Lecture 5: Statement of the main theorems.

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The goal for today is to state the two main theorems of this course, the *uniformization* and *algebraization* theorems.

We start with the uniformization theorem. It shows that, more or less, the study of Riemann surfaces can be reduced to that of the following three model cases:

1. $X = \mathbb{D}$, the unit disk in the complex plane. [Hyperbolic]
2. $X = \mathbb{C}$, the complex plane. [Euclidean]
3. $X = \mathbb{P}^1$, the Riemann sphere. [Spherical]

Before we give the precise statement, we recall the following topological preliminary:

Definition 0.1. *Let X be a Riemann surface. We say that X is* connected *if whenever X is written as the disjoint union of two open subsets $U, V \subset X$, then necessarily exactly one of U, V is empty.*

Thus if X is non-connected and non-empty, we have a decomposition $X = U \coprod V$ of X into two disjoint smaller Riemann surfaces. Then the study of X can be reduced to the separate study of U and V. In fact, every Riemann surface is uniquely the disjoint union of its *connected components*, which are also Riemann surfaces. In this way there is no real loss of generality in only considering connected Riemann surfaces.

An open subset $U \subset \mathbb{C}$ is connected in the above sense if and only if it so in the sense defined earlier, using paths. So there is no ambiguity in the terminology.

All three model Riemann surfaces above are connected, and the continuous image of any connected space is connected. Thus any quotient of any of the model spaces is also connected. The uniformization theorem gives the remarkable converse:

Theorem 0.2. *Let X be any connected Riemann surface. Then exactly one of the following possibilities holds:*

1. *There exists a group $\Gamma \subset Aut(\mathbb{D})$ acting freely and properly on $\mathbb{D}$ an isomorphism $X \cong \mathbb{D}/\Gamma$.*
2. *There exists a group $\Gamma \subset Aut(\mathbb{C})$ acting freely and properly on $\mathbb{C}$ an isomorphism $X \cong \mathbb{C}/\Gamma$.*
3. *There exists a group $\Gamma \subset Aut(\mathbb{P}^1)$ acting freely and properly on $\mathbb{P}^1$ an isomorphism $X \cong \mathbb{P}^1/\Gamma$.*

There are several remarks to make about this theorem; let me make a few of them now. The first is that, in the second and third cases, it's actually very easy to explicitly describe all the possible Γ acting freely and properly. For $\mathbb{P}^1$, it turns out that the only possibility is $\Gamma = \{id\}$, so that $X = \mathbb{P}^1$. That's kind of a degenerate case. For $\mathbb{C}$, either $\Gamma = \{id\}$ or Γ is generated by translation by some nonzero vector, or Γ is generated by translation by two $\mathbb{R}$-linearly independent vectors. This also leaves relatively few possibilities for the quotient.

On the other hand, for $\mathbb{D}$ there is a whole plethora of possibilities for Γ. For example, you can look at any of M. C. Escher's hyperbolic tessellations. The hyperbolic analogs of translations shifting these tessellations give a possible group Γ. You could say that Escher was drawing Riemann surfaces.

Intuitively, you can understand this heirarchy as follows: spherical geometry closes in on itself, and doesn't leave much room to play. Plane geometry is flat, and has a little more room to play. But hyperbolic geometry opens up wildly, leaving lots of room to play.

Speaking of geometry, another remark to make is that in all three cases one can see that such a Γ must in fact act by *isometries* in the relevant geometry. This is not something to take for granted: being an automorphism of the model Riemann surface a priori only guarantees that Γ preserves angles. But in fact it also preserves the relevant notion of distance as well. A corollary is the following:

Corollary 0.3. *Every connected Riemann surface can be given the structure of $\mathcal{G}$-surface, for $\mathcal{G}$ one of $\{$hyperbolic, euclidean, spherical$\}$.*

Here is one last remark. Note that the uniformization theorem is a theorem of the type, "every so-and-so is isomorphic to one of these models". But if we really have a so-and-so and want to identify it with one of the models, we need to *choose* an isomorphism. Then it's good to ask, how unique is the choice of isomorphism? Can we parametrize all the possible choices?

There is a very general answer to this question, which is that if we fix a choice of isomorphism between X and a model M, then the set of all other choices, i.e. all isomorphisms between X and some model M' (maybe the same as M, maybe different), is in bijection with the set of isomorphisms from M to M'.

