

Lecture 7: Kähler differentials, smooth maps, and the de Rham complex

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Okay, now let's pretend this is a proper course and start over with the definition of algebraic de Rham cohomology. We'll do it in the "relative" setting, so we'll have a de Rham complex $\Omega_{B/A}^\bullet$ of A -modules attached to any map of commutative rings $A \rightarrow B$, or equivalently to any pair consisting of a commutative ring A and a commutative algebra B over A . When $A = \mathbb{C}$ and B is a smooth $\mathbb{C}$ -algebra this will recover the de Rham complex of smooth affine varieties discussed in the previous two lectures.

1 Kähler differentials

We'll start with a discussion of the B -module of Kähler differentials $\Omega_{B/A}^1$, the most basic term in the de Rham complex, in this general algebraic setting. But first, some geometric motivation. If $X = \text{Spec}(B)$ is an affine variety over the complex numbers $A = \mathbb{C}$, and $x \in X(\mathbb{C})$ is a point, then x can be equivalently encoded in the ideal $I_x \subset B$ of functions vanishing at x . Thus

$$B/I_x \simeq \mathbb{C}$$

via evaluation at x ; this encodes the 0^{th} -order information about a function f at the point x , namely just its value $f(x)$. The n^{th} order information is contained in the higher quotient

$$B/I_x^{n+1};$$

if X were affine space $\mathbb{A}^r$ and x were the origin this would be the ring of formal power series in r variables "up to order n ", so neglecting terms of total degree higher than n . In particular, the first order information is contained in

$$B/I_x^2.$$

This is not purely linear, though, because it still contains the zeroth order information. We can thus simplify by just taking

$$I_x/I_x^2,$$

which is now the universal recipient for linear first order data at the point x . Again if $X = \mathbb{A}^r$ and x is the origin, now we're just remembering the purely first order terms in the power series expansion of f , or equivalently the r partial derivatives of f . Formally, this is encoded in the map

$$d: B \rightarrow I_x/I_x^2$$

sending f to the class of $f - f(x)$ where $f(x)$ is viewed as a constant function.

The role of Kähler differentials is to globalize this picture: $\Omega_{B/\mathbb{C}}^1$ is to be a B -module such that for any point $x \in X(\mathbb{C})$, the fiber

$$\Omega_{B/\mathbb{C}}^1(x) = \Omega_{B/\mathbb{C}}^1 \otimes_B \mathbb{C} \simeq I_x/I_x^2$$

identifies with I_x/I_x^2 , where $\mathbb{C}$ is viewed as a B -algebra via the map $B \rightarrow \mathbb{C}$ of evaluation at x .

Back in the land of pure algebra, this translates to the following. For a map of commutative rings $A \rightarrow B$, we want a B -module $\Omega_{B/A}^1$ such that for any retraction $r : B \rightarrow A$ of $A \rightarrow B$, the base change

$$\Omega_{B/A}^1 \otimes_B A \simeq I_r/I_r^2$$

via this retraction identifies with I_r/I_r^2 , where $I_r = \ker(r)$.

This is not enough to pin down $\Omega_{B/A}^1$, but if we extend it appropriately it will be. Namely, instead of talking just about sections of $\text{Spec}(B) \rightarrow \text{Spec}(A)$, we need to talk about *families* of sections of $\text{Spec}(B) \rightarrow \text{Spec}(A)$, meaning sections of arbitrary pullbacks of $\text{Spec}(B) \rightarrow \text{Spec}(A)$, or equivalently lifts of an arbitrary $\text{Spec}(A') \rightarrow \text{Spec}(A)$ along $\text{Spec}(B) \rightarrow \text{Spec}(A)$. Namely, we would also like to require that for any such lift, corresponding to $r : B \rightarrow A'$, we have

$$\Omega_{B/A}^1 \otimes_B A' \simeq I_r/I_r^2,$$

where now I_r is the kernel of the induced map $B' \rightarrow A'$ where $B' = B \otimes_A A'$, which is a retraction of $A' \rightarrow B'$.

