Lecture 15: Deformation to the normal bundle and cycle classes

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Let S be our fixed base scheme. Last time we ended with a theory of Thom classes for vector bundles. Namely, to every vector bundle $V \to X$ of rank r over a smooth S-stack was assigned a class

$$Th_V \in H^{2r} F^{\geq r} dR_{V \text{ rel } V \smallsetminus 0_X}.$$

It was functorial under pullback and multiplicative under cartesian product (over S), and it was uniquely determined by these requirements plus the fact that for line bundles it corresponds to the first Chern class. In this lecture we'll use Thom classes to produce cycle classes associated to closed inclusions of smooth S-stacks.

Theorem 1. *There exists a unique assignment of a* cycle class

$$Cl_Y \in H^{2r} F^{\geq r} dR_{X \text{ rel } X \smallsetminus Y}$$

to every codimension r smooth closed substack $Y \subset X$ of a smooth S-stack X, such that:

1. *For the inclusion $B\mathbb{G}_m \subset \mathbb{G}_m \backslash\backslash \mathbb{A}^1$ of the zero section of the universal line bundle, we have*

   $$Cl_{B\mathbb{G}_m} = c_1,$$

   the first Chern class of the universal line bundle.

2. *If $Y \subset X$ and $Y' \subset X'$ are two closed inclusions of smooth S-stacks with product closed inclusion $Y \times Y' \subset X \times X'$, then*

   $$Cl_{Y \times Y'} = Cl_Y \otimes Cl_{Y'}$$

   via the Künneth isomorphism $dR_{X \times X' \text{ rel } X \times X' \smallsetminus Y \times Y'} \simeq dR_{X \text{ rel } X \smallsetminus Y} \otimes dR_{X' \text{ rel } X' \smallsetminus Y'}$.

3. *If $f : X' \to X$ is a morphism of smooth S-stacks which is transversal to $Y \subset X$, then*

   $$Cl_{f^{-1}Y} = f^* Cl_Y.$$

Moreover, assigning to $Y \subset X$ the induced class in the associated graded relative Hodge cohomology

$$Cl_Y \in H^r |\Omega^r|_{X \text{ rel } X \smallsetminus Y}$$

is also unique subject to the obvious analogs of the three axioms above.

Just as in the previous lecture on Chern classes and Thom classes, passing to the generality of stacks here is not just "for fun". The construction of the cycle classes, even just for schemes, passes through stacks, and I don't see why the uniqueness part of the theorem should hold in characteristic p if we just restrict to schemes. Actually, for schemes the existence of such cycle classes was given by Berthelot in his book "Cohomologie cristalline des schémas de characteristique $p > 0$", but he gives an explicit construction and doesn't prove any uniqueness claims.

To make sense of all the notions used here, we should discuss some preliminaries.

Lemma 2. *Suppose given a closed inclusion $i : Y \subset X$ of smooth S-schemes. Then:*

1. *Locally for the Zariski topology on X, there exists $r \geq 0$ and a smooth map $X \to \mathbb{A}^r$ with $Y = f^{-1}0$.*
2. *The conormal sheaf $C_{Y\subset X} := i^*\mathcal{I}$, where $\mathcal{I}$ is the ideal sheaf defining Y, is locally free of finite rank, and fits into a short exact sequence*

$$0 \to C_{Y\subset X} \to i^*\Omega^1_X \to \Omega^1_Y \to 0.$$

3. *The natural map $SymC_{Y\subset X} \to \oplus_{n\geq 0} i^*\mathcal{I}^n$ of graded quasi-coherent $\mathcal{O}_Y$-algebras is an isomorphism. (Note also that $\oplus_{n\geq 0} i^*\mathcal{I}^n$ can also be interpreted as the associated graded for the $\mathcal{I}$-adic filtration on $\mathcal{O}_X$.)*

The local rank of $C_{Y\subset X}$ is the same as the number r in part 1, and is also the difference of the relative dimensions of X and Y.

