Lecture 4: de Rham cohomology for complex manifolds

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Let's get back to de Rham cohomology, but move from real manifolds to complex manifolds. This is a big change. Recall that if a function between open subsets of $\mathbb{C}^n$'s is complex differentiable at every point in its domain, then in fact it is infinitely complex differentiable, and even more it is analytic: it has a convergent power series expansion at every point, where convergent means convergent on some polydisk centered at the point. A function satisfying these equivalent properties is called *holomorphic*.

So the wide range of different kinds of real manifolds going from C^1 to C^∞ to real analytic just collapses to one single possibility in the complex domain. This also means that there are often a variety of different approaches to the same result, from the more analytic, e.g. using Cauchy-Riemann equations, to the more algebraic, e.g. using power series expansions. The whole feel of the subject is sort of half way between analysis and algebra.

In any case, to a complex manifold M we can associate a de Rham complex

$$\Omega^\bullet(M) = \left[\Omega^0(M) \to \Omega^1(M) \to \ldots\right],$$

just as in the real case, characterized as a functor of M by the following:

1. $\Omega^\bullet(M)$ is a commutative differential graded algebra, contravariantly functorial in the manifold M;
2. $\Omega^0(M)$ identifies with the ring $\mathcal{O}(M)$ of holomorphic functions on M, functorially in M;
3. For arbitrary M and $k \geq 0$, the presheaf $U \mapsto \Omega^k(U)$ on open subsets of M is a sheaf;
4. If U is an open subset of $\mathbb{C}^n$ with coordinate functions $z_1, \ldots, z_n \in \mathcal{O}(U)$, then:
   (a) As a $\mathcal{O}(U)$-module, $\Omega^1(M)$ is free of rank n with basis $dz_1, \ldots, dz_n$;
   (b) The multiplication map $\Lambda^k_{\mathcal{O}(U)}\Omega^1(U) \to \Omega^k(U)$ is an isomorphism; in particular the latter is free of rank $\binom{n}{k}$ with basis the $dz_{i_1} \ldots dz_{i_k}$ for $1 \leq i_1 < \ldots < i_k \leq n$ and vanishes for $k > n$;
   (c) For $f \in \mathcal{O}(U)$ we have $df = (\partial_1 f)dx_1 + \ldots + (\partial_n f)dx_n$.

Again, this set of requirements is fairly clearly uniquely determined, and its consistency encodes three fundamental theorems in calculus of many variables: the equality of mixed partial derivatives, which gives that $d^2 = 0$; the product rule, which verifies the CDGA axioms; and the chain rule, which ensures that the coordinate-based description is indeed functorial.

The de Rham complex of $*$ is simply $\mathbb{C}$ concentrated in degree 0. Thus, by functoriality we deduce a functorial comparison map

$$\mathbb{C} \to \Omega^\bullet(M)$$

of complexes for an arbitrary complex manifold M. (Just as in the real case, the interpretation of this is that constant functions have vanishing derivative.) And once again, we have the Poincaré lemma. But this time it's not enough to take $\mathbb{C}^n$ as a local model, because the complex unit disk $\mathbb{D}$ is not biholomorphic with $\mathbb{C}$. However there is a basis of open neighborhoods of every point in $\mathbb{C}^n$ consisting of polydisks: products of disks of varius radii in $\mathbb{C}$. Any disk is biholomorphic to a unit disk by scaling and translating, so every point of a complex manifold has a neighborhood basis of open subsets biholomorphic with $\mathbb{D}^n$. This leads to the statement of the Poincaré lemma in this context.

Lemma 1. *Suppose $M = \mathbb{D}^n$ is a polydisk. Then the map of complexes*

$$\mathbb{C} \to \Omega^\bullet(\mathbb{D}^n)$$

is a quasi-isomorphism.

Proof. We can use the same tensor product trick as in the real case to avoid writing down too many indices and getting confused. The $\mathbb{C}$-vector space $\mathcal{O}(\mathbb{D}^n)$ can be endowed with the structure of a Frechet space. Just as in the real case, we can use the topology of uniform convergence of all partial derivatives of all orders on all compact subsets.[1] Morever, Schwartz's kernel theorem holds, so

$$\mathcal{O}(\mathbb{D}^n) \otimes \mathcal{O}(\mathbb{D}^m) = \mathcal{O}(\mathbb{D}^{n+m})$$

for the projective tensor product. By comparing the basis coming from the coordinate functions, it follows that

$$\Omega^\bullet(\mathbb{D}^n) \otimes \Omega^\bullet(\mathbb{D}^m) = \Omega^\bullet(\mathbb{D}^{n+m})$$

as well.

