Lecture 13: Fundamental groups and universal covers

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Well, we clearly ran out of time in this course, and we won't be proving either the geometrization theorem or the algebrization theorem. But we can still see how these theorems work in the special case of elliptic curves, the Riemann surfaces we studied in the past two lectures. The arguments we give in this case are explicit prototypes for the general arguments.

One of the keys to geometrization is the general topological notion of a *universal cover*, so we start by reviewing this. We'll stick to the context of Riemann surfaces, but the ideas are valid much more generally. This theory discussed here was invented by Poincare, who was also the first to notice the geometrization theorem. Probably not a coincidence.

Definition 0.1. *Let X be a Riemann surface, and x, y two points on X. A* path *from x to y is a piecewise continuously differentiable map*

$$\gamma : [0,1] \to X$$

such that $\gamma(0) = x$ and $\gamma(1) = y$. If γ and γ' are two such paths from x to y, then a homotopy *between γ and γ' is a "path of paths" connecting γ to γ', i.e. a piecewise continuously differentiable map*

$$h : [0,1] \times [0,1] \to X$$

such that for all $s \in [0,1]$ the map $h_s(t) = h(s,t)$ is a path from x to y, with $h_0 = \gamma$ and $h_1 = \gamma'$.

If there exists a homotopy from γ to γ', then we say that γ and γ' are homotopic, and write $\gamma \sim \gamma'$. This defines an equivalence relation on paths from x to y; we denote the set of equivalence classes by

$$\pi_X(x,y).$$

Thus, $\pi_X(x,y)$ is the set of homotopy classes of paths on X from x to y.

The magic of this definition is the following. If, say, X is connected, then there is an unfathomably large number of paths from x to y. But on the other hand, there is also an unfathomable number of homotopies possibly connecting these paths. However, these unfathomable infinities cancel each other out, and the resulting set $\pi_X(x,y)$ ends up being fathomable, even if infinite. This is a general phenomenon in homotopy theory.

Here are some simple examples:

1. Let X be the unit disk $\mathbb{D}$ or the complex plane $\mathbb{C}$, or, more generally, any *convex* open subset of $\mathbb{C}$. Then for any $x, y \in X$, the set $\pi_X(x,y)$ has exactly one element, represented by the straight-line path from x to y.

   Indeed, given any two paths γ, γ' from x to y, we can make a homotopy between them by defining $h(s,t) = s \cdot \gamma'(t) + (1-s) \cdot \gamma(t)$. This linearly interpolates between γ and γ'; it lies in X because X is convex.

2. Let $X = \mathbb{P}^1$. Also in this case, for any $x, y \in X$ we have that $\pi_X(x, y)$ has exactly one element. Indeed, every path from x to y must miss at least one point of $\mathbb{P}^1$; but $\mathbb{P}^1$ with a point removed identifies with $\mathbb{C}$, and we know there from the previous example that any two paths are homotopic.

3. Let $X = \mathbb{C} \setminus \{0\}$. At last, something interesting: for any two points $x, y \in X$, the set $\pi_X(x, y)$ can be identified with the set of integers $\mathbb{Z}$. Namely, to an integer $n \in \mathbb{Z}$ corresponds any path which wraps n times around the origin counterclockwise (say) while going from x to y. It's less trivial to verify this example, but we will have the tools for it by the end of the next lecture.

If $\pi_X(x, y)$ has exactly one element for any two points $x, y \in X$, then we say that X is *simply connected*. (Note that simply connected implies connected.) Thus, $\mathbb{D}, \mathbb{C}$, and $\mathbb{P}^1$ are simply connected, but $\mathbb{C} \setminus \{0\}$ is not.

Intuitively, X being simply connected means that there are no "holes" in X. If you have a hole, you can make paths around it in different (non-homotopic) ways.

Now we turn to the main business of this lecture. If X is an arbitrary Riemann surface and x is a point on X, then there is a canonical way to "grow" this point x into a simply-connected Riemann surface $\widetilde{X}$, which maps by a "covering map" to X. Furthermore, if X is connected, then we can recover X from $\widetilde{X}$ in the following way: the set $\Gamma = \pi_X(x, x)$ has a natural group structure, this group naturally acts on $\widetilde{X}$ properly and freely, and the quotient Riemann surface $\widetilde{X}/\Gamma$ canonically identifies with X via the covering map. Thus, the study of connected Riemann surfaces reduces to that of simply connected Riemann surfaces, and the actions of groups on the latter. Moreover, the relevant group is recovered "topologically" from the original Riemann surface.