The advantage now is that there is a *universal family of sections* of any map $\text{Spec}(B) \rightarrow \text{Spec}(A)$, given by the diagonal embedding

$$\text{Spec}(B) \rightarrow \text{Spec}(B) \times_{\text{Spec}(A)} \text{Spec}(B)$$

in the pullback of $\text{Spec}(B) \rightarrow \text{Spec}(A)$ along itself. Namely, this corresponds to the identity map from $\text{Spec}(B)$ to itself over $\text{Spec}(A)$, making the universality evident. Now, on the level of coordinate rings the diagonal amounts to the ring homomorphism

$$B \otimes_A B \rightarrow B$$

given by multiplication. Indeed, this is the identity on $B \otimes 1$ and on $1 \otimes B$, so it satisfies the requisite characterizing property. Thus, if our desired property of $\Omega_{B/A}^1$ is to be satisfied, we must have the following definition:

Definition 1. Let $A \rightarrow B$ be a map of commutative rings. We define $\Omega_{B/A}^1$ to be the B -module I/I^2 , where

$$I = \ker(B \otimes_A B \rightarrow B).$$

One may worry about exactly why or how I/I^2 is a B -module. A priori it is a $B \otimes_A B$ -module, which can be made into a B -module in two different ways, via the left or right factors. But actually the resulting B -module structures agree, as the difference

$$b \otimes 1 - 1 \otimes b$$

lies in I . One can also give descriptions of I/I^2 which make the B -module structure manifest. For example,

$$I/I^2 = B \otimes_{B \otimes_A B} I,$$

or alternatively:

Lemma 2. *Let $A \rightarrow B$ be a map of commutative rings. Then*

$$\Omega_{B/A}^1 = \text{Tor}_1^{B \otimes_A B}(B, B).$$

Proof. This follows from the short exact sequence $0 \rightarrow I \rightarrow B \otimes_A B \rightarrow B \rightarrow 0$ by noting that the middle term is free hence flat and that the quotient map induces an isomorphism mod I . $\square$

Now, we defined $\Omega_{B/A}^1$ so that our desired property was true in the universal case. But it doesn't quite follow formally that it's true in general, because it's not entirely clear that the formation of I/I^2 should be compatible with base-change in the appropriate sense: after all, in general if I is an ideal in a commutative ring R and $R \rightarrow R'$ is an arbitrary map, then it's not always true that $IR' = I \otimes_R R'$. In fact, this is satisfied for all I if and only if $R \rightarrow R'$ is flat. But we definitely don't want to restrict to flat base-changes because the inclusion of a point in a variety is never flat, unless the point is isolated. Nonetheless, everything works out in the situation at hand.

Proposition 3. *Let $A \rightarrow B$ be a map of commutative rings, let $A \rightarrow A'$ be an arbitrary A -algebra, and let $B' = B \otimes_A A'$ be the co-base change. For any map $f : B \rightarrow A'$ under A , we have*

$$\Omega_{B/A}^1 \otimes_B A' = I_f/I_f^2,$$

where I_f is the kernel of the map $r_f : B' \rightarrow A'$, the retraction of the structure map $A' \rightarrow B'$ corresponding to f .

Proof. Let's try proceeding by reduction to the universal case. Write down the commutative diagram

$$\begin{array}{ccccc} B' & \longleftarrow & B \otimes_A B & \longleftarrow & B \\ \uparrow & & \uparrow & & \uparrow \\ A' & \longleftarrow & B & \longleftarrow & A \end{array}$$

of co-base changes (relative tensor products), so that our retraction r_f is the co-base change of the universal one, the multiplication map $m : B \otimes_A B \rightarrow B$ used to define $\Omega_{B/A}^1 = I \otimes_{B \otimes_A B} B$ with $I = \ker(m)$. It suffices to show that I base-changes to I_f along the map $B \otimes_A B \rightarrow B'$. But because the left-square is a relative tensor product, this base-change can be equivalently calculated along $B \rightarrow A'$. But now, because m is a retraction of $B \otimes_A B$, as a B -module we can equivalently view I as the cokernel of $B \otimes_A B \rightarrow B$ and similarly for r_f and I_f , and this description obviously base-changes appropriately as tensor products preserve cokernels. $\square$

We can split this property up into two parts as follows.