Proof. Omitted. It's pretty easy to see that part 1 implies the rest; on the other hand the only way I know how to prove part 1 is by proving part 2 first. □

Next we discuss transversality.

Definition 3. *Let $i : Y \subset X$ be a closed inclusion of smooth S-schemes, and let $f : X' \to X$ be a map of smooth S-schemes. We say that f is* transverse *to Y if the map of locally free sheaves on $Y' = f^{-1}Y$*

$$i'^* f^*\Omega^1_X \to f^*\Omega^1_Y \oplus i'^*\Omega^1_{X'},$$

induced by functoriality of Kähler differentials, is a locally split inclusion; or equivalently, if when base-changed to every residue field of Y' it is an injective map of full rank.

Transversality interacts well with the concept of a smooth morphism.

Lemma 4. *Let $i : Y \subset X$ be a closed inclusion of smooth S-schemes, and let $f : X' \to X$ be a map of smooth S-schemes.*

1. *If f is smooth then f is transverse to Y.*
2. *More generally, if f is smooth, then $g : X'' \to X'$ is transverse to $f^{-1}Y$ if and only if $g \circ f : X'' \to X$ is transverse to Y.*
3. *If for some smooth surjective map $g : X'' \to X'$ we have that $g \circ f$ is transverse to Y, then f is transverse to Y.*

Using the graph construction, we can factor any map between smooth schemes as a locally closed inclusion followed by a smooth map. Using this we see that every transversal map is locally the composition of a transverse closed inclusion and a smooth map. Furthermore, transverse closed inclusions look, étale locally, like transverse intersections in a vector space. Using these facts it's not hard to check the following.

Proposition 5. *Suppose f is transverse to i as above. Then:*

1. *The scheme-theoretic pullback $Y \times_X X'$ is a smooth S-scheme, whose sheaf of Kähler differentials naturally identifies with the cokernel of the map in the definition of transversality.*
2. *For all $n \geq 1$, the natural map $f^*\mathcal{I}^n \to \mathcal{I}'^n$ of quasi-coherent sheaves on X' is an isomorphism, where $\mathcal{I}$ is the ideal sheaf on X defining Y and $\mathcal{I}'$ is the ideal sheaf on X' defining Y'. Moreover there are no higher Tor's in this pullback, i.e. the statement also holds for derived pullback.*
3. *The natural map $f^*C_{Y\subset X} \to C_{Y'\subset X'}$ is an isomorphism.*

In particular, passing to quotients:

Corollary 6. *If f is transverse to i then the conormal sheaf construction C_i commutes with f-pullback.*

Next we extend all these notions to stacks.

Definition 7. *A map of smooth S-stacks $Y \to X$ is called a closed inclusion if for every map $X' \to X$ from a scheme, the pullback $X' \times_X Y \to X'$ is a closed inclusion of schemes.*

Since a smooth stack admits a smooth atlas by smooth schemes, we can test this on smooth maps from smooth schemes. Because in the smooth scheme setting the conormal sheaf is compatible with smooth pullback by the above discussion of transversality, we deduce by descent that the conormal sheaf also makes sense in the stack setting.

Definition 8. *A map of smooth S-stacks $X' \to X$ is called* smooth *if it makes X' into a smooth X-stack, i.e. for every pullback to a scheme we get a smooth stack over that scheme.*

Again because smooth stacks admit smooth atlases, we deduce that the conormal sheaf is also compatible with pullback along smooth maps of stacks. We also extend the definition of a transverse map by copying the scheme definition. Note that transversality can be checked locally for the fpqc topology, so everything reduces to the scheme case. In brief, the extension to sooth stacks is straightforward by checking everything smooth-locally.

With that background out of the way, let's return to the problem of constructing our cycle classes

$$Cl_Y \in H^{2r} F^{\geq r} dR_{X \text{ rel } X \smallsetminus Y}$$

satisfying the axioms above: $Cl_{B\mathbb{G}_m} = c_1$ for the zero section of the universal line bundle, multiplicativity, and base-change compatibility for transversal pullbacks. First note that if these axioms hold, then for zero sections of vector bundles the cycle class must be the Thom class of the previous lecture. Indeed, the axioms are basically the same: we just have to note that every vector bundle pullback is transverse to the zero section.

