Lecture 16: Poincaré duality

Dustin Clausen

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Poincaré duality is not supposed to be this hard to prove, but I think it really is. The statement, I recall, is that if X is a proper smooth S-scheme of relative dimension d, then the de Rham cohomology of X is self-dual up to a shift by $2d$; more precisely, there is a canonical duality pairing

$$dR_X \otimes dR_X \to \mathcal{O}_S[-2d]$$

in $Perf(S)$. Even more precisely, this should be a duality of filtered objects in $Perf(S)$, where dR_X has the Hodge filtration and we view the target as the filtered object $\mathcal{O}_S[-2d]\{d\}$ which has only one nonvanishing associated graded, in filtration degree d, equal to $\mathcal{O}_S[-2d]$. Even even more precisely, this duality of filtered objects should have the property that on associated gradeds, it recovers the duality on Hodge cohomology coming from Grothendieck-Serre duality.

We discussed this in the lecture on Deligne's theorem part 2, where we also showed that this statement was equivalent to one single vanishing of a differential in the Hodge-de Rham spectral sequence, namely that the map

$$H^d Rp_* \Omega^{d-1} \to H^d Rp_* \Omega^d$$

induced by the de Rham differential should be zero where $p : X \to S$ is the structure map for X. Indeed, when this condition is satisfied, we can promote the fundamental class in Hodge cohomology to a fundamental class in de Rham cohomology, encoded as a filtered map $dR_X \to \mathcal{O}_S[-2d]\{d\}$; composing with the product structure on de Rham cohomology gives a candidate duality pairing, and we check on associated graded that it is indeed a perfect duality pairing.

However, it is difficult to directly attack the question of the vanishing of this differential. There are two proofs of Poincaré duality I know of, one in Berthelot's book "Cohomologie cristalline des schémas en characteristique $p > 0$" and another in the Stacks Project, and they are both fairly intricate and roundabout. They also both proceed by a detailed study of the diagonal map $X \to X \times X$.

We will essentially do the same. The key idea is that it is actually easier to produce and study the *coevaluation map* for the proposed duality than the evaluation map. But the outline is the same, if dual: we produce the candidate coevaluation map, using the theory of cycle classes of the previous lecture; then we check on associated gradeds that it is indeed a perfect co-pairing, thereby proving Poincaré duality. Formally, here is the statement:

Theorem 1. *Let X be a smooth and proper S-scheme of relative dimension d. Classify the cycle class of the diagonal $\Delta : X \to X \times X$*

$$Cl_\Delta \in H^{2d} F^{\geq d} dR_{X\times X}$$

by a filtered map

$$\mathcal{O}_S[-2d]\{d\} \to dR_{X\times X} \simeq dR_X \otimes dR_X,$$

using also the Künneth theorem.

Then this co-pairing is a perfect co-pairing, i.e. the transposed map

$$dR_X^\vee \otimes \mathcal{O}_S[-2d]\{d\} \to dR_X$$

is a filtered isomorphism.

We can check this claim on associated graded, so it suffices to verify the analogous assertion for Hodge cohomology: that the cycle class of the diagonal gives a perfect co-evaluation pairing. The strategy for that will be to produce a different construction of cycle classes in Hodge cohomology, based instead on Grothendieck-Serre duality. For that construction of cycle class, it will be a more-or-less straightforward chase to see that the co-evaluation coming from the cycle class of the diagonal does correspond to the evaluation we produced earlier using Grothendieck-Serre duality.

More precisely, the following three lemmas put together will prove the theorem, as we will explain after stating the lemmas.

Lemma 2. *Suppose $i : Y \to X$ is a closed immersion of smooth S-stacks, of codimension d. Then the right adjoint $i^!$ to $i_* : D(Y) \to D(X)$ commutes with transverse base change and satisfies the projection formula $i^!(-) \simeq i^!\mathcal{O} \otimes i^*(-)$. Moreover there is a natural isomorphism*

$$i^!\mathcal{O} \simeq (\Lambda^d C_i[d])^\vee,$$

where C_i is the conormal bundle. This isomorphism is compatible with transverse pullback, and is also multiplicative, in the sense that if $i' : Y' \to X'$ is another such closed immersion, then it intertwines the natural isomorphisms

$$(i \times i')^!\mathcal{O} \simeq i^!\mathcal{O} \boxtimes i'^!\mathcal{O}$$

and

$$C_{i\times i'} \simeq C_i \boxtimes C_{i'}.$$

Lemma 3. *Let X be a smooth and proper S-scheme. Suppose $\mathcal{F}, \mathcal{G} \in Perf(X)$ are in duality via a perfect coevaluation pairing*