Corollary 4. *Let $A \rightarrow B$ be a map of commutative rings.*

1. *For any $A \rightarrow A'$ with $B' := A' \otimes_A B$, we have*

$$\Omega_{B'/A'}^1 = \Omega_{B/A}^1 \otimes_B B' = \Omega_{B/A}^1 \otimes_A A',$$

in other words the module of Kähler differentials commutes with base-change.

2. *For any retraction $r : B \rightarrow A$ with $I_r := \ker(r)$, we have*

$$\Omega_{B/A}^1 \otimes_B A = I_r/I_r^2,$$

so our desired property we used to motivate the definition holds.

Proof. Part 2 is the special case of the proposition where $A' = A$. On the other hand, part 1 follows by writing out the definition of $\Omega_{B'/A'}^1$ and recognizing it as I_f/I_f^2 for $f : B \rightarrow B'$. (Conversely, it's clear that parts 1 and 2 imply the previous proposition, so corollary this is a reformulation.) $\square$

Next we discuss the differential. Recall in the motivation that for a variety $X = \text{Spec}(B)$ with rational point $x \in X(A)$ we had a map

$$d : B \rightarrow I_x/I_x^2$$

defined on $f \in B$ by subtracting off the constant function with value $f(x)$ then taking the class modulo I_x^2 . This should be induced by a single map

$$d : B \rightarrow \Omega_{B/A}^1$$

defined in the completely analogous way using the universal family of sections. This unwinds to the following.

Proposition 5. *Let $A \rightarrow B$ be a map of commutative rings. Define a map*

$$d : B \rightarrow \Omega_{B/A}^1$$

by $d(f) = [f \otimes 1 - 1 \otimes f]$. Then:

1. $d(a) = 0$ for $a \in A$;
2. $d(fg) = fd(g) + gd(f)$ for $f, g \in B$.

Proof. Part 1 is because in the relative tensor product $B \otimes_A B$ we can move elements of A from the right to the left by construction, so even the representative $a \otimes 1 - 1 \otimes a$ is zero. For part 2, let's temporarily abusively write $d(f)$ for the representative $1 \otimes f - f \otimes 1$ and not its class in $\Omega_{B/A}^1$. Then (thanks Maxime!)

$$\begin{aligned} d(fg) - d(f) \cdot d(g) &= fg \otimes 1 - 1 \otimes fg - (f \otimes 1 - 1 \otimes f)(g \otimes 1 - 1 \otimes g) \\ &= f \otimes g + g \otimes f \\ &= f(1 \otimes g - g \otimes 1) + g(1 \otimes f - f \otimes 1) \\ &= fdg + gdf. \end{aligned}$$

As $d(f) \cdot d(g)$ obviously lies in the square of the ideal, this finishes the proof. $\square$

Clearly, under property 2, property 1 is equivalent to d being A -linear. In other words, this proposition shows that d is an A -derivation from B to $\Omega_{B/A}^1$ in the sense of the following definition.

Definition 6. *Let $A \rightarrow B$ be a map of commutative rings and M a B -module. A map $D : B \rightarrow M$ is called an A -derivation if the following properties hold:*

1. D is an A -linear map (between B -modules);
2. $D(fg) = fD(g) + gD(f)$ for $f, g \in B$.

In fact our $d : B \rightarrow \Omega_{B/A}^1$ is the universal A -derivation:

Proposition 7. Let $A \rightarrow B$ be a map of commutative rings, M a B -module, and $D : B \rightarrow M$ an A -derivation. Then there is a unique linear map $h : \Omega_{B/A}^1 \rightarrow M$ with $h \circ d = D$.

