Lecture 4: More examples of Riemann surfaces

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In the last lecture we defined the notion of a Riemann surface, and the notion of an isomorphism between Riemann surfaces. We also saw two examples: the complex plane $\mathbb{C}$ itself, and the Riemann sphere $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. We also noted that any open subset of a Riemann surface is also a Riemann surface.

Today, though, we want to talk about more varied examples, both of Riemann surfaces and of isomorphisms between Riemann surfaces. To discuss the examples properly, though, it will be handy to have some more language at our disposal. So here are some more definitions.

Definition 0.1. *Let X and Y be two Riemann surfaces. A map $f : X \to Y$ is called* holomorphic *if for every $x \in X$, there exists a chart $U \leftrightarrow V$ of X with $x \in U$ and a chart $U' \leftrightarrow V'$ of Y with $f(x) \in U'$, such that:*

- *$f(U) \subset U'$;*
- *When we transport $f|_U : U \to U'$ via the charts to a map $F : V \to V'$, then F is a holomorphic map between open subsets of $\mathbb{C}$.*

In a nutshell, f is holmomorphic if it is locally holomorphic; and this latter condition can be checked on charts. Here are some basic remarks:

1. The identity map from any Riemann surface to itself is holomorphic;
2. The composition of two holomorphic maps is holomorphic;
3. An isomorphism of Riemann surfaces is the same thing as a holomorphic bijection with holomorphic inverse.

Another definition:

Definition 0.2. *Let $f : X \to Y$ be a holomorphic map between Riemann surfaces. We say that f is a* local isomorphism *if for every point $x \in X$, there is an open subset U containing x such that $f(U)$ is open in Y and $f : U \to f(U)$ is an isomorphism of Riemann surfaces.*

An equivalent way of saying this would be, a holomorphic map is a local isomorphism if the local representatives $F : V \to V'$ on charts (as in the definition of holomorpic) can be taken to be *biholomorphisms*. Equivalently, by the inverse function theorem, it's enough to check that the derivative of F at every point of V is nonzero.

Again, some basic remarks:

1. Every isomorphism is a local isomorphism; in fact, an isomorphism is the same thing as a local isomorphism which is also a bijection.
2. If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are two composable maps between Riemann surfaces, then:

(a) If f and g are local isomorphisms, so is $g \circ f$.

(b) If $g \circ f$ and g are local isomorphisms, so is f.

(c) If $g \circ f$ and f are local isomorphisms and f is surjective, then g is a local isomorphism.

3. If $f : X \to Y$ is a local isomorphism, then an arbitrary map out of Y is holomorphic if and only if its composition with f is holomorphic.

There are many examples of local isomorphisms which are not isomorphisms themselves. Surjectivity could fail: for example, the inclusion of every open subset is a local isomorphism. Injectivity can also fail: for example, the n^{th} power map $\mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ is a local isomorphism, since its derivative is everywhere nonzero. But there are exactly n preimages for every point z in the target $\mathbb{C} \setminus \{0\}$, so the map is not itself an isomorphism.

Now we will give some more examples of Riemann surfaces. The first is related to our discussion of algebraic functions. Let $P(z, w) \in \mathbb{C}(z)[w]$: a polynomial in w whose coefficients are rational functions in z. Actually, there is little loss in assuming the coefficients are actually polynomials in z (e.g., we could always multiply through by denominators), so that $P(z, w) \in \mathbb{C}[z, w]$, a polynomial in two variables with complex coefficients. We will be interested in the *set of zeros* of P, meaning the set

$$C = \{(z_0, w_0) \in \mathbb{C}^2 : P(z_0, w_0) = 0\}.$$

This set C is close to being able to be given a natural Riemann surface structure; but it turns out that, for that, we'll need to throw out some bad points (generally finite in number). Thus, let

$$C' = \{c \in C : \partial_w P(c) \neq 0\}$$

and

$$C'' = \{c \in C : \partial_z P(c) \neq 0\}.$$

Theorem 0.3. *There is a unique Riemann surface structure on $X = C' \cup C''$ such that the z-projection*

$$(z, w) \mapsto z$$

and the w-projection

$$(z, w) \mapsto w$$

both restrict to holomorphic maps $X \to \mathbb{C}$. Moreover, the z-projection is a local isomorphism on C', and the w-projection is a local isomorphism on C''.

Here the uniqueness claim has to be understood in the following sense: given any two such Riemann surface structures, the identity map on X gives an isomorphism between them. That is, any chart from the one must be compatible with any chart from the other.