Thus it suffices to show that $\Omega^\bullet(\mathbb{D})$ is the direct sum of $\mathbb{C}[0]$ and a chain acyclic complex. But indeed

$$\Omega^\bullet(\mathbb{D}) = \left[\mathcal{O}(\mathbb{D}) \overset{d/dz}{\to} \mathcal{O}(\mathbb{D})\right],$$

and complementary summand for the constant functions is given by the space $\mathcal{O}(\mathbb{D})_0$ of functions vanishing at 0, and $d/dz : \mathcal{O}(\mathbb{D})_0 \to \mathcal{O}(\mathbb{D})$ is an isomorphism by integrating, or taking the anti-derivative of the power series expansion term-by-term and noting it still converges in the open unit disk. ☐

Since as mentioned there's a neighborhood basis of every point consisting of polydisks, we deduce:

Corollary 2. *Over an arbitrary complex manifold M, the comparison map $\mathbb{C} \to \Omega^\bullet$ induces an isomorphism on sheafified homology groups.*

So far, there hasn't been that much difference with the real case. But now we'll see a serious divergence. From the previous corollary, we get a spectral sequence in sheaf cohomology

$$E_1^{p,q} = H^q(M; \Omega^p) \Rightarrow H^{p+q}(M; \mathbb{C}).$$

[1] Actually, there's a fun side remark to make here. The collapsing of hierarchy of differentiability in the complex case also has a manifestation in the Frechet space. One actually gets the same topology if one the topology of uniform convergence of just the function, not its derivatives, on arbitrary compact subsets. This is because any compact subset is contained in the interior of a larger compact subset, and then Cauchy's integral theorem implies that a bound on the function on the larger compact implies a bound on the derivatives of the function on the smaller one. One can also equivalently take a power series perspective and take basically any family of sequence space norms limiting to the correct convergence criterion.

Let's review the construction of this spectral sequence. For any complex $C^\bullet$ of presheaves of abelian groups on M, let

$$R\Gamma(M; C^\bullet) \in D(\mathbb{C})$$

denote its hypercohomology, defined as $\Gamma(M; |C^\bullet|^{sh})$ the notation of the last lecture. Hypercohomology is a functor from the ordinary category of complexes of presheaves to the ∞-category $D(\mathbb{C})$. Moreover, it sends short exact sequences of complexes of presheaves to fiber-cofiber sequences in $D(\mathbb{C})$, and it inverts maps which induce an isomoprhism on sheafified homology groups.

Taking the brutal truncation of the de Rham complex of presheaves and applying hypercohomology, we deduce a filtration on $R\Gamma(M; \mathbb{C})$ given by

$$F^{\geq p} R\Gamma(M; \mathbb{C}) = R\Gamma(M; F^{\geq p}\Omega^\bullet),$$

with associated graded (meaning, the cofiber of $F^{\geq p+1} \to F^{\geq p}$) given by $R\Gamma(M; \Sigma^p \Omega^p) = \Sigma^p R\Gamma(M; \Omega^p)$. Then the spectral sequence of this filtered object, as constructed in "Higher algebra" in the general context of a stable ∞-category with t-structure, gives the above spectral sequence

$$E_1^{p,q} = H^q(M; \Omega^p) \Rightarrow H^{p+q}(M; \mathbb{C}),$$

known as the Hodge-de Rham spectral sequence.

In the real case, this was concentrated on the $q = 0$ line. But here it won't be, at least not in general. The study of this spectral sequence motivates us to look at the cohomology of vector bundles on complex manifolds, and thankfully there a bunch of fantastic theorems in that subject. Here by a vector bundle on a complex manifold we always mean a holomorphic vector bundle, so the gluing data for a local trivialization is given by holomorphic maps $U_{ij} \to GL_n(\mathbb{C})$. Holomorphic vector bundles are thereby equivalent to sheaves of $\mathcal{O}$-modules which are locally free of finite rank, via the functor of taking (holomorphic) sections, and it is in that guise that we will be considering them. Each Ω^p gives a prime example.

For an arbitrary (paracompact) complex manifold, there is the following general bound on the cohomology, which I suppose is due to Dolbeault since it follows from the so-called Dolbeault resolution.

