Lecture 14: Chern classes and Thom classes

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Let's pick up where we left off last time. Let S be a scheme. We've been considering the associated graded for the Hodge filtration on equivariant de Rham cohomology over S, which is accessible by Totaro's theorem:

$$\oplus_{p\geq 0} gr^p dR_{G\backslash\backslash X} \simeq f_*|Kos(\Omega^1_X \to \mathfrak{g}^*)|.$$

Here G is a smooth group scheme acting on a smooth scheme X and $f : G\backslash\backslash X \to S$ is the structure map. When $G = \mathbb{G}_m$ and $X = S$, only the pure symmetric power terms in the Koszul CDGA contribute, and we saw that the pushforward f_* just amounts to taking the underlying non-equivariant object, giving the result

$$\oplus_{p\geq 0} gr^p dR_{B\mathbb{G}_m} = \oplus_{p\geq 0} c_1^p \cdot \mathcal{O}_S[-2p]$$

where $c_1 \in H^2 gr^1 dR_{B\mathbb{G}_m}$ is the class corresponding to the canonical generator of $Sym^1\mathfrak{g}^* = \mathfrak{g}^*$, namely the pullback of $dT \in \Omega^1_{\mathbb{A}^1}$ along the identity section element of $\mathbb{G}_m$.

We would like to understand this class c_1 more explicitly. Abstractly, the data of c_1 is the same as the data of a map of fpqc sheaves of groupoids on smooth S-schemes

$$B\mathbb{G}_m \to Map_{D(-)}(\mathcal{O}, \Omega^1[1]).$$

In fact, and this should really be specified as part of the data of c_1, on the cocycle level this class c_1 vanishes on pullback along $S \to B\mathbb{G}_m$ (i.e. the corresponding characteristic class canonically vanishes on trivial principal $\mathbb{G}_m$-bundles), so it is really a canonical class in relative Hodge cohomology. For the above map of sheaves of groupoids, this means that it naturally promotes to a *pointed* map, where the target is pointed by 0. Giving a pointed map from a groupoid of the form BG to a pointed groupoid (X, x) is the same as giving a homomorphism from G to the automorphism group of x in X, i.e. $\pi_0\Omega X$. Thus the data of our c_1 is the same as the data of a map of presheaves of groups

$$\mathbb{G}_m \to \Omega^1$$

on smooth S-schemes.

Lemma 1. *The homomorphism $\mathbb{G}_m \to \Omega^1$ corresponding to c_1 is given (up to a sign which is probably a matter of convention) by the "logarithmic derivative" homomorphism*

$$\lambda \mapsto \frac{d\lambda}{\lambda},$$

so-called because formally speaking $\frac{d\lambda}{\lambda} = dlog(\lambda)$.

Proof. We have to unwind the construction of c_1. Although we could do everything in the universal case, it's probably easier to think in terms of c_1 being a characteristic class of principal $\mathbb{G}_m$-bundles $\widetilde{T} \to T$. Recall, in that guise, that it came from the short exact sequence of quasi-coherent sheaves on T

$$0 \to \Omega^1_T \to \Omega^1_{\widetilde{T}} \to \mathfrak{g}^* \to 0,$$

namely we take the image of the canonical generator of $\mathfrak{g}^*$ under the associated boundary map for this short exact sequence. To unwind the corresponding homomorphism $\mathbb{G}_m \to \Omega^1$, we're supposed to recall that this short exact sequence gets a canonical splitting when the bundle is trivialized, thus giving a lift of our canonical generator of $\mathfrak{g}^*$ to the middle term $\Omega^1_{\widetilde{T}}$. Given an element of $\mathbb{G}_m$ we can change the trivialization, then take the difference of the two lifts to get an element of Ω^1_T.