Proposition 9. *Supose $V \to X$ is a vector bundle over a smooth S-stack X, and $i : X \subset V$ is the inclusion of the zero section. Then $Cl_X = Th_V$.*

Now, the idea of the proof of existence and uniqueness of Gysin classes is to reduce to this case of the zero section of a vector bundle by means of a construction known as the *deformation to the normal bundle*. Intuitively you can think about this as follows. Let's consider the simplest case, the inclusion of a point in a smooth manifold. Imagine the smooth manifold embedded in some ambient Euclidean space, and scale it up by a factor λ centered at your point. As $\lambda \to \infty$, in any fixed region of space your embedded manifold looks more and more like the tangent space of your manifold at that point. Thus it makes sense to add a limiting object, the tangent space, at the point $\lambda = \infty$. Algebraically, it's a bit more conventient to reparametrize with $T = 1/\lambda$, and formally speaking the construction takes the following form.

Theorem 10. *To every closed inclusion $Y \subset X$ of smooth S-stacks can be assigned a closed inclusion*

$$Y \times \mathbb{G}_m \backslash\backslash \mathbb{A}^1 \subset \mathcal{D}$$

of smooth $\mathbb{G}_m\backslash\backslash\mathbb{A}^1$-stacks, where:

1. *The pullback to $\mathbb{G}_m\backslash\backslash\mathbb{G}_m = S$ identifies with the original $Y \subset X$, and this pullback is transverse;*
2. *The pullback to $\mathbb{G}_m\backslash\backslash 0 = B\mathbb{G}_m$ identifies with the inclusion of the zero section in the vector bundle $N_{Y\subset X} = Spec_Y(C_{Y\subset X})$ over Y, viewed as $\mathbb{G}_m$-equivariant by weight -1 scalar multiplication on C_i, and this pullback is transverse.*

Moreover, this association sends smooth maps $X' \to X$ to smooth maps $\mathcal{D}' \to \mathcal{D}$, and analogously for étale maps, smooth surjective maps, and transverse maps. Moreover smooth fiber products go to smooth fiber products.

Finally, if $Y \subset X$ is already the inclusion of the zero section of a vector bundle $V = X$ over Y, then the associated inclusion $Y \times \mathbb{G}_m\backslash\backslash\mathbb{A}^1 \to \mathcal{D}$ is naturally isomorphic to the inclusion of the zero section of the vector bundle over $Y \times \mathbb{G}_m\backslash\backslash\mathbb{A}^1$ given by pulling back the vector bundle $\mathbb{G}_m\backslash\backslash V$ over $Y \times B\mathbb{G}_m$ where the $\mathbb{G}_m$-equivariance given by weight -1 on V.

Proof. By descent, we can reduce to the construction to the scheme case, provided we check the claims about smooth maps and smooth fiber products. So let's work in schemes. We'll construct $\mathcal{D}$ over $\mathbb{A}^1$; the $\mathbb{G}_m$-equivariance will come from the grading which will be part of the construction. This $\mathcal{D}$ will in fact be relatively affine over $X \times \mathbb{A}^1$ (however, the structure map to X is not part of the data specified above!), given by the coordinate quasi-coherent algebra

$$\ldots \oplus \mathcal{I}^2 T^{-2} \oplus \mathcal{I} T^{-1} \oplus \mathcal{O}_X \oplus \mathcal{O}_X T \oplus \mathcal{O}_X T^2 \oplus \ldots$$

with "obvious" graded multiplication law with the T^n term living in degree n. This is known as the *Rees algebra*. In particular the multiplication by T map from degree n to degree $n+1$ is the identity on $\mathcal{O}_X$ for $n \geq 0$, and for $n < 0$ is the inclusion $\mathcal{I}^{-n} \to \mathcal{I}^{-n-1}$. It follows that when we invert T we get