Theorem 3. *Let $\mathcal{E}$ be a vector bundle on a complex manifold M of dimension d. Then*

$$H^q(M; \mathcal{E}) = 0$$

for $q > d$.

Here d stands for the complex dimension of M. So the real dimension is $2d$, and that is also the covering dimension of M or any open subset of M. By an exercise to the previous lecture, it follows that the cohomology of an arbitrary sheaf on M vanishes above degree $2d$. So on a complex manifold we can cut that in half for a vector bundle. On a C^∞ real manifold we could even cut it all the way down to 0 with bump functions, but we will soon see that in the complex case one can indeed have a non-vanishing cohomology group in degree d, so Dolbeault's result is sharp in that sense.

This theorem gives a nice picturesque feel to the E_1 page of our spectral sequence

$$E_1^{p,q} = H^q(M; \Omega^p) \Rightarrow H^{p+q}(M; \mathbb{C}).$$

Namely, the de Rham complex has length d so on the p-variable we only have entries from 0 to d, and now we also know the same for the q variable. Thus the E_1-page is concentrated in a d-by-d box.

This actually gibes well with the fact that the cohomology of M lives only in degrees 0 to $2d$; in fact it gives another proof of that fact, at least for cohomology with $\mathbb{C}$-coefficients. Speaking of which, by staring at the spectral sequence we see that there's always a surjective edge map

$$H^d(M;\Omega^d) \to H^{2d}(M;\mathbb{C}),$$

so if M is compact and connected, whence the top cohomology group has dimension one (note that a complex manifold always carries an orientation because holomoprhic maps preserve orientations), it follows that $H^d(M;\Omega^d) \neq 0$, justifying the claim we made earlier that there can indeed be cohomology in the top dimension d. But actually one can be much more precise about this, as we will now see in our discussion of the cohomology of vector bundles on compact complex manifolds.

The first theorem is the Cartan-Serre theorem.

Theorem 4. *Let M be a compact complex manifold and $\mathcal{E}$ a vector bundle on M. Then for all $q \geq 0$, the $\mathbb{C}$-vector space*

$$H^q(M;\mathcal{E})$$

is finite dimensional.

This is in stark contrast to the real case, where these vector spaces are zero for $q > 0$, but huge infinite dimensional Frechet spaces for $q = 0$. On the other hand, the $H^q(M;\Omega^p)$ are the E_1 page of a spectral sequence converging to $H^q(M;\mathbb{C})$, which is a finite dimensional vector space by compactness of M. So in some sense, even though there are potentially many nontrivial differentials in the complex case, still the E_1 page is much closer to the final answer than in the real case, even though there's only one nonzero differential in the real case, the first one.

In fact, one can also use the Cartan-Serre theorem to see that at least some of the differentials have to vanish for non-formal reasons.

Corollary 5. *If M is a connected compact complex manifold, then $\mathcal{O}(M) = \mathbb{C}^{\pi_0(M)}$, the ring of locally constant functions on M. Hence, all differentials leaving the $(0,0)$ spot in the spectral sequence must vanish.*

Proof. By the Cartan-Serre theorem, $\mathcal{O}(M)$ is finite dimensional. Thus it is a reduced finite-dimensional commutative $\mathbb{C}$-algebra, hence is isomorphic as a $\mathbb{C}$-algebra to a finite product of copies of $\mathbb{C}$. Thus $\mathcal{O}(M)$ is spanned by idempotents; however idempotent functions are clearly locally constant. $\square$

Further information is provided by the following theorem, known as Serre duality:

Theorem 6. *Let M be a connected compact complex manifold of dimension d. Then the edge map $H^d(M;\Omega^d) \to H^{2d}(M;\mathbb{C}) \simeq \mathbb{C}$ is an isomorphism, and for any vector bundle $\mathcal{E}$ on M and $q \geq 0$ the pairing*

$$H^q(M;\mathcal{E}) \otimes H^{d-q}(M;\mathcal{E}^\vee \otimes \Omega^d) \to H^d(M;\Omega^d) \simeq \mathbb{C}$$

is a perfect pairing.

The first statement is equivalent to saying that any differential landing at the (d,d) entry must be 0. (Actually, for degree reasons the only possible nonzero differential is the first one, so the claim is just that $H^d(M;\Omega^{d-1}) \to H^d(M;\Omega^d)$ is zero.)

