Lecture 6: Grothendieck's theorem, part two

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Let's start with a recap of last time. Let X be a smooth algebraic variety over $\mathbb{C}$. Grothendieck's theorem says that the natural comparison map

$$R\Gamma(X;\Omega^\bullet) \to R\Gamma(X^{an};\Omega^\bullet) \simeq R\Gamma(X^{an};\mathbb{C})$$

is an isomorphism. Recall that the first map comes from the natural map from the de Rham complex of X to the de Rham complex of its analytification X^{an}, and the last isomorphism comes from the Poincaré lemma, which implies that the de Rham complex of X^{an} is a resolution of the constant sheaf $\mathbb{C}$.

We saw that the case where X is proper, hence X^{an} is compact, was a simple consequence of Serre's GAGA theorem identifying algebraic cohomology of vector bundles with analytic cohomology. Indeed, GAGA implies that the comparison map

$$R\Gamma(X;\Omega^p) \to R\Gamma(X^{an};\Omega^p)$$

is an isomorphism for all p. Thus, in the proper case, Grothendieck's theorem even holds on the level of associated gradeds for the Hodge filtration (the brutal filtration on the de Rham complex). This is very far from being true in the affine case, for example when $p = 0$ and $q = 0$ we have $H^0(\mathbb{A}^1;\mathcal{O}) = \mathbb{C}[z]$ whereas $H^0((\mathbb{A}^1)^{an};\mathcal{O})$ is the ring of entire holomorphic functions on the complex plane, so we need a different argument to prove Grothendieck's theorem.

In general, we also saw by a local-global argument that we can reduce to the case where $X = U$ is affine. Actually, though, we will give an argument that works whenever U is just separated. The goal will be to still somehow make use of GAGA, by choosing an appropriate compactification of U. The simplest possible kind of compactification comes from the following theorem of Hironaka, a consequence of his work on resolution of singularities.

Theorem 1. *Let U be a smooth separated variety over $\mathbb{C}$. Then there exists an open immersion $j : U \to X$ with X a smooth proper variety over $\mathbb{C}$, such that the boundary variety $D = X \smallsetminus U$ is a* normal crossings divisor *on X.*

To say that $D \subset X$ is a normal crossings divisor means that for every $x \in X$ there is an open subset $U \subset X$ containing x, a etale map $f = (f_1, \dots, f_n) : U \to \mathbb{A}^n$ for some $n \geq 0$, and a $0 \leq k \leq n$ such that $D \cap U = Z(f_1 \cdot \dots \cdot f_k)$. Thus D is locally the preimage of the union of k coordinate hyperplanes in affine space under an etale map. When we analytify, this implies that $D^{an} \subset X^{an}$ is a normal crossings divisor in the analytic sense, meaning it is locally biholomorphic to the inclusion of a union of coordinate hyperplanes in affine space.

Remark 2. *One might very naively hope for an even better situation, where D is a smooth subvariety of X. But alas, this cannot be arranged in general. First of all, if U is affine, then D has to have codimension one, as otherwise one could see working locally that $\mathcal{O}(U) = \mathcal{O}(X)$. The simplest case $U = \mathbb{A}^n$ might give cause for optimism because it compactifies to $\mathbb{P}^n$ with smooth boundary divisor. However, $U = (\mathbb{A}^1 \setminus \{0,1\}) \times \mathbb{A}^1$ admits no such compactification. This can be proved using mixed Hodge theory. Probably there's a more elementary way; can you find one?*

From now on we will fix such a compactification $U \subset X$ with boundary divisor D. To prove the comparison map $R\Gamma(U;\Omega^\bullet) \to R\Gamma(U^{an};\Omega^\bullet)$ from the algebraic de Rham cohomology of U to its analytic de Rham cohomology is an isomorphism, we will fit it into a commutative diagram with maps of the following form:

$$R\Gamma(U;\Omega^\bullet) \leftarrow R\Gamma(X;\Omega^\bullet_{\langle D\rangle}) \to R\Gamma(X^{an};\Omega^\bullet_{\langle D^{an}\rangle}) \to R\Gamma(U^{an};\Omega^\bullet).$$

Then we will prove, by separate arguments, that all three maps are isomorphisms. The outer two maps take place completely in the algebraic, respectively analytic realms, and the middle map is where the comparison occurs between algebraic invariants and analytic ones. Since X is proper, we can hope to use GAGA there. Let's start with the purely analytic situation of the right hand map.