$$c : \Omega^d[d] \to \mathcal{F} \otimes \mathcal{G}.$$

Then $p_*\mathcal{F}, p_*\mathcal{G} \in Perf(S)$ are in duality via the perfect coevaluation pairing

$$\mathcal{O} \to p_*\mathcal{F} \otimes p_*\mathcal{G}$$

described as follows: applying the previous lemma to the diagonal, rewrite c as the map

$$\Omega^d[d] \to \Delta^*(\mathcal{F} \boxtimes \mathcal{G}) \simeq \Delta^!(\mathcal{F} \boxtimes \mathcal{G}) \otimes \Omega^d[d],$$

then cancel the invertible $\Omega^d[d]$ and adjoint over to a map

$$\Delta_*\mathcal{O} \to \mathcal{F} \boxtimes \mathcal{G},$$

then apply $(p \times p)_*$, precompose with the unit $\mathcal{O} \to p_*\mathcal{O}$ and use Künneth to get the desired

$$\mathcal{O} \to p_*\mathcal{F} \otimes p_*\mathcal{G}.$$

Moreover, this coevaluation is the mate of the evaluation pairing we discussed in the second lecture on Deligne's theorem, determined by the mate of our original coevaluation pairing c.

Lemma 4. *Let X be a smooth S-scheme of relative dimension d, and let p,q with $p+q=d$. Then the map*

$$\Omega_X^d[d] \to \Omega_X^p[p] \otimes \Omega_X^q[q],$$

induced by taking d^{th} exterior powers in the map

$$C_\Delta = \Omega_X^1 \to \Delta^*\Omega_{X\times X}^1$$

and projecting to the appropriate Künneth component, identifies with the coevaluation map for the perfect pairing

$$\Omega_X^p[p] \otimes \Omega_X^q[q] \to \Omega_X^d[d]$$

induced by multiplication.

So, let's explain why these lemmas imply Theorem 1. To start, let's use Lemma 2 to make a cycle class map in Hodge cohomology. Given a closed immersion of smooth S-stacks

$$i : Y \to X$$

of codimension d, Let C denote the conormal bundle and consider the map

$$C \to i^*\Omega_X^1.$$

Take d^{th} exterior powers and shift by d to deduce a map

$$\Lambda^d C[d] \to i^*\Omega_X^d[d] \simeq \Lambda^d N[d] \otimes i^!\Omega_X^d[d].$$

Cancelling $\Lambda^d C[d]$ and adjointing over, we get

$$i_*\mathcal{O} \to \Omega_X^d[d].$$

Applying p_* and using the natural map from $\mathcal{O}$, we get a natural map classifying a class in Hodge cohomology of degree d, weight d on X.

Now, note that, in contrast to our discussion of cycle classes in de Rham cohomology which were really only classes (i.e. only defined up to homotopy), this construction is natural on the ∞-level. (The only equivalence in Lemma 2 which has to be produced by hand is the one $i^!\mathcal{O} \simeq (\Lambda^d C_i[d])^\vee$, and this is an equivalence between objects in a single degree, so it reduces to a 1-categorical coherence problem.) The compatibility of this cycle map with transverse pullbacks follows from the analogous statement in Lemma 2; and since i becomes $\varnothing$ on transverse pullback to $X \smallsetminus Y$ it follows that our cycle map naturally promotes to a map to Hodge cohomology of X relative to $X \smallsetminus Y$, exactly as required.

The mutiplicativity and compatibility with transverse pullback follow from the analogous statements in Lemma 2, and it is a simple computation to check in the case of the universal line bundle that this cycle class is induced by c_1. Thus, by the uniqueness claim, this cycle class map agrees with the one produced in the previous lecture; in particular it promotes to the cycle class map in filtered de Rham cohomology used in the statement of Theorem 1.

Thus, to prove Theorem 1, it suffices to show that this specific cycle class in Hodge cohomology, applied to the diagonal, recovers the coevaluation map for the Poincaré duality in Hodge cohomology which we produced using Grothendieck-Serre duality. However, unwinding the definitions, this exactly follows by combining Lemmas 3 and 4.

Now let's say something about the proofs of these three Lemmas. The last one, Lemma 4, is the most elementary. Since both the evaluation pairing and the claimed co-evaluation pairing are étale local and we're working with quasi-coherent sheaves, we can reduce to affine space. Then we can just calculate in coordinates. Details omitted.

