Lecture 5: Grothendieck's theorem on algebraic de Rham cohomology

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Recall from the previous lecture that if M is a Stein manifold, then there is a natural isomorphism

$$R\Gamma(M;\mathbb{C}) \simeq |\Omega^\bullet(M)|,$$

in $D(\mathbb{C})$, i.e. the holomorphic de Rham complex of M calculates the topological cohomology of M with complex coefficients. Recall also that examples of Stein manifolds are given by the analytifications $M = X^{an}$ of smooth affine algebraic varieties X over the complex numbers. Grothendieck's theorem says that for such a Stein manifold, one can use the much smaller complex of algebraic differential forms.

Theorem 1. *Let X be a smooth affine algebraic variety over $\mathbb{C}$. Then the natural map*

$$\Omega^\bullet(X) \to \Omega^\bullet(X^{an}),$$

from the algebraic de Rham complex of X to the holomorphic de Rham complex of its analytification, is a quasi-isomoprhism.

Thus, algebraic de Rham cohomology of a smooth affine variety computes the ordinary topological cohomology of the underlying topological space of $\mathbb{C}$-valued points. We will of course define and study algebraic de Rham cohomology from a systematic perspective later, but for now let's explore this theorem in a more hands-on way. First, here are some examples.

Example 2. *Let $X = \mathbb{A}^1 = \mathrm{Spec}(\mathbb{C}[t])$. Then the algebraic de Rham complex is*

$$\left[\mathbb{C}[t] \overset{d/dt}{\to} \mathbb{C}[t]\right].$$

The only polynomials with 0 derivative are the constants, and every polynomial $\sum_{n\geq 0} c_n x^n$ has an antiderivative $\sum_{n\geq 0} c_n \frac{x^{n+1}}{n+1}$. Thus the cohomology is $\mathbb{C}$ in degree 0 and vanishes in all other degrees. This matches the topological cohomology of $\mathbb{A}^1(\mathbb{C}) = \mathbb{C}$ as the latter is contractible.

Example 3. *Let $X = \mathbb{G}_m = \mathbb{A}^1 \setminus \{0\} = \mathrm{Spec}(\mathbb{C}[t,t^{-1}])$. Then the algebraic de Rham complex is*

$$\left[\mathbb{C}[t,t^{-1}] \overset{d/dt}{\to} \mathbb{C}[t,t^{-1}]\right].$$

Again the H^0 term is $\mathbb{C}$, but now the H^1 term is also $\mathbb{C}$ because t^{-1} has no anti-derivative. This matches the topological cohomology of $\mathbb{G}_m(\mathbb{C}) = \mathbb{C} \setminus \{0\} \sim S^1$.

Remark 4. *If, as in the above two cases, our variety X is defined over $\mathbb{Q}$, then a very interesting phenomenon appears in the topological cohomology of its complex points. Namely, the cohomology with $\mathbb{C}$-coefficients carries two different rational structures: the one coming from topological cohomology with $\mathbb{Q}$-coefficients, and the other coming from algebraic de Rham cohomology with $\mathbb{Q}$-coefficients. The fact that they're different can be measured by integrating algebraic differential forms defined with $\mathbb{Q}$-coefficients along cycles in singular homology defined with $\mathbb{Q}$ coefficients, and noting that more often than not, these so-called* periods *are irrational numbers. The most basic example is the familiar contour integral $\int \frac{dz}{z}$ around the unit circle, which gives $2\pi i$.*

Example 5. *Fix constants $a, b \in \mathbb{C}$ such that the polynomial $x^3 + ax + b$ has no repeated roots, and consider the punctured elliptic curve*

$$X = Spec(\mathbb{C}[x,y]/(f(x,y)))$$

where $f(x,y) = y^2 - x^3 - ax - b$. This is a smooth affine curve, and the de Rham complex is the two-term complex

$$[\mathcal{O}(X) \to \mathcal{O}(X)\{dx, dy\}/\partial_1 f dx + \partial_2 f dy],$$

where the first term is $\mathcal{O}(X) = \mathbb{C}[x,y]/(f(x,y))$, the second term is the quotient of the free $\mathcal{O}(X)$-module on two generators dx, dy by the single relation which appears, and the differential is $\mathbb{C}$-linear, satisfies the Leibniz rule, and sends x and y to the formal generators dx and dy.

The topological space $X(\mathbb{C})$ is a punctured 2-torus, so Grothendieck's theorem tells us to expect a one-dimensional H^0 and a two-dimensional H^1. You will verify this as an exercise by producing a global one-form $\omega \in \Omega^1(X)$ such that the classes of ω and $x\omega$ give a basis for the first algebraic de Rham cohomology.