Lemma 3. *Let M be a complex manifold and $N \subset M$ a normal crossings divisor, meaning a closed subset such that the inclusion $N \subset M$ is locally on M isomorphic to a union of coordinate hyperplanes in affine space. The subsheaf of $\mathcal{O}_M$ consisting of holomorphic functions which vanish on N is locally free of rank one.*

Proof. We need to prove that on the unit polydisk $\mathbb{D}^n$ with coordinates $z_1,\dots,z_n$, if a holomorphic function f vanishes on the union of the first k coordinate hyperplanes then f is a multiple of $z_1\cdot\ldots\cdot z_k$. This is clear from the Taylor expansion. For example, suppose $n = 2$ and

$$f = \sum_{n,m} c_{n,m} z_1^n z_2^m$$

vanishes when $z_1 = 0$. Then the holomorphic function of one variable $f(0,z_2) = \sum_m c_{0,m} z_2^m$ is zero, hence all its coordinates are zero, hence f is divisible by z_1. □

A local generator for this ideal sheaf is called a local defining equation for N; such a local generator is therefore well-defined up to unit multiple. Now we can make the definition.

Definition 4. *Let M be a complex manifold and $N \subset M$ a normal crossings divisor. Write $j : M\setminus N \to M$ for the corresponding open inclusion. For $p \geq 0$, let*

$$\Omega^p_{\langle N\rangle} \subset j_*\Omega^p$$

denote the subsheaf with $\Omega^p_{\langle N\rangle}(U)$ defined as the subset of $j_*\Omega^p(U) = \Omega^p(U \setminus N)$ consisting of those p-forms which locally on U satisfy the following condition: they can be written in the form $g^{-k}\omega$ where ω extends to a p-form on U and g is a local defining equation for N.

In other words, these are meromorphic p-forms with at worst poles along the divisor D, as opposed to "essential singularities". Note that $d(g^{-1}) = -dg/g^2$. It follows that the de Rham differential on $j_*\Omega^\bullet$ restricts to a differential on these $\Omega^\bullet_{\langle N\rangle}$, making the latter a complex of sheaves on M.

By applying $|\cdot|$ to the inclusion defining $\Omega^\bullet_{\langle N\rangle}$, we get a comparison map of presheaves on M with values in $D(\mathbb{C})$

$$|\Omega^\bullet_{\langle N\rangle}| \to j_*|\Omega^\bullet|.$$

Theorem 5. *The induced map*

$$|\Omega^\bullet_{\langle N\rangle}| \to j_*(|\Omega^\bullet|^{sh})$$

exhibits the target as the sheafification of the source. Hence, on global sections, we get

$$R\Gamma(M;\Omega^\bullet_{\langle N\rangle}) \tilde{\to} R\Gamma(M\setminus N;\Omega^\bullet).$$

Proof. The target is a sheaf because pushforwards preserve sheaves. Thus it will suffice to show that the map is an isomorphism on stalks at any point. By definition, $N\subset M$ locally looks like a union of coordinate hyperplanes inside a polydisk $\mathbb{D}^m$. In that case $M\setminus N \simeq (\mathbb{D}^\times)^k\times\mathbb{D}^{m-k}$ is Stein, hence there is no higher sheaf cohomology for the Ω^p, so

$$(j_*|\Omega^p|^{sh})(M) = |\Omega^p|^{sh}(M\setminus N) = |\Omega^p(M\setminus N)|$$

and hence using the brutal filtration we find also $j_*(|\Omega^\bullet|^{sh})(M\setminus N) = |\Omega^\bullet(M\setminus N)|$. Thus we reduce to to the following Poincaré-type lemma, proved by Atiyah-Hodge. □

Lemma 6. *Let $n\geq 0$ and $0\leq k\leq n$. Then the inclusion*

$$\bigcup_n \frac{1}{(z_1\cdot\ldots\cdot z_k)^n}\Omega^\bullet(\mathbb{D}^n)\subset\Omega^\bullet\left((\mathbb{D}^\times)^k\times\mathbb{D}^{n-k}\right)$$

is a quasi-isomorphism.

