

Lecture 13: Totaro's theorem on equivariant Hodge cohomology

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June 10, 2021

In this lecture we'll fix a base scheme S , a smooth S -group scheme G , and a smooth S -scheme X with an action of G . Recall from last time that we ended up with a definition/description of equivariant de Rham cohomology (implicitly, over the base S) of the following form:

$$dR_{G \backslash X} = \varprojlim_{T \rightarrow G \backslash X} dR_T.$$

Here the index category consists of all smooth S -schemes with a map to the quotient stack $G \backslash X$ (we also saw that we can equivalently restrict to the Čech nerve of a single smooth atlas, for example the one coming from $X \rightarrow G \backslash X$.) Each dR_T is naturally a filtered object in the derived quasicoherent ∞ -category $D(S)$ via the Hodge filtration, and thus so is $dR_{G \backslash X}$. For $p \geq 0$, the p^{th} associated graded for this filtration is

$$gr^p dR_{G \backslash X} = \varprojlim_{T \rightarrow G \backslash X} f_* |\Omega_T^p[-p]|$$

where $f : T \rightarrow S$ is the structure map.

In the scheme case, this associated graded was Hodge cohomology, the derived pushforward to S of the $-p$ shift of a quasi-coherent sheaf Ω^p . The above expression is not of the same form, because Ω^p is not quasi-coherent on the site of all smooth schemes. Nonetheless, it's easy and formal to find a derived quasi-coherent sheaf

$$“\Omega_{G \backslash X}^p” \in D(G \backslash X)$$

with the property that $gr^p dR_{G \backslash X}$ is the pushforward of “ $\Omega_{G \backslash X}^p$ ” $[-p]$ to S , namely we just set

$$“\Omega_{G \backslash X}^p” := \varprojlim_{g: T \rightarrow G \backslash X} g_* |\Omega_T^p|$$

which tautologically fills the desired role.

This object is intrinsic to the stack $G \backslash X$, but it is not obvious how to get an explicit tractable expression for it, or even decide in which degrees it lives (rarely will it live in degree zero). Our goal today is to state and prove Totaro's theorem, which gives, in terms of G acting on X , an explicit and functorial complex of quasi-coherent sheaves on $G \backslash X$ which realizes to “ $\Omega_{G \backslash X}^p$ ” $\in D(G \backslash X)$.

The key to the construction of this complex is to look at the situation of a free action, or more specifically a principal G -bundle $\tilde{T} \rightarrow T$ over a smooth S -scheme T . In this case we're looking for a complex of quasi-coherent sheaves on T which is a resolution of the quasi-coherent sheaf $\Omega_{T/S}^p$, and the key property we want this complex to have is that, in its incarnation as a complex of G -equivariant quasi-coherent sheaves on $\tilde{T}$, it should make sense for an arbitrary smooth scheme with G -action replacing $\tilde{T}$.

Of course, we should start with $p = 1$, i.e. the Kähler differentials or cotangent sheaf. For intuition, we can think in terms of the dual question and ask how to understand the tangent sheaf of the base of a principal G -bundle in terms of the tangent sheaf of the total space. Differentiating the group action at the identity $e \in G$ gives a natural map

$$\mathfrak{g} \rightarrow \mathcal{T}_{\tilde{T}}.$$

This is G -equivariant for the adjoint action, it is injective, and the quotient identifies with $\mathcal{T}_T$. This should be geometrically reasonable; we're collapsing along the orbits of the G -action. This gives a two-term complex resolving $\mathcal{T}_T$, and in the dual picture with differentials, it suggests the following.

Proposition 1. *Let G be a smooth group scheme acting on the smooth scheme X , all over the implicit base S . Then:*

1. *The sheaf of Kähler differentials $\Omega_X^1 \in QCoh(X)$ naturally descends to an object of $QCoh(G \backslash X)$.*
2. *There is a natural map $d : \Omega_X^1 \rightarrow \mathfrak{g}^*$ in $QCoh(G \backslash X)$, where $\mathfrak{g}^*$ is defined as follows: let $\mathfrak{g}^* = e^* \Omega_G^1$, with e the identity section. A priori this lives in $QCoh(S)$, but via the adjoint action of G on itself we can descend it to $QCoh(BG)$ and then pull it back to $G \backslash X$.*
3. *If the G -action on X makes X into a principal G -bundle over a smooth scheme $X' = G \backslash X$, then the map d extends to a natural short exact sequence*

$$0 \rightarrow \Omega_{X'}^1 \rightarrow \Omega_X^1 \xrightarrow{d} \mathfrak{g}^* \rightarrow 0$$

in $QCoh(G \backslash X)$.