Proof. First let us show existence, by specifying defining charts. Let $(z_0, w_0) \in X$. Then either $(z_0, w_0) \in C'$ or $(z_0, w_0) \in C''$.

If we're in the first case, then by the implicit function theorem (c.f. Lecture 2) there is an open neighborhood $U \subset \mathbb{C}$ of z_0 and an open neighborhood $V \subset \mathbb{C}$ of w_0 such that there is a unique holomorphic function $f : U \to V$ with $f(z_0) = w_0$ and $P(z, f(z)) = 0$ for all $z \in U$. In fact, we can even arrange it so that

$$(U \times V) \cap C' = \{(z, f(z)) : z \in U\}.$$

Then we can take the chart

$$(U \times V) \cap C' \leftrightarrow U$$

given by the z-projection $(z, w) \mapsto z$ and its inverse $z \mapsto (z, f(z))$.

If we're in the second case instead, then we can just play the same trick with projection to the w-axis instead.

Taken together, these defining charts obviously cover X. Moreover, they're rigged so that the z-projection is a local isomorphism on C' and the w-projection is a local isomorphism on C''. Thus, to ensure that we have a Riemann surface structure, we need only check the compatibility and the Hausdorff condition.

Between two charts of the "project to z" type, the transition function is given by the inverse to projection to z, followed by projection to z. This is the identity map, hence clearly biholomorphic. Thus the compatibility is clear in that case. Ditto between two charts of the project to w type. Finally, if one chart is of the project to z type and the other of the project to w type, then the transition function is given by the inverse to projection to z followed by projection to w: this is just the function f. But f is holomorphic, and the symmetric argument shows that its inverse is holomorphic too. Thus it is biholomorphic, and we have the compatibility in all cases.

Lastly, the Hausdorff condition. Note that for every $x \in X$, the defining chart around x cuts out a subset of X of the form $(U \times V) \cap X$, with U and V open subsets of $\mathbb{C}$. By shrinking U and V, we can make this as small as we like. But two distinct points in $\mathbb{C}^2$ can always be separated by small enough open sets of the form $U \times V$. That verifies the Hausdorff condition.

Finally, we should argue for uniqueness. Suppose we have some other Riemann surface structure on X for which the z-projection and w-projection are holomorphic. Consider the sets $(U \times V) \cap X$ of the defining charts we used above. We can decompose the identity map on such a set as projection to the z-axis followed by the inverse to the projection to the z-axis. The first map is holomorphic in the new Riemann surface structure and the second map is holomorphic in the old one; thus the composite, i.e. the identity, is a holomorphic map from the old to the new. Now we can invoke the fact (mentioned, but not yet proved) that a holomorphic bijection automatically has a holomorphic inverse, and hence is an isomorphism, to conclude. $\square$

This theorem gives us a bunch of new examples of Riemann surfaces. In general, it turns out that they look kind of interesting: they are g-holed donuts with finitely many points removed. Sometimes, however, we can recognize them as old friends in disguise.

For example, take $P(z, w) = w^n - z$, with n a natural number. Here we have $\partial_z P = -1$. This is always nonzero, so there are no points to remove, and

$$X = \{(z_0, w_0) : w_0^n = z_0\}.$$

Furthermore we see that projection to the w-axis is a local isomorphism. But in fact it's a global isomorphism, since it has inverse given by $w \mapsto (w^n, w)$. Thus

$$X \cong \mathbb{C}.$$

Note that, via this isomorphism, the projection of X to the z-axis turns into the n^{th} power map $\mathbb{C} \to \mathbb{C}$. This is only a local isomorphism when we remove zero: that matches with the fact that $\partial_w P = n \cdot w$.

Now we will discuss another source of Riemann surfaces: *quotients by free actions.*

Definition 0.4. *Let X be a Riemann surface. An* automorphism *of X is an isomorphism from X to itself. The set of automorphisms of X is denoted $Aut(X)$.*

Let's give an example, which we'll justify later. It turns out that $Aut(\mathbb{C})$ consists exactly of the maps of the form

$$z \mapsto a \cdot z + b,$$

where a is a nonzero complex number and b is an arbitrary complex number. Thus, any biholomorphism from $\mathbb{C}$ to itself is the composition of rotation, scaling, and translation.