Proof. As usual, we would like to use a tensor product trick as usual to reduce to the case of one variable. The slight concern is that we have left the world of Frechet spaces, since the terms in the subcomplex are sequential colimits of Frechet spaces. If we had a context where there was a tensor product defined on such sequential colimits of Frechet spaces, such that it gave the usual completed tensor product on nuclear Frechet spaces and commuted with these sequential colimits, we would conclude that our subcomplex, and indeed the inclusion itself, is a tensor product of the special cases where $n=k=1$ or $n=1,k=0$. There is indeed such a context, called liquid vector spaces, but let's just take it as a black box.

In the second special case the inclusion is the identity and the claim is vacuous. As for the first case, the subcomplex is just the two-term complex where both terms are the meromorphic functions on $\mathbb{D}$ with at worst a pole at 0, and the map is differentiation; the bigger complex, the de Rham complex of $\mathbb{D}^\times$, is the analogous thing but where an essential singularity at 0 is allowed. In either case we can use Laurent series to explicitly see that the cohomology is one-dimensional spanned by the constants in degree 0, and is one-dimensional spanned by dz/z in degree 1. The point, I guess, is that the anti-derivative of a function with at worst poles at 0 also has at worst poles at 0. □

This concludes our purely analytic work. Now let's do the same thing in the algebraic context. Suppose $D\subset X$ is a normal crossings divisor in a smooth variety X over $\mathbb{C}$. Again this implies that the sheaf of functions vanishing on D is locally generated by a single non-zerodivisor, unique up to units; but now this is more of a tautology than a claim that needs to be proved. This condition we just stated is simply the definition of D being a so-called *effective Cartier divisor*, and for the algebraic argument that's all we'll need; the normal crossings condition is overkill.

Once again we can define

$$\Omega^p_{\langle D\rangle}\subset j_*\Omega^p$$

to be the subsheaf consisting of those p-forms defined away from D which are locally of the form $g^{-k}\omega$ where ω extends over D and g is a local defining equation. But now, actually, this inclusion is an equality. Indeed, if we have a closed subscheme of an affine scheme $Spec(A)$ cut out by a single equation $g \in A$, then the open complement is also affine, with coordinate ring

$$A[1/g] = colim(A \xrightarrow{g} A \xrightarrow{g} \dots).$$

Moreover $\Omega^p(Spec(A[1/g])) = \Omega^p(Spec(A)) \otimes_A A[1/g]$. (In other words, Ω^p is a quasi-coherent sheaf of $\mathcal{O}$-modules. In terms of our axiomatic definition of Ω^p, this follows on general grounds from the fact that it is locally free, but with a better definition it is easy to prove directly.) This proves the claimed equality $\Omega^p_{\langle D \rangle} = j_*\Omega^p$.

Thus our desired complex $\Omega^\bullet_{\langle D \rangle}$ of sheaves on X is just the pushforward of the de Rham complex of $U = X \setminus D$ along the inclusion $j : U \to X$. Again we get an induced comparison map of presheaves on X with values in $D(\mathbb{C})$

$$|\Omega^\bullet_{\langle D \rangle}| \to j_*|\Omega^\bullet|,$$

and we claim:

Lemma 7. *The composite map*

$$|\Omega^\bullet_{\langle D \rangle}| \to j_*(|\Omega^\bullet|^{sh})$$

identifies the target as the sheafification of the source. In particular, on global sections we deduce

$$R\Gamma(X; \Omega^\bullet_{\langle D \rangle}) \xrightarrow{\sim} R\Gamma(U; \Omega^\bullet).$$

Proof. Again the target is a sheaf as pushforwards preserves sheaves. To finish the proof, we claim the composite map is an isomorphism on sections over any affine open. Let's abuse notation and replace X by this affine open and check the map is an isomorphism on global sections. Since $U = X \setminus D$ is affine, we have that $|\Omega^p|^{sh}(U) = \Omega^p(U)$ sitting in degree 0 as there is no higher cohomology for quasi-coherent sheaves on the affine scheme U; hence also $|\Omega^\bullet|^{sh}(U) = |\Omega^\bullet(U)|$ by filtering using the brutal truncation. Thus

$$(j_*|\Omega^\bullet|^{sh})(X) = |\Omega^\bullet|^{sh}(U) = |\Omega^\bullet(U)|.$$

But we verified above that $\Omega^\bullet_{\langle D \rangle}(X) = \Omega^\bullet(U)$, whence the claim. $\square$

In the analytic case, $\Omega^p_{\langle N \rangle}$ was not the same as $j_*\Omega^p$ and to get a handle on the former we did a local calculation on a polydisk. In the algebraic case they are the same which lets us avoid the local calculations. And thankfully so, because Zariski-local models aren't substantially simpler than global models.