Proof. Even the first claim is not so obvious, because at least naively we can't simply argue that $\Omega_{X/S}^1$ is G -equivariant "by functoriality" as we could if the group G were discrete. Although it won't be part of the formal proof, let's first give a "sanity check" abstract argument for why such equivariant structure should exist. Then we'll give the "official" argument by directly producing the descent data encoding equivariance.

Remember the definition of $\Omega_{X/S}^1$. We consider the diagonal immersion

$$\Delta : X \rightarrow X \times_S X,$$

take its second infinitesimal neighborhood $X^{(2)}$ viewed as an augmented relatively affine X -scheme (via the first projection, augmentation given by the diagonal), then take the complementary summand of the augmentation of the associated quasicoherent $\mathcal{O}_X$ -algebra. To promote $\Omega_{X/S}^1$ to a G -equivariant quasicoherent sheaf, by this description it's enough to promote the splitting diagram

$$X \rightarrow X^{(2)} \rightarrow X$$

to a diagram of objects with G -action. By functoriality of the construction $G(S)$ certainly acts, and in particular it acts on the S -valued points. But as the construction is compatible with pullbacks along maps $T \rightarrow S$ we get compatible actions of $G(T)$ on the T -valued points for all T , whence the required action.

Okay, now for the official description. It's based on the fact that Kähler differentials satisfy a Künneth formula. Namely, if X and Y are S -schemes and $p : X \times Y \rightarrow X$ and $q : X \times Y \rightarrow Y$ are the

two projections (all products are implicitly taken in the category of S -schemes, so they're actually fiber products over S), then

$$p^*\Omega_X^1 \oplus q^*\Omega_Y^1 \xrightarrow{\sim} \Omega_{X \times Y}^1.$$

To produce the equivariance we need an isomorphism $a^*\Omega_X^1 \simeq q^*\Omega_X^1$ satisfying the cocycle identity, where $a, q : G \times X \rightarrow X$ are the action and second projection maps. However, note that $G \times X$, besides being expressed as the product of G and X in the tautological manner via the first and second projections, can also be expressed as the product of G and X via the first projection and the action map. Thus, by Künneth, $a^*\Omega_X^1$ and $q^*\Omega_X^1$ are both complementary summands to $p^*\Omega_G^1$ and are therefore canonically identified. One checks the cocycle condition using triple Künneth on $G \times G \times X$.

To define the map $d : \Omega_X^1 \rightarrow \mathfrak{g}^*$ of underlying quasicoherent sheaves on X , we consider the composition

$$a^*\Omega_X^1 \rightarrow \Omega_{G \times X}^1 \rightarrow p^*\Omega_G^1,$$

where the first map comes from functoriality of Kähler differentials, the second map is projection to the appropriate Künneth component. Then we pull back the composite map along $(e, id) : X \rightarrow G \times X$ to get our desired d . One checks routinely that this commutes with the equivariance structures.

Finally, to check the third claim, we define the map $\Omega_{X'}^1 \rightarrow \Omega_X^1$ using functoriality of Kähler differentials. To check we get the required short exact sequence, since all the constructions are local on X' we can reduce to the case where $X = G \times X'$ with G -action on the first factor, and then the claim results from the Künneth formula, which actually gives a natural splitting for our desired short exact sequence. (Here one should identify Ω_G^1 with the constant bundle $\mathfrak{g}^*$ using left translation.) $\square$

Thus we get our desired complex when $p = 1$: the two-term complex $d : \Omega_X^1 \rightarrow \mathfrak{g}^*$ on $G \backslash X$ always exists, and when X is the total space of a principle G -bundle it is quasi-isomorphic to $\Omega_{G \backslash X}^1$. To pass from Ω^1 to Ω^p , we want to know how, given a short exact sequence of finitely generated projective R -modules

$$0 \rightarrow K \rightarrow M \rightarrow N \rightarrow 0$$

over a commutative ring R , we can write $\Lambda^\bullet K$ in terms of $M \rightarrow N$, in such a way that the description also makes sense without assuming the map is surjective, and moreover commutes with arbitrary basechange along ring maps $R \rightarrow R'$. The solution for this is to use the following general algebraic machinery of “Koszul complexes” as explained in Illusie’s “Complexe cotangent et déformations”.

