Lecture 9: Globalization and Deligne's theorem on degeneration

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Recall that to a smooth map of commutative rings $A \to B$, we associated a complex of flat A-modules $\Omega^\bullet_{B/A}$, where each term is a finitely generated projective B-module. This complex is functorial in B. Moreover, if $B \to B'$ is a map of commutative rings corresponding to a Zariski-open inclusion $Spec(B') \subset Spec(B)$, then for all $p \geq 0$

$$\Omega^p_{B/A} \otimes_B B' \simeq \Omega^p_{B'/A};$$

furthermore if $A \to A'$ is an arbitrary map, then

$$\Omega^\bullet_{B/A} \otimes_A A' \simeq \Omega^\bullet_{B'/A'}.$$

It follows that we can globalize as follows.

Definition 1. *Let $f : X \to S$ be a smooth map of schemes and $p \geq 0$.*

1. *Define $\Omega^p_{X/S}$ to be the quasicoherent sheaf on X determined as follows: if $Spec(B) \subset X$ is an affine open whose image under f factors through the affine open $Spec(A) \subset S$, then*

$$\Omega^p_{X/S}(Spec(B)) = \Omega^p_{B/A}$$

   compatibly with restriction. (This is independent of A by the second property recalled above, and it is quasi-coherent by the first property.)

2. *Define the de Rham differential $d : \Omega^p_{X/S} \to \Omega^{p+1}_{X/S}$, a map of sheaves of $f^{-1}\mathcal{O}_S$-modules, by taking its value on sections over $Spec(B)$ as above to be the de Rham differential $\Omega^p_{B/A} \to \Omega^{p+1}_{B/A}$.*

Thus we have a de Rham complex of sheaves of $f^{-1}\mathcal{O}_S$-modules on X

$$\Omega^\bullet_{X/S}.$$

By the correspondence between complexes and filtered objects in the derived category (with n^{th} associated graded concentrated in degree n), we can equivalently encode $\Omega^\bullet_{X/S}$ as a filtered object

$$|\Omega^\bullet_{X/S}|$$

in the ∞-category $Mod_{f^{-1}\mathcal{O}_S}(Sh(X; D(\mathbb{Z})))$, where the p^{th} associated graded

$$F^{\geq p}|\Omega^\bullet_{X/S}|/F^{\geq p+1}|\Omega^\bullet_{X/S}| \simeq |\Omega^p|[-p]$$

is the $-p$ shift of the derived quasicoherent sheaf corresponding to the ordinary quasicoherent sheaf Ω^p on X. Here and from now on we follow the convention that we automatically sheafify everything unless otherwise indicated.

This lets us go further and define de Rham cohomology. In the setting of complex manifolds, we took the hypercohomology of the de Rham complex of sheaves, which is the same thing as the global sections, = pushforward to the point, of the associated sheaf in the derived sense. This motivates the following.

Definition 2. *Let $f : X \to S$ be a smooth map of schemes. Define the* de Rham cohomology *of X/S to be*

$$\mathrm{dR}_{X/S} := f_* |\Omega^\bullet_{X/S}| \in Mod_{\mathcal{O}_S}(Sh(S; D(\mathbb{Z}))).$$

Via the brutal filtration on $\Omega^\bullet_{X/S}$, we can view this as a filtered object in $Mod_{\mathcal{O}_S}(Sh(S; D(\mathbb{Z})))$; the resulting filtration $F^{\geq -}\mathrm{dR}_{X/S}$ is called the Hodge filtration.

The following lemma reduces the study of $\mathrm{dR}_{X/S}$ to the affine case.

Lemma 3. *Let $f : X \to S$ be a smooth map of schemes.*

1. *As $U \subset X$ varies over open subschemes, the functor $U \mapsto \mathrm{dR}_{U/S}$ is a sheaf on X with values in $Mod_{\mathcal{O}_S}(Sh(S; D(\mathbb{Z})))$.*
2. *For $V \subset S$ open, $\mathrm{dR}_{X/S}|_V = \mathrm{dR}_{f^{-1}V/V}$.*
3. *If $X = Spec(B)$ and $S = Spec(A)$ are both affine, then $\mathrm{dR}_{X/S}$ identifies with the derived quasi-coherent sheaf associated to the object $|\Omega^\bullet_{B/A}| \in D(A)$.*

Moreover, these claims are also valid on the level of filtered objects.