$$\ldots \oplus \mathcal{O}_X T^{-2} \oplus \mathcal{O}_X T^{-1} \oplus \mathcal{O}_X \oplus \mathcal{O}_X T \oplus \mathcal{O}_X \oplus \mathcal{O}_X T^2 \oplus \ldots,$$

i.e. $\mathcal{O}_X[T, T^{-1}] = \mathcal{O}_X \otimes \mathcal{O}_{\mathbb{G}_m}$, whereas when we mod out by T we get

$$\ldots \oplus \mathcal{I}^2/\mathcal{I}^3 T^{-2} \oplus \mathcal{I}/\mathcal{I}^2 T^{-1} \oplus \mathcal{O}_X/\mathcal{I} \oplus 0T \oplus 0T^2 \oplus \ldots,$$

i.e. $Spec(Sym C_{Y\subset X})$ but with the grading reversed to be in negative degrees. This gives our total space $\mathcal{D}$ with the required properties; moreover, we have the homomorphism to $i_*\mathcal{O}_Y[T]$ given by the quotient map $\mathcal{O}_X \to i_*\mathcal{O}_Y$ in non-negative degrees and zero in negative degrees which gives the closed inclusion satisfying the properties 1 and 2.

As for the claimed identification when X is a vector bundle over Y with the inclusion given by the zero section, let's explain it in coordinates associated to a trivialization of the vector bundle with the implicit understanding that the identification is linear in the coordinates. We should also remember in any case that this identification will not respect the structure map to X. If X is a trivial vector bundle with coordinates $x_1, \dots, x_n$ over the affine scheme Y, then one checkss that the coordinate ring for $\mathcal{D}$ is the polynomial ring $\mathcal{O}_Y[z_1, \dots, z_n, T]$ where $z_i = x_i \cdot T^{-1}$, and this gives the claims.

To see compatibility with smooth morphisms and so on, the key is that multilication by T is injective on the coordinate algebra of $\mathcal{D}$, and this lets us check the claims on reduction mod T and inverting T, when they become plain. We omit the details. □

We'll need one more trick with this construction to be able to phrase our key claim. For every $n \geq 1$, there is an endomorphism of the stack $\mathbb{G}_m\backslash\backslash\mathbb{A}^1$ induced by $z \mapsto z^n$ on both $\mathbb{A}^1$ and $\mathbb{G}_m$. We can in fact make a tower of stacks over $\mathbb{G}_m\backslash\backslash\mathbb{A}^1$, indexed by n under divisibility, all identified with $\mathbb{G}_m\backslash\backslash\mathbb{A}^1$, where the transition map from n to m is $z \mapsto z^{n/m}$. Let us write $(\mathbb{G}_m\backslash\backslash\mathbb{A}^1)_n$ for the n^{th} term in this stack, and write $\mathcal{D}_n$ for the pullback to $(\mathbb{G}_m\backslash\backslash\mathbb{A}^1)_n$ of the deformation in the above theorem.

The key result is the following.

Theorem 11. *Let $Y \subset X$ be a closed inclusion of smooth S-stacks, $Y \times \mathbb{G}_m\backslash\backslash\mathbb{A}^1 \subset \mathcal{D}$ the associated deformation to the normal bundle, and $Y \times (\mathbb{G}_m\backslash\backslash\mathbb{A}^1)_n \subset \mathcal{D}_n$ the pullback along the n-power map on $\mathbb{G}_m\backslash\backslash\mathbb{A}^1$. Then the map*

$$\varinjlim_n dR_{\mathcal{D}_n \text{ rel } \mathcal{D}_n \smallsetminus Y \times (\mathbb{G}_m\backslash\backslash\mathbb{A}^1)_n} \to \varinjlim_n dR_{\mathbb{G}_m\backslash\backslash_{-n}N_{Y\subset X} \text{ rel } \mathbb{G}_m\backslash\backslash_n N_{Y\subset X}\smallsetminus(B\mathbb{G}_m)_n},$$

induced by the inclusion of the pullback along $0 : B\mathbb{G}_m \subset \mathbb{G}_m\backslash\backslash\mathbb{A}^1$, is an isomorphism. Here $\mathbb{G}_m\backslash\backslash_{-n}N_{Y\subset X}$ means the quotient by the weight $-n$ action on $N_{Y\subset X}$, and we consider both sides as filtered objects with respect to the Hodge filtration.

