherent due to the base-change property for etale maps.*

Note that we do not *get fpqc descent for $dR_{-/S}$ in general, only étale descent, so for non-smooth G this is a more subtle form of equivariant de Rham cohomology than the naive one.*

The main point is then the following.

Theorem 21. *Let S be a scheme, $G \to S$ a flat finitely presented group scheme over S, and X a smooth S-scheme with G-action. Then*

$$dR_{G\backslash\backslash X/S} \simeq \varprojlim_{T\to G\backslash\backslash X, T \text{ smooth over } S} dR_{T/S}.$$

In other words, in the limit defining $dR_{G\backslash\backslash X/S}$ we can restrict to schemes smooth over S.

Proof. By Artin's theorem, there is a smooth atlas $Y \to G\backslash\backslash X$. We want to show that Y itself is smooth over S; then all the elements of the Cech nerve will be smooth, and this is cofinal in the limit in the statement of the theorem and the proof will be complete. But the map $X \to G\backslash\backslash X$ is finitely presented faithfully flat, thus the pullback to Y is faithfully flat over Y, but on the other hand it's smooth over X which is smooth. Thus Y admits a finitely presented faithfully flat map from a smooth S-scheme, and it follows, not obviously, that Y itself is smooth. $\square$

Now, this all sounds fine, especially for smooth G, which is our main case of interest. But there is a fly in the ointment. Namely, the associated gradeds of $dR_{G\backslash\backslash X/S}$ for the Hodge filtration are calculated as

$$\Omega^p_{/S}[-p](G\backslash\backslash X) = \varprojlim_{\text{smooth } T\to G\backslash\backslash X} p_*\Omega^p_{T/S}[-p]$$

where p_* denotes the pushforward from T to S. There is a big difference here from the scheme case in that Ω^p itself is not a quasi-coherent sheaf. Indeed, for a quasicoherent sheaf, once you know the sections on one affine the sections on another affine mapping to it are formally determined as the base-change. But this is not the case for Ω^p as we have smooth schemes of different dimension floating around.

In the next lecture we'll state and prove Totaro's theorem, which gives a fix for this: a complex of quasi-coherent sheaves on $G\backslash\backslash X$ which also computes $\Omega^p[-p](G\backslash\backslash X/S)$.

Exercise 22. *Suppose given a commutative ring R and an R-module M which is locally free of rank n. Explicitly write down the coordinate ring of the corresponding principal GL_n-bundle.*