Proof. Let $U \subset X$ be open. Note that on the level of complexes of presheaves, $\Omega^\bullet_{U/S}$ and $\Omega^\bullet_{X/S}|_U$ have the same sections on affine open subsets, by construction. Thus, by sheafification it follows that $|\Omega^\bullet_{U/S}| = |\Omega^\bullet_{X/S}||_U$.

If we write h_U in terms of colimits of h_{U_i}'s in the ∞-category $Sh(X; \mathcal{S})$, then we get a corresponding description of $(f_U)_*|\Omega^\bullet_{X/S}||_U$ in terms of limits of values of $(f_{U_i})_*|\Omega^\bullet_{X/S}||_{U_i}$. Claim 1 follows by combining with the previous.

Claim 2 follows because again on the level of complexes of presheaves, $\Omega^\bullet_{f^{-1}V/V}$ and $\Omega^\bullet_{X/S}|_V$ have the same restrictions to a basis of affine open subsets by construction, hence their sheafifications $|\Omega^\bullet_{U/S}|$ and $|\Omega^\bullet_{X/S}||_U$ agree, whence 2 by pushing forward.

For claim 3, inducting up the Hodge filtration it suffices to show that $f_*|\Omega^p|[-p]$ is the derived quasi-coherent sheaf associated to the object $\Omega^p[-p] \in D(A)$. This follows because f_* preserves quasi-coherence and corresponds to the forgetful functor $D(B) \to D(A)$ in terms of the global sections functor. $\square$

Why do 1 and 2 reduce us to the affine case, described in 3? Because $dR_{X/S}$ is a sheaf on S by definition, it suffices to understand its restriction to affine opens. Then by 2 we are reduced to where $S = Spec(A)$ is affine. Then by the first property, we can understand $dR_{X/S}$ by working locally on X, in particular we can reduce to where $X = Spec(B)$ is affine.

Corollary 4. *Let $f : X \to S$ be a smooth map of schemes.*

1. *If f is qcqs, then $\mathrm{dR}_{X/S} \in D(S)$, i.e. the de Rham cohomology is derived quasi-coherent.*

2. *If f is qcqs, then for any $g : S' \to S$ there is a natural identification $g^* \mathrm{dR}_{X/S} \simeq \mathrm{dR}_{X'/S'}$ where $X' = X \times_S S'$.*

3. *If f is proper, then $\mathrm{dR}_{X/S} \in Perf(S)$, i.e. the de Rham cohomology is a dualizable object of $D(S)$ under tensor product.*

Moreover, all these claims are also valid on the level of filtered objects.

Proof. If $V \subset S$ is affine open, then by parts 2 of the previous we have $\mathrm{dR}_{X/S}|_V = \mathrm{dR}_{f^{-1}V/V}$. Thus we can reduce to the case where $S = Spec(A)$ is affine, and in claim 2 we can reduce to where S' is affine.

In that case, if f is qcqs, then X is qcqs, so h_X can be written in terms of h_U for affine U using finite colimits. Thus by part 1 we can reduce to where X is also affine, since the collection of quasi-coherent objects is closed under finite limits. Then parts 1 and 2 follow from part 3 of the previous and the base-change property of the de Rham complex.

As for part 3, by induction up the Hodge filtration we reduce to proving that $f_*\Omega^p$ is perfect, but we recalled in the previous lecture, as a consequence of Grothendieck-Serre duality, that when f is smooth and proper we have that f_* preserves perfect complexes. Since Ω^p is locally free of finite rank, it is perfect, whence the claim. □

So we have a de Rham cohomology perfect complex when f is proper and smooth, which base-changes in the geometrically appropriate way by 2. We would like to state a theorem of Deligne showing that when S is a $\mathbb{Q}$-scheme, this perfect complex has homology that is as nice as possible. But first let's explain a bit about homology of objects of $D(S)$ for a general scheme S. In general, for a sheaf $\mathcal{F}$ of $\mathcal{O}_S$-modules in the derived sense on S (precisely, $\mathcal{F} \in Mod_{\mathcal{O}_S}(S; D(\mathbb{Z}))$), we would define $H_n\mathcal{F}$ to be the sheaf of $\mathcal{O}_S$-modules in the underived sense on S (precisely, $H_n\mathcal{F} \in Mod_{\mathcal{O}_S}(S; Ab)$) given by sheafifying the homology presheaves of $\mathcal{F}$. But we have the following simple lemma showing this sheafification is very innocuous in the quasi-coherent setting.