Let's first assume this result and give the proof of existence and uniqueness of cycle classes. First, we show uniqueness. Suppose we have two theories of cycle classes, Cl and Cl'. Let $Y \subset X$ be a closed inclusion of smooth S-stacks of codimension r, and consider the cycle classes

$$Cl_{Y\times\mathbb{G}_m}, Cl'_{Y\times\mathbb{G}_m} \in H^{2r}F^{\geq r}dR_{\mathcal{D} \text{ rel } \mathcal{D}\smallsetminus Y\times\mathbb{G}_m\backslash\backslash\mathbb{A}^1}$$

associated to the deformation to the normal bundle. By functoriality under transverse pullback and Proposition 9 above, both of these classes map to the Thom class of $\mathbb{G}_m\backslash\backslash_{-1}N_{Y\subset X}$ under pullback along $0 : B\mathbb{G}_m \to \mathbb{G}_m\backslash\backslash\mathbb{A}^1$. Thus, by the above theorem, the pullbacks of these two classes to $\varinjlim_n dR_{\mathcal{D}_n \text{ rel } \mathcal{D}_n\smallsetminus Y\times(\mathbb{G}_m\backslash\backslash\mathbb{A}^1)_n}$ must agree.

On the other hand, the tower $(\mathbb{G}_m\backslash\backslash\mathbb{A}^1)_n$ of stacks over $\mathbb{G}_m\backslash\backslash\mathbb{A}^1$ is constant when restricted to $\mathbb{G}_m\backslash\backslash\mathbb{G}_m$, so the restriction map

$$dR_{\mathcal{D} \text{ rel } \mathcal{D}\smallsetminus Y\times\mathbb{G}_m\backslash\backslash\mathbb{A}^1} \to dR_{X \text{ rel } X\smallsetminus Y}$$

along $\mathbb{G}_m\backslash\backslash\mathbb{G}_m \subset \mathbb{G}_m\backslash\backslash\mathbb{A}^1$ factors through $\varinjlim_n dR_{\mathcal{D}_n \text{ rel } \mathcal{D}_n\smallsetminus Y\times(\mathbb{G}_m\backslash\backslash\mathbb{A}^1)_n}$. By functoriality under transverse pullback again, these restrictions give Cl_Y and Cl'_Y, so those classes must agree, as desired.

Note that the same proof also works to show uniqueness of cycle classes in Hodge cohomology; this will be crucial for us in the next lecture.

For existence we just turn this around to make a definition. By the above theorem, there is a unique class in $\varinjlim_n dR_{\mathcal{D}_n \text{ rel } \mathcal{D}_n \smallsetminus Y \times (\mathbb{G}_m \backslash\backslash \mathbb{A}^1)_n}$ which maps to the image of the Thom class of $\mathbb{G}_m\backslash\backslash_{-1} N_{Y\subset X}$; then we take the image of this class in $dR_{X \text{ rel } X\smallsetminus Y}$ as the definition of our cycle class. The properties are all straightforward to verify, by using the properties of the deformation to the normal bundle to reduce to the analogous properties for Thom classes.

Now we will prove Theorem 11. First, note that by smooth descent of de Rham cohomology we can reduce to the case of smooth schemes. Then the next idea is to further reduce to the case where $Y \subset X$ is the 0-inclusion $0 \subset \mathbb{A}^1$. For this we follow the proof of Morel-Voevodsky's "purity theorem" in their "$\mathbb{A}^1$-homotopy theory of schemes", which also used the deformation to the normal bundle in a similar fashion. The key property behind Morel-Voevodsky's argument is the following "Nisnevich excision".