The canonical splitting came from the Künneth formula. To give the corresponding lift of $\omega_e \in \mathfrak{g}^*$, we extend ω_e to an invariant one-form $\omega \in \Omega^1_G$ then pull back to $\widetilde{T}$ via the projection. In the case $G = \mathbb{G}_m = Spec(R[t, t^{-1}])$ the invariant one form corresponding to the canonical generator is $\frac{dt}{t}$. When we change the trivialization by a $\lambda \in \mathbb{G}_m$ this replaces t by λt, and the difference is indeed

$$\frac{d(\lambda t)}{\lambda t} - \frac{dt}{t} = \frac{d\lambda}{\lambda}$$

as claimed. □

Corollary 2. *The class $c_1 \in H^2 gr^1 dR_{B\mathbb{G}_m}$ canonically lifts to $c_1 \in H^2 F^{\geq 1} dR_{B\mathbb{G}_m}$.*

Proof. Since $d(\frac{d\lambda}{\lambda}) = 0$, we have a map of complexes of sheaves of abelian groups

$$\mathbb{G}_m \to \left(\Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \dots\right) = F^{\geq 1}\Omega^\bullet[1]$$

extending $dlog : \mathbb{G}_m \to \Omega^1$, whence the desired c_1 by shifting up by one and passing to hypercohomology. □

Corollary 3. *We have*

$$dR_{B\mathbb{G}_m} = \oplus_{n \geq 0} c_1^n \cdot \mathcal{O}_S,$$

where c_1 has homological degree -2. Moreover the Hodge filtration is the same as the c_1-adic filtration, i.e.

$$F^{\geq p} dR_{B\mathbb{G}_m} = \oplus_{n \geq p} c_1^n \cdot \mathcal{O}_S.$$

Now, let us move beyond the trivial $\mathbb{G}_m$ action on the terminal S-scheme S and go to some more general $\mathbb{G}_m$-actions. In the short term we'd like to present the calculation of the de Rham cohomology of projective bundles

$$\mathbb{P}(\mathcal{E}) = \mathbb{G}_m \backslash\backslash (\mathbb{A}(\mathcal{E}) \smallsetminus 0_X)$$

associated to locally free sheaves $\mathcal{E}$ of finite rank on smooth S-schemes X. Here $\mathbb{A}(\mathcal{E}) = Spec_X Sym\mathcal{E}$, so that points of $\mathbb{A}(\mathcal{E})$ amount to maps $\mathcal{E} \to \mathcal{O}$, i.e. global sections of $\mathcal{E}^*$, and points of $\mathbb{P}(\mathcal{E})$ amount to iso classes of locally free of rank one *quotients* $\mathcal{E} \to \mathcal{L}$ of $\mathcal{E}$. (Caution: many sources take the dual convention.)

But for this any several other purposes it's actually convenient to start with the variant

$$\mathbb{G}_m \backslash\backslash \mathbb{A}(\mathcal{E})$$

without removing the zero section. Note that when we remove the zero section the action is free, and indeed $\mathbb{A}(\mathcal{E}) \smallsetminus 0_X$ is the total space of a principal $\mathbb{G}_m$-bundle over the scheme $\mathbb{P}(\mathcal{E})$. But the action on 0 is trivial and this means $\mathbb{G}_m \backslash\backslash \mathbb{A}(\mathcal{E})$ is really just a stack, not a scheme.

But even more, we will also want to generalize our base S-scheme X to a base S-stack; in fact this extra generality will be absolutely crucial for us as we'll see when discussing weighted homotopy invariance. We've only talked about the so-called *smooth quotient stacks*, namely $G\backslash\backslash X$ for smooth group schemes G acting on smooth S-schemes X, and whenever we say smooth stack we will implicitly mean stack of that form. Of course there is a more general notion, but none of our constructions will leave the world of quotient stacks, and the universal case for all problems we consider will always be a smooth quotient stack anyway, so let's not get in to that extra generality.

As discussed in Lecture 12, concepts such as vector bundle, projective bundle, de Rham cohomology, and so forth are defined for stacks in the naive way: such an object over a stack X is the same as a compatible family of such objects on all schemes mapping to X. All concepts we use will have smooth descent, and then it's equivalent to just give the data of such a compatible family over the Cech nerve of some smooth cover.