Thus we can study the ambiguity in the choice of isomorphism, simply by studying the isomorphisms between the different models. In our case these automorphisms also have a nice description:

Theorem 0.4. *Let $\mathbb{M}$ be one of the three models $\mathbb{D}, \mathbb{C}, \mathbb{P}^1$, and let $\Gamma, \Gamma' \subset \mathbb{M}$ be two groups acting freely and properly on $\mathbb{M}$. Then the set of isomorphisms from $\mathbb{M}/\Gamma$ to $\mathbb{M}/\Gamma'$ is in natural bijection with the set of $\alpha \in Aut(\mathbb{M})$ such that $\alpha \cdot \Gamma \cdot \alpha^{-1} = \Gamma'$, taken up to the equivalence relation that $\alpha \sim \alpha'$ if $\alpha' = \alpha \circ \gamma$ for some $\gamma \in \Gamma$.*

More precisely, every isomorphism between quotients *lifts* to an automorphism of $\mathbb{M}$, and the only constraint on this automorphism is the evident one written, which expresses that the image of $\alpha(m)$ in the second quotient only depends on the image of m in the first quotient. And the ambiguity in the lift is multiplication by some $\gamma \in \Gamma$.

There is another geometric corollary. Outside of two special cases (namely $\Gamma = \{id\}$ acting on $\mathbb{C}$ or $\mathbb{P}^1$), every possible such α also acts by isometries on $\mathbb{M}$. Thus, not only can every Riemann surface be given a geometric structure, but the Riemann surface isomorphisms between these are, outside two special cases, exactly the same as the isometries (geometric isomophisms) between them. So the ambiguities in the choice of isomorphism to a model can also be understood completely in terms of geometry. In this sense we can very nearly say that the study of Riemann surfaces is the same as the study of geometric surfaces, either hyperbolic, euclidean, or spherical.

Next we discuss the algebraization theorem. This theorem applies to a more restrictive class of Riemann surfaces than the uniformization theorem. Here is the relevant definition.

Definition 0.5. *Let S be a subset of a Riemann surface X. We say that S is* compact *if whenever we have a collection $\{U_i\}_{i \in I}$ of open subsets of X such that S is contained in the union of the U_i over all $i \in I$, then there exists a finite subset $J \subset I$ such that S is contained in the union of only those U_i with $i \in J$.*

We say that X is compact if it is compact when viewed as a subset of itself.

Some examples: an open disk in the complex plane is non-compact. The complex plane is non-compact. A closed disk is compact. The Riemann sphere is compact. Intuitively, compactness signifies that you can't "wander off to infinity" in your space, without having a point in the space to end up at.

The following is not the algebraization theorem, but it is a theorem which gives a preliminary picture for compact Riemann surfaces.

Theorem 0.6. *Let X be a compact connected Riemann surface. Then, as a* differentiable *surface, X is isomorphic to some g-holed torus for a unique $g = 0, 1, 2, \ldots$.*

Recall that there are many more diffeomorphisms between open subsets of the plane than there are biholomorphisms. So it's reasonable to think that many more Riemann surfaces can be isomorphic as differentiable surfaces, than as Riemann surfaces. And indeed, except in the case $g = 0$, where the only possibility up to isomorphism is $X = \mathbb{P}^1$, there is always an uncountably infinite number of pairwise non-isomorphic Riemann surfaces which are all differentiably isomorphic to a g-holed torus (and hence to each other.) But the issue is not as wild as I'm making it sound: the collection of such isomorphism classes of Riemann surfaces can be organized into a beautiful finite dimensional space, the so-called *moduli space of curves of genus g*.

Now let's move towards stating the algebraization theorem. Here is another definition.

Definition 0.7. *Let X be a connected Riemann surface. A* meromorphic function *on X is by definition a holomorphic map $X \to \mathbb{P}^1$ which is not equal to the constant map with value ∞. The set of meromorphic functions on X is denoted $\mathcal{M}(X)$.*

Among the meromorphic functions are the holomorphic functions, which are just the holomorphic maps $X \to \mathbb{C} \subset \mathbb{P}^1$. But there are more meromorphic functions. The local picture is as follows: locally, X is isomorphic to a connected open subset of $\mathbb{C}$; and there a meromorphic function is just a quotient of two holomorphic functions, such that the denominator is not identically zero.

The set $\mathcal{M}(X)$ carries some natural algebraic structure. Namely, meromorphic functions can be added, multiplied and subtracted; and we can also divide by nonzero meromorphic functions. In total, the structure obtained is that of a *field*. More precisely, it is a *field over* $\mathbb{C}$: a field which contains $\mathbb{C}$ as a subfield (consider the constant functions).