What does the duality statement tell us about the Hodge cohomology groups $H^q(M;\Omega^p)$? Well, note that $(\Omega^p)^\vee \otimes \Omega^d = \Omega^{d-p}$ because of the perfect pairing $\Omega^p \otimes \Omega^{d-p} \to \Omega^d$ coming from multiplication.

Thus Serre duality implies that $H^q(M;\Omega^p)$ has is dual to $H^{d-q}(M;\Omega^{d-p})$. In terms of the E_1-page of the Hodge-de Rham spectral sequence, this is a duality which flips along the diagonal $p+q=d$.

In fact, there is a duality pairing on the whole spectral sequence which abuts to Poincare duality on the topological cohomology $H^*(M;\mathbb{C})$. We will discuss this phenomenon in more detail in the context of algebraic de Rham cohomology, so let's not get into it now.

At this point one may get optimistic and hope that all the differentials must vanish when M is a compact complex manifold. This is not true, but still it's true in many cases of interest, by the following result in Hodge theory:

Theorem 7. *Let M be a compact complex manifold which admits a Kähler metric. Then the Hodge-de Rham spectral sequence*

$$E_1^{p,q} = H^q(M;\Omega^p) \Rightarrow H^{p+q}(M;\mathbb{C})$$

degenerates at the E_1-page. Thus, there is a canonical filtration $F^{\geq p}H^*(M;\mathbb{C})$ on the topological cohomology, the Hodge filtration*, with associated graded given by $F^{\geq p}/F^{\geq p+1} = H^{*-p}(M;\Omega^p)$.*

Note that since the whole spectral sequence is functorial in the complex manifold M via pullback, this Hodge filtraiton on cohomology is also functorial (for holomoprhic maps, that is). The maps don't need to respect the Kähler metric in any sense; the metric is just a black box whose existence guarantees degeneration.

Example of Kähler manifolds abound. First of all and most importantly for us, the analytificiation of any smooth projective variety carries a Kähler metric by restricting the Fubini-Study metric on $\mathbb{P}^n$. But there are plenty of other examples as well, for instance any complex torus (quotient of $\mathbb{C}^d$ by a lattice) inherits the canonical flat metric from $\mathbb{C}^d$ which is Kähler. (Most complex tori are not algebraic, so this is a different set of examples from the previous.)

Degeneration means that all the differentials vanish, at all entries, on all pages. You might think of the differentials on page 7 as being kind of scary to try to study, so what does degeneration really mean? Actually, it's easy to rephrase it without mentioning the spectral sequence. It simply means that the Hodge filtration

$$F^{\geq p}R\Gamma(X;\mathbb{C})$$

on the object in $D(\mathbb{C})$ calculating the topological cohomology, coming from taking hypercohomology of the brutal filtration on the de Rham complex, satisfies the property that for all p, the fiber-cofiber sequence

$$F^{\leq p+1}R\Gamma(X;\mathbb{C}) \to F^{\leq p}R\Gamma(X;\mathbb{C}) \to R\Gamma(X;\Omega^p[p])$$

gives rise to a short exact sequence

$$H_*F^{\leq p+1}R\Gamma(X;\mathbb{C}) \to H_*F^{\leq p}R\Gamma(X;\mathbb{C}) \to H_*R\Gamma(X;\Omega^p[p])$$

on cohomology in every degree, i.e. all the boundary maps in the associated long exact sequence vanish.

In essence, it means that the Hodge filtration on $R\Gamma(M;\mathbb{C})$ is compatible with the canonical filtration, so it passes to the associated graded for the canonical filtration, that is the topological cohomology groups of M.

Moreover, in situations such as ours where the E_1 page is finite dimensional, degeneration simply amounts to a dimension count: the *Betti number* $h^k = dim H^k(M;\mathbb{C})$ should equal the sum of the *Hodge numbers* $h^{p,q} = dim H^q(M;\Omega^p)$ for $p+q=k$. Any non-trivial differential would cut into the Hodge cohomology groups and make the Betti numbers smaller.

There is an important addition to the Hodge-de Rham degeneration, known as the Hodge decomposition. To explain it, note that there's extra structure on $R\Gamma(M;\mathbb{C})$: an integral lattice coming from the integral homology. But let's just remember the cohomology with real coefficients, or equivalently the action of complex conjugation on $R\Gamma(M;\mathbb{C})$ induced by the complex conjugation on $\mathbb{C}$. Using this, we get a new filtration on $H^*(M;\mathbb{C})$, namely the complex conjugate of the old filtration.