Lemma 5. *Suppose $\mathcal{F} \in D(S)$ is a derived quasicoherent sheaf and $n \in \mathbb{Z}$. Then the sheaf $H_n\mathcal{F}$ of $\mathcal{O}_S$-modules is an ordinary quasi-coherent sheaf. The assignment $\mathcal{F} \mapsto H_n\mathcal{F}$ commutes with restriction to open subsets, and on an affine open $Spec(R)$ it corresponds on global sections to the operation $M \mapsto H_n(M)$ from $D(R)$ to Mod_R.*

In particular, no sheafification is necessary when considering sections on affine opens.

Proof. By definition, if $V \subset U$ is an inclusion of affine opens, then

$$\mathcal{F}(U) \otimes_{\mathcal{O}(U)} \mathcal{O}(V) \xrightarrow{\sim} \mathcal{F}(V).$$

As $\mathcal{O}(U) \to \mathcal{O}(V)$ is flat, we deduce

$$H_n(\mathcal{F}(U)) \otimes_{\mathcal{O}(U)} \mathcal{O}(V) \xrightarrow{\sim} H_n(\mathcal{F}(V)).$$

Thus, the presheaf $H_n\mathcal{F}$ on, restricted to any affine open U, agrees, on affine opens, with the quasi-coherent sheaf associated to the $\mathcal{O}(U)$-module $H_n(\mathcal{F}(U))$. Sheafifying, we get the claim. □

More generally and for the same reasons, there are canonical truncation functors $\tau_{\leq n} : D(S) \to D(S)$, defined as the sheafification of the section-wise truncation functors on presheaves, and on affine opens they correspond to the usual truncation functors on global sections.

Now, the homology groups of a general perfect complex have no general reason to be locally free of finite rank, as simple examples such as $cofib(\mathbb{Z} \xrightarrow{n} \mathbb{Z})$ show. In fact they need not themselves even be perfect when viewed as complexes concentrated in a single degree, i.e. the full subcategory $Perf(R) \subset D(R)$ is not closed under canonical truncation. Even more, the homology groups of a general perfect complex over a general ring R need not even be finitely generated as R-modules! In general, this requires R to be a *coherent* ring, for example a noetherian ring.

But yet, when S is an arbitrary $\mathbb{Q}$-scheme and $f : X \to S$ is proper and smooth, the perfect complex $dR_{X/S} \in Perf(S)$ does have locally finite free homology as a consequence of the following theorem of Deligne, proved in his paper "Criteres de degenerances..." which is our reference for this lecture and the next.

Theorem 6. *Let $f : X \to S$ be a smooth and proper map of $\mathbb{Q}$-schemes. Then:*

1. *For all $p \geq 0$ and $q \geq 0$, the quasicoherent sheaf $R^q f_*(\Omega^p) = H_{-q} f_* |\Omega^p|$ on S is locally free of finite rank.*

2. *The local ranks of $R^q f_*(\Omega^p)$ and $R^p f_*(\Omega^q)$ are the same.*

3. *The spectral sequence of quasi-coherent sheaves on S*

$$E_1^{p,q} = R^q f_* \Omega^p \Rightarrow H_{-p-q} \, \mathrm{dR}_{X/S},$$

associated to the Hodge filtration of $\mathrm{dR}_{X/S}$, degenerates at E_1. Thus each $H_n \, \mathrm{dR}_{X/S}$ has a functorial filtration with associated graded the $R^q f_*(\Omega^p)$ for $p + q = -n$ and in particular is also locally free of finite rank.

In the rest of this lecture we'll give some algebraic preliminaries on the proof, then in the next lecture we'll finish the proof and derive some consequences.

Definition 7. *Let R be a commutative ring, and $M \in D(R)$. The following conditions are equivalent:*

1. *The homology R-module $H_n(M)$ vanishes for all n outside some finite range, and for all n is finitely generated projective.*

2. *M is isomorphic to a finite direct sum of shifts of finitely generated projective R-modules.*

3. *M is represented by a complex of finitely generated projective R-modules with zero differential.*

When these conditions hold we'll say that M is split-perfect.