Example 6. *For the algebraic de Rham complex, we have a Künneth formula $\Omega^\bullet(X) \otimes_{\mathbb{C}} \Omega^\bullet(Y) \simeq \Omega^\bullet(X \times Y)$ for the algebraic tensor product. Using this one can see that Grothendieck's theorem holds for a product of varieties if it holds for each factor. In particular we get higher-dimensional examples from the above elementary one-dimensional examples.*

Grothendieck's theorem in this naive form only holds for affine varieties. There is a more general form which holds for all smooth varieties, but that requires hypercohomology. On the other hand, the proof of the a priori simpler affine case crucially uses compactification and therefore requires a treatment of general varieties, including hypercohomology! We'll explain a part of this now and finish up in the next lecture.

Let's start by making a discussion of the de Rham complex of smooth varieties, in parallel to our discussion of the de Rham complex in the analytic settings. Again, we'll give a more thorough discussion later where we'll adopt a different definition and see that it satisfies the properties we'll use here to characterize the theory.

To every smooth algebraic variety X over the complex numbers we can assign a complex

$$\Omega^\bullet(X) = \left[\Omega^0(X) \to \Omega^1(X) \to \dots\right],$$

characterized as a functor of X by the following:

1. $\Omega^\bullet(X)$ is a commutative differential graded algebra, contravariantly functorial in the variety X;
2. $\Omega^0(X)$ identifies with the ring $\mathcal{O}(X) = Hom(X, \mathbb{A}^1)$ of (algebraic) functions on X, functorially in X;

3. For arbitrary X and $k \geq 0$, the presheaf $U \mapsto \Omega^k(U)$ on Zariski open subsets of X is a sheaf;

4. If U admits an etale map $U \to \mathbb{A}^n$ with induced coordinate functions $x_1, \ldots, x_n \in \mathcal{O}(U)$, then:
   (a) As a $\mathcal{O}(U)$-module, $\Omega^1(U)$ is free of rank n with basis $dx_1, \ldots, dx_n$;
   (b) The multiplication map $\Lambda^k_{\mathcal{O}(U)}\Omega^1(U) \to \Omega^k(U)$ is an isomorphism; in particular the latter is free of rank $\binom{n}{k}$ with basis the $dx_{i_1} \ldots dx_{i_k}$ for $1 \leq i_1 < \ldots < i_k \leq n$ and vanishes for $k > n$.

Note the slight differences from the analytic case:

1. First of all, we have to use the very coarse Zariski topology since that's the only one that makes algebraic sense.

2. Second of all, we have to use etale maps $U \to \mathbb{A}^n$ in our local description, since it's not true that any smooth variety is locally isomorphic to an open subset of $\mathbb{A}^n$. (Here "locally" and "open subset" are meant in the sense of the Zariski topology.) Indeed, any smooth affine curve has a unique compactification to a smooth proper curve — just by naively adding the missing points at infinity, basically — so any nonempty Zariski open subset of, e.g., an elliptic curve "knows" the elliptic curve it came from and can't be isomorphic to an open subset of $\mathbb{A}^1$ since that would complete to $\mathbb{P}^1$.

   On the other hand it is true that any smooth variety has a Zariski open cover by U which admit an etale map to $\mathbb{A}^n$. We'll review this and say a bit more later, but for now the key point is that if $U \to \mathbb{A}^n$ is etale then the induced map $U^{an} \to \mathbb{C}^n$ of analytic spaces is a local isomorphism. This is why the $dx_1, \ldots, dx_n$ can be expected to form a basis even if U^{an} is not literally an open subset of $\mathbb{C}^n$.

3. Finally, and this is a nice simplification in the algebraic context, we no longer have to specify "by hand" that $df = \partial_1 f dx_1 + \ldots \partial_n f dx_n$ when $f \in \mathcal{O}(\mathbb{A}^n)$, since such an f is by definition a polynomial in $x_1, \ldots x_n$ and so this follows from the Leibniz rule in the CDGA structure.

Now, the theme here is the comparison with the holomorphic de Rham complex of the associated complex analytic manifold X^{an}. There is a unique natural transformation

$$\Omega^\bullet(X) \to \Omega^\bullet(X^{an})$$

from the algebraic de Rham complex of a smooth variety to the holomorphic de Rham complex of its analytification, such that in degree zero it identifies with the inclusion of algebraic maps into holomorphic maps.