Proof. First let's show uniqueness. It suffices to see that $\Omega_{B/A}^1$ is generated, as a B -module, by the elements db . Suppose $\sum_i f_i \otimes g_i$ lies in the kernel of the multiplication map, so $\sum_i f_i g_i = 0$. Then

$$\sum_i f_i \otimes g_i = \sum_i df_i \cdot g_i - \sum_i 1 \otimes f_i g_i = \sum_i df_i \cdot g_i,$$

as desired. For existence, we'd like to recognize any B -module M as an example an I/I^2 , so that our desired h will come from functoriality. There is a nice way to do this by considering the *square zero extension* $B \oplus M$. This direct sum gets a commutative ring structure by

$$(b, m)(b', m') = (bb', bm' + b'm).$$

This definition is formally determined by the requirement that the inclusion of the first factor $i_0 : B \rightarrow B \oplus M$ be a ring map and that the second factor $M \subset B \oplus M$ should form an ideal of square zero on which the B -module structure induced via i_0 is the original one. Note that the projection $B \oplus M \rightarrow B$ is also a ring map, so $B \oplus M$ is an augmented B -algebra.

On the other hand, giving an A -derivation $D : B \rightarrow M$ is equivalent to giving another A -algebra section $i : B \rightarrow B \oplus M$ of this projection map, of course potentially different from the original one i_0 which corresponds to the zero derivation. The two maps i and i_0 however do agree when restricted to A , so we get an induced map

$$B \otimes_A B \rightarrow B \oplus M,$$

and moreover it makes a commutative triangle with $B \otimes_A B \rightarrow B$ by multiplication and $B \oplus M \rightarrow B$ by the projection. Taking the induced maps on I/I^2 's for I the kernel of the map to B we get our candidate map $h : \Omega_{B/A}^1 \rightarrow M$. A simple unwinding shows it satisfies the desired property. $\square$

This gives a different possible definition of $\Omega_{B/A}^1$, namely as the free B -module generated by formal symbols df for $f \in B$, modulo the submodule generated by da for $a \in A$ and $d(fg) - fdg - gdf$ for $f, g \in B$. This B -module tautologically satisfies the previous proposition, hence it identifies with the module of Kähler differentials as we defined it via I/I^2 .

As usual with such tautological presentations, it's very inefficient. As an exercise, you will prove the following.

Proposition 8. Suppose

$$B = A[x_i \mid i \in I]/(f_j \mid j \in J)$$

is presented as the quotient of a polynomial ring in variables x_i by the ideal generated by a set of polynomials f_j in the variables x_i . Then

$$\Omega_{B/A}^1 = B\{dx_i \mid i \in I\}/B\{\sum_i \partial_i f_j dx_i \mid j \in J\},$$

the quotient of the free B -module on the elements dx_i by the submodule generated by the expressions $\sum_i \partial_i f_j dx_i$ for $j \in J$. Here $\partial_i f_j$ is the partial derivative of f_j with respect to the i^{th} coordinate.

In particular, if $A \rightarrow B$ is finitely presented as an A -algebra, then $\Omega_{B/A}^1$ is finitely presented as a B -module.

2 Smooth maps

Our next topic is smooth maps and their relation to Kähler differentials. We adopt the following definition.

Definition 9. An A -algebra $A \rightarrow B$ is called smooth if there are elements $f_i \in B$ generating the unit ideal such that each $B[f_i^{-1}]$ admits an étale A -algebra map from some finitely generated polynomial ring $A[x_1, \dots, x_n]$.