What remains to be seen is that if X is a proper smooth scheme over $\mathbb{C}$ with a normal crossings divisor $D \subset X$, then the algebraic-to-analytic comparison map

$$R\Gamma(X; \Omega^\bullet_{\langle D \rangle}) \to R\Gamma(X^{an}; \Omega^\bullet_{\langle D^{an} \rangle})$$

is an isomorphism. Again, this will not need the normal crossings condition; it's enough for D to be an effective Cartier divisor. (Although, to be fair, we only defined the complex in the analytic case under the normal crossings assumption, but that was just for expository purposes.)

This amounts to a version of GAGA with quasi-coherent sheaf coefficients, which follows from the classical GAGA for coherent sheaves by a limiting argument. Instead of giving a general treatment, let's just explain the argument in this case we're interested in. It amounts to the following lemmas.

Lemma 8. *For $n \geq 0$ and $p \geq 0$, let $\Omega^p(nD)$ denote the sheaf of $\mathcal{O}_X$-modules on X defined as the subhseaf of $\Omega^p_{\langle D\rangle}$ where we require $k \leq n$ in the local expression $g^{-k}\omega$. Then:*

1. *$\Omega^p(nD)$ is locally free of finite rank, hence corresponds to a vector bundle.*
2. *The analytification of this vector bundle identifies with the analogously-defined vector bundle $\Omega^p(nD^{an})$ in the analytic context.*
3. *$\varinjlim_n \Omega^p(nD) = \Omega^p_{\langle D\rangle}$ in the category of sheaves on X.*
4. *$\varinjlim_n \Omega^p(nD^{an}) = \Omega^p_{\langle D^{an}\rangle}$ in the category of sheaves on X^{an}.*

Proof. For part 1, note that there is a local isomorphism $\Omega^p(nD) \simeq \Omega^p$ given by multiplication by g^n where g is a local defining equation for D. For part 2, we have the analogous local isomorphism in the analytic case, and as an algebraic local defining equation determines an analytic one and the Ω^p analytifies to Ω^p, this implies the claim. For 3 and 4, the maps along which the colimit is taken are the natural inclusions, and the claims simply amount to the definitions of $\Omega^p_{\langle D\rangle}$ and $\Omega^p_{\langle D^{an}\rangle}$. □

Lemma 9. *Let X be a topological space which either:*

1. *Is quasi-compact and has a basis of quasi-compact open subsets closed under intersection; or*
2. *Is compact Hausdorff.*

Then for all $q \geq 0$, the functor $H^q(X;-)$ from sheaves of abelian groups on X to abelian groups commutes with filtered colimits.

Proof. It suffices to prove the following ∞ analog: the global sections functor

$$\Gamma : Sh(X; D(\mathbb{Z})) \to D(\mathbb{Z})$$

commutes with filtered colimits.[1] In case 1, we can use the fact that the restriction functor

$$Sh(X; D(\mathbb{Z})) \xrightarrow{\sim} PSh_{MV}(\mathcal{B}_{qc}; D(\mathbb{Z})),$$

from sheaves on X to presheaves on quasi-compact open subsets of X satisfying Mayer-Vietoris, is an equivalence. This is HTT 6.5.4.4. Since colimits are calculated objectwise in presheaf categories and the Mayer-Vietoris condition, being a finite limit, is stable under filtered colimits, this implies case 1. (One can also argue more directly using the description of sheafification as a transfinite iteration of Cech constructions: one notes that the value of the Cech construction on a quasi-compact open is a filtered colimit of finite limits of values of the origin presheaf on quasi-compact opens.)