Claim 7. *This natural transformation induces a natural map on hypercohomology*

$$R\Gamma(X; \Omega^\bullet) \to R\Gamma(X^{an}; \Omega^\bullet).$$

Proof. This is perhaps not entirely obvious, because the underlying topological spaces are different on each side. Nonetheless, it is completely formal. If $p : X^{an} \to X$ denotes the natural continuous map for which $p^{-1}(U) = U^{an}$ for a Zariski open $U \subset X$, then for a presheaf of sheaf $\mathcal{F}$ on X^{an} with values in any ∞-category we have $p_*\mathcal{F}(U) = \mathcal{F}(U^{an})$, so what we can literally deduce from our above natural transformation is a map of presheaves on X with values in $D(\mathbb{C})$

$$|\Omega^\bullet(-)| \to p_*|\Omega^\bullet((-)^{an})|.$$

We can then compose with the sheafification map $|\Omega^\bullet((-)^{an})| \to |\Omega^\bullet((-)^{an})|^{sh}$ and use that p_* preserves sheaves to deduce a comparison map

$$|\Omega^\bullet(-)|^{sh} \to p_*(|\Omega^\bullet((-)^{an})|^{sh}),$$

whence the claim by taking global sections. □

Now we can state Grothendieck's theorem in the general case.

Theorem 8. *Let X be a smooth variety over $\mathbb{C}$. Then the comparison map*

$$R\Gamma(X;\Omega^\bullet) \tilde{\to} R\Gamma(X^{an};\Omega^\bullet)$$

from the algebraic de Rham cohomology to the analytic de Rham cohomology is an isomorphism. In particular, the algebraic de Rham cohomology calculates the topological cohomology $R\Gamma(X^{an};\mathbb{C})$, as the analytic de Rham cohomology calculates this by the Poincare lemma for complex manifolds, as explained in the previous lecture.

Note that we are defining the algebraic and complex-analytic de Rham cohomology as the *hyper*cohomology of the corresponding de Rham complexes of sheaves, not as the cohomology of the de Rham complex itself (the global sections of the complex of sheaves). This is the appropriate definition in general.

Lemma 9. *The two forms of Grothendieck's theorem are equivalent. More specifically:*

1. *if X is affine, then*
$$R\Gamma(X;\Omega^\bullet) = |\Omega^\bullet(X)|$$
and
$$R\Gamma(X^{an};\Omega^\bullet) = |\Omega^\bullet(X^{an})|,$$
so the first form of Grothendieck's theorem is equivalent to the special case of the second from where X is affine; and

2. *the second form of Grothendieck's theorem reduces to the affine case.*

Proof. To prove the first claim, we need to show that $|\Omega^\bullet(-)| \to |\Omega^\bullet(-)|^{sh}$ is an isomorphism on X. Filtering by the brutal truncation, it suffices to show the same for the associated gradeds, so we need to see that

$$|\Omega^p(-)| \to |\Omega^p(-)|^{sh}$$

is an isomorphism on X. But this holds true because $\Omega^p(-)$ is a quasicoherent sheaf of $\mathcal{O}$-modules on X (indeed, it is locally free of finite rank), hence it has no higher cohomology as X is affine. (This can be proved by a faithfully flat descent argument which we'll revisit later.)

In down-to-earth terms (provided spectral sequences count as being down-to-earth), we're saying that the spectral sequence

$$H^q(X;\Omega^p) \Rightarrow H^{p+q}(X;\Omega^\bullet)$$

sits along the $q=0$ line hence the $q=0$ line calculates the hypercohomology on the right.

Similarly in the analytic context, $|\Omega^\bullet(-)| \to |\Omega^\bullet(-)|^{sh}$ is an isomorphism on X^{an} because there is no higher cohomology of vector bundles on Stein manifolds.

Finally, the second claim, that Grothendieck's theorem reduces to the affine case, follows by applying the following lemma to $\mathcal{F} = |\Omega^\bullet(-)|$ and $\mathcal{G} = p_*|\Omega^\bullet((-)^{an})|^{sh}$, in the notation of Claim 7. □

The following local-global lemma was used in the proof:

Lemma 10. *Let X be a scheme, and let $\mathcal{F} \to \mathcal{G}$ be a map of sheaves on X with values in any target ∞-category. If $\mathcal{F}(U) \tilde{\to} \mathcal{G}(U)$ for any affine $U \subset X$, then $\mathcal{F} \tilde{\to} \mathcal{G}$.*

Proof. By Yoneda we can reduce to where the target ∞-category is $\mathcal{S}$. It suffices to show that for any open subset U of X, the representable sheaf h_U lies in the smallest co-complete full subcategory of $Sh(X;\mathcal{S})$ generated by the h_V for V an affine open. First assume U is separated. Then the intersection of any two affine open subset of U is also an affine open, hence any cover of U by affines will prove the claim. Now assume U is general. Any intersection of two affine open subsets of U is not necessarily affine but it is separated, so again choosing any affine cover we can reduce to the separated case which was already handled. ☐

