Lecture 6: Local structure of holomorphic maps.

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Today we will talk about the local structure of holomorphic maps between Riemann surfaces. There will be two theorems: one which holds for arbitrary maps, and another stronger one which holds only for *proper* maps.

We can already state and prove the first one:

Theorem 0.1. *Let $f : X \to Y$ be a holomorphic map between Riemann surfaces. Let also $x \in X$, with image $y := f(x)$ in Y. Suppose that it is not the case that f is constant in some neighborhood of x. Then there exist a natural number $n \in \{1, 2, \ldots\}$ and charts around x and y in which f is represented by the holomorphic map*

$$z \mapsto z^n$$

from an open disk around 0 in $\mathbb{C}$ to an open disk around 0 in $\mathbb{C}$, where the 0 in the source corresponds to the point x and the 0 in the target corresponds to y.

Thus, every (non-constant) holomorphic map looks locally like $z \mapsto z^n$ for some n.

Proof. We start by representing f by charts in an arbitrary manner, but making sure to translate the charts so that x and y both correspond to $0 \in \mathbb{C}$. Then f in the charts admits a power series expansion:

$$f(z) = c_0 + c_1 \cdot z + \ldots.$$

Since f is not constant in any neighborhood of x, not all the coefficients c_i are zero; therefore there is some smallest nonzero coefficient, call it c_n. It follows that we can write

$$f(z) = z^n \cdot g(z),$$

where $g(z)$ is a holomorphic function with $g(0) \neq 0$. Since $g(0) \neq 0$, we know exists is a holomorpic n^{th} root function defined in a neighborhood of $g(0)$. Since g is continuous, for small enough values of z the value $g(z)$ will lie in this neighborhood. It follows that on a neighborhood of 0 we can write

$$f(z) = z^n \cdot h(z)^n,$$

where $h(z)$ is a holomorphic function with $h(0) \neq 0$. Now, let $\varphi(z) = z \cdot h(z)$. Then $\varphi'(0) \neq 0$, so by the inverse function theorem φ is a biholomorphism in a neighborhood of zero. If we use this biholomorphism to change the chart on the source, then f gets represented by the function

$$f(z) = z^n,$$

as desired. $\square$

Note that the proof showed how to find the number n: if ever you locally write f as $z^m \cdot g(z)$ with $g(0) \neq 0$, then $n = m$. But the proof doesn't exactly show that the number n depends only on f and the point $x \in X$, since the power series expansion of f depends on the chart chosen. However, it is true that n is uniquely determined by f and x. One could check this just by investigating the effect of a change of coordinates, but there's also a slicker way: n can be characterized as the unique natural number such that there exists a neighborhood U of x such that for any $x' \in U$ different from x, the cardinality of $f^{-1}(f(x')) \cap U$ is equal to n. (This follows from the above theorem, together with the fact that there are exactly n n^{th} roots of any nonzero complex number.) In other words, the map f is n-to-1 in a neighborhood of x.

Since the n only depends on f and x, we give it notation that reflects this: we call it $v_x(f)$. In words, it is the *valuation (or multiplicity) of f at x.*

The above theorem has a number of corollaries. The first concerns the set of points which were excluded from the statement, the ones where f is constant in a neighborhood.

Corollary 0.2. *Let $f : X \to Y$ be a non-constant holomorphic map of Riemann surfaces, with X connected. Then there are no points $x \in X$ such that f is constant in a neighborhood of x.*

Thus, for most maps that we'll want to consider, the above theorem applies without qualification: around every point f is represented by some $z \mapsto z^n$.

Proof. Suppose for contradiction that f takes the constant value $C \in Y$ in a neighborhood of x_0. Let S denote the set of points $x \in X$ such that $f = C$ in a neighborhood of x. I claim S is both open and closed (closed means, the complement is open). That S is open is clear: if $x \in S$ then every point of the neighborhood of x on which we know $f = C$ will also lie in S. As for closedness, let $\overline{S}$ denote the closure of S (intersection of all closed subsets containing S), and let $x \in \overline{S}$. I claim first that f is constant in a neighborhood of x. Indeed, otherwise we can apply the theorem and see that f is represented by $z \mapsto z^n$ in a neighborhood of x. But that can't be: every neighborhood of x intersects S, and hence f takes the same value at infinitely many points in any chart around x. This is clearly not the case for the map $z \mapsto z^n$.

Thus f is constant in a neighborhood of x. But every neighborhood of x intersects S, so this constant value must be C. And hence $x \in S$. We conclude that S is closed in addition to being open.

On the other hand, S is nonempty since $x_0 \in S$. Thus, by definition of connectedness, we must have $S = X$, and so f is globally constant, a contradiction. □

Note the similarity of this corollary to the identity principle for connected open subsets of $\mathbb{C}$, which we proved using paths. In fact, either style of proof works to prove either statement. I gave this one here for variety.