In other words, thinking geometrically, a smooth map is one which Zariski locally admits étale coordinates. If the notion of étale is still obscure for you, don't worry, we will review it shortly. Anyway, the following properties follow immediately from the definition, once you know analogous facts for étale maps:

1. If $A \rightarrow B$ and $B \rightarrow C$ are smooth, then the composition $A \rightarrow B \rightarrow C$ is also smooth.
2. If $A \rightarrow B$ is smooth and $A \rightarrow A'$ is arbitrary, then $A' \rightarrow B' = B \otimes_A A'$ is also smooth.
3. Any smooth map is flat and finitely presented.

Consider the special case of 2 where A' is a residue field of A . Thinking geometrically, this says that the fibers of a smooth map are smooth. But smoothness is more than that; it's the analog of a submersion in differential geometry.

For present purposes we would like to show that for a smooth map $A \rightarrow B$, the B -module $\Omega_{B/A}^1$ is locally free of finite rank as a B -module. For this the key is the following.

Proposition 10. Let $A \rightarrow B$ be a map of commutative rings, and let $B \rightarrow C$ be an étale map of commutative rings. Then

$$\Omega_{B/A}^1 \otimes_B C \simeq \Omega_{C/A}^1.$$

Proof. The intuition is that étale maps of schemes behave as much like local isomorphisms as possible without actually being local isomorphisms in the algebraic sense with the Zariski topology. Now, as definition of étale let's take weakly étale + finitely presented, where weakly étale means that both $B \rightarrow C$ and $C \otimes_B C \rightarrow C$ are flat.¹ Then we claim that in fact the proposition holds true with weakly étale instead of étale.

To prove it, first note that since $B \rightarrow C$ is flat we have

$$\mathrm{Tor}_*^{B \otimes_A B}(B, B) \otimes_B C = \mathrm{Tor}_*^{B \otimes_A B}(B, C) = \mathrm{Tor}_*^{C \otimes_A C}(C \otimes_B C, C),$$

the last equality coming because $B \otimes_{B \otimes_A B} C \otimes_A C = C \otimes_B C$ (easiest to think in terms of Spec).

But as $C \otimes_B C \rightarrow C$ is flat, we have

$$\mathrm{Tor}_*^{C \otimes_A C}(C \otimes_B C, C) \otimes_{C \otimes_B C} C = \mathrm{Tor}_*^{C \otimes_A C}(C, C).$$

¹The condition that $B \rightarrow C$ be flat just imposes some very mild cohesion among the fibers geometrically, but since $C \otimes_B C \rightarrow C$ is a surjection, corresponding geometrically to a closed immersion, the condition that it be flat is very very strong: indeed, it forces that any C -module which is flat as a B -module automatically be flat as a C -module. Taking for example $B = k = \bar{k}$ an algebraically closed field, this rules out any higher-dimensional varieties as well as any non-reducedness, making the only example of an étale $\bar{k}$ -algebra a finite product of copies of $\bar{k}$, corresponding to just a finite set of points geometrically.

On the other hand, both $C \otimes_B C$ and C are quotients of $C \otimes_A C$, the latter a further quotient of the former. Seeing as the $C \otimes_B C$ -module structure coming from the left in the *Tor* and the C -module structure coming from the right agree as $C \otimes_A C$ -modules on general grounds, the $C \otimes_B C$ -module structure must factor through the C -module structure. But for the same reason – $\otimes_{C \otimes_B C} C$ is the identity on C -modules as C is a quotient of $C \otimes_B C$. Thus we can cancel the $\otimes_{C \otimes_B C} C$ in the above equation, and then specializing to $*$ = 1 proves the claim. $\square$

Corollary 11. *Let $A \rightarrow B$ be a map of commutative rings.*

1. *If $B \rightarrow B'$ is a localization, e.g. if $B' = B[f^{-1}]$ for some $f \in B$, then $\Omega_{B/A}^1 \otimes_B B' = \Omega_{B'/A}^1$, e.g. $\Omega_{B/A}^1[f^{-1}] = \Omega_{B[f^{-1}]/A}^1$.*