Definition 0.5. *Let Γ be a subset of $Aut(X)$. We say that Γ is a* group of automorphisms acting properly freely on X *if the following conditions are satisfied:*

1. *Γ contains the identity map id, is closed under composition, and is closed under inverses (Group condition);*
2. *For any point $x \in X$, there exists an open set U containing x such that $\gamma(U) \cap U = \emptyset$ whenever $\gamma \neq id$ (Proper freeness condition).*

Here is an example. Take $\Gamma \subset Aut(\mathbb{C})$ to consist of all translations by an integer multiple of $2\pi i$. The group condition is satisfied, because composing these translations is the same thing as adding the integers. The proper freeness condition is also satisfied: we can take for U any open disk of radius $\leq \pi$ around $z \in \mathbb{C}$. So is the properness condition: we can choose for U and V open disks around x and y of any radius less than or equal to half the distance between y and the closest translate of x by a multiple of $2\pi i$.

Note, however, that if we had taken all *rational* multiples of $2\pi i$ instead of all integer multiples, the proper freeness condition would not be satisfied. The elements of Γ have to give the points room to breathe. Another remark is that if $\Gamma \subset Aut(\mathbb{C})$ is to satisfy the proper freeness condition, then Γ must consist entirely of translations. That's because every non-translation γ has a fixed point, and fixed points of non-identity maps certainly violate the condition.

Theorem 0.6. *Let Γ be a group of automorphisms acting properly freely on a Riemann surface X. Consider the set*

$$X/\Gamma$$

obtained from X by identifying x with $\gamma(x)$ for all $x \in X$ and $\gamma \in \Gamma$. There is a unique Riemann surface structure on X/Γ such that the natural projection

$$p : X \to X/\Gamma$$

is holomorphic; moreover, p is in fact a local isomorphism.

Proof. Let us start by being more precise about the set X/Γ and the map $p : X \to X/\Gamma$. We can define X/Γ to be the set of subsets of X of the following form:

$$[x] = \{\gamma(x) : \gamma \in \Gamma\},$$

for x a point in X. Then the map $p : X \to X/\Gamma$ sends x to $[x]$. The following properties of p are all that we will need:

1. p is surjective; this follows from the definition.
2. p is not necessarily injective, but we can understand the extent of its failure to be injective: we have $p(x) = p(y)$ if and only if there is a $\gamma \in \Gamma$ with $\gamma(x) = y$ (this follows from the group condition).

Now let us produce defining charts on X/Γ. Let $x \in X$ be arbitrary; we will produce a chart around $p(x)$. Choose an open subset $U \subset X$ containing x as in the proper freeness condition. I claim that p gives a bijeciton

$$U \leftrightarrow p(U).$$

The map is surjective by definition. For injectivity, suppose $y, y' \in U$ satisfy $p(y) = p(y')$. Then there is a $\gamma \in \Gamma$ with $y' = \gamma(y)$. But this would mean that $y' \in U \cap \gamma(U)$; thus we must have $\gamma = id$ by topological freeness, and hence $y = y'$.

This gives a "chart on X/Γ" in the sense that it gives a bijection from a subset of X/Γ to an open subset of the Riemann surface X. That wasn't the precise definition of what a chart should be: the bijection should have gone to an open subset of $\mathbb{C}$. But we already have charts on X which do that, so by composing this quasi-chart with a real chart on X around x we can get a real chart on X/Γ.

The covering condition follows from the surjectivity of p. As for the compatibility, the transition maps in X/Γ are always given by applying γ (and composing with the charts on X), so are biholomorphic. As for the Hausdorff condition, suppose given two points $x, x' \in X$ with $p(x) \neq p(x')$. Choose U around x and U' around x' as in the proper freeness condition. If $\gamma(U) \cap U'$ is empty for all γ, then $p(U)$ and $p(U')$ will be disjoint neighborhoods of $p(x)$ and $p(x')$, verifying the Hausdorff condition. So suppose on the contrary that $\gamma(U) \cap U'$ is nonempty for some γ. Then this γ is unique, because the $\{\gamma(U)\}$ are all disjoint. But U' contains x' and $\gamma(U)$ contains $\gamma(x)$ and these are two distinct points, so by Hausdorffness of X we can shrink U and U' to ensure that the open sets don't intersect anymore. Then we've placed ourselves in the previous case, so in any case X/Γ is Hausdorff.