Theorem 8. *Let M be a compact complex manifold which admits a Kähler metric. Then the filtrations $F^{\geq p}$ and $\overline{F^{\geq p}}$ on each cohomology group $H^k(M;\mathbb{C})$ are* opposed*, and therefore induce a canonical direct sum decomposition*

$$H^k(M;\mathbb{C}) = \oplus_{p+q=k} H^q(M;\Omega^p).$$

What does it mean for two filrations of the same length to be opposed? This is a fun bit of linear algebra. If you have a vector space V with a filtration F of length n, then splitting the filtration and thereby getting a direct sum decomposition of V is completely equivalent to specifying another filtration G of length n which is *opposed* to F, which means that $F^i \oplus G^{n-i} = V$ for all $0 \leq i \leq n$.[2]

It's quite remarkable that the fact that two filtrations are needed to get a decomposition somehow matches with the fact that the complex numbers have two continuous automorphisms, identity and complex conjugation. Moreover, since the two filtrations are related by an automorphism, their terms and associated gradeds have matching dimensions. It therefore follows from the Hodge decomposition that for a complex Kähler manifold we have the *Hodge symmetry*

$$h^{p,q} = h^{q,p}$$

of the Hodge numbers, in addition to the symmetry

$$h^{p,q} = h^{d-p,d-q}$$

coming from Serre duality.

The structure on the cohomology of a Kähler manifold provided by the degeneration and decomposition theorems is extremely powerful and tells you a lot about your complex manifold. To give a simple example, in the case of complex tori of dimension one (elliptic curves), it completely recovers the elliptic curve! Let's take a look at the Hodge cohomology groups $H^q(E;\Omega^p)$ of an elliptic curve E. Since E is one dimensional, they vanish except for when (p,q) is $(0,0)$, $(1,0)$, $(0,1)$, or $(1,1)$. In bidegrees $(0,0)$ and $(1,1)$ we get the value $\mathbb{C}$ just as for a general compact complex manifold. But on an elliptic curve $\Omega^1 = \mathcal{O}$ because of the nonvanishing holomorphic differential coming from the universal cover. Thus all the rows are isomoprhic, so all the groups have to be $\mathbb{C}$. Now, the Hodge decomposition theorem is trivial in all degrees in this case as for any Riemann surface, and similarly the Hodge decomposition theorem is trivial in cohomological degrees 0 and 2.

But in degree one the Hodge decomposition theorem is really saying something, namely that a nonvanishing holomorphic 1 form ω cannot represent a real cohomology class, or in other words if you integrate it along a basis of $H_1(E;\mathbb{Z})$, then at least one of the values will be non-real. Or, since we can always multiply ω by a nonzero constant, we're really saying that these two integrals will never be real multiples of each other.

[2]You can also think of this in terms of the subgroups of $GL(V)$ stabilizing the given data. The stabilizer of a filtration is a so-called parabolic subgroup; splitting the filtration reduces this parabolic to a Levi. Then the algebraic group phrasing of this linear algebraic fact is that if you have a parabolic subgroup, then giving a Levi is the same thing as giving an opposite parabolic subgroup. (If your first parabolic subgroup is block upper triangular, the opposite one will be block lower triangular.)

This means that the map

$$H_1(E;\mathbb{C}) \to H^0(E;\Omega^1)^\vee$$

dual to the edge map $H^0(E;\Omega^1) \to H^1(E;\mathbb{C})$ restricts to an embedding of $H_1(E;\mathbb{Z})$ as a full lattice inside $H^0(E;\Omega^1)^\vee \simeq \mathbb{C}$. The elliptic curve is recovered as the quotient.

So indeed, the moduli of elliptic curves is the same as the moduli of all the cohomological structure you get on H^1 coming from Hodge theory. Even for more complicated complex manifolds, the so-called Hodge structure you obtain contains an amazing amount of information about the moduli.

The last topic I'd like to touch on is that of Stein manifolds, a kind of complex manifold whose overall behavior is more analogous to that of C^∞ real manifolds, in contrast to the compact case. There are many many many many different definitions of Stein manifolds and it takes a book to prove they're all equivalent. But let's give the following naive definition.

Definition 9. *A complex manifold is called* Stein *if it is biholomorphic to a closed complex submanifold of some $\mathbb{C}^n$.*

If you're familiar with real manifolds you may be asking yourself whether this is any condition at all, because every real manifold is a closed submanifold of some $\mathbb{R}^d$. But any compact Stein manifold must just be a finite set of points, because any holomorphic map from a compact complex manifold to $\mathbb{C}^n$ is locally constant as we showed above from the Cartan-Serre theorem, hence it can't be an embedding unless the manifold is finite.