The following "weighted homotopy invariance" is an important remark of Totaro's. It shows that even in characteristic p, the problem of the affine line having huge de Rham cohomology disappears provided one takes into account equivariance for scalar multiplication.

Lemma 4. *Let X be an S-stack and $\mathcal{E}$ a locally free sheaf of finite rank r on X, and for $k \in \mathbb{Z}$ let $\mathbb{G}_m$ act on $\mathbb{A}(\mathcal{E})$ via $(\lambda, x) \mapsto \lambda^k x$ and x a section of $\mathcal{E}$. Then if $k \neq 0$, the map*

$$\mathbb{G}_m \backslash\backslash \mathbb{A}(\mathcal{E}) \to B\mathbb{G}_m \times X$$

is an isomorphism on de Hodge cohomology, hence on filtered de Rham cohomology.

Proof. By descent, we can reduce to the case where X is a scheme and $\mathcal{E} \simeq \mathcal{O}^r$. Then $\mathbb{A}(\mathcal{E}) = \mathbb{A}^r \times X$, and by Künneth we can reduce to the case $X = S$. By Totaro's theorem, we have

$$\oplus_{p\geq 0} gr^p dR_{\mathbb{G}_m\backslash\backslash\mathbb{A}^r} \simeq f_* Kos(\Omega^1_{\mathbb{A}^r} \to \mathfrak{g}^*)$$

with $f : \mathbb{G}_m\backslash\backslash\mathbb{A}^r \to S$. We can factor this as the composition

$$\mathbb{G}_m\backslash\backslash\mathbb{A}^r \to B\mathbb{G}_m \to S,$$

where the first map is induced by the equivariant projection $\mathbb{A}^r \to S$. Pushforward along the first map is just remembering equivariance on the pushforward along $\mathbb{A}^r \to S$, which is just global sections, with no higher cohomology as $\mathbb{A}^r$ is affine. On the other hand, pushforward along the second map just picks out the weight zero part in terms of the equivalence between quasicoherent sheaves on $B\mathbb{G}_m$ and graded quasi-coherent sheaves on S. So we just need to look at the Koszul complex and pick out the weight zero part, i.e. the part on which the $\mathbb{G}_m$ action is trivial. On the $Sym^q \mathfrak{g}^*$ terms the action is trivial, so those have weight zero. However, for

$$\omega \in \Omega^p_{\mathbb{A}^r}$$

we have that ω is a linear combination of wedges of terms dt_i where t_i are the coordinates and the coefficients in this linear combination are polynomials in the t_i. By definition the t_i have weight k, so the weight of each dt_i is also k and the weight of a homogeneous polynomial of degree d in the t_i is $d \cdot k$, hence the weight of any ω with $p > 0$ is > 0 if $k > 0$ and < 0 if $k < 0$. In particular only the $Sym^q \mathfrak{g}^*$ terms remain after passing to weight zero components, whence the answer is the same as for $B\mathbb{G}_m$, as claimed. $\square$

Using this we can treat the cohomology of projective bundles.

Theorem 5. *Let X be a smooth S-stack and $\mathcal{E}$ a locally free sheaf of rank r over X. Then as a filtered dR_X-module, $dR_{\mathbb{P}(\mathcal{E})}$ is free of rank r on the classes*

$$1, c_1(\mathcal{O}(1)), \ldots c_1(\mathcal{O}(1))^{r-1}$$

with $c_1(\mathcal{O}(1))^i$ in homological degree $-2i$ and filtration $\geq i$.