In general, the field $\mathcal{M}(X)$ is huge. For example, when $X = \mathbb{C}$, we have not only the rational functions of z, but also $exp(z)$, rational functions of this, and even worse, e.g. we can keep nesting exponentials however many times we want. The problem is that we have no control over what happens at ∞. In contrast, when X is compact the field $\mathcal{M}(X)$ is fairly tame. This is a theorem which is logically only a preliminary to the statement of the algebraization theorem, but in fact constitutes the major part of the difficulty of the algebraization theorem:

Theorem 0.8. *Let X be a compact connected Riemann surface. Then the field $\mathcal{M}(X)$ is finitely generated and of transcendence degree one over $\mathbb{C}$.*

The meaning of this is that there exist two functions $z, f \in \mathcal{M}(X)$ such that:

1. The element z is transcendental: for every nonzero polynomial P with $\mathbb{C}$-coefficients, $P(z) \neq 0$ inside $\mathcal{M}(X)$.

2. The element f is algebraic over $\mathbb{C}(z)$: there exists a nonzero polynomial P whose coefficients are rational functions in z and such that $P(f) = 0$ inside $\mathcal{M}(X)$.

3. The two elements z and f generate $\mathcal{M}(X)$: every $g \in \mathcal{M}(X)$ can be written as a rational function (quotient of polynomials) of the two elements z and f.

Here are two examples. In the case $X = \mathbb{P}^1$, we have $\mathcal{M}(X) = \mathbb{C}(z)$, the set of rational functions in one variable z (which corresponds to the identity map from $\mathbb{P}^1$ to itself). Thus, e.g. the exponential function does not extend to a meromorphic function on $\mathbb{P}^1$: it is not meromorphic at ∞.

For another example, recall the Riemann surface X associated to a polynomial $P(z,w)$ in two variables: X was the set of smooth points of the zeroes of P in $\mathbb{C}^2$. There are two natural meromorphic functions on X: the projection to the z-axis, which we call z, and the projection to the w-axis, which we call w. Then (for generic choices of P) the element $z \in \mathcal{M}(X)$ is transcendental, and the element w is algebraic over $\mathbb{C}(z)$, since indeed $P(z,w) = 0$ inside $\mathcal{M}(X)$. Now, these two elements do not generate $\mathcal{M}(X)$: since X is not compact, we have a similar problem to what we had with $\mathbb{C}$. However, there exists a natural *compactification* $\overline{X}$ of X, obtained by adding finitely many points; and the functions z and w extend to meromorphic functions on $\overline{X}$, and indeed generate $\mathcal{M}(\overline{X})$.

Now we are ready to state the algebraization theorem. It says, in essence, that a compact Riemann surface is totally controlled by its field of meromorphic functions.

Theorem 0.9. *Let X and Y be two compact connected Riemann surfaces. Consider the set $\{X \to Y\}$ of all non-constant holomorphic maps $X \to Y$, and the set $\{\mathcal{M}(Y) \to \mathcal{M}(X)\}$ of all field homomorphisms $\mathcal{M}(Y) \to \mathcal{M}(X)$ which restrict to the identity on $\mathbb{C}$. Then the natural map*

$$\{X \to Y\} \to \{\mathcal{M}(Y) \to \mathcal{M}(X)\}$$

is a bijection.

The natural map is as follows: given an $f : X \to Y$, by composing with f we can go from meromorphic functions on Y to those on X.

An immediate corollary is the following, which reduces classification of compact Riemann surfaces to an algebraic problem:

Corollary 0.10. *Two compact connected Riemann surfaces are isomorphic if and only if their fields of meromorphic functions are isomorphic as fields over $\mathbb{C}$.*

In fact, we get something stronger: the set of isomorphisms $X \cong Y$ is in exact bijection with the set of isomorphisms $\mathcal{M}(X) \cong \mathcal{M}(Y)$. Thus, just as with the uniformization theorem, we can use algebra not just to parametrize the isomorphism classes of Riemann surfaces, but also to understand the isomorphisms themselves.

If it's not obvious to you why this all follows from the theorem, you should definitely take the time to prove it. It comes a useful abstract fact ("functors preserve isomorphisms; fully faithful functors preserve and detect isomorphisms.").

There is also a companion theorem to the algebraization theorem, which shows that you can also go backwards from algebra to Riemann surfaces. In practice, both directions are actually pretty useful.

Theorem 0.11. *Let F be any field which is finitely generated and of transcendence degree one over $\mathbb{C}$. Then there exists a compact connected Riemann surface X such that $\mathcal{M}(X)$ is isomorphic to F as a field over $\mathbb{C}$.*

Actually, we will do slightly better than what's suggested by the statement of this theorem. In fact, given such a field F, we will directly and canonically construct the corresponding Riemann surface X. Then if you start with a compact connected Riemann surface X and apply this procedure to the field $\mathcal{M}(X)$, you will recover a purely algebraic model for the original Riemann surface X. That is why the word "algebraization" is used.