Proof. It is clear that $2 \Rightarrow 3 \Rightarrow 1$. For $1 \Rightarrow 2$, by induction up the Postnikov tower it suffices to show that if $M \in D(R)_{\geq 0}$ has $H_0 M$ projective, then $M \simeq \tau_{\geq 1} M \oplus H_0 M[0]$. But for N projective we have $\pi_0 Map(N, M) = Hom(N, H_0 M)$ (true for $N = R$ by definition, hence true for anything which is a summand of a direct sum of copies of R) which gives the splitting. $\square$

We will now present some Grothendieck-style reductions permitting to reduce the question of split-perfectness to the case where R is an artinian local ring.

Lemma 8. *Let R be a commutative ring and $M \in D(R)$.*

1. *If M is split perfect, then for any map $R \to R'$, the object $M \otimes_R R' \in D(R')$ is also split-perfect, and*
$$H_n(M) \otimes_R R' = H_n(M \otimes_R R').$$
2. *If $R \to R_i$ is a finite set of flat maps such that $Spec(R_i) \to Spec(R)$ is jointly surjective, then if each $M \otimes_R R_i$ is split-perfect, so is M.*

Proof. Part 1 is clear using condition 2 of the definition. For part 2, note that for all $n \in \mathbb{Z}$ we have

$$H_n(M) \otimes_R R_i = H_n(M \otimes_R R_i)$$

as $R \to R_i$ is flat. Thus we reduce to showing the claim that the condition of an ordinary R-module of being finitely generated projective is flat-local, which is a consequence of faithfully flat descent. □

A particular case of 2 is an open cover of $Spec(R)$ by affine opens. This lets us globalize, saying that an object $M \in D(S)$ is split-perfect if it is so on any affine open subset. We deduce that the property of being split-perfect is local on S and preserved by base-change. We caution that a split-perfect object of $D(S)$ need not be *globally* split, meaning isomorphic to the direct sum of its homology with the appropriate shifts; you will give an example in the exercises.

Lemma 9. *Let R be a commutative ring and R_i a filtered colimit diagram of R-algebras. If $M \in Perf(R)$ and $M \otimes_R \varinjlim_i R_i$ is split-perfect, then $M \otimes_R R_i$ is split-perfect for some i.*

Proof. Write each $H_n(M \otimes_R \varinjlim_i R_i)$ explicitly as the image of an idempotent square matrix with entries in $\varinjlim_i R_i$. All this data is finitary in nature: we need to given N^2 elements of $\varinjlim_i R_i$ satisfying the explicit polynomial equations which result from the condition that the matrix be idempotent. Because the colimit is filtered, it follows that we can realize this data over some R_i, and replacing R by this R_i we can therefore assume these modules are base-changed from finitely generated projective R-modules. The data of the isomorphism

$$\oplus_n H_n(M \otimes_R \varinjlim_i R_i)[n] \simeq M \otimes_R \varinjlim_i R_i$$

of perfect complexes over $\varinjlim_i R_i$ is similarly finitary in nature as we see by choosing a representing complex for M and noting that isomorphisms in $Perf(-)$ correspond to chain-homotopy equivalences among representing complexes of projective modules. Thus this isomorphism can also be realized over some R_i, whence the conclusion. □

Remark 10. *We can also say this from a more high-brow perspective, without discussing individual elements of R and without representing our objects by chain complexes. The functor $Perf(-)$ from commutative rings to ∞-categories commutes with filtered colimits; this follows formally from the fact that $D(-)$ sends filtered colimits to inverse limits via the forgetful functors, using that $Perf(-)$ is the full subcategory of compact objects in $D(-)$. Similarly for finitely generated projective modules replacing perfect complexes. Filtered colimits of ∞-categories are calculated object-wise and map-wise, so we can also proof the above lemma using this other perspective but following the same basic outline.*

Lemma 11. *Let R be a commutative ring and $I \subset R$ an ideal with $R \xrightarrow{\sim} \varprojlim_n R/I^n$. If $M \in Perf(R)$, each $M \otimes_R R/I^n$ is split-perfect, and each homology module of $M \otimes_R R/I$ is free, then M is split-perfect and moreover each homology module of M is free.*

Proof. First we claim that $M \tilde{\to} \varprojlim_n M \otimes_R R/I^n$, where now this all takes place in $D(R)$. Since M is perfect, this reduces to the analogous claim with $M = R$, but that follows because $R \tilde{\to} \varprojlim_n R/I^n$ in the underived sense by hypothesis, and the transition maps $R/I^{n+1} \to R/I^n$ are surjective so there is no lim^1 term, hence it's also true in the derived sense.