2. *If $A \rightarrow B$ is smooth, then $\Omega_{B/A}^1$ is a locally free B -module of finite rank.*

Proof. For part 1, note that localizations are examples of weakly etale maps, as they are flat and idempotent ($B' \otimes_B B' = B'$). For part 2, using part 1 and the previous proposition we are reduced to seeing that $\Omega_{A[x_1, \dots, x_n]/A}^1$ is free of rank n , but it's free on the dx_i by Proposition 8. $\square$

Part 1 can be interpreted as follows. To any B -module M is associated a unique quasi-coherent sheaf of $\mathcal{O}$ -modules on $\text{Spec}(B)$, and part 1 implies that when $M = \Omega_{B/A}^1$, the sections of this quasi-coherent sheaf on any affine open $\text{Spec}(B') \subset \text{Spec}(B)$ identify with $\Omega_{B'/A}^1$.² That's a nice consistency property.

As for part 2, an important addendum is that the local rank of $\Omega_{B/A}^1$ as a B -module is the same as the relative dimension of B/A . (Meaning, at a point $x \in \text{Spec}(B)$ lying over $y \in \text{Spec}(A)$, take the dimension of the fiber $\text{Spec}(B)_y$ at the point x .) This is clear from the fact that etale maps preserve relative dimension.

Now, if you've never seen smooth algebras before you may wonder why there's a sufficient supply of them, or in other words why there are enough etale maps to affine n -space even though there are not enough Zariski open subsets of affine n -space. The main result producing examples of smooth (and etale) maps is the following *Jacobian criterion*:

Theorem 12. *Let $B = A[x_1, \dots, x_n]/(f_1, \dots, f_m)$ be a finite presentation of an A -algebra as indicated. Suppose that the Jacobian matrix*

$$(\partial_i f_j)$$

has full rank, i.e., rank = m , when base-changed to the residue field at any point of $\text{Spec}(B)$. Then $A \rightarrow B$ is smooth of relative dimension $n - m$.

Note that this Jacobian matrix of partial derivatives, viewed as a map from a free B -module of rank m to a free B -module of rank n , was exactly what we took the cokernel of in Proposition 8 to get $\Omega_{B/A}^1$. The Jacobian criterion then says that if the presentation makes $A \rightarrow B$ look smooth from the perspective of $\Omega_{B/A}^1$, then $A \rightarrow B$ really is smooth; in particular, and this is the least obvious aspect, $A \rightarrow B$ is flat and really has the naively calculated relative dimension $n - m$. Recall also that the same Jacobian criterion is what ensures that a finite set of C^∞ -functions cuts out a smooth submanifold of Euclidean space in classical manifold theory, so that matches as well.

²It's not true that in such a situation B' is necessarily a localization of B , but that's just a technical nuisance: after a further localization we'll be a localization of both B and B' and that can be used to conclude. In any case there is a basis of the topology of $\text{Spec}(B)$ given by $\text{Spec}(B')$ with B' a localization of B .

It's also true that any smooth map $A \rightarrow B$ is, locally on $\text{Spec}(B)$, of the form described in the Jacobian criterion, but I don't know how important that is.

3 The de Rham complex

Now we can define the terms of the de Rham complex.

Definition 13. For a map of commutative rings $A \rightarrow B$ and $p \geq 0$, define $\Omega_{B/A}^p = \Lambda^p \Omega_{B/A}^1$, the p^{th} exterior power of the B -module $\Omega_{B/A}^1$.

It's worthwhile giving a reminder on exterior powers. For a commutative ring R and an R -module M , the p^{th} exterior power R -module $\Lambda^p M$ is defined as the quotient of the p^{th} tensor power $\otimes^p M$ by the R -submodule generated by the pure tensors

$$m_1 \otimes \dots \otimes m_p$$

such that $m_i = m_{i+1}$ for some $1 \leq i \leq p-1$. In other words, $\Lambda^p M$ represents the functor of alternating R -multilinear maps $f : M \times \dots \times M \rightarrow N$ out of a p -fold product of copies of M , where alternating means that if any two adjacent entries are equal then f sends that tuple to zero.