Proof. We can make a natural comparison map

$$\oplus_{i=0}^{r-1} dR_X \otimes \mathcal{O}_S[-2i]\{i\} \to dR_{\mathbb{P}(\mathcal{E})}$$

by pulling back and multiplying with $c_1(\mathcal{O}(1))^i$. To check it's an isomorphism, we can work locally and thus assume X is a scheme and $\mathcal{E} \simeq \mathcal{O}^r$. Then by Künneth we reduce to $X = S$. Thus we want to show that

$$\oplus_{i=0}^{r-1} \mathcal{O}_S[-2i]\{i\} \to dR_{\mathbb{P}^{r-1}}$$

is an iso.

We will in fact prove the following more refined statement: there is a canonical nullhomotopy of $c_1(\mathcal{O}(1))^r$ on $\mathbb{P}^{r-1}$ (recall that c_1 was defined on the cocyle level, so the statement makes sense), whence a lift to a class in relative de Rham cohomology

$$Th_r \in dR_{\mathbb{G}_m \backslash\backslash \mathbb{A}^r \text{ rel } \mathbb{P}^{r-1}}$$

in homological degree degree $-2r$ and filtration $\geq r$; and moreover $dR_{\mathbb{G}_m \backslash\backslash \mathbb{A}^r \text{ rel } \mathbb{P}^{r-1}}$ is a free filtered $dR_{B\mathbb{G}_m}$-module on Th_r. By the definitional long exact sequence expressing this relative de Rham cohomology as the fiber of the restriction

$$dR_{\mathbb{G}_m \backslash\backslash \mathbb{A}^r} \to dR_{\mathbb{P}^{r-1}}$$

and the calculation of $dR_{\mathbb{G}_m \backslash\backslash \mathbb{A}^r} \simeq dR_{B\mathbb{G}_m}$ from the previous lemma, this refined statement will indeed imply the desired claim.

To produce Th_r and prove it's a free generator as claimed, note that the Künneth theorem for relative de Rham cohomology yields

$$dR_{\mathbb{G}_m \backslash\backslash \mathbb{A}(\mathcal{E}\oplus\mathcal{E}') \text{ rel } \mathbb{P}(\mathcal{E}\oplus\mathcal{E}')} \simeq dR_{\mathbb{G}_m \backslash\backslash \mathbb{A}(\mathcal{E}) \text{ rel } \mathbb{P}(\mathcal{E})} \otimes dR_{\mathbb{G}_m \backslash\backslash \mathbb{A}(\mathcal{E}') \text{ rel } \mathbb{P}(\mathcal{E}')},$$

whence

$$dR_{\mathbb{G}_m \backslash\backslash \mathbb{A}^r \text{ rel } \mathbb{P}^{r-1}} \simeq \otimes^r dR_{\mathbb{G}_m \backslash\backslash \mathbb{A}^1 \text{ rel } \mathbb{P}^0}$$

reducing all claims to the case $r = 1$. But then $\mathbb{P}^0 = S$ and we can just take our first chern class definitionally viewed as a class in relative de Rham cohomology of $B\mathbb{G}_m$, as discussed above. ☐

Now we can produce the theory of Chern classes.

Theorem 6. *There exists an association*

$$\mathcal{E} \mapsto c_i(\mathcal{E}) \in H^{2i} F^{\geq i} dR_X$$

for all locally free sheaves of finite rank $\mathcal{E}$ on smooth S-stacks X and all $i \geq 1$, satisfying the following properties:

1. *If $\mathcal{L}$ is locally free of rank one, then $c_1(\mathcal{L})$ is the first Chern class as defined earlier using $dlog$.*

2. *If $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ is a short exact sequence of locally free sheaves of finite rank, then*

$$c_i(\mathcal{E}) = \sum_{j+k=i} c_j(\mathcal{E}') \cdot c_k(\mathcal{E}'')$$

   for all $i \geq 0$, where we set $c_0 = 1$. (Cartan formula)

3. *$c_i(f^*\mathcal{E}) = f^*c_i(\mathcal{E})$ for all maps $f : Y \to X$ and $i \geq 1$.*

Moreover, such an association satisfying these axioms is unique, even if we replace 2 by the following weaker variant: that the conclusion holds if $\mathcal{E} \simeq \mathcal{E}' \oplus \mathcal{E}''$.