Proposition 2. *Suppose R is a commutative ring and $d : M \rightarrow N$ is a map of R -modules.*

1. *There is a unique structure of commutative differential graded R -algebra on the graded commutative R -algebra*

$$\Lambda^\bullet M \otimes \text{Sym}^\bullet N,$$

M in degree -1 and N in degree -2 , such that the differential agrees with our given d on elements of M , and kills elements of N . Denote this CDGA by $\text{Kos}(d)$.

2. *If $d' : M' \rightarrow N'$ is another map of R -modules, then $\text{Kos}(d \oplus d') \simeq \text{Kos}(d) \otimes \text{Kos}(d')$.*
3. *If N is flat and d is surjective, then the natural map of CDGA's*

$$\Lambda^\bullet K \rightarrow \Lambda^\bullet M \otimes \text{Sym}^\bullet N$$

is a quasi-isomorphism, where $K = \ker(d)$ and the differential is trivial on the source.

Proof. For part 1, the uniqueness is clear as the CDGA is generated by N and M . Well-definedness is a routine check similar to defining the differentials in the de Rham complex. For 2, on underlying graded commutative algebras this follows from the universal property of alternating and symmetric power algebras, and then the differentials must agree by uniqueness. For 3, by Lazard we can write N as a filtered colimit of finite free modules; pulling back to these and using the fact that all operations commute with filtered colimits, we reduce to $N = R^{\oplus n}$. Then the short exact sequence is split, so we can write it as a direct sum of $K = K \rightarrow 0$ and $0 \rightarrow N = N$. By 2 we are thus reduced to these separate cases. In the first case the result is trivial. In the second case, by 2 again we can reduce to $N = R$ and then the result is visibly correct. $\square$

Remark 3. *There is an extra graded commutative algebra structure on $Kos(d)$, where M and N both have degree one. The differential preserves this extra grading, so $Kos(d)$ breaks up as a direct sum according to these extra degrees. In the situation of 3, this extra grading matches the grading on the exterior algebra where K has degree one. Thus we get a resolution of $\Lambda^p K$ of the form*

$$\Lambda^p M \rightarrow \Lambda^{p-1} M \otimes N \rightarrow \dots \rightarrow Sym^p N$$

in degrees 0 to $-p$.

Corollary 4. *Let X be a smooth scheme with an action of the smooth group scheme G . There is a functorially attached extra-graded commutative differential graded quasicoherent sheaf on $G \backslash X$*

$$Kos(d : \Omega_X^1 \rightarrow \mathfrak{g}^*)$$

such that if X is the total space of a principal G -bundle over $X' = G \backslash X$, there is a natural graded quasi-isomorphism

$$\oplus_{p \geq 0} \Omega_{X'}^p[-p] \xrightarrow{\sim} Kos(d : \Omega_X^1 \rightarrow \mathfrak{g}^*)$$

Proof. The Koszul complex construction commutes with base-changes along maps $R \rightarrow R'$, so it automatically passes to quasi-coherent sheaves. Thus this follows by combining the previous two propositions. $\square$

In extra grading degree p , this gives a resolution of $\Omega_{X'}^p[-p]$ which looks like this: it is a complex of the form

$$\Omega_X^p \rightarrow \Omega_X^{p-1} \otimes \mathfrak{g}^* \rightarrow \dots \rightarrow Sym^p \mathfrak{g}^*$$

in degrees $-p$ to $-2p$.

Now we can state and prove Totaro's theorem.