The image of $m_1 \otimes \dots \otimes m_p$ in $\Lambda^p M$ is denoted

$$m_1 \wedge \dots \wedge m_p.$$

It follows that for any permutation $\sigma \in S_p$, we have

$$m_{\sigma(1)} \wedge \dots \wedge m_{\sigma(p)} = \text{sign}(\sigma) m_1 \wedge \dots \wedge m_p.$$

Indeed, it suffices to show this for σ a transposition, and then taking $p = 2$ for simplicity it follows from expanding out the relation

$$(m + n) \wedge (m + n) = 0$$

using $m \wedge m = 0$ and $n \wedge n = 0$.

However, that more naive alternating relation involving signs of permutations does not recover the more basic relation $m \wedge m = 0$ which defines the exterior power. Indeed, all we would be able to deduce would be that $m \wedge m = -m \wedge m$, or $2m \wedge m = 0$. When 2 is not a unit in R , this does not imply $m \wedge m = 0$. In the extreme case where R is an $\mathbb{F}_2$ algebra then the naive exterior algebra defined using signs is the same as the symmetric algebra, so for example if M is free of rank one then $\Lambda^p M$ would also be free of rank one for all $p \geq 0$. Thus the behavior of $\Lambda^p(-)$ would be highly dependent on the characteristic, and we don't like that.

On the contrary, the true exterior power behaves nicely over arbitrary commutative rings. This is articulated in the following.

Proposition 14. Let R be a commutative ring and M a free R -module of rank n . Then for $p \geq 0$, the R -module $\Lambda^p M$ is free of rank $\binom{n}{p}$. More precisely, if $m_1, \dots, m_n$ is a basis of M , then the

$$m_{i_1} \wedge \dots \wedge m_{i_p}$$

form a basis of $\Lambda^p M$, where $1 \leq i_1 < \dots < i_p \leq n$.

In particular, the top exterior power $\Lambda^n M$ is free of rank one. This gives rise to the theory of the determinant, and it's important that that works in arbitrary characteristic.

It can be a bit of a pain to prove this proposition directly. But if we consider the structure of the $\Lambda^p M$ all together it becomes easier. Namely, $\Lambda^p M$ is the p^{th} graded piece of the *exterior algebra* $\Lambda^\bullet M$, the quotient of the tensor algebra $\otimes^\bullet M$ by the two-sided ideal generated by the $m \otimes m$ in degree two. Because the basic alternating property implies the naive alternating property, $\Lambda^\bullet M$ is automatically a graded commutative algebra. But even more, it is a *strict* graded commutative algebra, which means that for all odd elements x we have $x^2 = 0$. Again, from the axioms of a graded commutative algebra it only follows that $2x^2 = 0$.

Note that in a general graded commutative R -algebra, the set of odd elements x for which $x^2 = 0$ is closed under addition and multiplication by even graded elements. It follows that if $A^\bullet$ and $B^\bullet$ are strict graded commutative R -algebras, then so is their graded tensor product $A^\bullet \otimes_R B^\bullet$. Since a graded commutative ring is just a commutative monoid object in the tensor category of graded R -modules, on general grounds this tensor product is also the coproduct. Thus we see that the coproduct in strict graded commutative R -algebras is calculated by the usual graded tensor product. With these remarks in hand we can actually prove Proposition 14.