The picture you can have is that $\widetilde{X}$ serves to unravel the loops around the holes of X. For example, if you take $X = \mathbb{C} \setminus \{0\}$ and $x = 1 \in X$, then it turns out that $\widetilde{X}$ can be identified with $\mathbb{C}$, mapping to $\mathbb{C} \setminus \{0\}$ via the exponential map. If you consider the pre-image of a loop going once around 0 in $\mathbb{C} \setminus \{0\}$, you will see an infinite spiral staircase mapping down to the loop. This is the "unraveling" of that non-trivial loop. Note that we've already seen that the exponential map identifies $\mathbb{C} \setminus \{0\}$ with the quotient of $\mathbb{C}$ by translation by $2\pi i \cdot \mathbb{Z}$; this matches with the fact that $\pi_X(1, 1) = \mathbb{Z}$.

Before defining $\widetilde{X}$, we need to introduce some important structure which exists on the collection of sets $\pi_X(x, y)$ as x and y vary in X. This is the collection of *composition laws*: for any three points $x, y, z \in X$, there is a natural map

$$\pi_X(x, y) \times \pi_X(y, z) \to \pi_X(x, z)$$

denoted by $(f, g) \mapsto f \cdot g$ and defined as follows: if f, g are respectively represented by paths $\gamma, \gamma' : [0, 1] \to X$, then since $\gamma(1) = \gamma'(0)$ we can stick γ and γ' side-by-side to get a piecewise smooth map $[0, 2] \to X$; then we can reparametrize the interval $[0, 2]$ to identify it with $[0, 1]$. The homotopy class of the resulting path from x to z is, by definition, $f \cdot g$.

This is independent of the choice of paths γ, γ' representing f, g, as well as of the choice of reparametrization. (Exercise: the homotopy class of a path is not changed by reparametrizing the unit interval.) Intuitively, the composition of two paths is obtained by first going around the first path, then going around the second path. This is what gives us the maps

$$\pi_X(x, y) \times \pi_X(y, z) \to \pi_X(x, z).$$

These maps satisfy some axioms, which express that the set of points of X, together with the maps $\pi_X(-,-)$ and these composition laws, form what's called a *groupoid*:

1. The composition law is associative: $(f \cdot g) \cdot h = f \cdot (g \cdot h)$;
2. For every $x \in X$, there is an identity element $id_x \in \pi_X(x,x)$, such that composing on either side with id_x doesn't change anything. (id_x is represented by the constant path with value x.)
3. For every element $f \in \pi_X(x,y)$, there is an element $f^{-1} \in \pi_X(y,x)$ such that $f^{-1} \cdot f = id_x$, $f \cdot f^{-1} = id_y$. (This f^{-1} can be gotten by flipping the interval $[0,1]$ on any path representing f, i.e. do the same path, but backwards.)

Exercise: write down the homotopies which prove that these axioms are satisfied.

This structure π_X is known as the *fundamental groupoid* of X. Note that if we fix a point $x \in X$, then it follows from the above that the set $\Gamma_x := \pi_X(x,x)$ has a natural group structure. Exercise: if x and y are in the same path component of X, then the groups Γ_x and Γ_y are isomorphic. An isomorphism can be gotten by choosing a path from x to y. If you choose a different path, the isomorphism changes by a *inner* automorphism of Γ_x (or Γ_y).

Now we can define the universal cover, $\widetilde{X}$, associated to a Riemann surface X with basepoint x. As you read about this unversal cover and its basic properties, you should recall something similar we've seen before, namely the Riemann surface of an analytic function. This is basically that, but without the function.

Definition 0.2. *Let X be a Riemann surface, and $x \in X$ a point. Let $\widetilde{X}$ denote the set consisting of all homotopy classes of paths $f : [0,1] \to X$ such that $f(0) = x$, where the homotopies are required to fix the endpoints (as in Definition 0.1). Sending f to $f(1)$ defines a map*

$$p : \widetilde{X} \to X.$$

Thus, for $y \in X$, the fiber $p^{-1}(y)$ identifies with the set $\pi_X(x,y)$. The niceness of $p : \widetilde{X} \to X$ stems from the fact that its fibers over nearby points can be canonically identified. More precisely:

Lemma 0.3. *Let U be a simply connected open subset of X (e.g., U could be isomorphic to a disk). Define an equivalence relation on $p^{-1}(U)$ as follows: say that $f_1, f_2 \in p^{-1}(U)$ are* horizontally situated *if $f_2 \cdot f_1^{-1} \in \pi_X(y_1, y_2)$ can be represented by a path in U. Denote the set of equivalence classes by S_U. Then the map*

$$p^{-1}(U) \to S_U \times U$$

which is the natural quotient on the first factor and the projection p on the second factor, is a bijection.