Proof. Since the rank of a vector bundle is constant on some disjoint union decomposition, and de Rham cohomology takes disjoint unions to products, the theory is uniquely determined from the case of vetor bundles of constant rank. So let's focus on that.

Suppose given X and $\mathcal{E}$ of rank r over X. By the version of the splitting principle which we'll prove as a lemma afterwards, there are maps $X \xrightarrow{g} Y \xleftarrow{f} Y'$ and a vector bundle $\mathcal{F}$ on Y such that:

1. $g^*\mathcal{F} \simeq \mathcal{E}$,

2. $f^*\mathcal{F} \simeq \mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_r$ with each $\mathcal{L}_i$ locally free of rank one;

3. $f^* : F^{\geq p} dR_Y \to F^{\geq p} dR_{Y'}$ is injective on H^* for all p.

Using axiom 3 for Chern classes it follows that Chern classes in general are determined by the case of direct sums of line bundles. Then axioms 1 and 2 show that the theory of direct sums of line bundles is uniquely determined as well.

Now let's show existence. We'll use the Grothendieck construction of Chern classes. Given $\mathcal{E}$ of rank r over X, consider the projective bundle $\mathbb{P}(\mathcal{E})$. By the lemma above the class $c_1(\mathcal{O}(1))^r \in H^{2r} F^{\geq r} dR_{\mathbb{P}(\mathcal{E})}$ can uniquely be written as a polynomial in the previous powers of $c_1(\mathcal{O}(1))$ with coefficients in dR_X. We define $c_i(\mathcal{E})$ to be $(-1)^{i+1}$ multiplied by the $(r-i)^{th}$ coefficient.

It is clear that axiom 3 is satisfied, and axiom 1 is trivially verified ($\mathbb{P}(\mathcal{L}) = S$, but the tautological line bundle $\mathcal{O}(1)$ identifies with $\mathcal{L}$.) For 2, again by the splitting principle we can reduce to the case where the short exact sequence is split, and then to the case where all vector bundles occurring are direct sums of line bundles. Thus we reduce to showing that on $\mathbb{P}(\mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_r)$, we have

$$\prod_{i=1}^{r} (c_1(\mathcal{O}(1)) - c_1(\mathcal{L}_i)) = 0.$$

This is proved in exactly the same way we proved the relationship $c_1(\mathcal{O}(1))^r = 0$ on $\mathbb{P}^{r-1}$ in the previous proof, namely by using Künneth in relative de Rham cohomology of stacks to reduce to the case $r = 1$. □

We used the following variant of the splitting principle. In part 1 we don't need stacks, but for part 2 it's crucial we pass to stacks even if X is a scheme!

Lemma 7. 1. *Suppose given a smooth S-stack X and a vector bundle $\mathcal{E}$ of constant rank on X. Then there is a map of stacks $f : X' \to X$ such that:*

(a) $f^* : F^{\geq p} dR_X \to F^{\geq p} dR_{X'}$ injective on cohomology for all p;

(b) $f^* \mathcal{E}$ admits a full flag, i.e. a filtration by sub-bundles where each quotient is a line bundle.

2. *Furthermore, suppose given a smooth S-stack X and a filtered vector bundle $\mathcal{E}$ on X. Then there exist maps of stacks $X \xrightarrow{g} Y \xleftarrow{f} Y'$ and a filtered vector bundle $\mathcal{F}$ on Y such that:*

   (a) $g^* \mathcal{F} \simeq \mathcal{E}$ as filtered vector bundles,

   (b) The pullback filtered vector bundle $f^* \mathcal{F}$ admits a splitting;

   (c) $f^* : F^{\geq p} dR_Y \to F^{\geq p} dR_{Y'}$ is injective on H^* for all p. (In fact, we'll even get it to be an iso.)