On the other hand, there are many examples of Stein manifolds:

1. If X is a smooth affine algebraic variety over $\mathbb{C}$, then its analytification X^{an} is Stein. This is clear, because there's an algebraic closed embedding of X into $\mathbb{A}^n_{\mathbb{C}}$, so X^{an} is cut out of $\mathbb{C}^n$ by a finite set of polynomials satisfying the Jacobian criterion.

2. The product of two Stein manifolds is Stein, and any closed submanifold of a Stein manifold is Stein. This is obvious.

3. If two open subsets U and V of a complex manifold M are Stein, then so is their intersection $U \cap V$. This follows from the previous point, because $U \cap V$ is also the intersection of $U \times V$ with the diagonal $M \subset M \times M$.

4. Any polydisk is a Stein manifold. This is not at all obvious; we need to turn an open embedding into a closed embedding somehow. But actually any non-compact Riemann surface is a Stein manifold, again for non-obvious reasons; it follows that the open disk is Stein, hence so is any polydisk.

5. By the previous two points, any complex manifold has a basis for the topology closed under finite intersection consisting of Stein manifolds. This is useful for local-global reductions of general complex manifolds to Stein manifolds.

The main result on cohomology of vector bundles on a Stein manifold is called *Cartan's Theorem B*:

Theorem 10. *Let $\mathcal{E}$ be a vector bundle on a Stein manifold M. Then*

$$H^q(M;\mathcal{E}) = 0$$

for $q > 0$.

In particular, the Hodge-de Rham spectral sequence sits along the $q = 0$ line just as in the real case, and the topological cohomology is computed by the de Rham complex:

Corollary 11. *For a Stein manifold M, we have*

$$R\Gamma(M;\mathbb{C}) \simeq |\Omega^\bullet(M)|.$$

This has some concrete meaning: for example, a closed one-form ω on M has an anti-derivative $f \in \mathcal{O}(M)$ if and only if its integral over any cycle in $H_1(M;\mathbb{Z})$ vanishes.

Example 12. *Let $M = \mathbb{C}^\times$, the punctured complex plane. We can use the everywhere nonvanishing global one form dz to trivialize the cotangent bundle, and we deduce that a holomorphic function $g \in \mathcal{O}(\mathbb{C}^\times)$ has an antiderivative if and only if the integral $\int g dz$ once around the origin vanishes. Of course, this is a familiar fact from complex analysis.*

One can also directly calculate the holomorphic de Rham complex using Laurent series and see that it is one-dimensional in degree one, generated by dz/z.

On the other hand, in the Stein case just as in the real manifold case the terms in the de Rham complex are infinite dimensional so there has to be a lot of cancellation in the differentials. (Many Stein manifolds, though not compact, do have finite dimensional cohomology: for example this is the case for any algebraic variety.) Actually, another theme of Stein manifolds, closely related to Theorem B, is that the global sections tell you everything you need to know. For example, there is the following result of Forster, the analog of the Serre-Swan theorem:

Theorem 13. *Let M be a Stein manifold. Then the functor $\mathcal{E} \mapsto \mathcal{E}(M)$ of global sections gives an equivalence of categories from the category of vector bundles on M to the category of finitely generated projective $\mathcal{O}(M)$-modules.*

In this and many other ways, the ring of holomorphic functions on a Stein manifold controls the whole manifold. Thus, Stein manifolds are also the analog of affine schemes in the world of complex manifolds.

Exercise 14.

1. *Calculate the cohomology $H^q(\mathbb{C}^2 \smallsetminus \{(0,0)\};\mathcal{O})$ using the Stein cover by $U = \mathbb{C}^\times \times \mathbb{C}$ and $V = \mathbb{C} \times \mathbb{C}^\times$. Deduce that $\mathbb{C}^2 \smallsetminus \{(0,0)\}$ is not Stein.*
2. *Consider the* Hopf surface, *the quotient of $\mathbb{C}^2 \smallsetminus \{(0,0)\}$ by the action of the group $\mathbb{Z}$ via $k\cdot(a,b) = (2^k a, 2^k b)$. Calculate the topological cohomology and Hodge cohomology of the Hopf surface, and deduce that Hodge degeneration holds, but Hodge decomposition does not.*