Proof. By definition, $\Lambda^\bullet M$ is the free strict graded associative algebra generated by M in degree one. Being automatically graded-commutative, it is also the free strict graded commutative algebra generated by M in degree one. It follows that $\Lambda^\bullet(-)$ sends coproducts to coproducts, in particular

$$\Lambda^\bullet(M \oplus N) = \Lambda^\bullet(M) \otimes \Lambda^\bullet(N).$$

Now, clearly when M is free of rank one we get that $\Lambda^\bullet(M)$ is R in degree zero and M in degree one. Taking the p -fold graded tensor product we deduce the claim. $\square$

In particular:

Corollary 15. *If $A \rightarrow B$ is a smooth map of relative dimension n , then $\Omega_{B/A}^p$ is a free B -module of rank $\binom{n}{p}$.*

Now we discuss the de Rham differential.

Proposition 16. *Let $A \rightarrow B$ be a map of commutative rings. There are unique A -linear maps $d : \Omega_{B/A}^p \rightarrow \Omega_{B/A}^{p+1}$ promoting $\Omega_{B/A}^\bullet$ into a commutative differential A -algebra such that d is the universal derivation discussed above when $p = 0$.*

Proof. I think this has to be done “by hand”. We have a presentation of $\Omega_{B/A}^1$ as a B -module, the tautological one coming from the fact that it is the recipient of the universal derivation. This gives rise to a presentation of $\Omega_{B/A}^p$ as a B -module: it is generated by symbols

$$db_1 \wedge \dots \wedge db_p$$

for $b_i \in B$ subject to the rules that this symbol vanishes if two adjacent b_i are equal or any b_i lies in A , the expression is multilinear in the b_i , and the evident Leibniz rule holds if we write any b_i as a product $b_i = fg$.

Since the differential is to be A -linear, not B -linear, this presentation as a B -module doesn't immediately help us. But any presentation as a B -module induces a presentation as an A -module we just

add extra formal symbols parametrized by the elements of B . We deduce that as an A -module, $\Omega_{B/A}^p$ is generated by symbols of the form

$$b_0 db_1 \wedge \dots \wedge db_p$$

for $b_i \in B$ subject to the rules that this symbol vanishes if two adjacent b_i for $i > 0$ are equal or any b_i for $i > 0$ lies in A , that multiplying ab_0 using the A -module structure is the same as treating ab_0 as a formal symbol, that expression is multilinear in the b_i , and the Leibniz rule holds if we write any b_i for $i > 0$ as a product, where this Leibniz rule now also incorporates the b_0 part.

Now we can just brutally check that defining

$$d(b_0 db_1 \wedge \dots \wedge db_p) = db_0 \wedge db_1 \wedge \dots \wedge db_p$$

respects these relations and defines a differential $d : \Omega_{B/A}^p \rightarrow \Omega_{B/A}^{p+1}$ satisfying the desired properties. $\square$

From this base existence and our previous work we can formally deduce (thanks Maximel!) the following universal property of the de Rham complex.

Proposition 17. *Let $A \rightarrow B$ be a map of commutative rings. Then the de Rham complex $\Omega_{B/A}^\bullet$ is the initial strict commutative differential graded A -algebra with a map from B to the degree zero part.*

Proof. Let $C^\bullet$ be an arbitrary strict commutative differential graded A -algebra with an A -algebra map $B \rightarrow C^0$. Then the composition $B \rightarrow C^0 \rightarrow C^1$ is an A -derivation, thus it factors uniquely through $B \rightarrow \Omega_{B/A}^1$. Now, forgetting the differential the universal property of the exterior algebra gives unique maps $\Omega_{B/A}^p \rightarrow C^p$ respecting multiplication. It suffices to show that they also respect the differential. But Ω^p is generated under multiplication by the b and db 's on which the differential is determined by the Leibniz rule and our first observation, so this is automatic. $\square$

Exercise 18. *Prove Proposition 8.*

Exercise 19. *Suppose $A \rightarrow B$ is smooth. For $p \geq 0$, produce a natural isomorphism $\Omega_{B/A}^p \simeq \text{Tor}_p^{B \otimes_A B}(B, B)$. For $p = 2$ give a counterexample if $A \rightarrow B$ is not smooth.*