Proof. For part 1 we can follow the standard argument: first pass to the projective bundle on $\mathcal{E}$. Pullback to the projective bundle is injective on cohomology as in 2 by the projective bundle formula. On pullback this gets you a short exact sequence

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{O}(1) \to 0,$$

and then you continue inductively with the projective bundle of $\mathcal{E}'$, etc. Essentially, you build the object on which you tautologically have a full filtration as desired, i.e. the moduli space of full flag, and check that the injectivity 2 is satisfied.

For part 2, we could try the same thing and build the object on which you tautologically have a splitting. For a filtration of length two, i.e. a short exact sequence $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}''$, splittings exist locally and the difference between any two splittings is a homomorphism $\mathcal{E}'' \to \mathcal{E}'$. Thus in general the moduli space of splittings is a principal homogeneous space for the vector bundle $Hom(\mathcal{E}'', \mathcal{E}')$. In trying to verify the requisite injectivity on cohomology, we run into the issue that due to the lack of homotopy invariance of de Rham cohomology, we can't easily control the de Rham cohomology of such a principal homogeneous space.

But weighted homotopy invariance saves the day. We set $Y = X \times B\mathbb{G}_m$ and extend $\mathcal{E}$ to a filtered vector bundle $\mathcal{F}$ on Y by means of the weight one $\mathbb{G}_m$-action by scalar multiplication. Proceeding by induction on the length of the filtration, we can reduce to the situation of a short exact sequence as above. We find that the moduli space of splittings is a principal homogeneous space for the vector bundle over $X \times B\mathbb{G}_m$ which is $Hom(\mathcal{E}'', \mathcal{E}')$ as an underlying vector bundle over X, with $\mathbb{G}_m$-equivariance given by scalar multiplication by λ^2, so the weight 2 action. In any case, by weighted homotopy invariance it follows that the induced pullback map on filtered de Rham cohomology is an isomorphism, whence the desired construction. $\square$

Now, following Totaro, we can see that the de Rham cohomology of BGL_r is "correct".

Theorem 8. *Let $r \geq 1$. Then dR_{BGL_r} is the polynomial algebra over $\mathcal{O}_S$ on the classes c_i for $1 \leq i \leq r$, the Chern classes of the tautological rank r vector bundle:*

$$H^* dR_{BGL_r} = \mathcal{O}_S[c_1, \dots, c_r].$$

The class c_i lives in homological degree $-2i$ and filtration $\geq i$; actually the Hodge filtration on dR_{BGL_r} is given by saying that $F^{\geq p}$ consists of the free sums of monomials $\prod_i c_i^{n_i}$ with $\sum_i i \cdot n_i \geq p$.

Proof. The Chern classes themselves make a filtered multiplicative map from right to left; we claim it is an iso. Consider the map

$$f : B\mathbb{G}_m^r \to BGL_r$$

corresponding to the inclusion of diagonal matrices, or equivalently classifying

$$(\mathcal{L}_1, \ldots, \mathcal{L}_r) \mapsto \mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_r.$$

We claim:

1. f^* is injective on $H^* F^{\geq p} dR$ for all p;
2. The image of f^* lands inside the S_r-invariants on $H^* F^{\geq p} dR_{B\mathbb{G}_m^r}$ for all p, where the symmetric group S_r acts by permuting the factors.

For 1, let's apply the version of the splitting principle used above to the universal bundle. We find that there is a stack Y containing BGL_r as a retract such that there is a map of stacks $Y' \to Y$, injective on cohomology, such that the retraction $Y \to BGL_r$ lifts along f when restricted to Y'. The conclusion follows.

For 2, it suffices to note that f is S_r-invariant up to homotopy because of homotopy-commutativity of direct sum.

