

Dirichlet L-series at $s = 1$

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Recall from last time the formula

$$\zeta_{\mathbb{Q}(\zeta_p)} = \zeta_{\mathbb{Q}}(s) \cdot \prod_{\chi \neq 1} L(s, \chi).$$

This was established “formally”, by treating each of the terms as a formal Dirichlet series $\sum_{n=1}^{\infty} a_n \cdot n^{-s}$, a book-keeping device for the coefficients a_n .

Now we will forget this formal approach. We’ll “do analysis” on each of these terms separately, and compare the results. Today we focus on the Dirichlet L-series $L(s, \chi)$ for $\chi \neq 1$; then afterwards we’ll analyze the general Dedekind zeta function $\zeta_F(s)$ for a number field F . Of course, this covers the other two factors $\zeta_{\mathbb{Q}(\zeta_p)}$ and $\zeta_{\mathbb{Q}}(s)$.

More specifically, we will prove the following two results. The first is “qualitative”, and the second is “quantitative”:

Theorem 0.1. *Let $\chi : (\mathbb{Z}/p\mathbb{Z})^{\times} \rightarrow \mathbb{C}^{\times}$ be an arbitrary character. Then the Dirichlet L-series*

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$$

converges absolutely for $\operatorname{Re}(s) > 1$. If $\chi \neq 1$, then it furthermore converges conditionally for $\operatorname{Re}(s) > 0$. Moreover, in both cases the limit is an analytic function.

Recall that to converge “conditionally” just means to converge. But the point is that with conditional convergence (as opposed to absolute convergence), it matters what order you take the sum in.

By the way, maybe now is the time to introduce the convenient notation $\sigma = \operatorname{Re}(s)$. The reason this is the quantity that tends to come up is that $|n^{-s}| = n^{-\sigma}$, seeing as any positive real raised to a purely imaginary power lives on the unit circle.

Theorem 0.2. *Let $\chi : (\mathbb{Z}/p\mathbb{Z})^{\times} \rightarrow \mathbb{C}^{\times}$ be a nontrivial character. Then there is an explicit finitary formula for $L(1, \chi)$ (we’ll write it down later). This formula takes a different form according as to whether $\chi(-1) = -1$ or $\chi(-1) = 1$. If $\chi(-1) = -1$, then the formula shows that $L(1, \chi)$ can be expressed in terms of ζ_p , ζ_{p-1} , i , and π . If $\chi(-1) = 1$, then the formula shows that $L(1, \chi)$ can be expressed in terms of ζ_p , ζ_{p-1} , and some values $\log(x)$ for x a positive real unit in $\mathbb{Z}[\zeta_p]$.*

When $\chi(-1) = -1$, we say that χ is *odd*. When $\chi(-1) = 1$, we say that χ is *even*. These are the only two possibilities, because the fact that -1 has order 2 in $(\mathbb{Z}/p\mathbb{Z})^{\times}$ implies that $\chi(-1)$ must be a second root of unity. This odd/even dichotomy will be important to us moving forward. Basically, for

now you should just think that we like odd characters, but we don't like even characters. In terms of the calculation of $L(1, \chi)$, this is reflected in the fact that π is simpler to deal with than logarithms of cyclotomic units.

OK, so let's get going on the first theorem, the qualitative one. We'll start with a more general question. Given a Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} a_n \cdot n^{-s}$$

with arbitrary complex coefficients $a_n \in \mathbb{C}$, what can we say about absolute convergence? What about conditional convergence? Certainly, the region of convergence will depend on how fast the coefficients a_n grow. Here is a nice general result in that direction:

Proposition 0.3. *Let $F(s) = \sum_{n=1}^{\infty} a_n \cdot n^{-s}$ be as above. Recall $\sigma = \operatorname{Re}(s)$.*

1. *Suppose we have an estimate of the form $|a_n| \leq C \cdot n^r$, where C and r are positive constants. Then the sum defining $F(s)$ converges absolutely to an analytic function for $\sigma > r + 1$.*
2. *Let $A_N = \sum_{n=1}^N a_n$. Suppose we have an estimate of the form $|A_N| \leq C \cdot N^r$, where C and r are positive constants. Then the sum defining $F(s)$ converges conditionally to an analytic function for $\sigma > r$.*

Note that the two sequences $\{A_N\}$ and $\{a_n\}$ completely determine one another. The first is the “discrete integral” of the second (that was the definition), and the second is the “discrete derivative” of the first (discrete derivative means, take the difference between consecutive terms).