Another corollary is a long-promised fact:

Corollary 0.3. *Let $f : X \to Y$ be a holomorphic map between Riemann surfaces. If f is a bijection, then it is a isomorphism (biholomoprhism).*

Proof. We need to see that the inverse of f is holomorphic. Actually, it's enough to see that f is a local isomoprhism: then the inverse will be locally holomorphic, hence holomorphic. For that, we apply the theorem. Certainly f, being injective, is not constant in a neighborhood of any point. Thus we can everywhere locally represent f by $z \mapsto z^n$ for some n. But in fact, again because of injectivity, n must be equal to 1. Therefore f is locally represented by the identity, and hence is certainly a local isomorphism. □

The same proof shows the following stronger fact: if we just assume f is holomorphic and injective, then the image $f(X)$ is open in Y, and f gives an isomorphism from X to $f(X)$.

Note that the corollary is interesting even for open subsets of the complex plane; yet the proof used Riemann surface ideas.

Now we work towards the second theorem. The difference between the two theorems will be the following: in the first theorem, we had to work in a chart around a point of the *source*. But in the second theorem, we will only work in a chart around a point of the *target*, and we'll get a full description of the restriction of our map to the preimage of this chart. But this requires stronger assumptions on f. The key concept is the following:

Definition 0.4. *Let $f : X \to Y$ be a holomorphic map of Riemann surfaces. We say that f is* proper *if for every compact subset $K \subset Y$, the inverse image $f^{-1}(K)$ is a compact subset of X.*

Here are some "abstract" examples:

1. If X is compact, then every holomorphic map $f : X \to Y$ to an arbitrary Y is proper. Indeed, this is essentially an exercise in point-set topology: if $K \subset Y$ is compact, then since Y is Hausdorff, K is closed. Then since f is continuous, $f^{-1}(K)$ is a closed subset of X. But a closed subset of a compact space is compact.

2. If $f : X \to Y$ is proper and $U \subset Y$ is an arbitrary open subset, then $f^{-1}(U) \to U$ is also proper. This is clear from the definitions.

The intuition is that a proper map "sends ∞ to ∞". In other words, if you wander off towards a point that isn't there in X, then your image under f should wander off to a point that isn't there in Y. To get an example of a map which is not proper, you can take your favorite proper map and just remove a point from the source. Also, the exponential map $exp : \mathbb{C} \to \mathbb{C}$ is not proper, since if you let $Im(z) \to -\infty$ the value $exp(z)$ stays close to 0. (Formally, the inverse image of a closed disk around 0 is not bounded, hence not compact.)

Here is a more concrete example: every non-constant polynomial map $f : \mathbb{C} \to \mathbb{C}$ is proper. It's possible to prove this straight from the definition: since the compact subsets of $\mathbb{C}$ are exactly the closed and bounded subsets, one needs only see that f^{-1} sends closed and bounded sets to closed and bounded sets. But we'll adopt another approach, which will eventually give more information.

The key is that we can extend f to a holomorphic map $\mathbb{P}^1 \to \mathbb{P}^1$ by setting $f(\infty) = \infty$. This will imply the claim, since the source $\mathbb{P}^1$ is compact and we recover the original f by restricting to the open subset $\mathbb{C}$ of the target $\mathbb{P}^1$.

So let us explain why this extended f is holomorphic. Certainly, it is holomorphic at every point of $\mathbb{C} \subset \mathbb{P}^1$, since there it's given by the original polynomial. Thus we need to check holomorphicity at ∞. For this, we should look on the chart at ∞. On both the source and target $\mathbb{P}^1$'s this chart is given by $z \mapsto 1/z$ (with $\infty \mapsto 0$). Thus we need to check that the function $g(z)$ defined for $z \neq 0$ by

$$g(z) = 1/f(1/z)$$

and for $z = 0$ by

$$g(0) = 0,$$

is holomorphic at 0. But since f is a nonconstant polynomial, we can write

$$f(z) = c_n z^n + \ldots + c_0$$

with $c_n \neq 0$ and $n \geq 1$. Then

$$g(z) = \frac{z^n}{c_n + c_{n-1}z + \ldots + c_0 z^n},$$

for $z \neq 0$ as well as for $z = 0$. But now this expression for g is clearly holomorphic (complex differentiable) in a neighborhood of 0: it is the quotient of two holomorphic functions, such that the denominator doesn't vanish at 0.

We remark at the same time that this expression for g shows that $v_\infty(f) = n = deg(f)$: the multiplicity of a (nonconstant) polynomial at ∞ is equal to its degree.

Now, to work towards the second theorem, let's prove some lemmas about proper maps.