Theorem 5. *Let X be a smooth scheme with an action of the smooth group scheme G . Write*

$$|Kos(d : \Omega_X^1 \rightarrow \mathfrak{g}^*)|$$

for the graded commutative algebra object of $D(G \backslash X)$ given by realizing the indicated graded CDGA. Then there is a natural identification

$$\oplus_{p \geq 0} \Omega_{G \backslash X}^p[-p] \simeq |Kos(d : \Omega_X^1 \rightarrow \mathfrak{g}^*)|,$$

and in particular

$$f_* |Kos(d : \Omega_X^1 \rightarrow \mathfrak{g}^*)| \simeq \oplus_{p \geq 0} gr^p dR_{G \backslash X},$$

where $f : G \backslash X \rightarrow S$. Thus we have an explicit quasi-coherent model for Hodge cohomology of quotient stacks.

Proof. For every map $T \rightarrow G \backslash X$ from a smooth scheme T , we have the natural and functorial isomorphism $|Kos(d : \Omega_{\tilde{T}}^1 \rightarrow \mathfrak{g}^*)| \simeq \oplus_{p \geq 0} \Omega_T^p[-p]$ in $D(T)$ by Corollary 4, where $\tilde{T} \rightarrow T$ is the principal G -bundle obtained by pulling back $X \rightarrow G \backslash X$. By definition “ $\Omega_{G \backslash X}^p[-p]$ ” is the inverse limit of the values of $g_* \Omega_T^p[-p]$ as we run over all maps $g : T \rightarrow G \backslash X$ from a smooth scheme T . By the previous lecture, we also get the same result by taking the limit just with g running over the Čech nerve of some fixed smooth cover, and we will make this simplification. We can even choose the canonical one coming from $X \rightarrow G \backslash X$ to ensure functoriality in the initial data.

In any case, it therefore suffices to show the analogous limit claim for $|Kos(d : \Omega_X^1 \rightarrow \mathfrak{g}^*)|$, namely that

$$|Kos(d : \Omega_X^1 \rightarrow \mathfrak{g}^*)| \xrightarrow{\sim} \varprojlim_{g:T \rightarrow G \backslash X} g_* |Kos(d : \Omega_T^1 \rightarrow \mathfrak{g}^*)|,$$

where again we can restrict to the Čech nerve of the canonical atlas if desired.

Because $\pi : X \rightarrow G \backslash X$ is a flat cover, we can test such equivalences in $D(G \backslash X)$ on pullback to X . By cohomology and base change,¹ π commutes with the pushforwards on the right. Additionally, π being flat, π^* also commutes with the inverse limit on the right, because all the objects in question are homologically bounded above, and in any given range of degrees a limit over Δ behaves like a finite limit. In total this means we simply need to show the above claim on underlying objects of $D(X)$ without the equivariance, replacing g_* by g'_* where $g' : \tilde{T} \rightarrow X$ is the pullback of g .

Now, filtering by bidegree we reduce to the analogous claim for each $\Omega_X^p \otimes Sym^q \mathfrak{g}^*$. By the projection formula we can pull out the second factor, so we reduce to checking

$$|\Omega_X^p| \xrightarrow{\sim} \varprojlim_{g:T \rightarrow G \backslash X} g'_* |\Omega_{\tilde{T}}^p|.$$

But now as g runs over the Čech nerve of a smooth atlas, its pullback g' runs over the Čech nerve of a smooth cover. So the claim follows from smooth descent for Ω^p , which, as we explained in the previous lecture, follows from étale descent which follows from quasi-coherence on the étale site. $\square$

In the most basic special case of the trivial G -action on S , all terms in the Koszul complex except the last one drop out as $\Omega_S^1 = 0$. Thus we get:

Corollary 6. *Let G be a smooth group scheme. Then there is an isomorphism of graded algebra objects of $D(S)$, graded by $p \geq 0$:*

$$\oplus_p gr^p dR_{BG} \simeq \oplus_p f_* |Sym^p \mathfrak{g}^*|[-2p],$$

where $f : BG \rightarrow S$ is the structure map.