As for Lemma 3, we need to see that our claimed co-evaluation pairing is the mate to our known evaluation pairing coming from Grothendieck-Serre duality. This is a formal, if slightly intricate, diagram chase in Grothendieck duality. Well, it is formal except for one point which we should discuss, namely a compatibility between the Grothendieck duality for closed immersions stated in Lemma 2 and the Grothendieck-Serre duality for smooth proper maps we stated earlier in the course. Suppose $p : X \to S$ is a smooth proper map. Consider the composition

$$X \to X \times_S X \to X$$

of the diagonal Δ and the first projection p_1. Then certainly $id = p_1 \circ \Delta$, so

$$\mathcal{O} = \Delta^! p_1^! \mathcal{O}.$$

On the other hand using Grothendieck-Serre duality and Lemma 2 we can alternatively calculate the right-hand side to be identified with

$$\Omega^d[d] \otimes (\Omega^d[d])^\vee \simeq \mathcal{O}.$$

We want these two identifications to be the same identification.

Now, we didn't actually explain how to produce the identification $p^!\mathcal{O} \simeq \Omega^d[d]$ in Grothendieck-Serre duality. On the other hand, we will explain how to produce the identification in Lemma 2, giving $\Delta^!\mathcal{O} \simeq (\Omega^d[d])^\vee$. It turns out that the first identification is actually produced using the second identification and the desired compatibility property, stated above, that we require. So really, the fact that this compatibility property holds follows from the definition of Grothendieck-Serre duality, which we never discussed in detail.

Behind this sleight-of-hand lies an important point which also pervades the discussion in this lecture. Grothendieck-Serre duality, defined using an evaluation pairing, is very tricky to produce directly. One of the reasons for this is that the evaluation pairing only exists when X is proper. On the other hand, the coevaluation pairing can be produced just when X is smooth; only the *property* of it being a perfect co-pairing requires the properness hypothesis. Intuitively speaking, this makes sense, because we know in the non-proper case that cohomology is dual not to cohomology, but to compactly supported cohomology; thus the perfect co-duality pairing should land in cohomology tensored with compactly supported cohomology, but compactly supported cohomology maps to cohomology, so we can just as well make a co-pairing of cohomology with itself which is only a duality in the proper case, and that is what we get from the cycle class of the diagonal. On the other hand, the map goes the wrong way to make a candidate pairing which is only a duality in the proper case.

Similarly, Grothendieck duality for closed immersions is much more explicit and easy to produce than Grothendieck-Serre duality. The reason is basically that we can work locally, which is not possible in the Grothendieck-Serre setting.

To finish, then, let's sketch the proof of this Lemma 2 giving Grothendieck duality for closed immersions. Since we're proving transverse base-change compatibility, it suffices to check all these conditions and produce the required identifications locally for the smooth topology. Thus we can assume $X = Spec(A)$ is an affine scheme and Y is the preimage of 0 under a smooth map

$$f = (f_1, \ldots, f_d) : X \to \mathbb{A}^d.$$

In particular, the map f is flat, and this implies that $(f_1, \ldots, f_d)$ is a regular sequence. Thus the coordinate ring of Y identifies with the ring

$$A//f_1, \ldots, f_d$$

obtained by successively *derived* modding out by $f_1, \ldots, f_d$. (It is concentrated in degree zero and thus equal to the ordinary modding out, but the important thing is that it is equal to the derived modding out.)

Now, it is fairly formal to see that a formula for $i^! : D(A) \to D(A//f_1, \ldots, f_d)$ is given by

$$i^! M = RHom_A(A//f_1, \ldots, f_d, M),$$

compared with

$$i^* M = M \otimes_A A//f_1, \ldots, f_d.$$

Let's produce, by induction on d, an identification

$$i^! M \simeq i^* M[-d]$$

which however depends on the choice of the regular sequence $f_1, \ldots, f_d$. This reduces to $d = 1$, where it comes from noting that

$$A//f = cof(A \xrightarrow{f} A),$$

so that on RHom out we find that

$$i^! M = fib(M \xrightarrow{f} M)$$

whereas on tensor product we find

$$i^* M = cofib(M \xrightarrow{f} M).$$

Then we conclude by the fact that fiber is the -1 shift of cofiber.

To finish the proof, we need to see that changing the regular sequence generating the ideal induces the same effect on this twist as the induced change on the top exterior power of the conormal bundle. This can be proved by a brutal calculation. There is probably also a more enlightened approach to all of this, where you use the deformation to the normal bundle again to reduce to producing your data for the zero section of a vector bundle, and then there you use a Koszul complex to make the canonical identification.

This finishes our discussion of Poincaré duality, and finishes the course too!

Exercise 5. *Use the deformation to the normal bundle and first Chern class to produce a canonical Gysin map $dR_Y \otimes \mathcal{O}_S[-2]\{1\} \to dR_X$ associated to a codimension one closed inclusion of smooth S-schemes. Take the cofiber: it is a filtered object in $D(S)$. When X is affine, show that this filtered object corresponds to a cochain complex, i.e. its n^{th} associated graded lives only in degree $-n$, and identify the corresponding cochain complex with the de Rham complex with log poles discussed in the exercises to the lecture on Grothendieck's theorem.*