Proof. Suppose we know $|a_n| \leq C \cdot n^r$, where C and r are positive constants. Then

$$|F(s)| \leq \sum_{n=1}^{\infty} |a_n| n^{-\sigma} \leq C \cdot \sum_{n=1}^{\infty} n^{r-\sigma}.$$

But by comparison with the integral $\int_1^{\infty} x^{r-\sigma} dx$, this latter sum converges for $\sigma > r + 1$. That justifies that $F(s)$ (or rather, to be precise, the series defining $F(s)$) converges for $\sigma > r + 1$. The fact that the limit is an analytic function can be justified by quoting the theorem from complex analysis which says that the limit of a sequence of analytic functions is analytic, provided the convergence is uniform on compact subsets of the region in question.

Now suppose we know $|A_N| \leq C \cdot N^r$, where C and r are positive constants. We will use the “summation by parts” identity (discrete analog of integration by parts):

$$\sum_{n=1}^N a_n \cdot n^{-s} = A_N \cdot (N+1)^{-s} - \sum_{n=1}^N A_n \cdot ((n+1)^{-s} - n^{-s}).$$

Now let's look at the absolute values of each of the terms on the right, assuming $\sigma > r > 0$. The first term has absolute value

$$|A_N| \cdot (N+1)^{-\sigma} \leq C \cdot N^r (N+1)^{-\sigma}.$$

This tends to zero when $N \rightarrow \infty$, so there's no problem there. As for the second term, using $(n+1)^{-s} - n^{-s} = \int_n^{n+1} (-s) \cdot x^{-s-1} dx$, we obtain

$$\left| \sum_{n=1}^N A_n \cdot ((n+1)^{-s} - n^{-s}) \right| \leq \sum_{n=1}^N C \cdot n^r \cdot |s| \cdot n^{-\sigma-1}.$$

We can pull C and $|s|$ out of the sum, and we see that we have the same sort of expression as before, but with $-\sigma$ replaced by $-\sigma-1$. So we conclude, again by comparison with the integral, that we have convergence to an analytic function for $\sigma > r$, as desired. $\square$

Now, to prove our first theorem, we just need to see that the criteria of this proposition apply to our case. In other words, it suffices to see that:

1. If $\chi : (\mathbb{Z}/p\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$ is an arbitrary character, then $|\chi(n)| \leq C$ for some $C > 0$.
2. If furthermore $\chi \neq 1$, then $|\sum_{n=1}^N \chi(n)| \leq C'$ for some $C' > 0$.

The first claim is trivial, since $|\chi(n)| = 1$ for all n (it's a root of unity). As for the second claim, the point is that the sum of $\chi(n)$ over p adjacent values of n is always zero. This is because such a sum is the same thing as summing $\chi(x)$ over all $x \in (\mathbb{Z}/p\mathbb{Z})^\times$. But since χ is a homomorphism, its image is a subgroup of the $(p-1)^{st}$ roots of unity, and hence $\text{im}(\chi)$ consists of the f^{th} roots of unity for some $f|(p-1)$. Furthermore, again since χ is a homomorphism, it hits each value (i.e. each f^{th} root of unity) equally often. Note that $f > 1$, since χ is non-trivial. Thus we need only apply the following general fact, which I leave as an exercise:

Lemma 0.4. *Let $f \in \mathbb{N}$. Then the sum of all the f^{th} roots of unity in $\mathbb{C}$ is equal to 0 if $f > 1$, and is equal to 1 if $f = 1$.*

So we have seen that the sum of $\chi(n)$ over p adjacent values of n is always 0. Then to see that the values $\sum_{n=1}^N \chi(n)$ are bounded in absolute value, we can maximally break the region $[1, N]$ into consecutive subregions of length p . The sum over each of these regions will give 0. Then there's at most $p-1$ terms left over. Each term has absolute value 1, so actually the whole sum is bounded by $p-1$ (independently of N).