Thus understanding Hodge cohomology of classifying stacks reduces to understanding $f_* Sym^p \mathfrak{g}^*$. In general, f_* is the derived analog of G -fixed points, and it can introduce higher cohomology. But one of Totaro's remarks is that for GL_n over a general base this doesn't happen for these specific coefficient systems $Sym^p \mathfrak{g}^*$. We'll prove this later, but for now let's just look at what it gives us. It means that

$$\oplus_p gr^p dR_{BGL_n} \simeq (\oplus_p Sym^p \mathfrak{g}^\vee[-2p])^G,$$

where $\mathfrak{g} = Lie(GL_n)$. Let's take $S = Spec(R)$ affine. This Lie algebra is $M_{n \times n}$, the R -module of $n \times n$ -matrices with R -coefficients, and the adjoint action of G is by conjugation. Thus, $\mathfrak{g}^\vee$ is the

¹which holds for $g : T \rightarrow G \backslash X$ as a formal consequence of the fact that it holds after pullback along any map $T' \rightarrow G \backslash X$, which in turn is the usual scheme case.

linear functions on $M_{n \times n}$, and accordingly $\oplus_p \text{Sym}^p \mathfrak{g}^\vee$ is the polynomial functions on $M_{n \times n}$, concretely just the polynomials in n^2 variables interpreted as the coefficients of a generic matrix. Thus the ring of invariants $(\oplus_p \text{Sym}^p \mathfrak{g}^\vee)^G$, which gives the Hodge cohomology of BGL_n , amounts to the ring of invariant polynomials, i.e. those polynomials in the entries of a matrix which are independent of the basis, i.e. which make sense purely for an endomorphism of a free module of rank n without choosing a basis.

The result ends up being that every such polynomial is uniquely a polynomial in the coefficients of the characteristic polynomial of the endomorphism, which correspond to the elements

$$c_i \in (\oplus_p \text{Sym}^p \mathfrak{g}^\vee)^G$$

for $1 \leq i \leq n$ defined by

$$c_i(f) = \text{tr}(\Lambda^i f)$$

for f an endomorphism of a free module of rank n . For example c_n gives the determinant and c_1 gives the trace. So for the Hodge cohomology we'll have

$$\oplus_p \text{gr}^p dR_{BGL_n} \simeq R[c_1, \dots, c_n]$$

with c_i in homological degree $-2i$ and graded degree i .

Remark 7. Note that in any case there is a natural map $(\text{Sym}^p \mathfrak{g}^*)^G \rightarrow f_*(\text{Sym}^p \mathfrak{g}^*)$ by canonical truncation, since the former is the top degree, degree zero part of the latter. So we have the classes $c_1, \dots, c_n$ in the Hodge cohomology of BGL_n even if we haven't given the argument that they freely generate. By pullback along $X \rightarrow BGL_n$, these give Chern classes of vector bundles with values in Hodge cohomology, namely given a locally free sheaf of finite rank $\mathcal{E}$ on a smooth scheme X over a commutative ring R we get natural classes

$$c_i(\mathcal{E}) \in H^i(X; \Omega_{X/R}^i).$$

In the next lecture we will lift these to de Rham cohomology:

$$c_i(\mathcal{E}) \in F^{\geq i} H^{2i} dR_{X/R}.$$

To finish this lecture we'll treat the case $G = GL_1 = \mathbb{G}_m$. We'll show that there's no higher cohomology by establishing an analog of the fact that the representation theory of $\mathbb{G}_m$ is semi-simple, and the irreducible representations are just the integer powers of the tautological one-dimensional representation.

Lemma 8. For every $M \in D(B\mathbb{G}_m)$, the unit map $M \rightarrow e_* e^* M$ admits a functorial splitting, where $e: S \rightarrow B\mathbb{G}_m$ is the quotient map.

Proof. Let $\mathcal{O}(1) \in Q\text{Coh}(B\mathbb{G}_m)$ denote the universal line bundle, and for $n \in \mathbb{Z}$ set $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n}$, or the pullback of $\mathcal{O}(1)$ along the map $B\mathbb{G}_m \rightarrow B\mathbb{G}_m$ induced by functoriality from the multiplication by n map in the abelian group sheaf $\mathbb{G}_m$.