It follows that there is a Milnor short exact sequence expressing H_*M in terms of the limit and lim^1 of the $H_*(M \otimes_R R/I^n)$. But these are finitely generated projective modules which base-change to each other by the first lemma above, so the transition maps are surjective and hence there is no lim^1. We deduce that

$$H_k M = \varprojlim_n H_k(M \otimes_R R/I^n),$$

for all $k \in \mathbb{Z}$, and moreover the transition maps are surjective.

The first term $H_k(M \otimes_R R/I)$ is free of finite rank by assumption. By the above limit claim we can lift a basis all the way to $H_k M$ and encode this as a map $f : R^d \to H_k M$. By the limit claim again, to show this is an iso it suffices to show that the induced map $(R/I^n)^d \to H_k(M \otimes_R R/I^n)$ is an iso for all n. But this is true for $n = 1$ and that implies it's true for all n by the following lemma, whence the claim. $\square$

We used the following lemma in the proof:

Lemma 12. *Let R be a ring and $I \subset R$ a nilpotent ideal. If $f : M \to N$ is a map in $D(R)$ such that $f \otimes_R R/I$ is an iso, then f is an iso.*

Proof. Filtering by powers of I, we can inductively reduce to the case $I^2 = 0$. Passing to the fiber of f, it suffices to show that if $M \otimes_R R/I = 0$, then $M = 0$. By the fiber-cofiber sequence $I \otimes_R M \to M \to R/I \otimes_R M$, it suffices to argue that $IR \otimes_R M = 0$. But I is an R/I-module as $I^2 = 0$. Thus it lies in the full subcategory of $D(R)$ generated under colimits by R/I, whence the claim. $\square$

Now we can prove the reduction theorem.

Theorem 13. *Suppose R is a noetherian ring and $M \in Perf(R)$. If for all artinian local R-algebras R' the R'-module $M \otimes_R R'$ is split-perfect, then M is split-perfect.*

Proof. Let $x \in Spec(R)$. Write R_x the local ring at x and $R_{\widehat{x}}$ for its completion at the maximal ideal. We apply the previous lemma to $R_{\widehat{x}}$ and I the maximal ideal. The first quotient is the residue field, so the freeness hypothesis in that lemma is automatic, and each higher quotient is artinian local so by our hypothesis here we can apply the previous lemma. We deduce that $M \otimes_R R_{\widehat{x}}$ is split-perfect. As R is noetherian, the map $R_x \to R_{\widehat{x}}$ is faithfully flat, thus from another lemma we deduce $M \otimes_R R_x$ is split-perfect. As R_x is the filtered colimit of $R[f^{-1}]$'s where the $D(f)$'s are a neighborhood basis for x, from another lemma we deduce that M is split-perfect in some neighborhood of x. Thus M is split-perfect over some open cover, hence it is split-perfect again by faithfully flat descent. $\square$

Finally, in the artinian local case, there is a numerical criterion for split-perfectness. This is based on the notion of *length* of a finitely generated R-module, which we recall here.

Definition 14. *Let R be an artinian local ring. Recall that in particular R is noetherian, so the collection of finitely generated R-modules forms an abelian subcategory of Mod_R, closed under extensions, submodules, and quotients.*

There is a unique way to assign a non-negative integer $l_R(M)$ to every finitely generated R-module M, such that:

1. length is additive in short exact sequences;

2. the length of the residue field k is 1.

The idea is that every finitely generated R-module has a canonical finite filtration, by powers of the maximal ideal, and each associated graded is a finitely generated k-module hence isomorphic to a finite direct sum of copies of k. This determines what the length has to be assuming 1 and 2 are satisfied, then one checks that 1 is indeed satisfied with that definition (or, invoke Quillen's devissage theorem in algebraic K-theory!). It follows also from this description that $l(M) = 0$ if and only if $M = 0$.

If there is an algebra splitting of the quotient $R \to k$, making R into a k-algebra, then this theory of length is simple: the length is simply the dimension over k. Cohen's structure theorem implies that such a splitting exists when k has characteristic zero. That hypothesis will be satisfied for us when we use this theory. So you can make that simplifying assumption if you like.