In total, we've just proved the qualitative theorem. In particular, we see that $L(s, \chi)$ converges at $s = 1$. So now we can turn to the quantitative result, which calculates $L(1, \chi)$. But first, there is also a qualitative result about $L(1, \chi)$ which is interesting to mention. It is the following:

$$L(1, \chi) \neq 0$$

for $\chi \neq 1$. This fact is interesting for two reasons (at least). First, it is the crucial ingredient in Dirichlet's proof that there are infinitely many primes congruent to $a \bmod n$, where a and n are arbitrary relatively prime natural numbers. This theorem, as far as we know, can't be proved without using the analysis of L-functions. But second, as far as I know, the explicit finitary formula which we'll give for $L(1, \chi)$ is of no help in proving that $L(1, \chi) \neq 0$. When χ is odd, the formula reduces the claim $L(1, \chi) \neq 0$ to a purely algebraic fact. But again, as far as I know, there is no known algebraic proof of this algebraic fact.

We can prove this fact $L(1, \chi) \neq 0$ if we're willing to borrow from the future. Namely, in the future we'll show that $\zeta_F(s)$ has a meromorphic continuation to a neighborhood of $s = 1$, with only a simple pole at $s = 1$ (we'll actually show something much more precise). Now, if we recall that

$$\zeta_{\mathbb{Q}(\zeta_p)}(s) = \zeta_{\mathbb{Q}}(s) \cdot \prod_{\chi \neq 1} L(s, \chi),$$

we see that none of the $L(1, \chi)$ can be zero, because otherwise we'd be canceling the pole of $\zeta_{\mathbb{Q}}(s)$ at $s = 1$, leaving no pole left for $\zeta_{\mathbb{Q}(\zeta_p)}(s)$.

OK, now let's get going on calculating $L(1, \chi)$. We won't get anywhere by setting $s = 1$ right away though. Instead we'll mess with the defining formula a lot until it starts to look like a good idea to set $s = 1$ (or rather, to let $s \rightarrow 1^+$, since by messing with the formula we'll destroy the conditional convergence!).

We start with a more general discussion, of Dirichlet series with periodic coefficients: let $F(s) = \sum_{n=1}^{\infty} a(n) \cdot n^{-s}$, and assume that

$$a(n+p) = a(n)$$

for all $n \in \mathbb{N}$. This is a pretty drastic assumption. In fact there's only a finite-dimensional space of possibilities for such sequences $\{a(n)\}$. This is a precise claim: the set of such sequences is the same as the set of functions

$$\mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{C}.$$

This is a complex vector space of dimension p . An obvious basis is given by the *characteristic functions* of the $k \in \mathbb{Z}/p\mathbb{Z}$: namely, let $c_k(n)$ be 1 if n is congruent to $k \pmod{p}$, and 0 otherwise. Our arbitrary sequence $a(n)$ can be uniquely expressed in terms of this basis by the formula

$$a(n) = \sum_{k=0}^{p-1} a(k) \cdot c_k(n).$$

What this means is that if we want to find a finitary formula for the Dirichlet series defined by the $a(n)$, it would suffice to find a finitary formula for the Dirichlet series defined by the $c_k(n)$, for every $k \in \mathbb{Z}/p\mathbb{Z}$.

But actually, this won't get us too far either! Instead, it pays to use a more tricky basis. The first basis just used that $\mathbb{Z}/p\mathbb{Z}$ is a set. There is also a "Fourier-dual" basis, which uses that $\mathbb{Z}/p\mathbb{Z}$ is an abelian group. Here the basis elements will be the characters of the *additive* group $\mathbb{Z}/p\mathbb{Z}$, or equivalently the p^{th} roots of unity ζ in $\mathbb{C}$. We define

$$b_{\zeta}(n) = \zeta^n.$$

Now, there are fancy ways of seeing a priori that these b_{ζ} also form a basis of the set of functions $\mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{C}$. For example, it is a general fact from representation theory that the set of class functions on a finite group G has a basis given by the characters of the irreducible representations of G . When $G = \mathbb{Z}/p\mathbb{Z}$ this gives exactly what we want. But we'll adopt a more explicit approach, and just write each of the previous basis vectors c_k in terms of these b_{ζ} . Namely:

$$c_k(n) = \frac{1}{p} \sum_{\zeta^{p=1}} \zeta^{-k} \cdot b_{\zeta}(n).$$

If you unwind the definitions, this claim just reduces to our lemma above about summing over roots of unity.