Then $e: S \rightarrow B\mathbb{G}_m$ is relatively affine with coordinate algebra $\oplus_{n \in \mathbb{Z}} \mathcal{O}(n)$, i.e. on pullback to any affine scheme it is affine with coordinate ring the pullback of $\oplus_{n \in \mathbb{Z}} \mathcal{O}(n)$. This follows by unwinding the correspondence between line bundles and $\mathbb{G}_m$ -torsors, a special case of the exercise in the previous lecture.

From the projection formula it follows $e_* e^* M \simeq M \otimes e_* e^* \mathcal{O} = M \otimes (\oplus_{n \in \mathbb{Z}} \mathcal{O}(n))$ where the unit map we want to split comes from the unit map $\mathcal{O} \rightarrow \oplus_{n \in \mathbb{Z}} \mathcal{O}(n)$ of the algebra. So it suffices to project to the degree zero part. $\square$

Lemma 9. *Let $n \in \mathbb{Z}$. Then for $f : B\mathbb{G}_m \rightarrow S$, we have $f_*\mathcal{O}(n) = \mathcal{O}$ for $n = 0$ and 0 for $n \neq 0$.*

Proof. We have maps

$$\mathcal{O} \rightarrow f_*\mathcal{O} \rightarrow f_*e_*\mathcal{O} = \mathcal{O}$$

giving $\mathcal{O}$ as a summand of $f_*\mathcal{O}$. On the other hand $\mathcal{O} \rightarrow e_*\mathcal{O} = \bigoplus_n \mathcal{O}(n)$ is also the inclusion of a summand, hence the same is true on f_* . It follows that the summand $\mathcal{O} \rightarrow f_*\mathcal{O}$ must be the whole thing, and simultaneously we see that the complementary summand $f_* \bigoplus_{n \neq 0} \mathcal{O}(n)$ is zero, whence the claims. $\square$

Theorem 10. *For every $M \in D(B\mathbb{G}_m)$ the natural map*

$$\bigoplus_{n \in \mathbb{Z}} M_n \rightarrow M$$

is an iso, where $M_n = f^(f_*(M \otimes \mathcal{O}(-n))) \otimes \mathcal{O}(n)$.*

In fact, this description furnishes an equivalence $D(B\mathbb{G}_m) \simeq \text{Fun}(\mathbb{Z}^\delta, D(S))$ between representations of $\mathbb{G}_m$ (in this sense) and $\mathbb{Z}$ -graded objects in $D(S)$. Under this equivalence, f_ corresponds to picking out the degree zero component.*

Proof. By the lemma, M is the summand of an object of the form $e_*N = \bigoplus_n N \otimes \mathcal{O}(n)$, so we reduce to that case. Then the first claim reduces, by a simple calculation, to the second lemma above.

The second claim is a simple consequence of the first claim. $\square$

Corollary 11. *For $N \in D(S)$, the natural map $N \rightarrow f_*f^*N$ is an iso.*

In other words, if the action of $\mathbb{G}_m$ is trivial, then the cohomology f_* just gives the underlying object e^* . Thus for the Hodge cohomology of $B\mathbb{G}_m$ we deduce:

Theorem 12. *Let $S = \text{Spec}(R)$. Then*

$$\bigoplus_{p \geq 0} gr^p dR_{B\mathbb{G}_m} = \bigoplus_{p \geq 0} R \cdot c_1^p$$

is a polynomial algebra on one generator c_1 of homological degree -2 and graded degree 1 .

Proof. By Totaro's theorem and the above corollary,

$$\bigoplus_{p \geq 0} gr^p dR_{B\mathbb{G}_m} = \bigoplus_{p \geq 0} \text{Sym}^p \mathfrak{g}^*[-2p].$$

It suffices to note that $\mathfrak{g}^*$ is free of rank one with canonical generator coming from $\mathbb{G}_m \subset \mathbb{A}^1$ and pulling back dT to $1 \in \mathbb{G}_m$. $\square$

Exercise 13. *Let $n \geq 1$. Apply Totaro's theorem to the presentation of $B\mu_n$ as $\mathbb{G}_m \backslash \mathbb{G}_m$ for the action of $\mathbb{G}_m$ on itself given by $(\lambda, x) \mapsto \lambda^n \cdot x$ to calculate the Hodge cohomology of $B\mu_n$ over an arbitrary commutative ring R .*