For case 2 we can use an analogous result, HA 5.5.5.3, which gives

$$Sh(X; D(\mathbb{Z})) \xrightarrow{\sim} PSh^{cont}_{MV}(\mathcal{B}_c; D(\mathbb{Z})),$$

where the right hand side denotes the presheaves on compact subsets of X satisfying Mayer-Vietoris and the continuity condition

$$\mathcal{F}(K) = \varinjlim \mathcal{F}(K')$$

[1] In fact, it is equivalent to say it commutes with all colimits, since a functor between stable ∞-categories which commutes with finite limits must commute with finite colimits, and all colimits are built from finite colimits and filtered colimits.

where the colimit runs over all compact K' containing an open neighborhood of K. Here the functor sends a sheaf $\mathcal{F}$ on X to the presheaf on compact subsets defined by $\mathcal{F}(K) = \varinjlim \mathcal{F}(U)$ where the colimit runs over all open neighborhoods of K. (It's also the same as the global sections of the pullback of $\mathcal{F}$ to K, but this is not quite obvious.) Again the Mayer-Vietoris and continuity conditions are preserved under filtered colimits, whence the conclusion. □

Putting the lemmas together and using GAGA we get the comparison isomorphism

$$H^p(X;\Omega^p_{\langle D\rangle}) = \varinjlim_n H^p(X;\Omega^p(nD)) = \varinjlim_n H^p(X^{an};\Omega^p(nD^{an})) = H^p(X^{an};\Omega^p_{\langle D^{an}\rangle}).$$

Filtering by the brutal truncation we deduce that

$$R\Gamma(X;\Omega^\bullet_{\langle D\rangle}) = R\Gamma(X^{an};\Omega^\bullet_{\langle D^{an}\rangle}),$$

finishing the proof of Grothendieck's theorem.

Exercise 10. *In this exercise you will see that there is a much more "efficient" replacement for the complex of sheaves $\Omega^\bullet_{\langle D\rangle}$ used in the proof of Grothendieck's theorem.*

Let M be a complex manifold and $N \subset M$ a normal crossings divisor. For $p \geq 0$, define a sheaf $\Omega^p(logN)$ on M as the subcomplex of $j_*\Omega^p$ consisting of those p-forms on the complement which locally are $\mathcal{O}$-linear combinations of wedges of forms of the following type:

$$\frac{dz_1}{z_1},\ldots,\frac{dz_k}{z_k},dz_{k+1},\ldots dz_n,$$

where the z_i are local coordinates in which N appears as the zero set of $z_1\cdot\ldots\cdot z_k$.

It suffices to do 2 of the following 5 parts.

1. *Prove that $\Omega^p(logN)$ is a vector bundle locally isomorphic to Ω^p; note that part of this should be some argument ensuring independence of the local coordinates in the description. Prove also that the de Rham differential sends $\Omega^p(logN)$ to $\Omega^{p+1}(logN)$.*

2. *Mimicking the argument in the lecture with a version of the Atiyah-Hodge lemma, show that $R\Gamma(M;\Omega^\bullet(logN)) \xrightarrow{\sim} R\Gamma(M \smallsetminus N;\Omega^\bullet)$.*

3. *Suppose $N \subset M$ is the analytification of a normal crossings divisor $D \subset X$ in a smooth variety X over the complex numbers. Define an algebraic analog, a complex of sheaves $\Omega^\bullet(logD)$ on X, show that it has the same properties as in 1, and show that it analytifies to the analytic one.*

4. *Deduce from GAGA and Grothendieck's theorem that the algebraic analog of 2 also holds.*

5. *Can you find a purely algebraic proof for the algebraic analog of 2? If so you immediately get full credit for the course and I buy you a beer.*

Exercise 11. *Here you will give another proof that cohomology of abelian sheaves commutes with filtered colimits on compact Hausdorff spaces. Fix a compact Hausdorff space X. There exists a continuous surjection $p : T \to X$ where $T = \varprojlim_i T_i$ is a* profinite set*: an inverse limit, in topological spaces, of finite discrete sets. Indeed one can take T to be the Stone-Cech compactification of the underlying set of X.*

1. *Note that T has a basis for the topology consisting of clopen subsets. Deduce that on T, and every clopen subset T' of T, every sheaf of abelian groups is acyclic, and $H^0 : Sh(T'; Ab) \to Ab$ preserves filtered colimits.*

2. *Deduce that $Rp_* = p_* : Sh(T; Ab) \to Sh(X; Ab)$ and this functor also preserves filtered colimits.*

3. *For every $\mathcal{F} \in Sh(X; Ab)$, show that $\mathcal{F} \to p_* p^* \mathcal{F}$ is a functorial filtered colimit preserving injection from $\mathcal{F}$ into an acyclic sheaf.*

4. *Prove that $H^q : Sh(X; Ab) \to Ab$ commutes with filtered colimits for all $q \geq 0$.*