Lemma 12. *Suppose $Y \subset X$ is a closed inclusion of smooth S-schemes, and let $f : X' \to X$ be an étale map such that $Y' = f^{-1}Y \to Y$ is an isomorphism. Then the pullback map of filtered objects*

$$dR_{X \text{ rel } X\smallsetminus Y} \to dR_{X' \text{ rel } X'\smallsetminus Y'}$$

is an isomorphism.

Proof. In fact, this Nisnevich excision property holds for any étale sheaf, or even any Nisnevich sheaf. It follows by applying étale descent to the cover of X by $X \smallsetminus Y$ and X'. The only thing one needs to see is that the "extra" descent data over $X' \times_X X'$ is redundant. For this one considers the open cover of $X' \times_X X'$ by the diagonal $X' \to X' \times_X X'$ (an open immersion as f is étale) and $(X' \smallsetminus Y') \times_{X\smallsetminus Y} (X' \smallsetminus Y')$. These cover by the isomorphism condition.

To finish, we recall that Hodge cohomology is an étale sheaf by quasi-coherence. □

Now, to prove Theorem 11 we can work locally and thus assume that there is an étale map $X \to \mathbb{A}^n$ such that $Y = f^{-1}\mathbb{A}^{n-r}$. If the map $Y \to \mathbb{A}^{n-r}$ were an isomorphism so that we have a Nisnevich excision square, we would deduce the analogous Nisnevich excision property for the associated deformation to the normal bundle, and this would give our desired reduction. But $Y \to \mathbb{A}^{n-r}$ is not an isomorphism in general.

Nonetheless there is the following trick to relate $Y \subset X$ to $0 : Y \subset Y \times \mathbb{A}^r$ by a zig-zag of Nisnevich excision squares. Namely, consider the diagonal embedding $Y \subset X \times_{\mathbb{A}^n} (Y \times \mathbb{A}^r)$. Using the two projections we get two Nisnevich excision squares of the desired form. Applying the deformation to the normal bundle construction, this gives the reduction to the case of the zero section of a trivial vector bundle.

Then, using Künneth, we can reduce to the case $Y = S$ and then to the case $r = 1$. Thus, what we need to check is the following:

Lemma 13. *Consider the action of $\mathbb{G}_m$ on $\mathbb{A}^1 \times \mathbb{A}^1 = Spec_S(\mathcal{O}_S[x,T])$ by weight -1 on x and weight 1 on T and the map*

$$\mathbb{G}_m\backslash\backslash(\mathbb{A}^1 \times \mathbb{A}^1) \to \mathbb{G}_m\backslash\backslash\mathbb{A}^1$$

on quotients induced by projection to the second factor. Note that the pullback to $(\mathbb{G}_m\backslash\backslash\mathbb{A}^1)_n$ is given by the quotient of the action of $\mathbb{G}_m$ on

$$(\mathbb{A}^1 \times \mathbb{A}^1)_n = Spec_S(\mathcal{O}_S[x, T^{1/n}])$$

given by weight $-n$ on x and weight 1 on $T^{1/n}$.

Then the comparison map on de Rham cohomology

$$dR_{\mathbb{G}_m \backslash\backslash (\mathbb{A}^1 \times \mathbb{A}^1)_n \text{ rel } \mathbb{G}_m \backslash\backslash (\mathbb{G}_m \times \mathbb{A}^1)_n} \to dR_{\mathbb{G}_m \backslash\backslash (\mathbb{A}^1)_n \text{ rel } \mathbb{G}_m \backslash\backslash (\mathbb{G}_m)_n}$$

is an isomorphism in the colimit over n.

Proof. First let's see what happens when we rationalize, passing to $dR \otimes \mathbb{Q}$, or equivalently changing our base to the rationalization of S instead. Then in fact the map is an isomorphism at each level n. Indeed, we have $\mathbb{A}^1$-invariance and the de Rham cohomology of $\mathbb{G}_m$ is also "as expected", so we have a Thom isomorphism for arbitrary vector bundles. It follows that the comparison map in level n identifies, up to a shift both in homological index and in filtration, with the comparison map on de Rham cohomology associated to $0 : B\mathbb{G}_m \to (\mathbb{G}_m \backslash\backslash \mathbb{A}^1)_n$. But that again is an isomorphism by $\mathbb{A}^1$-invariance.