OK. Now if you combine the two formulas above, you obtain the following expression for a general periodic function $a : \mathbb{N} \rightarrow \mathbb{C}$ with period p in terms of its first p values and the p^{th} roots of unity:

$$a(n) = \frac{1}{p} \sum_{k=0}^{p-1} \sum_{\zeta^p=1} a(k) \cdot \zeta^{n-k}.$$

This is actually kind of nice: we've written an explicit finitary algebraic expression for a general periodic function. Now let's specialize to our $a(n) = \chi(n)$, and simultaneously choose to write our roots of unity as r^{th} powers of ζ_p^{-1} , for $\zeta_p = \exp(2\pi i/p)$. We find

$$\chi(n) = \frac{1}{p} \sum_{k=0}^{p-1} \sum_{r=0}^{p-1} \chi(k) \cdot \zeta_p^{rk} \cdot \zeta_p^{-rn}.$$

Now, here is a nearly trivial remark. We can exclude the $k = 0$ and $r = 0$ cases from the above double sum. For $k = 0$, this is because $\chi(k) = 0$ by definition. For $r = 0$, this is because, again by our lemma and the fact that $\chi \neq 1$, the sum of the values of χ is zero. Thus we actually have (after throwing in some rearranging)

$$\chi(n) = \frac{1}{p} \sum_{r=1}^{p-1} \zeta_p^{-rn} \sum_{k=1}^{p-1} \chi(k) \cdot \zeta_p^{rk}.$$

Now, although that was a nearly trivial step, in some sense it was a crucial one. We just went from the additive group $\mathbb{Z}/p\mathbb{Z}$ to the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^\times$, which is more friendly to our character χ . Using this we can remove the dependence on r in the last sum $\sum_{k=1}^{p-1} \chi(k) \cdot \zeta_p^{rk}$. Namely, since the values rk as k ranges over $(\mathbb{Z}/p\mathbb{Z})^\times$ are the same as the values k , we can rewrite

$$\sum_{k=1}^{p-1} \chi(k) \cdot \zeta_p^{rk} = \sum_{k=1}^{p-1} \chi(r^{-1}k) \cdot \zeta_p^k = \overline{\chi(r)} \sum_{k=1}^{p-1} \chi(k) \zeta_p^k.$$

Now the only dependence on r is the outside factor $\overline{\chi(r)}$. Let us give a special notation to the remaining sum:

$$g(\chi) := \sum_{k=1}^{p-1} \chi(k) \cdot \zeta_p^k.$$

This just depends on χ : no indices anymore. It is called a "Gauss sum". We would've run into it if we had decided to analyze Gauss's construction of regular n -gons by ruler and compass. But instead the Dirichlet L-series is just showing it to us. In any case, we conclude:

$$\chi(n) = \frac{g(\chi)}{p} \sum_{r=1}^{p-1} \overline{\chi(r)} \cdot \zeta_p^{-rn}.$$

This is just a formula for $\chi(n)$! But now it looks good to consider the whole L-series $L(s, \chi)$. Namely, we get

$$L(s, \chi) = \frac{g(\chi)}{p} \sum_{r=1}^{p-1} \overline{\chi(r)} \cdot \sum_{n=1}^{\infty} \zeta_p^{-rn} \cdot n^{-s}.$$

This will give a finitary formula for $L(1, \chi)$ as long as we can evaluate the limit as $s \rightarrow 1^+$ of the sum

$$\sum_{n=1}^{\infty} \zeta_p^{-rn} \cdot n^{-s}.$$

Now, if you naively set $s = 1$, this looks a lot like the power series expansion trying to converge to $-\log(1 - \zeta_p^{-r})$. With a little bit of work one can see that this is the case: the limit as $s \rightarrow 1^+$ of the above expression is indeed $-\log(1 - \zeta_p^{-r})$. This is a somewhat delicate statement, since $1 - \zeta_p^{-r}$ lies on the boundary of the disc of convergence of the power series expansion for $\log$. So the convergence is only conditional, not absolute. But it still works.

We deduce:

$$L(1, \chi) = -\frac{g(\chi)}{p} \sum_{r=1}^{p-1} \overline{\chi(r)} \cdot \log(1 - \zeta_p^{-r}).$$

That's an explicit finitary formula for $L(1, \chi)$. It works for all nontrivial characters χ ; the even/odd distinction never came up. But it will come up now, when we try to understand the term $\log(1 - \zeta_p^{-r})$ more explicitly.