Lemma 0.5. *Let $f : X \to Y$ be a non-constant proper map between Riemann surfaces, with X connected. For every $y \in Y$, the fiber $f^{-1}(y)$ is finite.*

Proof. Since $\{y\}$ is compact and f is proper, the fiber is compact. Thus, to show it is finite, it is enough to show that it's *discrete*, i.e. every point $x \in f^{-1}(y)$ has an open neighborhood which contains no other points of $f^{-1}(y)$. But by the local structure theorem, we can find a chart around x in which f looks like $z \mapsto z^n$ for some n. Note, however, that the fiber above 0 of that latter map is just the single point 0. We deduce that such a chart gives an open neighborhood of x meeting the specifications. $\square$

Lemma 0.6. *Let $f : X \to Y$ be a proper map between Riemann surfaces, and let $y \in Y$. If U is any open neighborhood of $f^{-1}(y)$, then there is an open neighborhood V of y such that $f^{-1}(V) \subset U$.*

Pictorally speaking, the fibers above all nearby points "fall into" the fiber above y.

Proof. Fix a chart around y, and use this to define the notion of "closed disk in Y centered at y". Let D be one such closed disk. Then D is compact; and since f is proper, $f^{-1}(D)$ is then also compact. Now consider the collection $\{D_i\}_{i \in I}$ of all closed disks centered at y and contained in D. The intersection of all of these disks is $\{y\}$: we can check this on the chart. It follows that U together with all of the $f^{-1}(Y \setminus D_i)$ form an open cover of Y, hence of $f^{-1}(D)$. But this latter is compact, so we have a finite subcover, say by U together with $f^{-1}(Y \setminus D_i)$ for i in a finite set I_0. Now, let V be any open disk centered at y and contained in all of the D_i for $i \in I_0$. It follows that $f^{-1}(V) \subset U$, since certainly no point of $f^{-1}(V)$ can lie in any $f^{-1}(Y \setminus D_i)$. $\square$

Now we can state the theorem.

Theorem 0.7. *Let $f : X \to Y$ be a proper map of Riemann surfaces which is not constant on any connected component of X. Let also $y \in Y$, and denote by $\{x_1, \ldots, x_k\}$ the fiber $f^{-1}(y)$ (it is a finite set, by the first lemma).*

Then there exists a chart around y (making y correspond to $0 \in \mathbb{C}$) and charts around each x_i (making x_i correspond to $0 \in \mathbb{C}$) with the following property: the inverse image of the chart around y is equal to the disjoint union of all the charts around the x_i's, and moreover in each chart around x_i the function f is represented by some $z \mapsto z^n$ (so, $n = v_{x_i}(f)$).

So we have a complete picture of what happens "above" our given chart around y, if we think of f as mapping from upstairs to downwards.

Proof. We start by selecting charts around each x_i as in the (first) local structure theorem. Note, e.g. from the proof of the local structure theorem, that we can use the same chart around y for all these different x_i. By shrinking the charts, we can also use the Hausdorff property of X to guarantee that these charts around the x_i are disjoint.

Now, by the second lemma we can, after shrinking the chart around y, assume that the inverse image of the chart around y is contained in the disjoint union of the charts around the x_i. To guarantee equality instead of just containment, we simply further shrink the charts around the x_i, replacing them with their intersection with the inverse image of the chart around y. $\square$

There is a fun corollary of this theorem.

Corollary 0.8. *$f : X \to Y$ be a non-constant proper map between connected Riemann surfaces. For every $y \in Y$, define a natural number*

$$deg_y(f) := \sum_{x \in f^{-1}(y)} v_x(f).$$

Then $deg_y(f)$ is independent of y. (It can thus be denoted $deg(f)$ and called the degree of f*.)*

Proof. We will show that the function $Y \to \mathbb{N}$ given by $y \mapsto deg_y(f)$ is continuous, i.e. the inverse image of every singleton $\{d\}$ is open in Y. Since Y is connected, the claim will follow from this.

Thus, suppose that we have a y with $deg_y(f) = d$. We want to show that every y' near y also satisfies $deg_{y'}(f) = d$. By the above structure theorem, we can assume that Y is a disk and X is a disjoint union of disks, indexed by the $x \in f^{-1}(y)$, where on the x^{th} disk the map f is given by $z \mapsto z^{v_x(f)}$. But then every point $y' \neq y$ has exactly $v_x(f)$ preimages in the x^{th} disk for all x, since there are exactly n n^{th} roots of any nonzero complex number. Summing over all x, we find that there are exactly $deg_y(f)$ preimages of y' under the map f. On the other hand the multiplicity at each of these preimage points is equal to 1, since $z \mapsto z^n$ is a local isomoprhism away from 0. This implies the claim, $deg_{y'}(f) = deg_y(f)$, and finishes the proof. $\square$

There is another fun corollary of the corollary. Consider again our non-constant polynomial, viewed as a map $f : \mathbb{P}^1 \to \mathbb{P}^1$. Then $deg_\infty(f)$ is the degree of the polynomial f, since the only point in the preimage of ∞ is ∞ and the multiplicity of f there is the degree of f, as we saw earlier. But on the other hand $deg_0(f)$ is, by definition, the number of zeros of the polynomial f, counted with multiplicity. Thus we deduce the fundamental theorem of algebra: a nonzero polynomial has exactly as many roots as its degree, if we count these roots with multiplicity.