Thus it suffices to show the analogous claim for the fiber of $dR \to dR \otimes \mathbb{Q}$, which is the filtered colimit over positive integers k under divisibility of shifts of copies of dR/k. Thus it suffices to show the analogous claim for dR/k. We can even assume $k = p$ is a prime if we like; this is maybe psychologically useful but it will be irrelevant for the argument.

Now, we can pass to the associated graded for the Hodge filtration to reduce to the analogous claim for mod p Hodge cohomology. Then we use Totaro's model for equivariant Hodge cohomology. The key remark is the following: all the terms in the Koszul complex containing a $Sym^q \mathfrak{g}^*$ factor with $q > 0$ will be killed when we take the colimit over n. Indeed, the n^{th} power map on $\mathbb{G}_m$ induces multiplication by n on $\mathfrak{g}^*$, and for a cofinal collection of n this is zero mod p.

Thus, Totaro's theorem shows that the filtered colimit of mod p Hodge cohomology, for all four terms $\mathbb{G}_m \backslash\backslash (\mathbb{A}^1 \times \mathbb{A}^1)_n$, $\mathbb{G}_m \backslash\backslash (\mathbb{G}_m \times \mathbb{A}^1)_n$, $\mathbb{G}_m \backslash\backslash (\mathbb{A}^1)_n$, and $\mathbb{G}_m \backslash\backslash (\mathbb{G}_m)_n$ we're interested in, is calculated by just looking at the degree 0 part of the de Rham complex and then taking the filtered colimit over n. But we can even simplify further, because in this de Rham complex, the terms of Hodge degree > 0 in the T variable will become 0 in the filtered colimit over n for the same reason. Thus we can calculate the same limiting object by just looking at the de Rham complex of the first factor $\mathbb{A}^1$ or $\mathbb{G}_m$ as the case may be, tensoring with $\mathcal{O}_S[T^{1/n}]$ in the cases where the second factor appears, taking the degree 0 part, then passing to the filtered colimit over n.

For $\mathbb{G}_m \backslash\backslash (\mathbb{A}^1)_n$ we just get the degree 0 part of the de Rham complex of $\mathbb{A}^1$, which is just the constants sitting in degree 0. For $\mathbb{G}_m \backslash\backslash (\mathbb{G}_m)_n$ we get the degree 0 part of the de Rham complex of $\mathbb{G}_m$, which is the constants in degree 0 and the constant multiples of dx/x in degree 1. Thus for the relative Hodge cohomology of the first relative to the second, we're free on one class in homological degree -2 and Hodge degree 1 coming from the dx/x.

For $\mathbb{G}_m \backslash\backslash (\mathbb{A}^1 \times \mathbb{A}^1)_n$, in degree 0 we're free on the classes $x^a T^a$ for $a \geq 0$ and in degree 1 we're free on the classes $x^a T^{a+1} dx$. For $\mathbb{G}_m \backslash\backslash (\mathbb{G}_m \times \mathbb{A}^1)_n$ it's the same except that we're allowed to have a be negative, provided only that $a + 1 \geq 0$. In degree zero this doesn't make any difference so we get the same thing in degree 0. But in degree 1 we get the extra free generator dx/x. Thus the relative theory is the same as before, proving the claim. □

Exercise 14. *Prove that any smooth group scheme "degenerates to an additive group scheme": more precisely, if G is a smooth S-group scheme, then there is a smooth group scheme over $\mathbb{A}^1_S$ whose fiber at every nonzero point is isomorphic to G but whose fiber at 0 is isomorphic to an additive group scheme, indeed a group scheme underlying a vector bundle. Can you have the opposite phenomenon: can for example the additive group scheme $\mathbb{G}_a$ degenerate to $\mathbb{G}_m$ in the same sense?*