On the other hand, Künneth for stacks (follows by a simple argument using smooth atlas and Künneth for schemes) shows that the cohomology of $B\mathbb{G}_m^r$ is the polynomial ring on r variables, the r different pullbacks of c_1. Thus the above shows that the cohomology of BGL_r identifies with some subring of the ring of symmetric polynomials in these r variables. However, we know a priori this subring contains the Chern classes. By the Cartan formula for Chern classes of direct sums, the c_i pull back to the elementary symmetric polynomials in these r variables. Now we conclude by the fact from commutative algebra that every symmetric polynomial is uniquely polynomial in the elementary symmetric polynomials. □

Finally, using this, we can produce a theory of Thom classes.

Theorem 9. *There is a unique way to assign, to every vector bundle $V \to X$ of rank r over a smooth S-stack X, a class*

$$Th_V \in H^{2r} F^{\geq r} dR_{V \text{ rel } V \smallsetminus 0_X}$$

such that:

1. *For the universal locally free of rank one sheaf $\mathcal{O}(1)$ on $B\mathbb{G}_m$, note that the corresponding total space $V = \mathbb{A}(\mathcal{O}(1))$ is $\mathbb{G}_m \backslash\backslash \mathbb{A}^1$, and the complement of the zero section is $\mathbb{G}_m \backslash\backslash \mathbb{G}_m = S$. Recall that c_1 is a class in de Rham cohomology of $B\mathbb{G}_m$ which is trivial on restriction to S. Thus by pullback to $\mathbb{G}_m \backslash\backslash \mathbb{A}^1$ it gives a candidate Thom class, and we demand that indeed $Th_{\mathbb{G}_m \backslash\backslash \mathbb{A}^1} = c_1$.*
2. *We have $Th_{V \oplus W} = Th_V \cdot Th_W$ via the Künneth isomorphism $dR_{V \oplus W \text{ rel } V \oplus \smallsetminus 0_X} \simeq dR_{V \text{ rel } V \smallsetminus 0_X} \otimes dR_{V \text{ rel } V \smallsetminus 0_X}$.*

Moreover, promoting V to the vector bundle $\mathbb{G}_m \backslash\backslash V \to X \times B\mathbb{G}_m$ using scalar multiplication as usual, multiplication by $Th_{\mathbb{G}_m \backslash\backslash V}$ gives an iso

$$dR_{\mathbb{G}_m \backslash\backslash V \text{ rel } \mathbb{G}_m \backslash\backslash V \smallsetminus 0_{X \times B\mathbb{G}_m}} \simeq dR_{X \times B\mathbb{G}_m}[-2r]\{r\},$$

i.e. we have a weighted Thom isomorphism.

Proof. Note that in the situation of the splitting principle lemma above, we also get injectivity on the relative de Rham cohomology where the Thom class is supposed to live by the same reasoning. Thus the theory is determined by the case where V is a direct sum of line bundles; by axiom 2 we reduce to line bundles themselves, and these are determined by axioms 1 and 3. Thus we have uniqueness.

For existence, we work in the universal case. For the tautological rank n vector bundle over BGL_r, the complement of the zero section identifies with the map $BGL_{r-1} \to BGL_r$ classifying direct summing with the trivial line bundle. The top Chern class c_r vanishes on restriction to BGL_{r-1} for example by the Whitney sum formula; thus it promotes to a class in relative de Rham cohomology (uniquely, as the cohomology is concentrated in even degrees), and we take that as the definition of the Thom class in the universal case. In general it is defined by pullback. By construction axioms 1 and 3 are satisfied. For axiom 2, we note that the Cartan formula shows that the top Chern class is multiplicative.

As for the weighted Thom isomorphism, we can work locally, and it follows from weighted homotopy invariance and the projective bundle formula. □

Exercise 10. *Recall from the end of the previous lecture that we directly produced candidate Chern classes in Hodge cohomology from Totaro's theorem. Show these Chern classes (in essence defined by Atiyah as an algebraic analog of Chern-Weil theory) satisfy the axioms for Chern classes stated in this lecture, hence are "the" Chern classes as defined for example using the cohomology projective bundles as in this lecture.*