Thus, via this bijection, the map $p : p^{-1}(U) \to U$ looks exactly like the projection $S_U \times U \to U$. In other words, above U, there are S_U-many "sheets" of $p^{-1}(U)$. Each "sheet" maps down to U bijectively, and consists of points which are horizontally situated with respect to each other.

Proof. That being horizontally situated is an equivalence relation follows from the facts that the identity path from a point in U to itself lies in U, that if a path lies in U then so does its reverse, and that a composition of paths in U lies in U. To see that the map is a bijection, we need to check that for every $y \in U$ and every $f_1 \in p^{-1}(U)$, there is a unique $f_2 \in p^{-1}(y)$ such that f_1 and f_2 are horizontally situated. And indeed, f_2 is uniquely determined as $f_1 \cdot g$, where $g \in \pi_X(y_1, y)$ is represented by a path in U. Such a g exists and is unique since U is simply connected. $\square$

Note that if we shrink U to a smaller simply connected open subset V, then we get two different bijections identifying $p^{-1}(V)$: namely the one for U restricted over V:

$$p^{-1}(V) \simeq S_U \times V,$$

and the one for V:

$$p^{-1}(V) \simeq S_V \times V.$$

However, it is easy to see that the natural map $S_V \to S_U$ is a bijection which renders these compatible.

It follows that we can use these bijections to equip $\widetilde{X}$ with the structure of a Riemann surface. Namely, each $S_U \times U$ is a Riemann surface in the obvious way (disjoint union of S_U copies of U), and we define charts on on $\widetilde{X}$ by requiring each of the bijections of the lemma be an isomorphism of Riemann surfaces. By the above discussion, if you vary U these bijections can only change by bijections $S \times U \simeq S' \times U$ coming from bijections of sets $S \simeq S'$; such bijections $S \times U \simeq S' \times U$ are certainly holomorphic, so the resulting charts on $\widetilde{X}$ are indeed compatible.

Proposition 0.4. *With these charts, $\widetilde{X}$ becomes a simply connected Riemann surface. There is a canonical proper free action of $\Gamma = \pi_X(x,x)$ on $\widetilde{X}$, and the quotient $\widetilde{X}/\Gamma$ is canonically isomorphic to the connected component of X containing x.*

Proof. We won't need the simple connectedness, so we leave that as an exercise. The action of Γ is defined as follows: for $\gamma \in \Gamma$, set

$$\gamma(f) = \gamma \cdot f.$$

In terms of the defining charts $S_U \times U$, it's easy to see from the groupoid axioms that Γ acts by simply transitively permuting the sets S_U, leaving the U factor alone. This gives the conclusion locally, hence globally. □

Corollary 0.5. *The geometrization theorem is equivalent to the following claim: every simply connected Riemann surface $\mathbb{M}$ is isomorphic to either $\mathbb{D}$, $\mathbb{C}$, or $\mathbb{P}^1$. Moreover, given a connected Riemann surface X, the group Γ acting on the model space $\mathbb{M}$ for which $\mathbb{M}/\Gamma \simeq X$ can be abstractly identified with the fundamental group $\pi_X(x,x)$ at any chosen point $x \in X$.*

This classification of simply connected Riemann surfaces is called the *Riemann mapping theorem*. Again, we won't have time to prove it. What we'll do instead is answer the following question: what structure on a connected X with basepoint x will allow us to recognize that $\widetilde{X}$ identifies with $\mathbb{C}$, and hence that X is uniformized by $\mathbb{C}$? The answer will be introduced in the next lecture.

There is something worth noting in this discussion: while the group Γ is recovered purely from the topology of X (it is the fundamental group), the manner of Γ's acting on the model $\mathbb{M}$ depends not just on the topology of X, but truly on its Riemann surface structure. We'll see this explicitly for elliptic curves.