Recall that for $z = re^{i\theta}$ with $-\pi < \theta < \pi$ and $r > 0$, we have

$$\log(z) = \log(r) + i\theta.$$

(This is the “principal” branch of the logarithm, the unique analytic function in the above region which agrees with the usual log on the positive real axis.) Thus to understand $\log(1 - \zeta_p^{-r})$ is to understand the absolute value and the argument of $1 - \zeta_p^{-r}$. This is a matter of trigonometry. I recommend doing it in steps, with pictures: first understand that $\text{Arg}(\zeta_p^{-r}) = -2\pi r/p$. Then deduce that $\text{Arg}(-\zeta_p^{-r}) = \pi - 2\pi r/p$. Then deduce that $\text{Arg}(1 - \zeta_p^{-r}) = \pi/2 - \pi r/p$. Finally, show that $|1 - \zeta_p^{-r}| = 2 \cdot \sin(\pi r/p)$. The conclusion is that

$$\log(1 - \zeta_p^{-r}) = \log(2 \cdot \sin(\pi r/p)) + \frac{\pi i}{2}(1 - 2r/p).$$

Now we can plug this back into our formula. Then we can try pairing up r with $-r$. What we find is that for χ odd, the $\log(\sin)$ terms will cancel, and for χ even, the πi terms will cancel. Let's work it like this. In our first formula for $L(1, \chi)$, replace r by $-r$. This gives

$$L(1, \chi) = -\frac{g(\chi)}{p} \sum_{r=1}^{p-1} \overline{\chi(-r)} \cdot \overline{\log(1 - \zeta_p^{-r})}.$$

Now average these two formulas together, getting

$$L(1, \chi) = -\frac{g(\chi)}{p} \sum_{r=1}^{p-1} \overline{\chi(r)} \cdot \frac{1}{2} \left(\log(1 - \zeta_p^{-r}) + \chi(-1) \cdot \overline{\log(1 - \zeta_p^{-r})} \right).$$

Thus when χ is odd we're picking up i times the imaginary part of $\log(1 - \zeta_p^{-r})$, and when χ is even we're picking up the real part of $\log(1 - \zeta_p^{-r})$. The conclusion is that for χ odd, we have

$$L(1, \chi) = -\frac{i \cdot \pi \cdot g(\chi)}{2p} \sum_{r=1}^{p-1} \overline{\chi(r)} \cdot (1 - 2r/p).$$

But since again the sum of all the values of χ is 0, this simplifies further to

$$L(1, \chi) = \pi \cdot \frac{i \cdot g(\chi)}{p^2} \sum_{r=1}^{p-1} \overline{\chi(r)} \cdot r.$$

That's our final formula for $L(1, \chi)$ when $\chi(-1) = -1$. It's purely algebraic (just p^{th} and $(p-1)^{\text{st}}$ and 4^{th} roots of unity), except for the factor of π .

On the other hand, when χ is even we get

$$L(1, \chi) = -\frac{g(\chi)}{p} \sum_{r=1}^{p-1} \overline{\chi(r)} \cdot \log(2 \cdot \sin(\pi r/p)).$$

But again since the sum of the values of $\chi(r)$ is 0, we can remove the factor of 2 and get the simpler

$$L(1, \chi) = -\frac{g(\chi)}{p} \sum_{r=1}^{p-1} \overline{\chi(r)} \cdot \log(\sin(\pi r/p)).$$

Now, actually, let's do something a little weird and put in an extra factor of $\sin(\pi/p)^{-1}$. We can do this for the same reason. This gives:

$$L(1, \chi) = -\frac{g(\chi)}{p} \sum_{r=1}^{p-1} \overline{\chi(r)} \cdot \log\left(\frac{\sin(\pi r/p)}{\sin(\pi/p)}\right).$$

This will be our final formula for $L(1, \chi)$ when $\chi(-1) = 1$. It's not as nice as when χ is odd, because the $\log(\sin)$ factors are a little mysterious. But there is something nice about them, which is also why we divided by $\sin(\pi/p)$. Namely, $\frac{\sin(\pi r/p)}{\sin(\pi/p)}$ is actually a unit in $\mathbb{Z}[\zeta_p]$. This is because, up to roots of unity, (at least for $p > 2$)

$$\frac{\sin(\pi r/p)}{\sin(\pi/p)} = \frac{1 - \zeta_p^r}{1 - \zeta_p},$$

and we saw in the course of the first proof of the irreducibility of Φ_p that these numbers are units. For now, let's just keep this in mind: though the $\log(\sin)$ factors are strange, we can think of them as coming from units in $\mathbb{Z}[\zeta_p]$. In this guise they will also match up with our corresponding analysis of $\zeta_{\mathbb{Q}(\zeta_p)}(s)$ and give us interesting information.