With this Riemann surface structure, the map p is clearly holomorphic, and even a local isomorphism: on the charts as defined above, p actually is represented by the identity map. We leave the uniqueness claim as an exercise: it's basically the same as the uniqueness claim in the previous theorem. □

Let's turn to our example, where Γ is the translations by integer multiples of $2\pi i$ on $\mathbb{C}$. Here is how you can picture $\mathbb{C}/\Gamma$. Consider the infinite horizontal strip

$$S = \{a + bi \in \mathbb{C} : 0 \leq b < 2\pi\}.$$

Then the translates of S by integer multiples of $2\pi i$ are disjoint and cover the whole plane $\mathbb{C}$; thus p restricted to S gives a bijection $S \leftrightarrow \mathbb{C}/\Gamma$. But we should be careful not to think that $\mathbb{C}/\Gamma$ looks exactly like S, since S is not an open subset. Indeed, it has a boundary piece, the horizontal line where $b = 0$. If you take an open disk around a point on that boundary, only the top half will lie in S. The bottom half will be outside S, and thus, if we want to think about it in $\mathbb{C}/\Gamma$, we have to translate it up by $2\pi i$. In a sense we should imagine that $\mathbb{C}/\Gamma$ is gotten from the infinite horizontal strip by gluing the bottom to the top by translation. Thus $\mathbb{C}/\Gamma$ is an infinite cylinder.

You may not think we've met an infinite cylinder before, but actually we have, in disguise. Indeed:

Proposition 0.7. *There is a natural isomorphism of Riemann surfaces $\mathbb{C}/2\pi i\mathbb{Z} \cong \mathbb{C} \setminus \{0\}$, given by the exponential map.*

Proof. Consider the exponential map $exp : \mathbb{C} \to \mathbb{C} \setminus \{0\}$; recall that if we use cartesian coordinates $a + bi$ on the source and polar coordinates (r, θ) on the target, then

$$exp(a + bi) = (e^a, b).$$

From this we see that $exp(z + 2\pi i) = exp(z)$ for all $z \in \mathbb{C}$, so that there is a unique map $f : \mathbb{C}/2\pi i\mathbb{Z} \to \mathbb{C} \setminus \{0\}$ such that $f(p(z)) = exp(z)$ for all $z \in \mathbb{C}$. We claim that this map f gives the isomoprhism.

Since the projection $p : \mathbb{C} \to \mathbb{C}/2\pi i\mathbb{Z}$ and exp are both local isomorphisms, and p is surjective, it follows that f is a local isomoprhism. On the other hand the description of exp in terms of polar coordinates clearly shows that f is bijective. Thus it is an isomorphism, as claimed. □

This identification of $\mathbb{C} \setminus \{0\}$ with a quotient of $\mathbb{C}$ is an example of what's known as *uniformization*. The reason for the name is (I think) the following. Note that $\mathbb{C} \setminus \{0\}$ inherits Euclidean geometry as a subset of the plane $\mathbb{C}$. But in a sense the Riemann surface $\mathbb{C} \setminus \{0\}$ is not "uniform" with respect to this Euclidean geometry: there are automorphisms of $\mathbb{C} \setminus \{0\}$, e.g. multiplication by nonzero complex numbers, which stretch distances and hence do not preserve the rigid geometry.

However, the presentation of $\mathbb{C} \setminus \{0\}$ as a quotient of $\mathbb{C}$ by a group of translations induces a new Euclidean geometry on $\mathbb{C} \setminus \{0\}$, since translations are Euclidean isometries. And in this new geometry, multiplication by a nonzero complex number indeed preserves the rigid geometry, since it just corresponds to translations on $\mathbb{C}$. (Or, in the cylinder picture, it is translation along the cylinder followed by rotation of the cylinder.) Exercise: what does the automorphism $z \mapsto 1/z$ look like on the cylinder?

Let's conclude with one last (similar) example, which we'll only justify later.

Instead of taking the quotient of $\mathbb{C}$ by vertical translations, we could take the quotient of $\mathbb{C}$ simultaneously by horizontal and vertical translations. For example, we could take

$$\mathbb{C}/\mathbb{Z}[i],$$

the quotient of $\mathbb{C}$ by all translations by numbers of the form $a + bi$ with $a, b \in \mathbb{Z}$. In this case, instead of getting a cylinder, we get a torus (called the *square torus*, for obvious reasons). It turns out that we've also (almost) encountered this Riemann surface already, too:

Theorem 0.8. *Let X' denote the Riemann surface obtained by removing from $\mathbb{C}/\mathbb{Z}[i]$ the single point which is the image of $0 \in \mathbb{C}$. Then X' is naturally isomorphic to the Riemann surface obtained from the polynomial*

$$P(z, w) = w^2 - z^3 - z.$$

Thus, the set of solutions $\{(z, w) : w^2 = z^3 - z\}$, once we add a missing point (and there are natural algebraic ways to do this), carries a hidden Euclidean geometry!