Remark 15. *As Maxime remarks, this result of Cohen is easy to prove if $k = \mathbb{Q}$. Namely, since every nonzero integer acts invertibly on the residue field, inducting up the filtration we see that every nonzero integer acts invertibly on R. Thus R is a $\mathbb{Q}$-vector space, hence a $\mathbb{Q}$-algebra as $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}$.*

On the other hand, the claim fails when k has characteristic p, for example we have $R = \mathbb{Z}/p^2\mathbb{Z}$ which has residue field $\mathbb{F}_p$ but is not an $\mathbb{F}_p$-algebra. Make sure you understand where the above argument fails with $\mathbb{F}_p$ replacing $\mathbb{Q}$ if you replace the condition of n acting invertibly by the condition that p acts by zero.

The numerical criterion is as follows.

Proposition 16. *Let R be an artinian local ring with residue field k, and let $M \in Perf(R)$. Then for all $n \in \mathbb{Z}$ we have*

$$l_R(H_n(M)) \leq l_R(R) \cdot dim_k H_n(M \otimes_R k).$$

Moreover, equality holds for all n if and only if M is split-perfect.

Proof. If M is split-perfect, it is clear that equality holds because we reduce to $M = R$ where the result is trivial. Thus let us prove the inequality and show that equality implies M is split-perfect.

By the lemma which follows, M can be represented by a complex (M_n, d) of finitely generated free R-modules such that $d \otimes_R k = 0$ in all degrees. It follows that $M \otimes_R k$ is represented by the complex with terms $M_n \otimes_R k$ and trivial differential. Thus $H_n(M \otimes_R k) = M_n \otimes_R k$, and since M_n is finitely generated free it follows that

$$l_R(R) \cdot dim_k H_n(M \otimes_R k) = l_R(M_n).$$

But $H_n(M) = ker d_n / im d_{n+1}$ is a subquotient of M_n, so we deduce

$$l_R(H_n(M)) \leq l_R(M_n)$$

whence the desired inequality. If equality holds, then by additivity of length and the fact that the length is zero if and only if the module is zero, the subquotient $H_n(M)$ must be all of M_n, so $H_n(M)$ is finitely generated free, whence M is split perfect if equality holds for all n, as desired. □

We used the following lemma:

Lemma 17. *Let M be a perfect complex over a local ring R with residue field k. Then M can be represented by a bounded complex of finitely generated free R-modules (M_n, d) with $d \otimes_R k = 0$ in all degrees.*

Proof. We describe how to build such a resolution of M inductively. By shifting we can assume the bottom homology group of M is in degree zero. Choose a basis of the finite dimensional k-vector space $H_0(M) \otimes_R k$, and lift to a map

$$f_0 : M_0 \to M$$

with M_0 a finitely generated free R-module, so that $H_0 f_0$ induces an iso on base change to k. In particular, by Nakayama's lemma, $H_0 f_0$ is surjective. Then we can continue in the usual manner to obtain a resoultion: pass to the fiber of f_0 and iterate this construction. □

Exercise 18.

1. *Suppose R is a noetherian ring such that every module has finite tor-amplitude (this is equivalent to saying that R is a regular noetherian ring by Serre's homological characterization of regularity). Show that every finitely generated R-module M, viewed as an object of $D(R)$ living in degree zero, is a perfect complex.*

2. *Let k be a field and $R = k[\epsilon]/\epsilon^2$. Consider the R-module $M = R/\epsilon$. Show that M viewed as an object of $D(R)$ living in degree zero is not a perfect complex.*

Exercise 19.

1. *Let S be a scheme and let $n \in \mathbb{Z}_{>0}$. Show that the set of isomorphism classes of objects $M \in D(S)$ equipped with fixed isomorphisms $H_0(M) \simeq \mathcal{O}_S$ and $H_n(M) \simeq \mathcal{O}_S$ and satisfying furthermore that $H_k(M) = 0$ for $k \neq 0, n$ is in bijection with $H^{n+1}(S; \mathcal{O}_S)$.*

2. *Let E and E' be any two elliptic curves over a field k. Show that $H^2(E \times_k E'; \mathcal{O}) = k$.*

3. *Deduce that there is a split-perfect complex on $E \times_k E'$ which is not isomorphic to the direct sum of its shifted homology groups.*