However, this reduction to the affine case is in some sense not the point, because the easiest case actually lies at the opposite extreme: the *proper* case! (Properness is an algebraic condition, generalizing projectiveness, which implies that the associated analytic space is compact.) That case can be handled very easily thanks to Serre's so-called *GAGA* theorem, which in the context of vector bundles over smooth varieties[1] says the following:

Theorem 11. *Let X be a smooth proper algebraic variety, and let $\mathcal{E}$ be a vector bundle over X with analytification $\mathcal{E}^{an}$ over X^{an}. Then the natural comparison map*

$$H^q(X;\mathcal{E}) \to H^q(X^{an};\mathcal{E}^{an})$$

is an isomorphism for all $q \geq 0$.

This is some sort of companion to the Cartan-Serre theorem on finiteness of cohomology groups of vector bundles on compact complex manifolds.

Example 12.

1. *For $q = 0$ and $\mathcal{E} = \mathcal{O}$, the claim is that any holomorphic map $X^{an} \to \mathbb{C}$ is an algebraic map $X \to \mathbb{A}^1$. Well, we saw from Cartan-Serre's theorem that such a map is constant if X^{an} connected, and constants are certainly algebraic. In general we'd still need some small argument to see that the connected components of X (meaning, in the Zariski topology) correspond to those of X^{an}.*

2. *For $q = 0$ and the line bundle $\mathcal{O}(n)$ on $\mathbb{P}^1$, the claim is that giving a meromorphic function on $\mathbb{P}^1$ with only one pole, at ∞, of pole order $\leq n$, is the same as giving a degree n polynomial.*

3. *As usual, the meaning on higher cohomology groups is a bit harder to tease out, but they are absolutely crucial for the proof even if you're only interested in $q = 0$!*

Using the GAGA theorem, we can show rather easily that Grothendieck's theorem holds in the proper case.

Theorem 13. *Let X be a smooth proper variety over $\mathbb{C}$. Then the map*

$$R\Gamma(X;\Omega^\bullet) \to R\Gamma(X^{an};\Omega^\bullet)$$

is an isomorphism.

[1] As with the theorems in the previous lecture, there is a more general version which applies to coherent sheaves over an arbitrary (proper) variety. But we don't want to take the time to get into coherent sheaves and non-smooth analytic spaces.

Proof. We can filter our de Rham complexes by their brutal truncations, and both on the algebraic and the analytic side $|\Omega^\bullet|$ is recovered as the limit of its truncated versions; indeed $|F^{\geq p}\Omega^\bullet|$ lives in degrees $\leq -p$ and hence only has homology in degrees $\leq -p$. The same statement holds after sheafification, because sheafification preserves upper bounds on homology.

The comparison map being functorial, this reduces us to the analogous assertion on the associated graded pieces, namely Ω^p sitting in a single degree. However, note that the comparison map $\Omega^p(-) \to \Omega^p((-)^{an})$ induces an identification $(\Omega^p)^{an} = \Omega^p((-)^{an})$, i.e., the analytification of the vector bundle over X^{an} of algebraic p-forms is the vector bundle over X of analytic p-forms. Indeed, this follows because locally they are free on the same basis over their respective coordinate rings. Thus the claim on associated graded reduces to GAGA. □

Exercise 14. *Go up and reread the example of the algebraic de Rham cohomology of a punctured elliptic curve $X = Spec(\mathbb{C}[x,y]/(y^2 - x^3 - ax - b))$. You may not use Grothendieck's theorem in the following, but anything else is fair game; in particular you are allowed to take the explicit description I gave of the de Rham complex for granted.*

1. *Prove that there is a unique one-form $\omega \in \Omega^1(X)$ which identifies with dx/y on the Zariski open subset where $y \neq 0$, and that $\Omega^1(X)$ is a free $\mathcal{O}(X)$-module of rank one on the class ω.*
2. *Prove that the classes of ω and $x\omega$ give a basis for the first de Rham cohomology $H^1_{dR}(X)$ as a $\mathbb{C}$-vector space.*
3. *Let E denote the elliptic curve with $X = E \smallsetminus \{\infty\}$. Prove that $H^1_{dR}(E) \xrightarrow{\sim} H^1_{dR}(X)$ (recall that the first term is defined as the hypercohomology of the de Rham complex of sheaves), and identify the Hodge filtration on $H^1_{dR}(E)$ in terms of your explicit descrition in 2.*